1 Track Number of Planar Graphs

1.1 Introduction

A *track layout* of a graph G consists of a vertex colouring and a total order on each colour class, such that no two edges between any two colour classes.

The *track number* of a graph is the minimum number of colours needed by a track layout of G.

A *partition* P of a graph G is a set of connected subgraphs of G, such that each vertex belongs to exactly one subgraph.

An *H-partition* of a graph G is a partition of $V(G)$ into disjoint bags $\{A_x : x \in V(H)\}$ indexed by the vertices of a graph H, such that for every edge $(u, v) \in E(G)$ one of the following holds:

1. $u, v \in A_x$ for some $x \in V(H)$ (intra-bag edge)
2. There is an edge $(x, y) \in E(H)$ with $u \in A_x$ and $v \in A_y$. (inter-bag edge)

A *layering* of a graph G is an ordered partition $(V_0, V_1, ...)$ of $V(G)$ such that for every edge $(v, w) \in E(G)$, if $v \in V_i$ and $w \in V_j$, then $|i - j| \leq 1$

The *layered width* of an H-partition of a graph G is the minimum integer l such that for some layering $(V_0, V_1, ...)$ of G we have $|A_x \bigcap V_i| \leq l, \forall x \in V(H), i \geq 0$.

A *BFS-layering* of a graph G is a layering of G such that if r is a vertex in a connected graph G, then $V_i = \{v \in V(G) | \operatorname{dist}_G(r, v) = i\}, \forall i \geq 0$.

For each $f \geq 3, s \geq 1$, a *planar (f, s)-tree* is an embedded planar graph defined recursively as follows: The smallest (f, s)-tree is a 2-connected planar graph on $f + s$ vertices with an embedding where each face (including the outer face) has size at most f. Every embedded graph that can be obtained from a planar (f, s)-tree G by doing the following operation is also a planar (f, s)-tree:

- Pick a face of G, say f, and add a set S of at most s new vertices to f. Add edges between some pairs of vertices of $V(f) \bigcup S$ such that that the resulting graph is 2-connected and each new face has size at most f.

Track layouts have been studied in the context of graph drawings[1, 4] as well as in graph layouts[5], but were formally introduced by Dujmovic, Morin, and Wood[3]. Track layouts see strong applications in three-dimensional low volume graph drawing. In particular, a graph G on n vertices has a 3D, straight-line drawing on a grid of size $O(1) \times O(1) \times O(n)$ if and only if G has constant track number.

1.2 Problem 2

We wish to improve the current best known bound on the track number of planar graphs. To this end, we investigate the best known bounds for the track number of planar (f, s)-graphs as an intermediate step, then use results on layered H-partitions to extend the result to planar graphs.

1.3 Related Work

The current best known bound on the track number of planar graphs is 255 as a result of Pupyrev[6], which was shown by using layered H-partitions directly with planar graphs. This improved the previous bound of 461,184,080, which was a consequence of the planar graph product structure theorem of Dujmovic et al.[2]

References

[1] Emilio Di Giacomo, Giuseppe Liotta, and Henk Meijer. Computing straight-line 3D grid drawings of graphs in linear volume. *Comput. Geom. Theory Appl.*, 32(1):26–58, 2005.

[2] Vida Dujmović, Gwenaël Joret, Piotr Micek, Pat Morin, Torsten Ueckerdt, and David R. Wood. Planar graphs have bounded queue-number. In *Proc. 60th Annual Symp. Foundations Comput. Sci.* (FOCS '19), pages 862–875. IEEE, 2019.

[3] Vida Dujmović, Pat Morin, and David R. Wood. Layout of graphs with bounded tree-width. *SIAM J. Comput.*, 34(3):553–579, 2005.

[4] Stefan Felsner, Giussepe Liotta, and Stephen K. Wismath. Straight-line drawings on restricted integer grids in two and three dimensions. *J. Graph Algorithms Appl.*, 7(4):363–398, 2003.

[5] Lenwood S. Heath and Arnold L. Rosenberg. Laying out graphs using queues. *SIAM J. Comput.*, 21(5):927–958, 1992.

[6] Sergey Pupyrev. Improved bounds for track numbers of planar graphs. *J. Graph Algorithms Appl.*, 24(3):323–341, 2020.