Solucion analitica

Punto 0.2

Para mostrar la discretizacion tomamos de entrada cosas que ya conocemos

$$\boxed{\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i-1,j}}{2hx}}$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2 x}}$$

Por otro lado del enunciado conocemos que

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \omega}$$

Ahora, reemplazando las dos primeras ecuaciones vamos a tener que

$$\boxed{\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} = \omega_{i,j}}$$

$$-4u_{i,j} = -u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} + h^2\omega_{i,j}$$

$$\boxed{u_{i,j} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} + h^2\omega_{i,j}\right)}$$

Ahora aplicando el operador nabla para ω vamos a tener que

$$\boxed{\nu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) = \frac{\partial u}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial \omega}{\partial y}}$$

Ahora reemplazamos los dos primeros calculos en la ecuación anterior

$$\nu\left(\frac{\omega_{i+1,j} - 2\omega_{i,j} + \omega_{i-1,j}}{h^2} + \frac{\omega_{i,j+1} - 2\omega_{i,j} + \omega_{i,j-1}}{h^2}\right) = \frac{[u_{i,j+1} - u_{i,j-1}][\omega_{i+1,j} - \omega_{i-1,j}]}{4h^2} - \frac{[u_{i+1,j} - u_{i-1,j}][\omega_{i,j+1} - \omega_{i,j-1}]}{4h^2}$$

$$-4\nu\omega_{ij} = \frac{[u_{i,j+1} - u_{i,j-1}][\omega_{i+1,j} - \omega_{i-1,j}]}{4} - \frac{[u_{i+1,j} - u_{i-1,j}][\omega_{i,j+1} - \omega_{i,j-1}]}{4} - \nu(\omega_{i+1,j} - \omega_{i-1,j} - \omega_{i,j+1} - \omega_{i,j-1})$$

$$\omega_{i,j} = \frac{R}{16}(u_{i+1,j} - u_{i-1,j})(\omega_{i,j+1} - \omega_{i,j-1}) - \frac{R}{16}(u_{i,j+1} - u_{i,j-1})(\omega_{i+1,j} - \omega_{i-1,j}) + \frac{1}{4}(\omega_{i+1,j} + \omega_{i-1,j} + \omega_{i,j+1} + \omega_{i,j-1})$$

finalmente, podemos evidenciar que

$$\boxed{u_{i,j} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} + h^2\omega_{i,j}\right)}$$

$$\boxed{\omega_{i,j} = \frac{R}{16}(u_{i+1,j} - u_{i-1,j})(\omega_{i,j+1} - \omega_{i,j-1}) - \frac{R}{16}(u_{i,j+1} - u_{i,j-1})(\omega_{i+1,j} - \omega_{i-1,j}) + \frac{1}{4}(\omega_{i+1,j} + \omega_{i-1,j} + \omega_{i,j+1} + \omega_{i,j-1})}$$

Punto 2

Primeramente partimos de la ecuación mostrada en el enunciado de la tarea

$$\omega = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}$$

$$= \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2}$$

Por condiciones de frontera conocemos v_y = constante por lo tanto vamos a tener que

$$\boxed{\omega = -\frac{\partial^2 u}{\partial y^2}}$$

Ahora aplicando expansión por series de Taylor para la funcion inicial

$$u(x, y+h) = u(x,y) + \frac{\partial u}{\partial y} + \frac{h^2}{2}\frac{\partial^2 u}{\partial y^2} + \Theta(h^3)$$

Ahora, con la condicion de frontera

$$u(x, y+h) = u(x,y) - \frac{h^2}{2}\omega_z = -\frac{2u(x,y+h) - u(x,y)}{h^2}$$

Por lo tanto vamos a tener que

$$\boxed{\omega_{i,j(R)} = -2\frac{u_{i,j+1} - u_{i,j}}{h^2}}$$