
GRID MINORS AND PRODUCTS

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ABSTRACT. Motivated by recent developments regarding the product structure of planar graphs, we study relationships between treewidth, grid minors, and graph products. We show that the Cartesian product of any two connected n -vertex graphs contains an $\Omega(\sqrt{n}) \times \Omega(\sqrt{n})$ grid minor. This result is tight: The lexicographic product (which includes the Cartesian product as a subgraph) of a star and any n -vertex tree has no $\omega(\sqrt{n}) \times \omega(\sqrt{n})$ grid minor.

1 Introduction

Treewidth¹ is a ubiquitous parameter in structural graph theory measuring how close a graph is to a tree. It was first introduced as *dimension* by Bertelè and Brioschi [2, pp. 37–38] in 1972, then rediscovered by Halin [35] in 1976. The parameter was popularized when it was once again rediscovered by Robertson and Seymour [48] in 1984 and has since been at the forefront of structural graph theory research. Graphs of bounded treewidth are of particular interest due to the wide implications of their tree-like structure. For example, Courcelle’s Theorem [9] implies that many NP-Complete problems can be solved in linear time on graphs of bounded treewidth.

Another key way to study the structure of graphs is through analysis of their minors.² Finding a highly structured minor of a graph G allows us to use properties of the minor to study G itself. Our focus is on grid minors. Let \boxplus_k be the $k \times k$ grid, which is the graph

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¹A *tree-decomposition* of a graph G is a collection $(B_x : x \in V(T))$ of subsets of $V(G)$ (called *bags*) indexed by the vertices of a tree T , such that (a) for every edge $uv \in E(G)$, some bag B_x contains both u and v , and (b) for every vertex $v \in V(G)$, the set $\{x \in V(T) : v \in B_x\}$ induces a non-empty (connected) subtree of T . The *width* of $(B_x : x \in V(T))$ is $\max\{|B_x| : x \in V(T)\} - 1$. The *treewidth* of a graph G , denoted by $\text{tw}(G)$, is the minimum width of a tree-decomposition of G .

²A graph G_1 is a *minor* of a graph G_2 if a graph isomorphic to G_1 can be obtained from a subgraph of G_2 through a sequence of edge contractions.

with vertex-set $\{1, \dots, k\}^2$ and edge-set $\{(x, y)(x', y') : |x - x'| + |y - y'| = 1, x, y, x', y' \in \{1, \dots, k\}\}$. For a graph G , let $\text{gm}(G)$ be the maximum integer k such that \boxplus_k is a minor of G . It is folklore that $\text{tw}(\boxplus_k) = k$; see [36] for a proof. This, combined with the fact that treewidth is minor-monotone, implies that for every graph G ,

$$\text{tw}(G) \geq \text{gm}(G). \quad (1)$$

A foundational result in this area is the *Grid Minor Theorem* (also known as the *Excluded Grid Theorem*) of Robertson and Seymour [46], which says there exists a function f such that for every positive integer k , $f(k)$ is the minimum integer such that every graph with treewidth at least $f(k)$ contains a $k \times k$ grid minor. Thus grid graphs are canonical examples of graphs with large treewidth.

This result had widespread impact on structural graph theory research and led to further investigation into the best possible bounds for the function f . Robertson and Seymour [46] proved the existence of $f(k)$, which they, along with Thomas, later showed to be in $2^{O(k^5)}$ [47]. Diestel, Jensen, Gorbunov, and Thomassen [15] showed that if G has treewidth $\Omega(k^{4m^2(k+2)})$ where k and m are integers, then G contains either K_m or the $k \times k$ grid as a minor. Leaf and Seymour [44] improved the upper bound to $f \in 2^{O(k \log k)}$. The first polynomial upper bound, stating that $f \in O(k^{98} \log k)$, was found by Chekuri and Chuzhoy [7]. Chuzhoy continued to work towards lowering this exponent, with the current state-of-the-art result by Chuzhoy and Tan [8] showing that $f \in O(k^9 \log^{O(1)} k)$. A lower bound of $f \in \Omega(k^2 \log k)$ was shown by Robertson et al. [47], and Demaine, Hajiaghayi, and Kawarabayashi [13] conjectured $f \in \Theta(k^3)$.

For particular classes of graphs, much stronger Grid Minor Theorems are known. Say a class \mathcal{G} has the *linear grid minor property* if, for some constant c , every graph in \mathcal{G} with treewidth at least ck contains \boxplus_k as a minor. For example, Robertson et al. [47] showed that every planar graph with treewidth at least $6k$ contains \boxplus_k as a minor. Thus the class of planar graphs has the linear grid minor property. More generally, Demaine and Hajiaghayi [12] proved that every proper minor-closed class has the linear grid minor property. The proof used the Graph Minor Structure Theorem, which in turn depends on the Grid Minor Theorem. Kawarabayashi and Kobayashi [41] gave an alternative self-contained proof. In particular, they showed that for any graph H there exists $c \leq |V(H)|^{O(|E(H)|)}$ such that every H -minor-free graph with treewidth at least ck contains \boxplus_k as minor. The linear grid minor property has been used to devise efficient polynomial time approximation schemes for many NP-hard problems on planar graphs and related graph families [10, 11, 13, 28, 34]. Note that the $\Omega(k^2 \log k)$ lower bound mentioned above shows that general graphs do not have the linear grid minor property.

In this paper, we study grid minors in graph products³. This is motivated both by the fact that the grid graph itself is isomorphic to the Cartesian product of two paths, and also

³See [6, 30, 31, 37, 42, 52] for related work on the treewidth of graph products, and see [5, 51] for related work on complete graph minors in graph products.

by recent developments in Graph Product Structure Theory. This area of research studies complex graph classes by modelling them as a product of simpler graphs and investigating the properties of these highly structured supergraphs. Before discussing Graph Product Structure Theory further, we first define the three types of graph products that we consider, each illustrated in Figure 1. Let G_1 and G_2 be graphs. The *Cartesian product* of G_1 and G_2 , denoted $G_1 \square G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ where two distinct vertices (u_1, v_1) and (u_2, v_2) are adjacent iff

- $u_1 = u_2$ and $v_1 v_2 \in E(G_2)$, or
- $v_1 = v_2$ and $u_1 u_2 \in E(G_1)$.

The *strong product* of G_1 and G_2 , denoted $G_1 \boxtimes G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ where two distinct vertices (u_1, v_1) and (u_2, v_2) are adjacent iff

- $u_1 = u_2$ and $v_1 v_2 \in E(G_2)$,
- $v_1 = v_2$ and $u_1 u_2 \in E(G_1)$, or
- $u_1 u_2 \in E(G_1)$ and $v_1 v_2 \in E(G_2)$.

The *lexicographic product* of G_1 and G_2 , denoted $G_1 \cdot G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ where two distinct vertices (u_1, v_1) and (u_2, v_2) are adjacent iff

- $u_1 v_1 \in E(G_1)$, or
- $u_1 = u_2$ and $v_1 v_2 \in E(G_2)$.

It follows from the above definitions that

$$G_1 \square G_2 \subseteq G_1 \boxtimes G_2 \subseteq G_1 \cdot G_2.$$

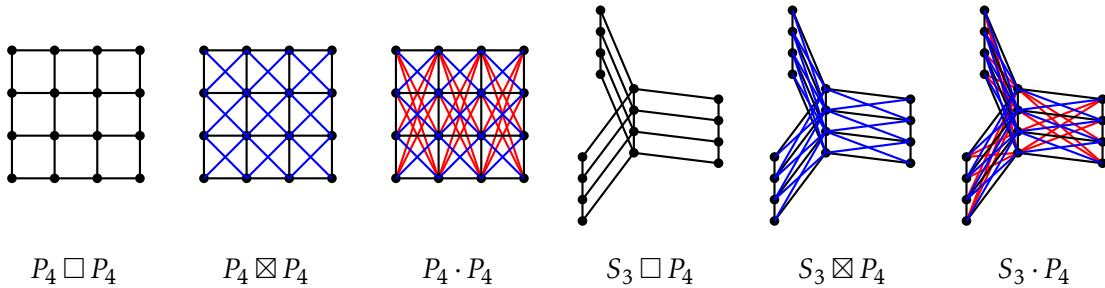


Figure 1: The products of two paths and of a star and a path.

The starting point for recent developments in Graph Product Structure Theory is the *Planar Graph Product Structure Theorem* of Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [23], which states that every planar graph G is isomorphic to a subgraph of the strong product of two very simple graphs, a graph H of treewidth⁴ at most 8 and a path P , written as $G \subseteq H \boxtimes P$. Although $H \boxtimes P$ is a supergraph of the original graph G , it often shares or inherits properties of G , and its rigid structure makes it easier to work with. For

⁴This treewidth bound was improved to 6 by Ueckerdt, Wood, and Yi [50].

example, any induced subgraph of $H \boxtimes P$ of diameter k has treewidth $O(k)$. For a more global example, any n -vertex subgraph of $H \boxtimes P$ has a balanced separator of size $O(\sqrt{n})$ (see [23, Lemma 6] and [25, Lemma 10]). The proofs of both of these facts are considerably simpler than the same result for planar graphs.

Product structure theorems have also been developed for other classes of graphs such as surface embeddable graphs [16, 23], graphs excluding an apex minor [22, 23, 39], graphs excluding any fixed minor [21, 23], and various non-minor-closed classes [1, 17, 26, 38]. These product structure theorems have been key in resolving many long-standing open problems on queue layouts [23], non-repetitive colourings [20], centred colourings [18, 22], adjacency labelling [19, 32], twin-width [3, 40, 43], vertex ranking [4], and box dimension [27]. This wide range of applications motivates the need for a deeper understanding of structural properties of graph products.

1.1 Our Results

It is known⁵ that for all n -vertex connected graphs G_1 and G_2 ,

$$\text{tw}(G_1 \square G_2) \geq n. \quad (2)$$

It thus makes sense for a grid minor theorem for graph products to be in terms of n . We show that this is in fact the case by proving the following results:

1. For any two n -vertex connected graphs G_1 and G_2 ,

$$\text{gm}(G_1 \cdot G_2) \geq \text{gm}(G_1 \boxtimes G_2) \geq \text{gm}(G_1 \square G_2) \in \Omega(\sqrt{n}) \text{ (see Theorem 7).}$$

2. There exists two n -vertex connected graphs G_1 and G_2 (a star and any tree) such that

$$\text{gm}(G_1 \square G_2) \leq \text{gm}(G_1 \boxtimes G_2) \leq \text{gm}(G_1 \cdot G_2) \in O(\sqrt{n}) \text{ (see Theorem 11).}$$

The previous best bound for the product of two n -vertex connected graphs comes from combining (2) with the state-of-the-art Grid Minor Theorem of Chuzhoy and Tan [8], giving $\text{gm}(G_1 \square G_2) \in \Omega(n^{1/9}/\text{polylog}(n))$. The first result above gives an excluded grid theorem for graph products that is stronger than what is possible for general graphs and much stronger than what can be proven for general graphs.

The second result shows that the first result is tight for the Cartesian, strong, and lexicographic product of two trees. A consequence of the second result and (2) is that

⁵Let G_1 and G_2 be connected graphs each with at least n vertices. For $i \in \{1, 2\}$, let v_i be a leaf of a spanning tree of G_i , and let $G'_i := G_i - v_i$, which is connected. For each $x \in V(G'_1)$, let B_x be the subgraph of $G_1 \square G_2$ induced by $\{x\} \times V(G'_2)$. For each $y \in V(G'_2)$, let B_y be the subgraph of $G_1 \square G_2$ induced by $V(G'_1) \times \{y\}$. Let B_1 be the subgraph of $G_1 \square G_2$ induced by $\{v_1\} \times V(G_2)$. Let B_2 be the subgraph of $G_1 \square G_2$ induced by $V(G'_1) \times \{v_2\}$. Let $\mathcal{B} := \{B_x \cup B_y : x \in V(G'_1), y \in V(G'_2)\} \cup \{B_1, B_2\}$. Then it is easily seen that \mathcal{B} is a bramble in $G_1 \square G_2$ of order at least $n+1$. By the Treewidth Duality Theorem [49], $\text{tw}(G_1 \square G_2) \geq n$. This result was extended by Wood [52] who showed that for all k -connected graphs G and H each with at least n vertices, $\text{tw}(G \square H) \geq k(n-2k+2)-1$.

there exists two trees whose Cartesian product has treewidth at least n but whose largest grid minor has size $O(\sqrt{n}) \times O(\sqrt{n})$. Thus, even these simple products do not have the linear (or even subquadratic) grid minor property.

The remainder of this paper is organized as follows: In [Section 2](#) we discuss background material. In [Section 3](#) we prove the above lower bound. In [Section 4](#) we prove the above upper bound. [Section 5](#) proves some exact bounds for the grid minor number of the products of stars and trees. Finally, [Section 6](#) concludes with directions for future work.

2 Preliminaries

For any standard graph-theoretic terminology and notation not defined here, we use the same conventions used in the textbook by Diestel [\[14\]](#). In this paper, every graph G is undirected and simple with vertex set $V(G)$ and edge set $E(G)$. The *order* of G is denoted $|G| := |V(G)|$. For two graphs G_1 and G_2 we use the notation $G_1 \leq G_2$ to indicate that G_1 is a minor of G_2 . We make use of the fact that the \leq relation is transitive. The following observation follows immediately from definitions.

Observation 1. *Let G_1 , G_2 , and H be graphs. If $G_1 \leq G_2$, then $G_1 \square H \leq G_2 \square H$.*

A *model* of a graph H in a graph G is a set $\mathcal{M} := \{B_x \subseteq V(G) : x \in V(H)\}$ of subsets of $V(G)$, called *branch sets*, indexed by the vertices of H and such that:

- (i) for each distinct pair $x, y \in V(H)$, $B_x \cap B_y = \emptyset$;
- (ii) for each $x \in V(H)$, $G[B_x]$ is connected and
- (iii) for each $xy \in E(H)$ there exists an edge $vw \in E(G)$ with $v \in B_x$ and $w \in B_y$.

It follows from definitions that $H \leq G$ if and only if there exists a model of H in G .

For each $n \in \mathbb{N}$, let S_n denote the *n -star*; the rooted tree with n leaves, each of which is adjacent to the root. For each $\ell, p \in \mathbb{N}$, let $S_{\ell,p}$ denote the star with ℓ leaves whose edges have been subdivided $p - 1$ times. More formally, $V(S_{\ell,p}) := \{v_0\} \cup \{v_{i,j} : (i,j) \in [\ell] \times [p]\}$ and $E(S_{\ell,p}) := \{v_0 v_{i,1} : i \in [\ell]\} \cup \{v_{i,j} v_{i,j+1} : (i,j) \in [\ell] \times [p-1]\}$. We call $S_{\ell,p}$ a *subdivided star*. Subdivided stars generalize both stars and paths: The n -vertex path P_n is isomorphic to $S_{1,n-1}$ and the n -leaf star S_n is isomorphic to $S_{n,1}$.

Lemma 2. *For any positive integer n and any n -vertex connected graph G , $K_n \leq G \square S_n$.*

Note that this lemma is implied by [\[51, Lemma 5.1\]](#); we include the proof here for the sake of completeness.

Proof. Let y_0 denote the root of S_n , let y_1, \dots, y_n denote the leaves of S_n . Let $V(K_n) = \{1, \dots, n\}$ and let v_1, \dots, v_n denote the vertices of G . We now construct a model $\mathcal{M} := \{B_x : x \in V(K_n)\}$ of K_n in $G \square S_n$. For each $i \in V(K_n)$, define the branch set

$$B_i := \{(v, y_i) : v \in V(G)\} \cup \{(v_i, y_0)\} .$$

We now show that \mathcal{M} is a model of K_n in $G \square S_n$. The induced graph $(G \square S_n)[B_i]$ is connected because $(G \square S_n)[\{(v, y_i) : v \in V(G)\}]$ is isomorphic to G and (v_i, y_0) is adjacent to (v_i, y_i) . For any $1 \leq i < j \leq n$, B_i and B_j are disjoint because $y_i \neq y_j$ and $v_i \neq v_j$. Furthermore, the vertex $(v_i, y_0) \in B_i$ is adjacent to $(v_i, y_j) \in B_j$. Therefore, there is an edge in $G \square S_n$ with one endpoint in B_i and one endpoint of B_j for each $1 \leq i < j \leq n$. \square

Note that, for any tree T with n leaves and at least one non-leaf vertex, $S_n \leq T$. In this case, [Lemma 2](#) and [Observation 1](#) imply that $K_n \leq G \square T$.

Finally, we mention the following upper bound on the treewidth of $G_1 \cdot G_2$. This result is stated in [37] for $G_1 \boxtimes G_2$ and is implicit in earlier works; we include the easy proof for completeness.

Lemma 3. *For any graphs G_1 and G_2 ,*

$$\text{tw}(G_1 \square G_2) \leq \text{tw}(G_1 \boxtimes G_2) \leq \text{tw}(G_1 \cdot G_2) \leq (\text{tw}(G_1) + 1)|V(G_2)| - 1.$$

Proof. The first two inequalities hold since $G_1 \square G_2 \subseteq G_1 \boxtimes G_2 \subseteq G_1 \cdot G_2$. For the final inequality, start with a tree-decomposition $(B_x : x \in V(T))$ of G_1 with bags of size at most $\text{tw}(G_1) + 1$. For each $x \in V(T)$ let $B'_x := \{(v, w) : v \in B_x, w \in V(G_2)\}$. Observe that $(B'_x : x \in V(T))$ is a tree-decomposition of $G_1 \cdot G_2$, and $|B'_x| \leq |B_x||V(G_2)| \leq (\text{tw}(G_1) + 1)|V(G_2)|$ for each $x \in V(T)$. The result follows. \square

[Equation \(2\)](#) and [Lemma 3](#) imply that for any trees T_1 and T_2 ,

$$\min\{|V(T_1)|, |V(T_2)|\} \leq \text{tw}(T_1 \square T_2) \leq \text{tw}(T_1 \boxtimes T_2) \leq \text{tw}(T_1 \cdot T_2) \leq 2 \min\{|V(T_1)|, |V(T_2)|\} - 1. \quad (3)$$

Thus the treewidth of the Cartesian, strong or lexicographic product of two trees is determined (up to a factor of 2) by the number of vertices in the two trees. The remainder of the paper shows that determining the largest grid minor in such a product is more nuanced.

3 The Lower Bound

3.1 Connected Graphs Contain Large Subdivided Stars

We first state some terminology that will be relevant in the following results. The *length* of a path v_0, \dots, v_r is the number, r , of edges in the path. The *depth* of a vertex v in a rooted tree T is the length of the path, in T , from v to the root of T . A path P in a rooted tree T is *vertical* if, for each $i \in \mathbb{N}$, $V(P)$ contains at most one vertex of depth i . The vertex of minimum depth in a vertical path is its *upper endpoint*, and the vertex of maximum depth in a vertical path is its *lower endpoint*. A vertex v is a *T-ancestor* of a vertex w if the vertical path from w to the root of T contains v . Two vertices of T are *unrelated* if neither is a T -ancestor of the other, otherwise they are *related*. A pair of paths P_1 and P_2 in T is *completely*

unrelated if v and w are unrelated, for each $v \in V(P_1)$ and each $w \in V(P_2)$. We say that P_1 and P_2 are *completely related* if v and w are related, for each $v \in V(P_1)$ and each $w \in V(P_2)$. The *height* $h_T(v)$ of a vertex v in T is the maximum order of a vertical path in T whose upper endpoint is v . For each $i \in \mathbb{N}$, let $H_i(T) := \{v \in V(T) : h_T(v) = i\}$ and $n_i(T) := |H_i(T)|$. We have the following observation:

Observation 4. *For any rooted tree T and any $i \in \mathbb{N}$, T contains a set of $n_i(T)$ pairwise completely unrelated vertical paths, each of order i . As a consequence, $S_{n_i(T),i} \leq T$ for each $i \in \mathbb{N}$.*

Proof. Let $v_1, \dots, v_{n_i(T)} := H_i(T)$ and observe that $v_1, \dots, v_{n_i(T)}$ are pairwise unrelated. For each $j \in \{1, \dots, n_i(T)\}$, let P_j be a path of order i that has v_j as an upper endpoint. (Such a path exists by the definition of $H_i(T)$.) Observe that, for distinct j and k , P_j and P_k are vertex-disjoint, and completely unrelated since v_j and v_k are unrelated. By contracting each edge that has both endpoints of depth less than i into a single vertex x and removing all vertices not in $\{x\} \cup \bigcup_{j \in \{1, \dots, n_i(T)\}} V(P_j)$ we obtain $S_{n_i(T),i}$. Thus $S_{n_i(T),i} \leq T$. \square

We will show that the product $G_1 \square G_2$ of two connected n -vertex graphs G_1 and G_2 contains an $\Omega(\sqrt{n}) \times \Omega(\sqrt{n})$ grid minor by studying the product $T_1 \square T_2$ of two spanning trees of G_1 and G_2 , respectively. [Lemma 2](#) allow us to dispense with the case when $n_i(T_b) \in \Omega(n)$ for some i and some $b \in \{1, 2\}$ since, if $n_i(T_b) \in \Omega(n)$, then T_b contains a $S_{\Omega(n)}$ -minor, so [Lemma 2](#) implies $K_{\Omega(n)} \leq T_1 \square T_2$, so $\boxplus_{\Omega(\sqrt{n})} \leq T_1 \square T_2$. The following lemma will be helpful when this is not possible. (In several places, including the following lemma, we make use of Euler's solution [\[33\]](#) to the Basel Problem: $\sum_{i=1}^{\infty} 1/i^2 = \pi^2/6$.)

Lemma 5. *Let T be a rooted tree with $n \geq 1$ vertices, and let $p \geq 1$ be an integer such that $n_i(T) \leq \frac{3}{2}n/(\pi i)^2$ for each $i \in \{1, \dots, p-1\}$. Then T contains pairwise-disjoint vertical paths $P_1, \dots, P_{\lceil n/4p \rceil}$, each of order p such that, for each $i \neq j$, P_i and P_j are either completely unrelated or completely related.*

Proof. Let $T' := T - (\bigcup_{i=1}^{p-1} H_i(T))$ be the subtree of T induced by vertices of height at least p . Then,

$$|T'| \geq |T| - \sum_{i=1}^{p-1} n_i(T) \geq n - \frac{3n}{2\pi^2} \sum_{i=1}^{p-1} \frac{1}{i^2} \geq n - \frac{n}{4} = \frac{3n}{4}.$$

Let L be the set of non-root leaves of T' . Each vertex in L is the upper endpoint of a vertical path in T of order p , as illustrated in [Figure 2](#). Therefore, if $|L| \geq \frac{n}{4p}$ then we are done, so assume that $|L| < \frac{n}{4p}$.

Let S be the set of vertices of T' that have two or more children in T' . In any rooted tree, the number of non-root leaves is greater than the number of non-leaf vertices with at least two children⁶ Therefore, $|S| < |L| \leq \frac{n}{4p}$.

⁶Let n_i be the number of vertices with i children in a rooted tree T . Thus $\sum_{i \geq 0} i n_i = |E(T)| < |V(T)| = \sum_{i \geq 0} n_i$. Hence, the number of non-root leaves $n_0 > \sum_{i \geq 1} (i-1)n_i \geq \sum_{i \geq 2} n_i$, as claimed.

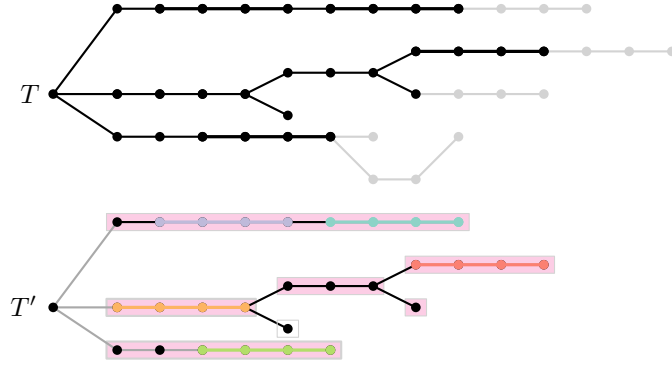


Figure 2: Finding paths in the tree T' induced by vertices of height at least $p = 4$. Note that the figures are drawn so that for each edge, the left vertex is the upper endpoint.

For each $v \in SUL$, let P_v be the vertical path of maximum length whose lower endpoint is v and that does not contain any vertex in $(SUL) \setminus \{v\}$. Then $\mathcal{P} := \{P_v : v \in SUL\}$ is a partition of $V(T')$ into at most $r := |S| + |L| \leq \frac{n}{2p}$ parts, each of which induces a vertical path in T' .

Let $\{P_1, \dots, P_r\} := \mathcal{P}$ and, for each $i \in \{1, \dots, r\}$, let P'_i be a subpath of P_i of order $p \lfloor |P_i|/p \rfloor$. (So P'_i has order rounded down to a multiple of p .) Then

$$\sum_{i=1}^r |P'_i| \geq \sum_{i=1}^r (|P_i| - (p-1)) = |T'| - (p-1)r \geq \frac{3n}{4} - \frac{n}{2} = \frac{n}{4}.$$

For each $i \in \{1, \dots, r\}$, P'_i can be partitioned into exactly $|P'_i|/p$ vertex-disjoint paths, each of order p . The set \mathcal{P}' of paths obtained this way has size $\ell := \sum_{i=1}^r |P'_i|/p \geq \frac{n}{4p}$. Therefore T contains ℓ pairwise vertex-disjoint paths, each of order p , where $\ell \geq \frac{n}{4p}$. Except for its lower endpoint, each vertex of a path in \mathcal{P}' has exactly one child in T . This ensures that each path in \mathcal{P}' is either completely related or completely unrelated to any other path in \mathcal{P}' . \square

3.2 The Product of Two Special Trees

Lemma 6. Let $s, p \geq 1$ be integers, let $\ell := 5s^2$, and let T be a rooted tree that contains s^2 pairwise-disjoint vertical paths, each of order $6p$ such that any pair of these paths is either completely related or completely unrelated. Then $\text{gm}(T \square S_{\ell, 2p}) \geq sp$.

Proof. Recall that $S_{\ell, 2p}$ has vertex set $\{v_0\} \cup \{v_{i,j} : (i,j) \in \{1, \dots, \ell\} \times \{1, \dots, 2p\}\}$. For each $i \in \{1, \dots, \ell\}$, let $A_i = S_{\ell, 2p}[\{v_{i,1}, \dots, v_{i,2p}\}]$ denote the i th *arm* of $S_{\ell, 2p}$, which is a path of order $2p$. Let P_1, \dots, P_{s^2} be pairwise vertex-disjoint paths in T , each of order $6p$, each pair of which is either completely related or completely unrelated. For each $i \in \{1, \dots, s^2\}$, let $P_i := p_{i,1}, \dots, p_{i,6p}$ where $p_{i,1}$ is the upper endpoint of P_i . Let $T_0 := T \square \{v_0\}$. For each $i \in \{1, \dots, \ell\}$ let $T_i := T \square A_i$, for each $j \in \{1, \dots, p\}$ let $T_{i,j} := T \square \{v_{i,j}\}$, for each $k \in \{1, \dots, s^2\}$ let $P_{k,i} := P_k \square A_i$ and $P_{k,i,j} := P_k \square \{v_{i,j}\}$.

Refer to Figure 3. Consider T_i for some $i \in \{1, \dots, \ell\}$. For visualization purposes, it is helpful to organize T_i into $2p$ rows $T_{i,1}, \dots, T_{i,2p}$. For each $j \in \{1, \dots, 2p-1\}$, $T_{i,j}$ and $T_{i,j+1}$ are ‘adjacent’ in the sense that each vertex $(a, v_{i,j}) \in V(T_{i,j})$ is adjacent to $(a, v_{i,j+1}) \in V(T_{i,j+1})$. We then organize T_1, \dots, T_ℓ into a sequence of blocks. These blocks are independent in the sense that there is no edge between T_i and T_j for any $i \neq j$. Moreover, there is an additional row T_0 that is adjacent to the first row, $T_{i,1}$, of each block T_i .

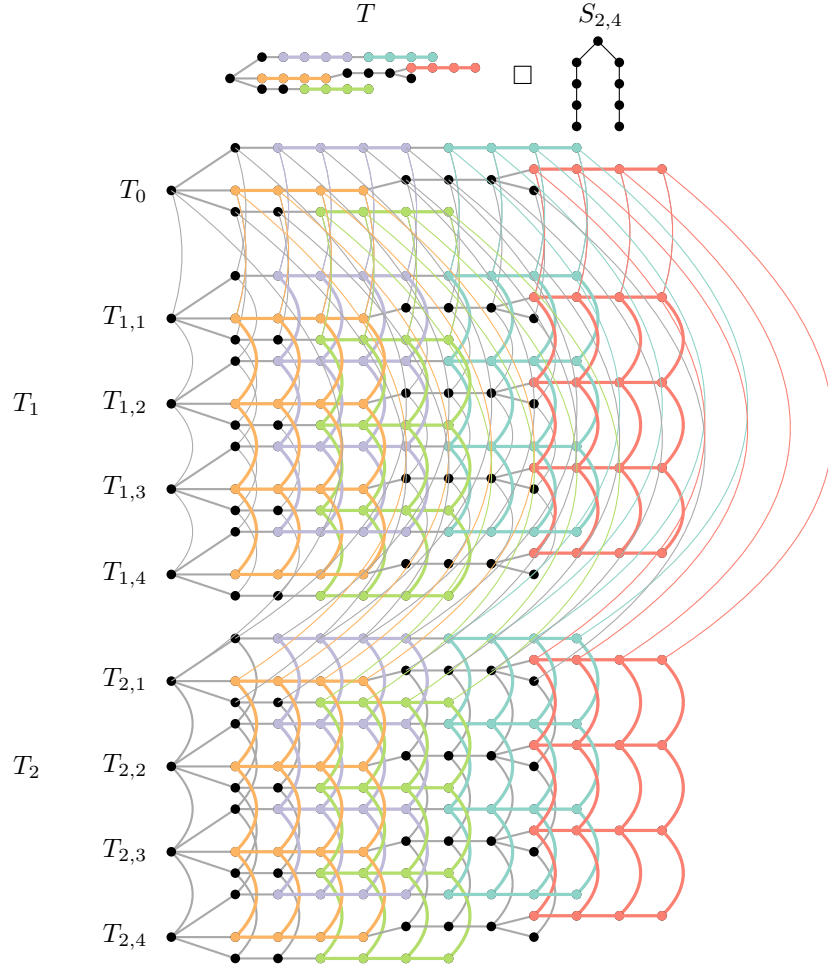


Figure 3: Visualizing the product in Lemma 6

Refer to Figure 4. We will construct a model of \boxplus_{sp} . We partition this model into s^2 subgrids each of which is isomorphic to \boxplus_p . Therefore, we need s^2 such subgrids G_1, \dots, G_{s^2} . The branch sets of each subgrid G_i will include a $p \times p$ grid within the $6p \times 2p$ grid $P_{i,i}$ (which is contained in the block T_i). The additional row T_0 will allow us to extend the branch sets of the $4p - 4$ boundary vertices of the G_i into $T_{i'}$ for any i' and from there they can be extended so that they are adjacent to any other subgrid G_j .

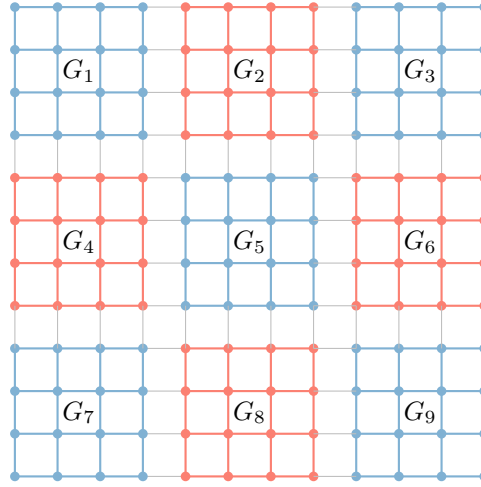


Figure 4: The $sp \times sp$ grid can be partitioned into $(sp/p)^2 = s^2$ subgrids, each of which is a $p \times p$ grid. (The case $sp = 12$ and $p = 4$ is shown here.)

Refer to Figure 5. To construct the branch sets for G_i we start with a $p \times p$ subgrid in $P_{i,i}$ whose top row is $(p_{i,p+1}, v_{i,1}), \dots, (p_{i,2p}, v_{i,1})$. This subgrid has $4p - 4$ ‘boundary’ vertices, four of which are ‘corner’ vertices. As illustrated in Figure 5, we create $4p$ disjoint paths in $P_{i,i}$ from the boundary vertices to $P_{i,i,1}$. We then add one vertex of $P_{i,0}$ to each path. In this way, the first $4p$ vertices of the path $P_{i,0}$ are partitioned into four subpaths, each of size p corresponding to edges coming out of the left, top, right, and bottom of boundary of G_i . The final $2p$ vertices of $P_{i,0}$ are not yet used (though we may add them to the branch sets of some boundary vertices later).

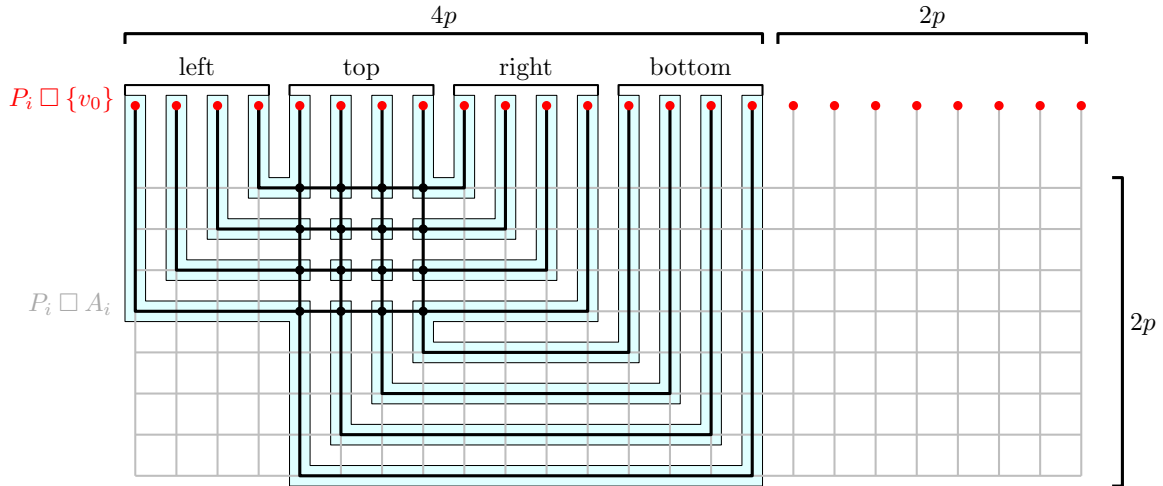


Figure 5: One of the $p \times p$ subgrids used in the proof of Lemma 6.

Our model is not yet complete. At this point, it is a model of a graph that consists of s^2 components, each of which is a $p \times p$ grid. At this point, the vertices in our (not yet-complete) model are all contained in T_0, T_1, \dots, T_{s^2} and, for each $i \in \{1, \dots, s^2\}$, the branch sets of vertices in G_i are contained $P_{i,i} \cup P_{i,0} \subseteq T_i \cup P_{i,0}$. This still leaves the vertices of $T_{s^2+1}, \dots, T_{5s^2}$ unused. We will use these to extend the branch sets of vertices on the boundary of each subgrid G_i to create the required adjacencies. For each $i \in \{1, \dots, s^2\}$, the vertices of $T_{s^2+4i-3}, \dots, T_{s^2+4i}$ will be reserved for the branch sets of G_i .

First, suppose that G_i is a subgrid that is immediately to the right of some subgrid G_j , so that the left boundary of G_i is adjacent to the right boundary of G_j . We will extend the branch sets for vertices on the left boundary of G_i so that they become adjacent to the branch sets for vertices in the right boundary of G_j . Let x_1, \dots, x_p be the vertices on the left boundary of G_i , ordered from top to bottom and, for each $k \in \{1, \dots, p\}$, let $x'_k := p_{i,k}$ so that (x'_k, v_0) is already included in the branch set for x_k . Let y_0, \dots, y_p be the vertices on the right boundary of G_j , ordered from top to bottom and, for each $k \in \{1, \dots, p\}$, let $y'_k := p_{j,2p+k}$ so that (y'_k, v_0) is already included in the branch set of y_k . There are two cases to consider. (We strongly urge the reader to refer to [Figures 6 and 7](#).)

- P_i and P_j are completely related (see [Figure 6](#)): We will extend the branch sets of x_1, \dots, x_p into T_{s^2+4i-3} . For each $k \in \{1, \dots, p\}$, we extend the branch set of x_k by adding the path

$$(x'_k, v_{s^2+4i-3,1}), \dots, (x'_k, v_{s^2+4i-3,k}) ,$$

the path in $T_{s^2+4i-3,k}$ from $(x'_k, v_{s^2+4i-3,k})$ to $(y'_k, v_{s^2+4i-3,k})$, and the path

$$(y'_k, v_{s^2+4i-3,k}), \dots, (y'_k, v_{s^2+4i-3,1}) .$$

The first vertex $(x'_k, v_{s^2+4i-3,1})$ of this path is adjacent to (x'_k, v_0) , which ensures that the branch set for x_k is connected. The last vertex $(y'_k, v_{s^2+4i-3,1})$ is adjacent to (y'_k, v_0) which ensures that the branch sets for x_k and y_k are adjacent.

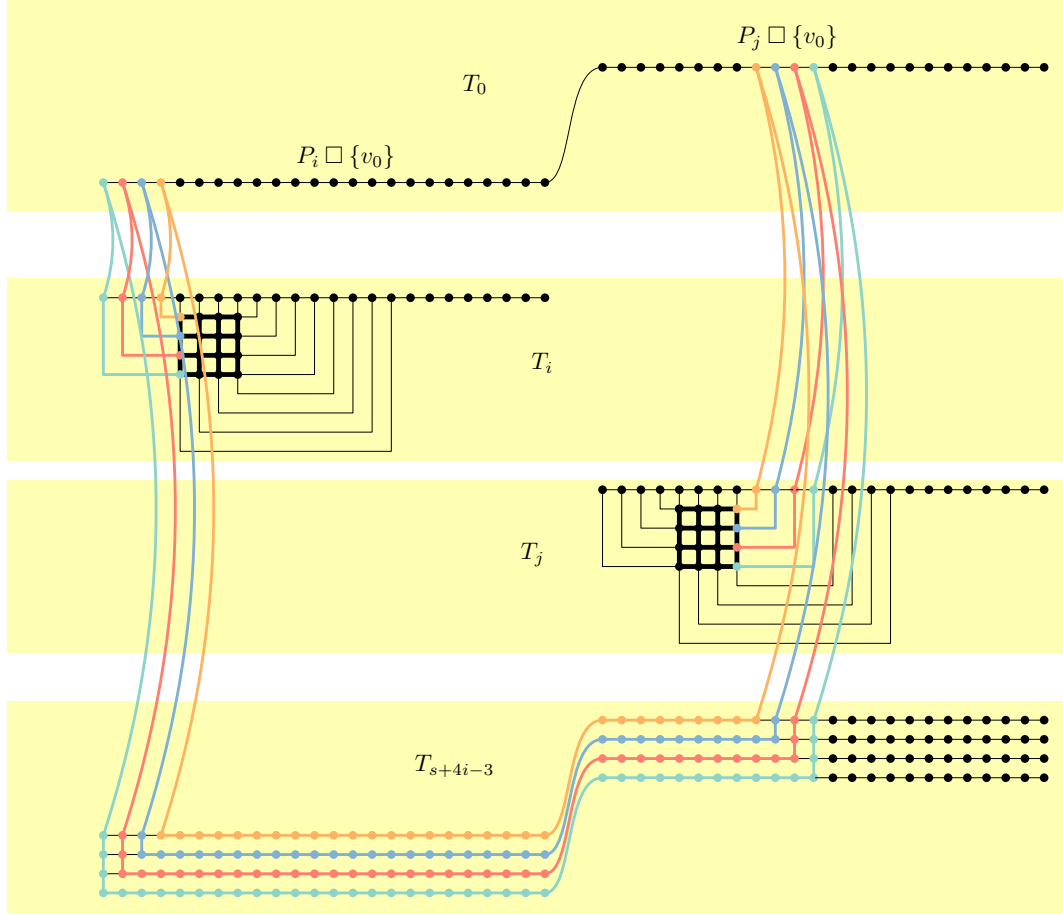


Figure 6: Connecting the left side of G_i to the right side of G_j when P_i and P_j are completely related.

- P_i and P_j are completely unrelated (see Figure 7): We will extend the branch sets of x_1, \dots, x_k into T_{s^2+4i-3} and T_{s^2+4i-2} . To make the connections between these two blocks we will use an additional p vertices of $P_{i,0}$. The need for a second block in this case is due to the fact that the obvious paths in T_{s^2+4i-3} that were used in the previous case would either intersect each other or reverse the order of connections so that the top-left vertex of G_i would become adjacent to the bottom-right vertex of G_j . Routing these paths through two trees allows us to make the connections in the right order using pairwise vertex-disjoint paths.

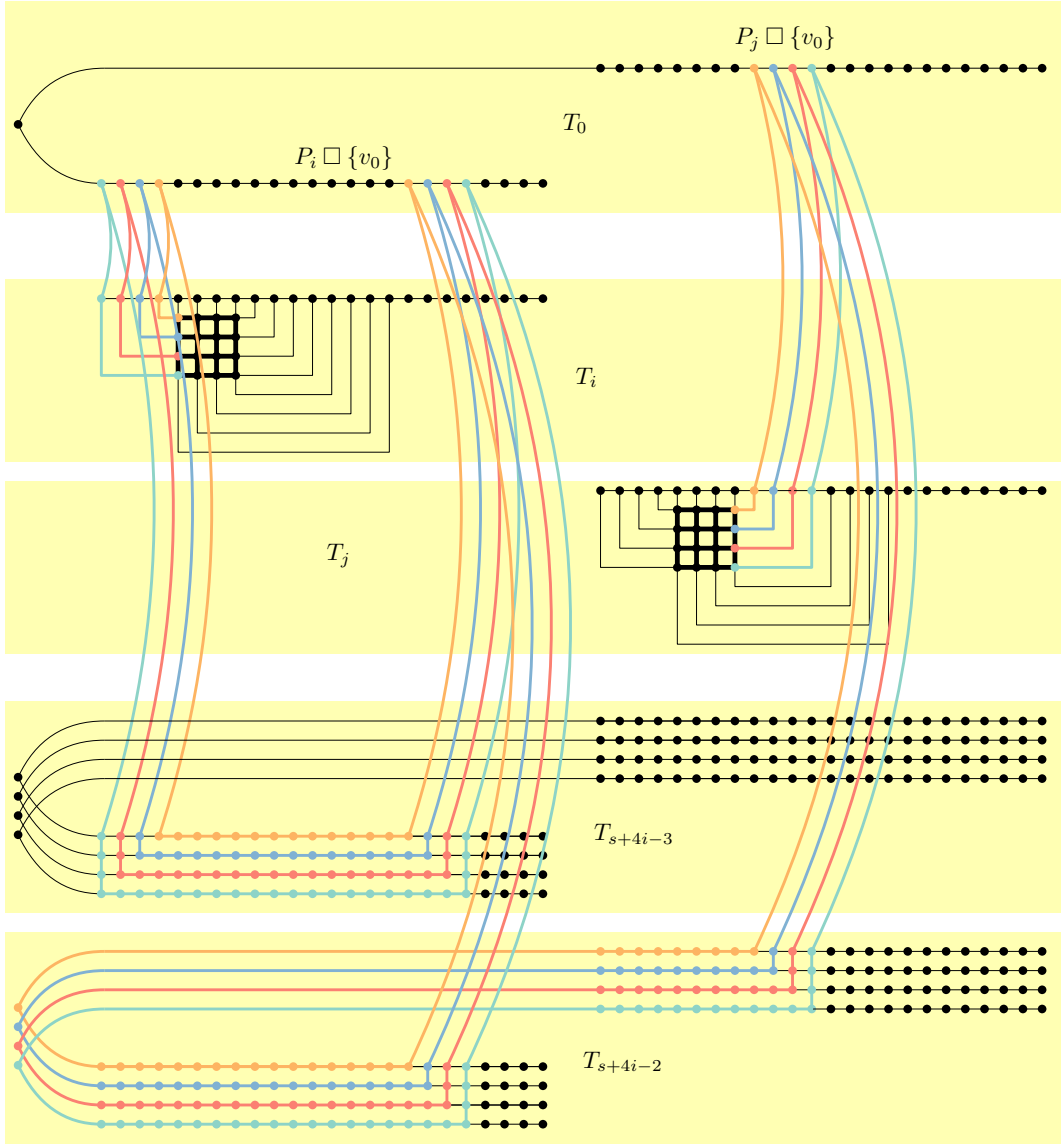


Figure 7: Connecting the left side of G_i to the right side of G_j when P_i and P_j are completely unrelated.

For each $k \in \{1, \dots, p\}$ we extend the branch set of x_k by adding the path,

$$(x'_k, v_{s^2+4i-3,1}), \dots, (x'_k, v_{s^2+4i-3,k}) ,$$

the path in $T_{s^2+4i-3,k}$ from $(x'_k, v_{s^2+4i-3,k})$ to $(p_{i,5p-k+1}, v_{s^2+4i-3,k})$, the path

$$(p_{i,5p-k+1}, v_{s^2+4i-3,k}), \dots, (p_{i,5p-k+1}, v_{s^2+4i-3,0}), (p_{i,5p-k+1}, v_{s^2+4i-2,1}), \dots, (p_{i,5p-k+1}, v_{s^2+4i-2,k})$$

the path in $T_{s^2+4i-2,k}$ from $(p_{i,5p-k+1}, v_{s^2+4i-2,k})$ to $(y'_k, v_{s^2+4i-2,k})$ and finally the path

$$(y'_k, v_{s^2+4i-3,k}), \dots, (y'_k, v_{s^2+4i-3,1}) .$$

As in the previous case, the first vertex of this path ensures that the branch set for x_k is connected and the last vertex ensures that the branch sets of x_k and y_k are adjacent.

So far, our model now models every horizontal grid edge but does not yet include the vertical edges between $p \times p$ subgrids. We now sketch how these can be included. Suppose that the subgrid G_i is directly below the subgrid G_j . Let x_1, \dots, x_p be the top boundary of G_i ordered so that x_1 is the leftmost vertex and x_p is the rightmost. For each $k \in \{1, \dots, p\}$, let $x'_k := (p_{i,p+k})$ so that the branch set of x_k includes x'_k . Let y_1, \dots, y_p be the bottom boundary of G_j ordered so that y_1 is the leftmost vertex and y_p is the rightmost. For each $k \in \{1, \dots, p\}$, let $y'_k := (p_{j,4p-k+1})$ so that the branch set of y_k includes y'_k . Observe that x'_1, \dots, x'_p occur in order along P_i but y'_1, \dots, y'_k occur in reverse order along P_j . The effect of this is to reverse the two cases that appear above, so that the straightforward case occurs when P_i and P_j are completely unrelated and the more complicated case occurs when they are completely related. Otherwise, the process of growing the branch sets for x_1, \dots, x_p is the same except that the vertices used to grow these new branch sets are contained in T_{s+4i-1} , T_{s+4i} , and $(p_{i,5p+1}, v_0), \dots, (p_{i,6p}, v_0)$. This ensures that these branch sets do not reuse vertices that are used to make G_i adjacent to the neighbour on its left.

Checking that the resulting collection of branch sets is indeed a model of the $r \times r$ grid is straightforward; both the disjointedness of the branch sets and the required adjacencies are guaranteed by the construction. \square

We now establish our lower bound on the largest grid minor in a Cartesian product:

Theorem 7. *For any connected graphs G_1 and G_2 each having at least $n \geq 1$ vertices,*

$$\text{gm}(G_1 \square G_2) \in \Omega(\sqrt{n}).$$

Proof. For each $b \in \{1, 2\}$, let T_b be a tree contained in G_b and having exactly n vertices (which can be constructed by successively deleting leaves starting with a spanning tree of G_b). For each $b \in \{1, 2\}$, let $p_b = \min\{i : n_i(T_b) \geq \frac{3n}{2(\pi i)^2}\}$. (This is well-defined since, otherwise $n = \sum_{i=1}^{\infty} n_i(T_b) < \sum_{i=1}^{\infty} \frac{3n}{2(\pi i)^2} = \frac{n}{4}$.) Without loss of generality, assume $p_2 \leq p_1$ and let $\ell := \lceil \frac{3n}{2(\pi p_2)^2} \rceil$. By [Observation 4](#), $S_{\ell, p_2} \leq T_2 \leq G_2$. If $p_2 \leq 5$ then $\ell > \frac{3n}{50\pi^2} \in \Omega(n)$ and by [Lemma 2](#) $K_\ell \leq G_1 \square S_{\ell, p_2}$. Since $\boxplus_{\sqrt{\ell}} \leq K_\ell$, this implies that $\text{gm}(G_1 \square G_2) \geq \sqrt{\ell} = \Omega(\sqrt{n})$ and we are done, so we may assume that $p_2 \geq 6$. Let $p := \lfloor p_2/6 \rfloor \geq 1$.

Since $p_1 \geq p_2$, $n_i(T_1) \leq \frac{3n}{2(\pi i)^2}$ for all $i \in \{1, \dots, p_2\}$. Therefore, [Lemma 5](#) implies that T_1 contains at least $n/4p_2$ pairwise disjoint paths $P_1, \dots, P_{\lceil n/4p_2 \rceil}$, each of length $p_2 \geq 6p$, such that each pair of paths is either completely related or completely unrelated. Let

$$s := \lfloor \min\{\sqrt{\ell/5}, \sqrt{n/4p_2}\} \rfloor = \Theta(\sqrt{n}/p)$$

so that $\ell \geq 5s^2$ and $\lceil n/4p_2 \rceil \geq s^2$. By [Lemma 6](#), $\text{gm}(T_1 \square S_{\ell, 6p}) \geq sp = \Theta(\sqrt{n})$. The lemma now follows from [Observation 1](#), the fact that $T_1 \leq G_1$, and the fact that $S_{\ell, 6p} \leq S_{\ell, p_2} \leq G_2$. \square

Our next result completes the relationships between grid minors and treewidth in Cartesian and strong products of trees.

Theorem 8. *For any two trees T_1 and T_2 ,*

$$\text{gm}(T_1 \square T_2) \leq \text{gm}(T_1 \boxtimes T_2) \leq \text{tw}(T_1 \boxtimes T_2) \in O(\text{gm}(T_1 \square T_2)^2) .$$

Proof. First note that $\text{gm}(T_1 \square T_2) \leq \text{gm}(T_1 \boxtimes T_2)$ since $T_1 \square T_2 \subseteq T_1 \boxtimes T_2$. Equation (1) shows that $\text{gm}(T_1 \boxtimes T_2) \leq \text{tw}(T_1 \boxtimes T_2)$. It remains to show that $\text{tw}(T_1 \boxtimes T_2) \in O(\text{gm}(T_1 \square T_2)^2)$.

Let $n_1 := |V(T_1)|$, let $n_2 := |V(T_2)|$, and assume without loss of generality that $n_1 \leq n_2$. By Lemma 3, $\text{tw}(T_1 \boxtimes T_2) \leq 2n_1 - 1$. By Theorem 7, $c \text{gm}(T_1 \square T_2) \geq \sqrt{2n_1}$ for some fixed positive constant c . Therefore,

$$(c \text{gm}(T_1 \square T_2))^2 \geq 2n_1 > \text{tw}(T_1 \boxtimes T_2) . \quad \square$$

It is worth pointing out that each of the inequalities in Theorem 8 is tight for certain trees T_1 and T_2 . The first two inequalities are tight for the product of two paths. Specifically, it is obvious that $\text{gm}(P_n \square P_n) = \text{gm}(P_n \boxtimes P_n) = n$, and $\text{tw}(P_n \boxtimes P_n) < 2n$ by (3). The last inequality is tight for S_n and P_n since $\text{tw}(S_n \boxtimes P_n) \in \Theta(n)$ by (2) and Lemma 3, and $\text{gm}(S_n \square P_n) \in \Theta(\sqrt{n})$ by Theorem 7 and Theorem 11 below.

4 Upper Bound

This section proves upper bounds of the form, $\text{gm}(G) \in O(\sqrt{n})$, where G is the product of various n -vertex graphs, as mentioned in Section 2.

Lemma 9. *Fix numbers $\Delta \geq c > 0$. Let \mathcal{G} be a graph class closed under minors and disjoint unions, such that $|E(H)| < c|V(H)|$ for every graph $H \in \mathcal{G}$. Let S be any star and H be any graph in \mathcal{G} . Let G be any graph with maximum degree Δ that is a minor of $S \cdot H$. Then*

$$|E(G)| < c|V(G)| + (\Delta - c)|V(H)|.$$

Proof. Let $(B_x : x \in V(G))$ be a model of G in $S \cdot H$. Let r be the root of S . Let R be the set of vertices x of G such that $(r, b) \in V(B_x)$ for some $b \in V(H)$. Let Q be the set of vertices x of G such that $V(B_x) \subseteq \{(v, b) : v \in V(S - r), b \in V(H)\}$. Thus $\{R, Q\}$ is a partition of $V(G)$. Moreover, $G[Q]$ is a minor of the disjoint union of n copies of H , implying $G[Q] \in \mathcal{G}$ and $|E(G[Q])| < c|Q|$. The number of edges of G incident to R is at most $\Delta|R|$. Thus $|E(G)| < c|Q| + \Delta|R| = c(|V(G)| - |R|) + \Delta|R| = c|V(G)| + (\Delta - c)|R| \leq c|V(G)| + (\Delta - c)|V(H)|$. \square

The class of graphs with treewidth at most t is closed under minors and disjoint unions, and $|E(H)| < t|V(H)|$ for every graph H with treewidth at most t . Lemma 9 implies:

Corollary 10. *Fix numbers $\Delta \geq t \geq 1$. Let S be any star and H be any graph with treewidth at most t . Let G be any graph with maximum degree Δ that is a minor of $S \cdot H$. Then*

$$|E(G)| < t|V(G)| + (\Delta - t)|V(H)|.$$

The next result completes the proof of the second part of our main theorem stated in [Section 1](#), showing that the lower bound in [Theorem 7](#) is optimal.

Theorem 11. *For any star S and any n -vertex tree T ,*

$$\text{gm}(S \square T) \leq \text{gm}(S \boxtimes T) \leq \text{gm}(S \cdot T) < \sqrt{3n+1} + 1.$$

Proof. The first two inequalities hold by definition. Let $k := \text{gm}(S \cdot T)$. Now apply [Corollary 10](#) with $t = 1$, and with $G := \boxplus_k$ and $\Delta = 4$. Thus

$$2k(k-1) = |E(G)| < |V(G)| + (\Delta-1)|V(T)| = k^2 + 3n.$$

Thus $k^2 - 2k < 3n$ and $k < \sqrt{3n+1} + 1$. □

We now show that results like [Theorem 11](#) can be concluded from results in the literature. (We include the above proofs for the sake of completeness and since [Lemma 9](#) and [Corollary 10](#) are of independent interest.)

A set S of vertices in a graph G is a *feedback vertex set* if $G - S$ is a forest. Luccio [45] proved that the minimum size of a feedback vertex set in \boxplus_k is $(\frac{1}{3} + o(1))k^2$. If \boxplus_k is a minor of $S \cdot T$ with $|V(T)| = n$, then \boxplus_k has a feedback vertex set of size at most n (consisting of the vertices of \boxplus_k whose branch sets intersect the copy of T corresponding to the root of S). Thus $n \geq (\frac{1}{3} + o(1))k^2$ and $k \leq (1 + o(1))\sqrt{3n}$, which implies [Theorem 11](#).

A result similar to [Theorem 11](#) can also be concluded from a more general result of Eppstein [29, Theorem 3], who proved that if $n > k/2$ and any set of n vertices are deleted from \boxplus_k , then the remaining graph contains a \boxplus_ℓ minor, where $\ell \geq \frac{k^2}{4n} - 1$. Say $k = \text{gm}(S \cdot T)$ where S is a star and T is any n -vertex tree. By the definition of $S \cdot T$, a set of at most n vertices can be deleted from \boxplus_k (corresponding to branch sets that intersect the copy of T at the root of S) so that each component of the remaining graph is a tree (a minor of T), and thus contains no \boxplus_2 minor. Note that $k \leq \text{tw}(S \cdot T) \leq 2n - 1$, so Eppstein's result is applicable. Hence $1 \geq \ell \geq \frac{k^2}{4n} - 1$, implying $k \leq \sqrt{8n}$. Eppstein's result is substantially stronger than [Theorem 11](#). For example, it implies:

Theorem 12. *For any star S and any n -vertex graph H ,*

$$\text{gm}(S \cdot H) \leq 2\sqrt{n(\text{tw}(H) + 1)}.$$

[Theorem 12](#) can be generalised to allow for arbitrary trees with bounded radius.

Theorem 13. *For any tree T with radius r and any n -vertex graph H with $E(H) \neq \emptyset$,*

$$\text{gm}(T \cdot H) \leq 5n^{1-1/2^r} \text{tw}(H)^{1/2^r}.$$

Proof. We proceed by induction on $r \geq 0$ (with H fixed). In the $r = 0$ case, $T = K_1$ and

$$\text{gm}(T \cdot H) \leq \text{tw}(H) = n^{1-1/2^0} \text{tw}(H),$$

as desired. Now assume $r \geq 1$ and the result holds for $r-1$. Let T be any tree T with radius r . Let $G := T \cdot H$ and let $k := \text{gm}(G)$. Let v the centre of T . Let T_1, \dots, T_p be the components of $T - v$. Let v_i be the neighbour of v in T_i . So v_i has eccentricity at most $r-1$ in T_i , and T_i has radius at most $r-1$. Let $k_i := \text{gm}(T_i \cdot H)$. By induction, for each $i \in \{1, \dots, p\}$,

$$\text{gm}(T_i \cdot H) \leq 5n^{1-1/2^{r-1}} \text{tw}(H)^{1/2^{r-1}}.$$

By the definition of $T \cdot H$, there is a set X of at most n vertices in G (corresponding to branch sets that intersect the copy of H at v) such that each component of $G - X$ is a minor of some $T_i \cdot H$. By Eppstein's result (which is applicable since $k \leq \text{tw}(G) \leq 2n-1$),

$$\frac{k^2}{4n} - 1 \leq \text{gm}(G - X) \leq \max_i \text{gm}(T_i \cdot H) \leq 5n^{1-1/2^{r-1}} \text{tw}(H)^{1/2^{r-1}}.$$

Since $n \geq 1$ and $\text{tw}(H) \geq 1$,

$$\frac{k^2}{4n} \leq 6n^{1-1/2^{r-1}} \text{tw}(H)^{1/2^{r-1}}$$

and

$$k \leq \sqrt{24n^{1-1/2^r} \text{tw}(H)^{1/2^r}} < 5n^{1-1/2^r} \text{tw}(H)^{1/2^r}.$$

The result follows. □

5 Product of Stars and Trees

Given the importance of products of stars and trees in the previous section, we now consider the following natural question: What is the least constant c such that for any star S and any n -vertex tree T ,

$$\text{gm}(S \square T) \leq (1 + o(1))\sqrt{cn} \quad (4)$$

Analogous questions are interesting for $S \boxtimes T$ and $S \cdot T$. [Theorem 11](#) gives an upper bound of $c \leq 3$ for $S \square T$, $S \boxtimes T$ or $S \cdot T$. For Cartesian products, we now show that $c = 2$ is the answer.

Lemma 14. *Let G be a bipartite graph with bipartition $\{A, B\}$. Let S be a star with at least $|A|$ leaves. Let T be any tree with at least $|B|$ vertices. Then G is a minor of $S \square T$.*

Proof. Let r be the root of S . Let f be an injection from A into the leaf-set of S . Let g be an injection from B into the vertex-set of T . For each vertex $v \in A$, define the branch set $B_v := \{f(v)\} \times V(T)$, which induces a connected copy of T in $S \square T$. For each vertex $w \in B$, define the branch set $B_w := \{(r, g(w))\}$. For each edge vw of G with $v \in A$ and $w \in B$, the edge $(f(v), g(w))(r, g(w))$ of $S \square T$ joins B_v and B_w . Hence $(B_v : v \in V(G))$ is a model of G in $S \square T$. □

Corollary 15. *For any tree T with n vertices, and for any star S with at least $n+1$ leaves,*

$$\text{gm}(S \square T) \geq \lfloor \sqrt{2n} \rfloor.$$

Proof. Let $k := \lfloor \sqrt{2n} \rfloor$. Let $\{A, B\}$ be the bipartition of \boxplus_k , where $|A| = \lceil \frac{k^2}{2} \rceil$ and $|B| = \lfloor \frac{k^2}{2} \rfloor$. So S has at least $n + 1 \geq \frac{k^2+1}{2} \geq |A|$ leaves, and T has at least $n \geq \frac{k^2}{2} \geq |B|$ vertices. By [Lemma 14](#), \boxplus_k is a minor of $S \square T$, and $\text{gm}(S \square T) \geq k$. \square

Lemma 16. *For any star S and any n -vertex tree T ,*

$$\text{gm}(S \square T) \leq \sqrt{2n} + 1.$$

Proof. Let $\mathcal{M} := \{B_x : x \in V(\boxplus_k)\}$ be a model of \boxplus_k in $G := S \square T$. Let v_0, \dots, v_μ denote the vertices of S , where v_0 has degree μ and v_1, \dots, v_μ are leaves. For each $i \in \{0, \dots, \mu\}$ let $R_i := \{v_i\} \square V(T)$, so that $T_i := G[R_i]$ is isomorphic to T . The idea behind the rest of this proof is that the vertices in R_0 are special because each vertex in R_0 is adjacent to the corresponding vertex in each of T_1, \dots, T_μ . This makes the vertices in R_0 a scarce resource that the model \mathcal{M} must use efficiently. Let $X_0 := \{x \in V(\boxplus_k) : B_x \cap R_0 \neq \emptyset\}$ be the set of vertices in \boxplus_k whose branch sets intersect R_0 .

Observe that $G - R_0$ is the vertex-disjoint union of trees T_1, \dots, T_μ . Therefore, for each component C of $\boxplus_k - X_0$, $G[\bigcup_{x \in V(C)} B_x]$ is a subtree of T_i for some $i \in \{1, \dots, \mu\}$. Let \mathcal{F} be the set of 4-cycles in \boxplus_k . (So \mathcal{F} is the set of inner faces in the usual plane drawing of \boxplus_k .) Observe that, for each $F \in \mathcal{F}$, $|V(F) \cap X_0| \geq 1$ since otherwise $V(F) \subseteq X \setminus X_0$, so $\bigcup_{x \in V(F)} B_x \subseteq V(G - R_0)$ which implies that F is a minor of $G - R_0$. This is not possible since F is a cycle and $G - R_0$ is a forest.

Next we will show that $|\bigcup_{x \in V(F)} B_x \cap R_0| \geq 2$ for each $F \in \mathcal{F}$. If $|V(F) \cap X_0| \geq 2$, then this is immediate, so suppose that $|V(F) \cap X_0| = 1$. Refer to [Figure 8](#). Let $F := xx_1x_2x_3$ where $x \in X_0$ and $x_1, x_2, x_3 \in X \setminus X_0$. Since $G[\{x_1, x_2, x_3\}]$ is connected, there exists an $i \in \{1, \dots, \mu\}$ such that $T'_j := G[B_{x_j}]$ is a subtree of T_i for each $j \in \{1, 2, 3\}$. Furthermore, the subtree T'_2 is “between” T'_1 and T'_3 in the sense that B_{x_1} and B_{x_3} are in different components of $T_i - B_{x_2}$. Since $N_{\boxplus_k}(x)$ contains both x_1 and x_3 , B_x contains vertices in $N_G(B_{x_1})$ and in $N_G(B_{x_3})$. Since $G[B_x]$ is connected, this implies that $G[B_x]$ contains a path from some vertex in $N_G(B_{x_1})$ to some vertex in $N_G(B_{x_3})$. Since $G[B_x]$ is a subgraph of $G - B_{x_2}$, this implies that any such path must contain at least two vertices of R_0 .⁷ Thus $|\bigcup_{x \in V(F)} B_x \cap R_0| = |B_x \cap R_0| \geq 2$, as claimed.

Now we can finish the proof with

$$\sum_{F \in \mathcal{F}} \sum_{x \in V(F)} |B_x \cap R_0| \geq \sum_{F \in \mathcal{F}} 2 = 2|\mathcal{F}| = 2(k-1)^2.$$

Each vertex in R_0 appears in B_x for at most one $x \in V(\boxplus_k)$, and each such x appears in $V(F)$ for at most four cycles $F \in \mathcal{F}$, so

$$\sum_{F \in \mathcal{F}} \sum_{x \in V(F)} |B_x \cap R_0| \leq 4 \sum_{x \in V(\boxplus_k)} |B_x \cap R_0| \leq 4|R_0| = 4n.$$

⁷This is true for the Cartesian product $S \square T$, but not true for the strong product $S \boxtimes T$, since $N_{S \boxtimes T}(B_{x_1})$ and $N_{S \boxtimes T}(B_{x_3})$ can have a common vertex in T_0 . This fact is used in the construction described in [Lemma 17](#).

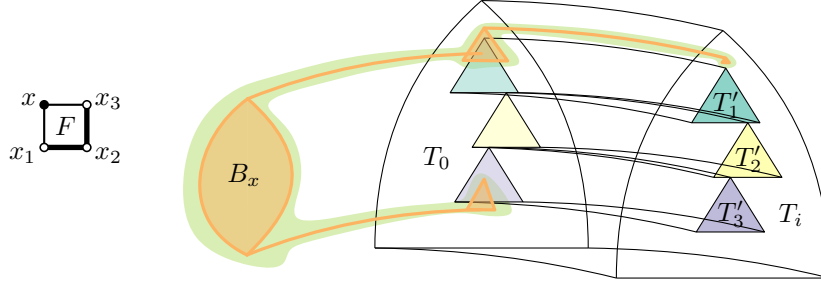


Figure 8: The proof of Lemma 16.

Combining the previous two equations gives $4n \geq 2(k-1)^2$, so $k \leq \sqrt{2n} + 1$. \square

Corollary 15 and Lemma 16 together show that $c = 2$ in (4).

Now consider (4) for the strong product $S \boxtimes T$. We now show that the answers for Cartesian and strong products are different. In particular, for $S \boxtimes T$ the minimum c in (4) satisfies $\frac{5}{2} \leq c \leq 3$.

Lemma 17. *For any star S with at least $\sqrt{10(n-2)} + 1$ leaves and any path P on at least n vertices,*

$$\text{gm}(S \boxtimes P) \geq \lfloor \sqrt{5(n-2)/2} \rfloor$$

Proof. Let $k := \lfloor \sqrt{5(n-2)/2} \rfloor$ and note that $n \geq 2k^2/5 + 2$. Let $G := S \boxtimes P$. Recall that $V(\boxtimes_k) = \{1, \dots, k\}^2$. For each $i \in \{0, \dots, 2k-2\}$, let $D_i := \{(x, y) \in V(\boxtimes_k) : y - x = k - i - 1\}$, and for convenience let $D_i = \emptyset$ for any $i \notin \{0, \dots, 2k-2\}$. (The vertices of each D_i are contained in a line of slope 1.) Let the vertices of S be v_0, \dots, v_{2k+1} , where v_0 is the vertex of degree $2k+1$ and v_1, \dots, v_{2k+1} are the leaves.

For each $\ell \in \{0, \dots, 4\}$, let $\alpha_\ell := |\bigcup_{a \in \mathbb{Z}} D_{\ell+5a+1} \cup D_{\ell+5a+2}|$. Observe that $\sum_{\ell=0}^4 \alpha_\ell = 2|V(\boxtimes_k)| = 2k^2$, since each diagonal D_i contributes to this sum exactly twice (when $i - \ell \equiv 1 \pmod{5}$ and when $i - \ell \equiv 2 \pmod{5}$). Therefore, $\alpha_\ell \leq 2k^2/5$ for some $\ell \in \{0, \dots, 4\}$. For the sake of brevity, assume that $\alpha_0 \leq 2k^2/5$. (The only reason for this assumption is to avoid having to include an ℓ term in many of the subscripts in the following paragraphs.)

We now explain how to embed a small (width-4) strip B_a of \boxtimes_k into a small subgraph $S \subseteq G[\{v_0, v_1\} \times V(P)]$ of G . This will allow us to decompose \boxtimes_k into disjoint independent strips and embed them all on $G[\{v_0, v_1\} \times V(P)]$. We will then complete the embedding by embedding the remaining (width-1) strips into $G[\{v_2, \dots, v_{k+1}\} \times V(P)]$. Fix some integer $a \in \{0, \dots, \lfloor 2k/5 \rfloor - 1\}$, and consider the induced subgraph $B_a := \boxtimes_k[D_{5a} \cup \dots \cup D_{5a+3}]$. Let $S' := S[\{v_0, v_1\}]$ and let P'_a be any subpath of P with $|D_{5a+1} \cup D_{5a+2}|$ vertices. As illustrated in Figure 9, $G' := S' \boxtimes P'_a$ contains a subgraph isomorphic to B_a , where the isomorphism $\varphi : B_a \rightarrow G'$ maps the vertices of $D_{5a} \cup D_{5a+3}$ onto the vertices in $\{v_0\} \times V(P'_a)$. (Note that this makes use of the fact that $|D_{5a} \cup D_{5a+3}| \leq |D_{5a+1} \cup D_{5a+2}|$, valid for all $a \in \{0, \dots, 2k-5\}$. Since P'_a uses only $|D_{5a+1} \cup D_{5a+2}|$ vertices of P , this immediately implies that $G[\{v_0, v_1\} \times V(P)]$

contains a subgraph isomorphic to $X := \boxplus_k - \bigcup_{a \geq 0} D_{5a+4}$.⁸ Furthermore, the vertex disjoint subpaths P'_a , $a \in \mathbb{Z}$ can be chosen so that P'_a and P'_{a+1} are always consecutive in P .

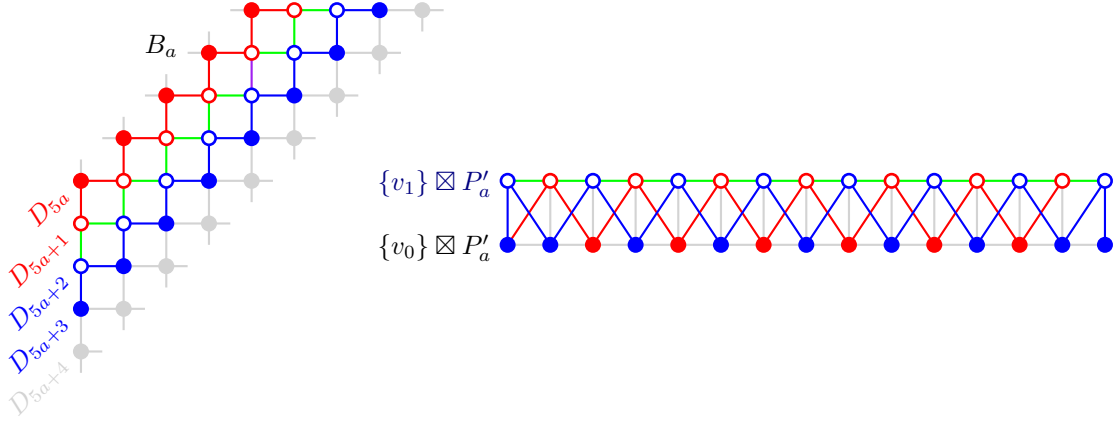


Figure 9: The construction in Lemma 17.

By setting $B_x := \{\varphi(x)\}$ for each $x \in V(X)$ we immediately obtain a model $\{B_x : x \in V(X)\}$ of X in G . We can complete this model of X to a model of \boxplus_k by mapping, for each $a \equiv 4 \pmod{5}$, the at most k vertices in D_{5a+4} to subsets of $\{v_2, \dots, v_{2k+1}\} \times P$. More precisely, let x_1, \dots, x_r be the vertices in D_{5a+4} . These vertices have neighbours in D_{5a+3} and in $D_{5(a+1)}$. The vertices in D_{5a+3} have branch sets in $\{v_0\} \times V(P'_a)$. The vertices in $D_{5(a+1)}$ have branch sets in $\{v_0\} \times V(P'_{a+1})$. If a is even then we set $B_{x_i} := \{v_{2i}\} \times (V(P'_a) \cup V(P'_{a+1}))$. If a is odd then we set $B_{x_i} := \{v_{2i+1}\} \times (V(P'_a) \cup V(P'_{a+1}))$. This ensures that B_{x_i} is adjacent to the branch sets of x_i 's neighbours in D_{5a+3} and $D_{5(a+1)}$. Using different subsets of v_2, \dots, v_{2k+1} for odd and even values of a ensures that the branch sets of vertices in D_{5a+4} are disjoint from those of vertices in $D_{5(a+1)+4}$. This completes the proof. \square

To complete this section, we now show that for lexicographic products $S \cdot T$, the answer to (4) is $c = 3$. By Theorem 11, it suffices to prove the following (where $P_3 = S_2$ is the star with two leaves):

Lemma 18. \boxplus_k is isomorphic to a subgraph of $P_3 \cdot P_n$, where $k = \lfloor \sqrt{3n-2} \rfloor$.

Proof. Recall that \boxplus_k has vertex-set $\{1, \dots, k\}^2$. For each $i \in \mathbb{Z}$, let $D_i := \{(x, x+i) : x, x+i \in \{1, \dots, k\}\}$. Each D_i is an independent set in \boxplus_k contained in a diagonal line of slope 1. As illustrated in Figure 10, let $S := \bigcup \{D_i : i \equiv 0 \pmod{3}\}$, which is an independent set in \boxplus_k . Let $A := \bigcup \{D_i : i \not\equiv 0 \pmod{3}, i > 0\}$ and $B := \bigcup \{D_i : i \not\equiv 0 \pmod{3}, i < 0\}$. Each of A and B induce linear forests in \boxplus_k . Note that S, A, B partitions $V(\boxplus_k)$, and

⁸Recall that P has $n \geq 2k^2/5 + 2 \geq \alpha_0 + 2$ vertices. The additional two vertices are required for two boundary cases where we can only guarantee that $|D_{\ell+5a+1} \cup D_{\ell+5a+2}| + 1 \geq |D_{\ell+5a} \cup D_{\ell+5a+3}|$. These cases can occur when $\ell + 5a = -2$ (because $|D_{-1} \cup D_0| = 1$ and $|D_{-2} \cup D_1| = 2$) and when $\ell + 5a = 2k - 3$ (because $|D_{2k-2} \cup D_{2k-1}| = 1$ and $|D_{2k-3} \cup D_{2k}| = 2$).

$\lfloor k^2/3 \rfloor \leq |S|, |A|, |B| \leq \lceil k^2/3 \rceil \leq n$. Consider the 3-vertex path $P_3 = (a, s, b)$. Injectively map S to the copy of P_n corresponding to s . Injectively and homomorphically map A to the copy of P_n corresponding to a . Injectively and homomorphically map B to the copy of P_n corresponding to b . This defines an injection from $V(\boxplus_k)$ to $V(P_3 \cdot P_n)$. Since there is no edge of \boxplus_k between A and B , each edge of \boxplus_k is mapped to an edge of $P_3 \cdot P_n$, and \boxplus_k is isomorphic to a subgraph of $P_3 \cdot P_n$. \square

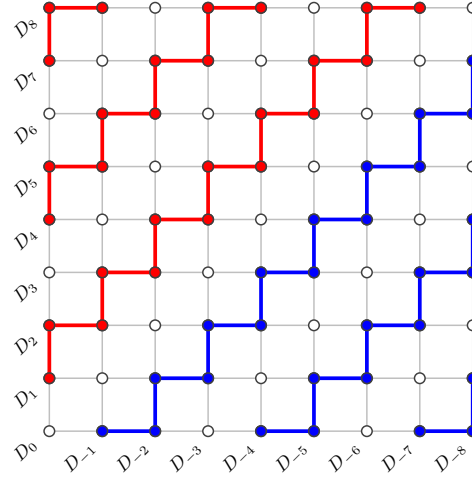


Figure 10: The proof of Lemma 18 with $k = 9$: vertices in S are white, red paths are in A , and blue paths are in B .

6 Open Problems

A first open problem is to tighten the bounds for $\text{gm}(S \boxtimes T)$ presented in Theorem 11 and Lemma 17, which combine to show that $\sqrt{5(n-2)/2} \leq \text{gm}(S \boxtimes T) \leq \sqrt{3n+1} + 1$. This would fully resolve the discussions resulting from (4).

Another area of future work is to further investigate the Planar Graph Product Structure Theorem, which we recall states that for every planar graph G , there exists a graph H of bounded treewidth and a path P such that $G \subseteq H \boxtimes P$. A specific area to investigate is identifying which properties of G can be preserved in $H \boxtimes P$. Several results of this type are known. For example, in the proof of Dujmović et al. [23], H is a minor of G , and so H is planar. An impossibility result in this area is the following: Even if G is planar and has maximum-degree 5, a result of the form $G \subseteq H \boxtimes P$ cannot guarantee that H has bounded treewidth and bounded degree [24].

A concrete question that remains open is whether the treewidth of G can be preserved in the product: Is it true that for every planar graph G , there exists a bounded treewidth graph H and a path P such that $G \subseteq H \boxtimes P$ and $\text{tw}(H \boxtimes P) \in O(\text{tw}(G))$? Note that

$$\Omega(\min\{|V(H)|, |V(P)|\}) \leq \text{tw}(H \boxtimes P) \leq O(\min\{|V(H)|, |V(P)|\}).$$

This upper bound follows from [Lemma 3](#) since both H and P have bounded treewidth. The lower bound follows from (2) since we may assume that G , H and P are connected. So this question really asks whether for every planar graph G , there exists a bounded treewidth graph H and a path P such that $G \subseteq H \boxtimes P$ and $\min\{|V(H)|, |V(P)|\} \leq O(\text{tw}(G))$. It is even open whether $\min\{|V(H)|, |V(P)|\} \leq f(\text{tw}(G))$ for some function f , or whether $\min\{|V(H)|, |V(P)|\} \leq O(\sqrt{|V(G)|})$ (which would be implied since $\text{tw}(G) \leq O(\sqrt{|V(G)|})$ for every planar graph G).

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