LITERATURE REVIEW: — Distributed Graph Colouring —

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October 6, 2021

1 Introduction

Graph colouring, the problem of assigning a colour to each vertex of a graph such that no two neighbouring vertices share the same colour, is one of the most important problems in graph theory. From the proposal of the Four Colour Theorem in the mid-1850s until today, graph colouring has been a fundamental problem in graph theory, and the focus of an incredible amount of research. The optimization aspect of this problem is identifying what the minimum amount of colours is needed to create such a proper colouring on a graph.

Furthermore, graph colouring sees many applications in scheduling and identifying groups in social graphs. Unfortunately, the problem of deciding if a graph has a k-colouring (can be coloured with k colours) is NP-complete, and thus no polynomial algorithms for finding graph colourings exist for general cases. This makes finding ways to improve current state-of-the-art algorithms incredibly important, as these algorithms need to run efficiently on graphs with massive scale.

Distributed Graph Colouring is an area looking to solve this problem by parallelizing algorithms for graph colouring. While this introduces additional considerations regarding the parallel model under consideration, the speed-ups obtained as a result are strongly beneficial. Thus, the development of simple, efficient distributed graph colouring algorithms is a large research area, with entire books dedicated solely to the topic [1].

2 Literature Review

2.1 Bounding Distributed Graph Colouring

Currently, the majority of work in distributed graph colouring focuses on finding a k-colouring such that $\Delta + 1 \leq k \leq O(\Delta^2)$, where Δ is the maximum valency of a vertex. The lower bound results from the common fact that any graph where the maximum degree is Δ can be coloured with $\Delta + 1$ colours, with simple greedy algorithms able to find such a colouring. The upper bound is a result of Linial, who proposed a model for distributed graph algorithms called the LOCAL model [3]. This model is round-based, allowing for each vertex to transmit information to each of its neighbours at the end of each round. Using this model, Linial shows that an $O(\Delta^2)$ colouring can be generated in $O(\log^* n)$ rounds, where $\log^*$ is the iterative logarithm.

2.2 Improving Colourings

While the $O(\Delta^2)$ bound proposed above is large, this is acceptable due to the presence of colour-reduction algorithms that can be used to reduce the amount of colours in the colouring. Working under the LOCAL model Linial proposed, many researchers focus on improving the efficiency of the colour-reduction algorithms so that, in combination with the colouring algorithms, more optimal graph colourings can be obtained without large trade-offs in speed. Linial himself once again sets strong foundations for this area, presenting an algorithm to reduce a k-colouring of a graph to a $O(\Delta^2 \log m)$-colouring in a single round under his model [3].

From there, much research focuses on one-round colour reduction algorithms that can reduce the colours significantly for certain classes of graphs. For example, one-round colour reduction algorithms were developed for directed paths that can reduce k-colourings to 3-colourings in $\frac{1}{2} \log^* n + O(1)$ rounds [2]. This was later reduced into a tight bound of $\frac{1}{2} \log^* n$ [4]. This result is an important due to its usefulness as a subroutine of other distributed graph colouring algorithms. Other improvements look towards specific classes of graphs [] or reductions from k-colourings under some set of assumptions about k. For example, Yannic proposes a one-round colouring algorithm that reduces an k-colouring to a $m(\Delta - m + 2)$-colouring, given that $k \geq m(\Delta - m + 3)$, removing m colours from the colouring, where $1 \leq m \leq \Delta/2 + 3/2$ [5].

2.3 Improving Colouring Algorithms

References

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