1 Maintaining Treewidth in Graph Products

1.1 Introduction

With the importance of treewidth and the many applications of product structure theory, a natural question is the following: Can the treewidth be maintained in some meaningful way through taking the product? Solving this problem in any sense would have deep impact on structural graph theory by giving a strong tool to study the treewidth of complex graph classes by using product structure theorems to simplify the problem and give a new approach to improving the treewidth bounds of complicated graph classes.

1.2 Problem 4

Is it true that for every planar graph G, there exists a bounded treewidth graph H and a path P such that $G \precsim H \boxtimes P$ and $\mathrm{tw}(H \boxtimes P) \in O(\mathrm{tw}(G))$?

1.3 Related Work

In our paper regarding Problem 1 [1, Lemma 3, Equation (2)], we show that

$$\Omega(\min\{|V(H)|, |V(P)|\}) \leq \mathrm{tw}(H \boxtimes P) \leq O(\min\{|V(H)|, |V(P)|\}).$$

Thus we can instead ask whether for every planar graph G, there exists a bounded treewidth graph H and a path P such that $G \precsim H \boxtimes P$ and $\min\{|V(H)|, |V(P)|\} \leq O(\mathrm{tw}(G))$.

Little research has been done studying product structures where the full product has bounded treewidth. In fact, it is even open whether $\min\{V(H)|, |V(P)|\} \leq O(f(\mathrm{tw}(G)))$ for some function f, or if $\min\{V(H)|, |V(P)|\} \leq O(\sqrt{|V(G)|})$. The second bound is weaker due to the planar separator theorem[2], which states that $\mathrm{tw}(G) \leq O(\sqrt{|V(G)|}\})$ for every planar graph G.

References

[1] Vida Dujmović, Pat Morin, David R. Wood, and David Worley. Grid minors and products, 2024.

[2] Richard J. Lipton and Robert E. Tarjan. A separator theorem for planar graphs. *SIAM J. Appl. Math.*, 36(2):177–189, 1979.