Doctoral Thesis Proposal

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1 Introduction

In this proposal we approach problems relating to structural graph theory, with the main focuses being on graph product structure theory as well as graph minor theory. The four problems we wish to tackle are as follows:

1. What is the size of the largest grid minor that can be found in a general graph product?

2. What is the best possible bound for the track number of planar graphs?

3. Can we embed any planar graph of max degree Δ in the product of $H \boxtimes P \boxtimes K_c$ where H has treewidth 2, P is a path, and c is bounded by a function of Δ?

4. Can we model the treewidth of a graph product as a function of the treewidth of the input graphs?

We begin by introducing structural graph theory background relevant to all four problems of interest. We then solve the first problem by presenting a tight bound on the size of the largest grid minor in the Cartesian, strong, and lexicographic product, presenting a constructive argument to show that the product of two n-vertex connected graphs will contain a $\Theta(\sqrt{n}) \times \Theta(\sqrt{n})$ sized grid minor. We conclude by presenting our preliminary work for problems 2 and 3, and the relevant literature pertaining to problem 4.

2 Background

2.1 Treewidth

A *tree-decomposition* of a graph G is a collection $(B_x : x \in V(T))$ of subsets of $V(G)$ (called *bags*) indexed by the vertices of a tree T, such that for every edge $uv \in E(G)$, some bag B_x contains both u and v, and for every vertex $v \in V(G)$, the set $\{x \in V(T) : v \in B_x\}$ induces a non-empty (connected) subtree of T. The *width* of $(B_x : x \in V(T))$ is $\max\{|B_x| : x \in V(T)\} - 1$.

The *treewidth* of a graph G, denoted by $\text{tw}(G)$, is the minimum width of a tree-decomposition of G.

It is hard to understate the importance of treewidth in the current literature. It is a ubiquitous parameter in structural graph theory, measuring, in loose terms, how close a graph is to a tree. Treewidth was first introduced as *dimension* by Bertelé and Brioschi[1, pp. 37–38] in 1972, then rediscovered by Hadlin[3] in 1976. The parameter was popularized when it was once again rediscovered by Robertson and Seymour[4] in 1984 and has since been at the forefront of structural graph theory research. Graphs of bounded treewidth are of particular interest due to the wide implications of their tree-like structure. For example, Courcelle's Theorem[2] proves that for many problems that are NP-Hard or NP-Complete on general graphs, they can be solved in linear time on graphs of bounded treewidth.

2.2 Graph Product Structure Theory

Another important area of structural theory is Graph Product Structure Theory. This area of research studies complex graph classes by modelling them as a product of simpler graphs and investigating the properties of these highly structured supergraphs to investigate their more complex subgraphs. The three graph products we focus on are the Cartesian product, the strong product, and the lexicographic product.

The *Cartesian product* of two graphs G_1 and G_2, denoted $G_1 \square G_2$, is the graph with vertex set $V(G_1 \square G_2) = V(G_1) \times V(G_2)$ where two distinct vertices (u_1, v_1) and (u_2, v_2) are adjacent iff

- $u_1 = u_2$ and $(v_1, v_2) \in E(G_2)$, or
- $v_1 = v_2$ and $(u_1, u_2) \in E(G_1)$.

The *strong product* of two graphs G_1 and G_2, denoted $G_1 \boxtimes G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ where two distinct vertices (u_1, v_1) and (u_2, v_2) are adjacent iff

- $u_1 = u_2$ and $(v_1, v_2) \in E(G_2)$,
- $v_1 = v_2$ and $(u_1, u_2) \in E(G_1)$, or
- $(u_1, u_2) \in E(G_1)$ and $(v_1, v_2) \in E(G_2)$.

The *lexicograhic product* of two graphs G_1 and G_2, denoted $G_1 \cdot G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ where two distinct vertices (u_1, v_1) and (u_2, v_2) are adjacent iff

- $(u_1, u_2) \in E(G_1)$
- $u_1 = u_2$ and $(v_1, v_2) \in E(G_2)$.

An important property that follows from the above definitions is that

$$G_1 \square G_2 \subseteq G_1 \boxtimes G_2 \subseteq G_1 \cdot G_2.$$

2.3 Grid Minors

The final area of structural graph theory that is of particular interest for this proposal is graph minor theory. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G through a sequence of edge contractions, edge deletions, and vertex deletions. Finding a highly structured minor of a graph G allows us to use properties of the minor to study G itself. In particular, we are interested in studying the largest k such that the $k \times k$ grid graph, denoted $\boxplus_k$, is a minor of G. We denote this number to be $\mathrm{gm}(G)$. The relation between grid minors and graph products is very natural due to the fact that $\boxplus_k$ is the Cartesian product of two k-vertex paths.

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3 Grid Minors of Graph Products

3.1 Introduction

A key theorem in the area of graph minor theory is the *Grid Minor Theorem* (also known as the *Excluded Grid Theorem*) of Robertson and Seymour [7], which shows that there exists a function f such that for every integer k, every graph with treewidth at least $f(k)$ contains a $k \times k$ grid minor. This result gave a new way to analyze grid structures in graphs and had widespread impact on structural graph theory research.

This result led to further investigation into what the best possible bounds were for the function f, with the current state of the art result being a lower bound of $\Omega(k^2 \log k)$ from Robertson et al. [8] and the best known upper bound of Chuzhoy and Tan which shows that $f \in O(k^9 \log^{O(1)} k)$[1].

For special classes of graphs, much stronger bounds on $f(k)$ have been shown. One such example is the *Excluded Grid Theorem for Planar Graphs*, which asserts that planar graphs have the *linear grid minor property*, i.e., that for any planar graph, $f(k) \in O(k)$. This property has been used to devise efficient polynomial time approximation schemes for many NP-hard problems on planar graphs and related graph families[2–6]. The $\Omega(n^2 \log n)$ lower bound mentioned above shows that general graphs do no have the linear grid minor property: there are graphs of treewidth $k^2 \log k$ whose largest grid minor is of size $O(k) \times O(k)$ [8].

However, For general graph products, no results are known that are stronger than the bounds gained from the Grid Minor Theorem. It is known that for two n-vertex graphs G_1 and G_2, $\text{tw}(G_1 \square G_2) \geq n$[], combining this with the state of the art grid minor theory upper bound above shows general graph products have a $\Omega(n^{1/9}/\operatorname{polylog}(n) \times \Omega(n^{1/9}/\operatorname{polylog}(n))$ grid minor. We investigate whether there exists a better Grid Minor Theorem for graph products, and answer the question in the affirmative, improving the bound to $\Theta(\sqrt{n})$ for Cartesian, strong, and lexicographic graph products.

3.2 Results

We improve the grid minor theorem for graph products in two steps. To do this, we show that:

1. For any two n-vertex connected graphs G_1 and G_2
$$\text{gm}(G_1 \cdot G_2) \geq \text{gm}(G_1 \boxtimes G_2) \geq \text{gm}(G_1 \square G_2) \in \Omega(\sqrt{n}) \tag{1}$$

2. There exists two n-vertex connected graphs G_1 and G_2 (a star and any tree) such that
$$\text{gm}(G_1 \square G_2) \leq \text{gm}(G_1 \boxtimes G_2) \leq \text{gm}(G_1 \cdot G_2) \in O(\sqrt{n}) \tag{2}$$

To show Equation (2), we make use the following lemma

Lemma 1. *Let S be any star and T be any tree. Let G be any graph with maximum degree Δ and minimum degree at least 3. If G is a minor of $S \cdot T$ then $|V(G)| < (\Delta+1)|V(T)|$.*

Proof. For disjoint $A, B \subseteq V(G)$, let $e(A,B)$ be the number of edges in G between A and B. Let $(B_x : x \in V(G))$ be a model of G in $S \cdot T$. Say $V(S) = \{a_0, a_1, \ldots, a_n\}$ where a_0 is the root of S. Let R be the set of vertices x of G such that $(a_0, b) \in V(B_x)$ for some $b \in V(T)$. So $|R| \leq |V(T)|$. For $i \in \{1, \ldots, n\}$, let X_i be the set of vertices x of G such that $V(B_x) \subseteq \{(a_i, b) : b \in V(T)\}$. By the definition of lexicographic product, $R, X_1, \ldots, X_n$ is a partition of $V(G)$, and no edge of G joins distinct X_i and X_j. For each $i \in \{1, \ldots, n\}$, by construction, $G[X_i]$ is a minor of T, implying $G[X_i]$ is a forest. Since G has minimum degree at least 3,

$$3|X_i| \leq \sum_{v \in X_i} \deg_G(v) = e(R, X_i) + 2|E(G[X_i])| < e(R, X_i) + 2|X_i|.$$

Hence $e(R, X_i) > |X_i|$. On the other hand, $e(R, X_1 \cup \cdots \cup X_n) \leq \Delta|R|$ since G has maximum degree Δ. Hence

$$|V(G)| - |R| = \sum_{i-1}^{n} |X_i| < \sum_{i=1}^{n} e(R, X_i) = e(R, X_1 \cup \cdots \cup X_n) \leq \Delta|R|,$$

implying $|V(G)| < (\Delta+1)|R| \leq (\Delta+1)|V(T)|$, as claimed. $\square$

Now, we prove our result by constructing a graph G with $\mathrm{gm}(G) \in O(n)$. Let $G = S_n \cdot T$ where T is any n-vertex tree. Now, assume that G contains a $\boxtimes_k$ minor. Let H be the graph obtained by contracting an edge adjacent to each corner of $\boxtimes_k$. Then H has minimum degree 3 and maximum degree 4, satisfying the preconditions for the lemma.

We know that $|V(H)| = k^2 - 4$, thus applying the result of the lemma gives

$$k^2 - 4 \leq \Delta|V(T)| = 5n \implies k \leq \sqrt{5n+4}$$

and so $\mathrm{gm}(S \square T) \leq \mathrm{gm}(S \boxtimes T) \leq \mathrm{gm}(S \cdot T) \in O(\sqrt{n})$.

The full proof of Equation (1) is long and involved. To summarize the approach taken, we choose to list only the required definitions, and to present the lemmas that build to the result without proof. The proofs of all results, as well as other supplementary information, can be found in the full paper [**?**]

For a rooted tree T, let $n_i(T)$ be the number of vertices of height i in T, where a vertex v has height i if the longest path with upper endpoint[1] v is of order i.

Lemma 2. *For any positive integer n and any n-vertex connected graph G, $K_n \preceq G \square S_n$.*

[1] The upper endpoint of a path P in a rooted tree T is the vertex $v \in P$ with minimum distance to the root

Observation 3. *For any rooted tree T and any $i \in \mathbb{N}$, T contains a set of $n_i(T)$ pairwise completely unrelated vertical paths, each of order i. As a consequence, $S_{n_i(T),i} \preceq T$ for each $i \in \mathbb{N}$.*

Lemma 4. *Let T be a rooted tree with $n \geq 1$ vertices, and let $p \geq 1$ be an integer such that $n_i(T) \leq \frac{3}{2}n/(\pi i)^2$ for each $i \in \{1, \dots, p-1\}$. Then T contains pairwise-disjoint vertical paths $P_1, \dots, P_{\lceil n/4p \rceil}$, each of order p such that, for each $i \neq j$, P_i and P_j are either completely unrelated or completely related.*

Lemma 5. *Let $s, p \geq 1$ be integers, let $\ell := 5s^2$, and let T be a rooted tree that contains s^2 pairwise-disjoint vertical paths, each of order $6p$ such that any pair of these paths is either completely related or completely unrelated. Then $\mathrm{gm}(T \,\square\, S_{\ell,2p}) \geq sp$.*

With these three pieces, the full result can be shown.

Theorem 6. *If G_1 and G_2 are connected graphs each having at least $n \geq 1$ vertices, then $\mathrm{gm}(G_1 \,\square\, G_2) \in \Omega(\sqrt{n})$*

Proof. For each $b \in \{1, 2\}$, let T_b be a tree contained in G_b and having exactly n vertices (which can be constructed by successively deleting leaves starting with a spanning tree of G_b). For each $b \in \{1, 2\}$, let $p_b = \min\{i : n_i(T_b) \geq \frac{3n}{2p\pi^2}\}$. (This is well-defined since, otherwise $n = \sum_{p=1}^{\infty} n_p(T_b) < \sum_{p=1}^{\infty} \frac{3n}{2p\pi^2} = \frac{n}{4}$.) Without loss of generality, assume $p_2 \leq p_1$ and let $\ell := \lceil \frac{3n}{2p_2\pi^2} \rceil^2$. By Observation 3, $S_{\ell,p_2} \preceq T_2 \preceq G_2$. If $p_2 < 6$ then $\ell > \frac{n}{4\pi^2} \in \Omega(n)$ and by Lemma 2 $K_\ell \preceq G_1 \,\square\, S_\ell$. Since $\boxplus_{\lfloor \sqrt{\ell} \rfloor} \preceq K_\ell$, this implies that $\mathrm{gm}(G_1 \,\square\, G_2) \geq \lfloor \sqrt{\ell} \rfloor = \Omega(\sqrt{n})$ and we are done, so we may assume that $p_2 \geq 6$. Let $p := \lfloor p_2/6 \rfloor \geq 1$.

Since $p_1 \geq p_2$, Lemma 4 implies that T_1 contains at least $n/4p$ pairwise disjoint paths $P_1, \dots, P_{\lceil n/4p \rceil}$, each of length $p_2 \geq 6p$, such that each pair of paths is either completely related or completely unrelated. Let

$$s := \lfloor \min\{\sqrt{\ell/5}, \sqrt{n/4p}\} \rfloor = \Theta(\sqrt{n}/p)$$

so that $\ell \geq 5s^2$ and $\lceil n/4p \rceil \geq s^2$. By Lemma 5, $\mathrm{gm}(T_1 \,\square\, S_{\ell,6p}) \geq sp = \Theta(\sqrt{n})$. The result now follows from the fact that $T_1 \preceq G_1$, and the fact that $S_{\ell,6p} \preceq S_{\ell,p_2} \preceq G_2$ combined with the observation that for any graphs G_1, G_2, and H, if $G_1 \preceq G_2$, then $G_1 \,\square\, H \preceq G_2 \,\square\, H$. □

Note that the full paper [**?**] further proves some exact bounds for special graph products, mainly the product of stars and trees.

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4 Track Number of Planar Graphs

4.1 Introduction

A *track layout* of a graph G consists of a vertex colouring and a total order on each colour class, such that no two edges between any two colour classes.

The *track number* of a graph is the minimum number of colours needed by a track layout of G.

A *partition* P of a graph G is a set of connected subgraphs of G, such that each vertex belongs to exactly one subgraph.

An *H-partition* of a graph G is a partition of $V(G)$ into disjoint bags $\{A_x : x \in V(H)\}$ indexed by the vertices of a graph H, such that for every edge $(u, v) \in E(G)$ one of the following holds:

1. $u, v \in A_x$ for some $x \in V(H)$ (intra-bag edge)
2. There is an edge $(x, y) \in E(H)$ with $u \in A_x$ and $v \in A_y$. (inter-bag edge)

A *layering* of a graph G is an ordered partition $(V_0, V_1, ...)$ of $V(G)$ such that for every edge $(v, w) \in E(G)$, if $v \in V_i$ and $w \in V_j$, then $|i - j| \leq 1$

The *layered width* of an H-partition of a graph G is the minimum integer l such that for some layering $(V_0, V_1, ...)$ of G we have $|A_x \bigcap V_i| \leq l, \forall x \in V(H), i \geq 0$.

A *BFS-layering* of a graph G is a layering of G such that if r is a vertex in a connected graph G, then $V_i = \{v \in V(G) | \operatorname{dist}_G(r, v) = i\}, \forall i \geq 0$.

For each $f \geq 3, s \geq 1$, a *planar (f, s)-tree* is an embedded planar graph defined recursively as follows: The smallest (f, s)-tree is a 2-connected planar graph on $f + s$ vertices with an embedding where each face (including the outer face) has size at most f. Every embedded graph that can be obtained from a planar (f, s)-tree G by doing the following operation is also a planar (f, s)-tree:

- Pick a face of G, say f, and add a set S of at most s new vertices to f. Add edges between some pairs of vertices of $V(f) \bigcup S$ such that that the resulting graph is 2-connected and each new face has size at most f.

Track layouts have been studied in the context of graph drawings[1, 4] as well as in graph layouts[5], but were formally introduced by Dujmovic, Morin, and Wood[3]. Track layouts see strong applications in three-dimensional low volume graph drawing. In particular, a graph G on n vertices has a 3D, straight-line drawing on a grid of size $O(1) \times O(1) \times O(n)$ if and only if G has constant track number.

4.2 Problem 2

We wish to improve the current best known bound on the track number of planar graphs. To this end, we investigate the best known bounds for the track number of planar (f, s)-graphs as an intermediate step, then use results on layered H-partitions to extend the result to planar graphs.

4.3 Related Work

The current best known bound on the track number of planar graphs is 255 as a result of Pupyrev[6], which was shown by using layered H-partitions directly with planar graphs. This improved the previous bound of 461,184,080, which was a consequence of the planar graph product structure theorem of Dujmovic et al.[2]

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5 Product Structure of Bounded Degree Planar Graphs

5.1 Introduction

The usage of product structure theory to study planar graphs has been a very active area of research since Dujmovic et al first showed that planar graphs have bounded queue number using product structure theory [3]. Since then, this result has gone on to lead to improvements in graph colouring[2], adjacency labelling[1, 5], and more. These results have pushed additional interest into the research of product structure theorems for other graph classes and for more specialized ones. We focus on improving the product structure theorems for bounded-degree planar graphs.

5.2 Problem 3

Given a planar graph G with maximum degree Δ, is it true that G is contained in the product

$$H \boxtimes P \boxtimes K_c$$

for a graph H with treewidth 3, a path P, and the complete graph K_c where c is bounded by some function of Δ?

5.3 Related Work

This problem looks to tighten the bound on $\mathrm{tw}(H)$ to close the bound on the product structure of bounded-degree planar graphs. The problem was initially shown to be true for $\mathrm{tw}(H) = 3$ by Dujmovic et al [3].It was then shown that the case for $\mathrm{tw}(H) = 1$ is false[4]. Tightening this bound would immediately improve results in [**?**].

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6 Maintaining Treewidth in Graph Products

6.1 Introduction

With the importance of treewidth and the many applications of product structure theory, a natural question is the following: Can the treewidth be maintained in some meaningful way through taking the product?

6.2 Problem 4

Is it true that for every planar graph G, there exists a bounded treewidth graph H and a path P such that $G \subsetapprox H \boxtimes P$ and $\mathrm{tw}(H \boxtimes P) \in O(\mathrm{tw}(G))$?

6.3 Related Work

In our paper regarding Problem 1, we show that

$\Omega(\min\{|V(H)|, |V(P)|\}) \leq \mathrm{tw}(H \boxtimes P) \leq O(\min\{|V(H)|, |V(P)|\})$. [1, Lemma 3, Equation (2)]

Thus this question really asks whether for every planar graph G, there exists a bounded treewidth graph H and a path P such that $G \subsetapprox H \boxtimes P$ and $\min\{|V(H)|, |V(P)|\} \leq O(\mathrm{tw}(G))$.

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