1 Grid Minors of Graph Products

1.1 Introduction

A key theorem in the area of graph minor theory is the *Grid Minor Theorem* (also known as the *Excluded Grid Theorem*) of Robertson and Seymour [8], which shows that there exists a function f such that for every integer k, every graph with treewidth at least $f(k)$ contains a $k \times k$ grid minor. This result gave a new way to analyze grid structures in graphs and had widespread impact on structural graph theory research.

This result led to further investigation into what the best possible bounds were for the function f, with the current state of the art result being a lower bound of $\Omega(k^2 \log k)$ from Robertson et al. [9] and the best known upper bound of Chuzhoy and Tan which shows that $f \in O(k^9 \log^{O(1)} k)$[1].

For special classes of graphs, much stronger bounds on $f(k)$ have been shown. One such example is the *Excluded Grid Theorem for Planar Graphs*, which asserts that planar graphs have the *linear grid minor property*, i.e., that for any planar graph, $f(k) \in O(k)$. This property has been used to devise efficient polynomial time approximation schemes for many NP-hard problems on planar graphs and related graph families[2–4, 6, 7]. The $\Omega(n^2 \log n)$ lower bound mentioned above shows that general graphs do no have the linear grid minor property: there are graphs of treewidth $k^2 \log k$ whose largest grid minor is of size $O(k) \times O(k)$ [9].

However, For general graph products, no results are known that are stronger than the bounds gained from the Grid Minor Theorem. It is known that for two n-vertex graphs G_1 and G_2, $\mathrm{tw}(G_1 \square G_2) \geq n^1$, combining this with the state of the art grid minor theory upper bound above shows general graph products have a $\Omega(n^{1/9}/\operatorname{polylog}(n) \times \Omega(n^{1/9}/\operatorname{polylog}(n))$ grid minor. We investigate whether there exists a better Grid Minor Theorem for graph products, and answer the question in the affirmative, improving the bound to $\Theta(\sqrt{n})$ for Cartesian, strong, and lexicographic graph products.

1.2 Results

We improve the grid minor theorem for graph products in two steps. To do this, we show that:

1. For any two n-vertex connected graphs G_1 and G_2

$$\mathrm{gm}(G_1 \cdot G_2) \geq \mathrm{gm}(G_1 \boxtimes G_2) \geq \mathrm{gm}(G_1 \square G_2) \in \Omega(\sqrt{n}) \qquad (1)$$

2. There exists two n-vertex connected graphs G_1 and G_2 (a star and any tree) such that

$$\mathrm{gm}(G_1 \square G_2) \leq \mathrm{gm}(G_1 \boxtimes G_2) \leq \mathrm{gm}(G_1 \cdot G_2) \in O(\sqrt{n}) \qquad (2)$$

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[1] A proof of this fact can be found in [5], though the authors make no claim to being the first to show this result.

To show Equation (2), we make use of the following lemma

Lemma 1. *Let S be any star and T be any tree. Let G be any graph with maximum degree Δ and minimum degree at least 3. If G is a minor of $S \cdot T$ then $|V(G)| < (\Delta + 1)|V(T)|$.*

Proof. For disjoint $A, B \subseteq V(G)$, let $e(A, B)$ be the number of edges in G between A and B. Let $(B_x : x \in V(G))$ be a model of G in $S \cdot T$. Say $V(S) = \{a_0, a_1, \ldots, a_n\}$ where a_0 is the root of S. Let R be the set of vertices x of G such that $(a_0, b) \in V(B_x)$ for some $b \in V(T)$. So $|R| \le |V(T)|$. For $i \in \{1, \ldots, n\}$, let X_i be the set of vertices x of G such that $V(B_x) \subseteq \{(a_i, b) : b \in V(T)\}$. By the definition of lexicographic product, $R, X_1, \ldots, X_n$ is a partition of $V(G)$, and no edge of G joins distinct X_i and X_j. For each $i \in \{1, \ldots, n\}$, by construction, $G[X_i]$ is a minor of T, implying $G[X_i]$ is a forest. Since G has minimum degree at least 3,

$$3|X_i| \le \sum_{v \in X_i} \deg_G(v) = e(R, X_i) + 2|E(G[X_i])| < e(R, X_i) + 2|X_i|.$$

Hence $e(R, X_i) > |X_i|$. On the other hand, $e(R, X_1 \cup \cdots \cup X_n) \le \Delta|R|$ since G has maximum degree Δ. Hence

$$|V(G)| - |R| = \sum_{i-1}^{n} |X_i| < \sum_{i=1}^{n} e(R, X_i) = e(R, X_1 \cup \cdots \cup X_n) \le \Delta|R|,$$

implying $|V(G)| < (\Delta + 1)|R| \le (\Delta + 1)|V(T)|$, as claimed. $\qquad\square$

Now, we prove our result by constructing a graph G with $\mathrm{gm}(G) \in O(n)$. Let $G = S_n \cdot T$ where T is any n-vertex tree. Now, assume that G contains a $\boxtimes_k$ minor. Let H be the graph obtained by contracting an edge adjacent to each corner of $\boxtimes_k$. Then H has minimum degree 3 and maximum degree 4, satisfying the preconditions for the lemma.

We know that $|V(H)| = k^2 - 4$, thus applying the result of the lemma gives

$$k^2 - 4 \le \Delta|V(T)| = 5n \implies k \le \sqrt{5n + 4}$$

and so $\mathrm{gm}(S \,\square\, T) \le \mathrm{gm}(S \boxtimes T) \le \mathrm{gm}(S \cdot T) \in O(\sqrt{n})$.

The full proof of Equation (1) is long and involved. To summarize the approach taken, we choose to list only the required definitions, and to present the lemmas that build to the result without proof. The proofs of all results, as well as other supplementary information, can be found in the full paper [5]

For a rooted tree T, let $n_i(T)$ be the number of vertices of height i in T, where a vertex v has height i if the longest path with upper endpoint[2] v is of order i.

Lemma 2. *For any positive integer n and any n-vertex connected graph G, $K_n \preceq G \,\square\, S_n$.*

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[2] The upper endpoint of a path P in a rooted tree T is the vertex $v \in P$ with minimum distance to the root

Observation 3. *For any rooted tree T and any $i \in \mathbb{N}$, T contains a set of $n_i(T)$ pairwise completely unrelated vertical paths, each of order i. As a consequence, $S_{n_i(T),i} \preceq T$ for each $i \in \mathbb{N}$.*

Lemma 4. *Let T be a rooted tree with $n \geq 1$ vertices, and let $p \geq 1$ be an integer such that $n_i(T) \leq \frac{3}{2}n/(\pi i)^2$ for each $i \in \{1, \dots, p-1\}$. Then T contains pairwise-disjoint vertical paths $P_1, \dots, P_{\lceil n/4p \rceil}$, each of order p such that, for each $i \neq j$, P_i and P_j are either completely unrelated or completely related.*

Lemma 5. *Let $s, p \geq 1$ be integers, let $\ell := 5s^2$, and let T be a rooted tree that contains s^2 pairwise-disjoint vertical paths, each of order $6p$ such that any pair of these paths is either completely related or completely unrelated. Then $\mathrm{gm}(T \square S_{\ell,2p}) \geq sp$.*

With these three pieces, the full result can be shown.

Theorem 6. *If G_1 and G_2 are connected graphs each having at least $n \geq 1$ vertices, then $\mathrm{gm}(G_1 \square G_2) \in \Omega(\sqrt{n})$*

Proof. For each $b \in \{1,2\}$, let T_b be a tree contained in G_b and having exactly n vertices (which can be constructed by successively deleting leaves starting with a spanning tree of G_b). For each $b \in \{1,2\}$, let $p_b = \min\{i : n_i(T_b) \geq \frac{3n}{2p\pi^2}\}$. (This is well-defined since, otherwise $n = \sum_{p=1}^{\infty} n_p(T_b) < \sum_{p=1}^{\infty} \frac{3n}{2p\pi^2} = \frac{n}{4}$.) Without loss of generality, assume $p_2 \leq p_1$ and let $\ell := \lceil \frac{3n}{2p_2\pi^2} \rceil^2$. By Observation 3, $S_{\ell,p_2} \preceq T_2 \preceq G_2$. If $p_2 < 6$ then $\ell > \frac{n}{4\pi^2} \in \Omega(n)$ and by Lemma 2 $K_\ell \preceq G_1 \square S_\ell$. Since $\boxplus_{\lfloor \sqrt{\ell} \rfloor} \preceq K_\ell$, this implies that $\mathrm{gm}(G_1 \square G_2) \geq \lfloor \sqrt{\ell} \rfloor = \Omega(\sqrt{n})$ and we are done, so we may assume that $p_2 \geq 6$. Let $p := \lfloor p_2/6 \rfloor \geq 1$.

Since $p_1 \geq p_2$, Lemma 4 implies that T_1 contains at least $n/4p$ pairwise disjoint paths $P_1, \dots, P_{\lceil n/4p \rceil}$, each of length $p_2 \geq 6p$, such that each pair of paths is either completely related or completely unrelated. Let

$$s := \lfloor \min\{\sqrt{\ell/5}, \sqrt{n/4p}\} \rfloor = \Theta(\sqrt{n}/p)$$

so that $\ell \geq 5s^2$ and $\lceil n/4p \rceil \geq s^2$. By Lemma 5, $\mathrm{gm}(T_1 \square S_{\ell,6p}) \geq sp = \Theta(\sqrt{n})$. The result now follows from the fact that $T_1 \preceq G_1$, and the fact that $S_{\ell,6p} \preceq S_{\ell,p_2} \preceq G_2$ combined with the observation that for any graphs G_1, G_2, and H, if $G_1 \preceq G_2$, then $G_1 \square H \preceq G_2 \square H$. $\square$

Note that the full paper [5] further proves some exact bounds for special graph products, mainly the product of stars and trees.

References

[1] Julia Chuzhoy and Zihan Tan. Towards tight(er) bounds for the excluded grid theorem. *J. Combin. Theory Ser. B*, 146:219–265, 2021.

[2] Erik D. Demaine, Fedor V. Fomin, MohammadTaghi Hajiaghayi, and Dimitrios M. Thilikos. Subexponential parameterized algorithms on bounded-genus graphs and H-minor-free graphs. *J. ACM*, 52(6):866–893, 2005.

[3] Erik D. Demaine and MohammadTaghi Hajiaghayi. Diameter and treewidth in minor-closed graph families, revisited. *Algorithmica*, 40(3):211–215, 2004.

[4] Erik D. Demaine, MohammadTaghi Hajiaghayi, and Ken-ichi Kawarabayashi. Algorithmic graph minor theory: improved grid minor bounds and Wagner's contraction. *Algorithmica*, 54(2):142–180, 2009.

[5] Vida Dujmović, Pat Morin, David R. Wood, and David Worley. Grid minors and products, 2024.

[6] David Eppstein. Diameter and treewidth in minor-closed graph families. *Algorithmica*, 27(3–4):275–291, 2000.

[7] Fedorr V. Fomin, Daniel Lokshtanov, and Saket Saurabh. Excluded grid minors and efficient polynomial-time approximation schemes. *J. ACM*, 65(2), jan 2018.

[8] Neil Robertson and Paul Seymour. Graph minors. V. Excluding a planar graph. *J. Combin. Theory Ser. B*, 41(1):92–114, 1986.

[9] Neil Robertson, Paul Seymour, and Robin Thomas. Quickly excluding a planar graph. *J. Combin. Theory Ser. B*, 62(2):323–348, 1994.