Geometry $\rightsquigarrow$ Algebra

Space $\longrightarrow$ Group

Floer cohomology

<u>Q</u>: How do geometric symmetries relate to algebraic symmetries?

<u>Fundamental</u> group π_1

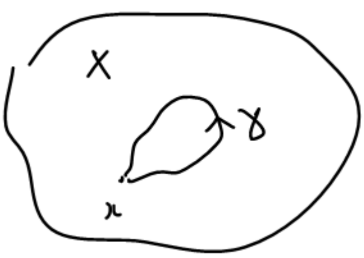
Space X, base point $x \in X$

$\pi_1(X, x)$ group

$\|$

$$\underbrace{\text{loops in } X, \text{ based at}}\ x \Big/ \text{homotopy}$$

continuous deformations



continuous maps $\gamma : [0,1] \to X$

$\gamma(0) = \gamma(1) = x$

Group operation is concatenation

Q: How is γ^{-1} the inverse

$$\gamma^{-1}(t) = \gamma(1-t)$$

Identity = [constant loop]

Homotopy is 1-parameter family.

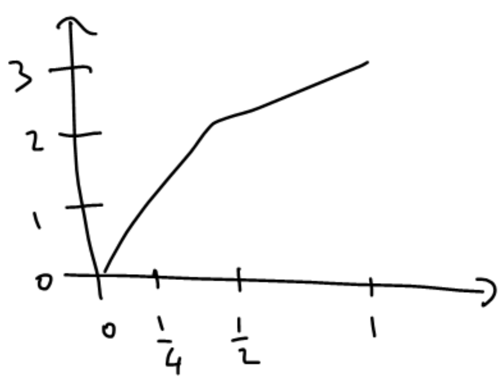
Homotopy from γ_0 to γ_1 should be a continuous map $[0,1]\times[0,1] \longrightarrow X$

$$\gamma(t,0) = \gamma_0(t), \quad \gamma(t,1) = \gamma_1(t)$$

$$\gamma(0,s) = \gamma(1,s) = x \quad \forall s$$

$(\gamma_1\gamma_2)\gamma_3 \qquad \gamma_1(\gamma_2\gamma_3)$

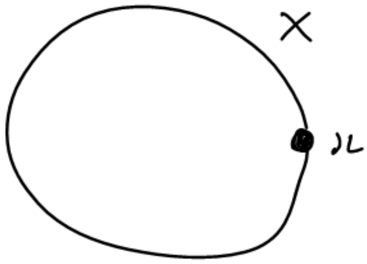
Not equal but homotopic.



$X = pt$ $\quad \pi_1 =$ trivial group

$X =$ contractible $\quad$ —— " ——

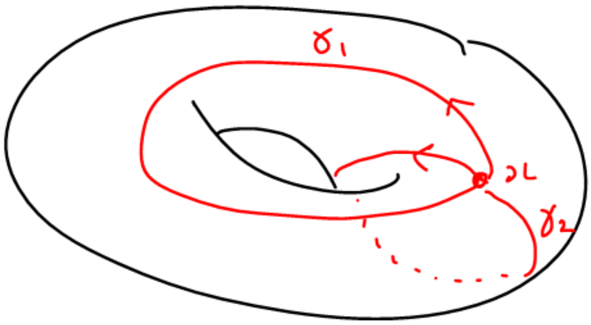
$X = S^1$



$\pi_1 = \mathbb{Z}$

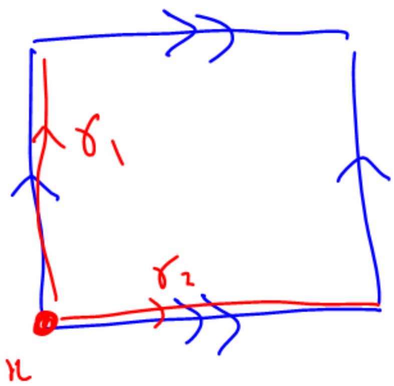
$$\pi_1 \ni (t \mapsto e^{2\pi i n t}) \longmapsto n \in \mathbb{Z}$$

$X = T^2$



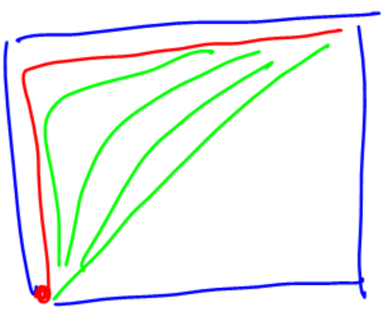
Claim: $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$

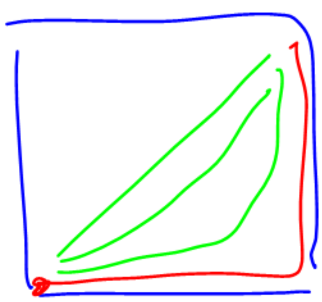
γ_1 $\quad$ γ_2



$$(T^2 = S^1 \times S^1)$$

$\gamma_1\gamma_2$

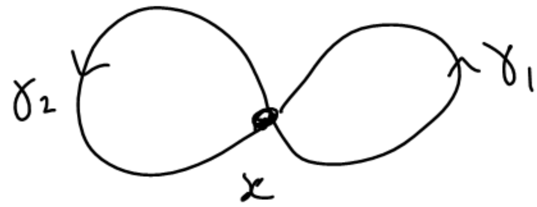




$\gamma_2\gamma_1$

$X = S^1 \vee S^1$

figure of 8



$$\pi_1 = \langle \gamma_1, \gamma_2 \rangle \quad \text{free group } F_2$$

Functoriality: maps between spaces give maps between associated algebraic objects.

Given $f : X \to Y$ continuous

Want $f_* : \pi_1(X, x) \longrightarrow \pi_1(Y, f(x))$ group hom

$$[\gamma] \longmapsto [f \circ \gamma]$$

$f = id : X \to X$

$id_* : \pi_1(X, x) \longrightarrow \pi_1(X, x)$ is identity on $\pi_1(X, x)$

$f : X \to Y, \quad g : Y \to Z$

$$(g \circ f)_* = g_* \circ f_*$$

If f is an automorphism of X has an inverse
then f_* is an automorphism of $\pi_1(X,x)$.

(Inverse of f_* is $(f^{-1})_*$)

Get map

$$\varphi : \underset{\text{of } (X,x)}{\text{Automorphism group}} \longrightarrow \underset{\text{of } \pi_1(X,x)}{\text{Automorphism group}}$$

$$f \longmapsto f_*$$

This a homomorphism.

Example $X = S^1$

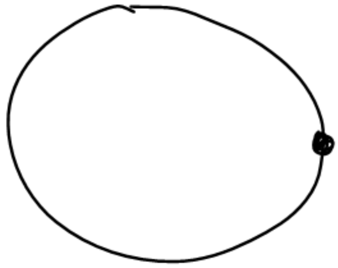
$\pi_1 = \mathbb{Z}$

$\mathrm{Aut}(\mathbb{Z}) = \{ id, \ n \mapsto -n \} = \mathbb{Z}/2$

algo

[$\mathrm{Aut}_{set}(\mathbb{Z})$ = infinite symmetric group

$\mathrm{Aut}_{ring}(\mathbb{Z}) = \{id\}$]



$S^1 \subset \mathbb{C}$

$z \longmapsto \bar{z}$

$e^{2\pi i n t} \longmapsto e^{2\pi i (-n) t}$

Q1: What is $\mathrm{Aut}_{\text{ab gp}}(\mathbb{Z}^2)$?

Is $\mathrm{Aut}(T^2) \longrightarrow \mathrm{Aut}(\mathbb{Z}^2)$ surjective?

Q2: ______________ $\mathrm{Aut}_{gp}(F_2)$?

______ $\mathrm{Aut}(S^1 \vee S^1) \longrightarrow \mathrm{Aut}_{gp}(F_2)$?

Q3: $S^1 \vee S^1 \longrightarrow T^2$

What is this map doing?

How does the commutativity relation arise?

Q4: Can you find a space X where $\pi_1(X)$ has torsion?

Ring R abgp under $+$, 0

multiplication $\times$

- $\times$ distributes over $+$

$$a \times (b+c) = (a\times b)+(b\times c)$$

- $\times$ is associative and commutative
- there's a 1

An R-Module M is a "vector space over R"

- $M, +, 0$ is an abgp
- can multiply by "scalars" in R

eg a $\mathbb{C}$-module is a complex v.s.

$\mathbb{Z}$-module is an abelian group

R ring, M, N R-modules

Define: The tensor product $M \otimes_R N$

is the R-module generated by

$m \otimes n \quad m \in M, \; n \in N$

modulo relations:

- $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$
- $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$
- $r(m \otimes n) = (rm) \otimes n = m \otimes (rn)$

Example 1: $M \otimes_R R \cong M$

$$m \otimes r \longmapsto rm$$

$$m \otimes 1 \longleftarrow\!\!| \; m$$

Example 2: $M \otimes_R R^n \cong M^n$

$$m \otimes (r_1, \dots, r_n) \longmapsto (r_1 m, \dots, r_n m)$$

$$\begin{aligned} &m_1 \otimes (1, 0, \dots, 0) \\ &+ m_2 \otimes (0, 1, \dots, 0) \\ &+ \dots + m_n \otimes (0, \dots, 0, 1) \end{aligned} \longleftarrow\!\!| \; (m_1, \dots, m_n)$$

Question 5: $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Q}$

Q6: $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Z}/p$ $\quad p$ prime

Q7: $\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n$

Q8: Is there a non-zero abgp A such that $A \otimes_{\mathbb{Z}} A = 0$?

Q9: $f: M \to N$ R-module map

get "$f \otimes \mathrm{id}_P$": $M \otimes_R P \longrightarrow N \otimes_R P$

$$m \otimes p \longmapsto f(m) \otimes p$$

(a) if f is surjective, must $f \otimes \mathrm{id}$ be surjective?

(b) __________ injective _____

__________ injective?