1. $GL(2,\mathbb{Z}) = \mathrm{Aut}(\pi_1)$

$$\cap$$

$\mathrm{Aut}(T^2)$

$\mathrm{Homeo}(T^2)$ homeomorphisms

$\mathrm{Diff}(T^2)$ diffeomorphisms

$$\mathrm{Aut}(T^2) \big/ \mathrm{Aut}_0(T^2) =: MCG(T^2)$$

$$GL(2,\mathbb{Z}) = MCG(T^2)$$

mapping class group

automorphisms homotopic to id

X space $\qquad \mathrm{Aut}(X) \big/ \mathrm{Aut}_0(X) \longrightarrow \mathrm{Aut}(\pi_1)$

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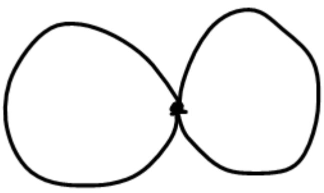
2. 

$\mathrm{Aut}(F_2)$ nasty

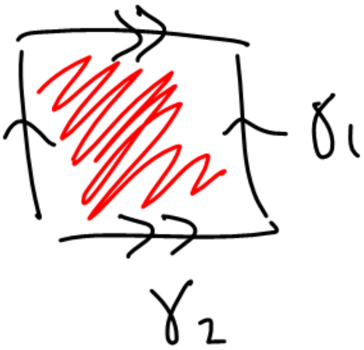
$$\cup$$

$$F_2$$

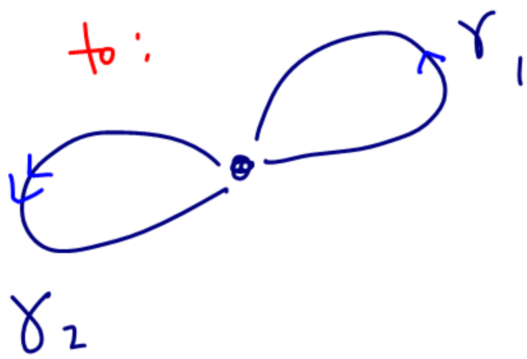
$$F_3 \hookrightarrow F_2$$

<u>Q</u>: Can you embed F_∞ in F_2?

3. $S^1 \vee S^1 \longrightarrow T^2$



Glue disk D^2 to:



$$\partial D^2 \rightsquigarrow \gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1}$$

$$\langle \underbrace{\gamma_1, \gamma_2}_{\pi_1(S^1 \vee S^1)} \mid \gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1} \rangle = \underbrace{\mathbb{Z}^2}_{\pi_1(T^2)}$$

CW complex / cell complex

Start with some points X_0

Glue some intervals to these points – obtain X_1

Glue some 2d disks to X_1 to obtain X_2

(via maps $\partial D^2 \to X_1$)

Glue in some 3d disks

⋮

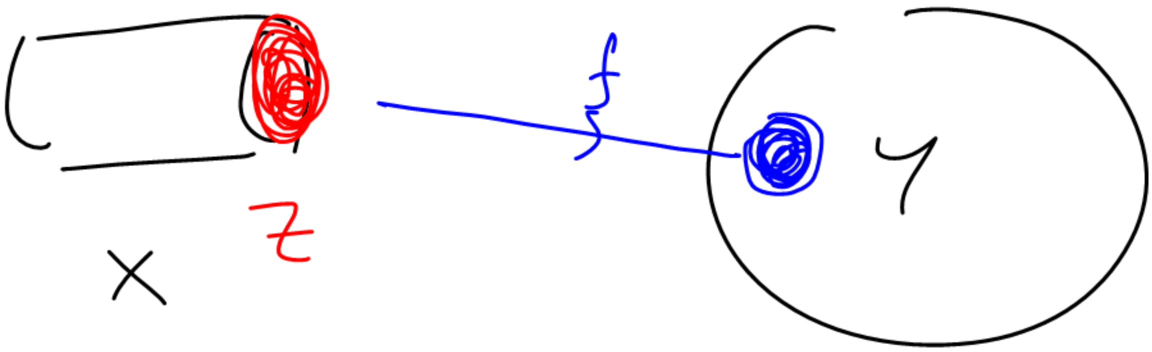
$\longrightarrow$ obtain a space X

Spaces X, Y, $Z \subset X$.

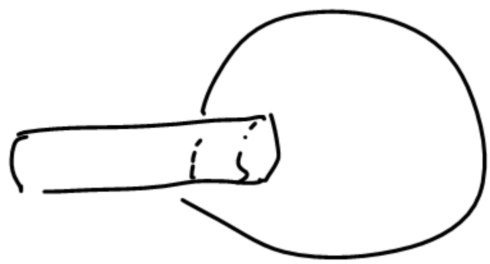
Map $f : Z \to Y$.

Can "glue X to Y along f":

$$X \cup Y \big/ z \sim f(z)$$



$\rightsquigarrow$



The i-dimensional disks are called i-cells

X_i = i-skeleton

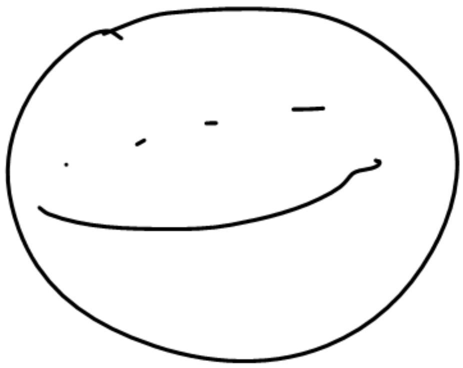
<u>Fact</u>: $\pi_1(X)$ depends only on X_2

Generators come from X_1

$\pi_1(X_1)$ is a free group

Relations come from 2-cells

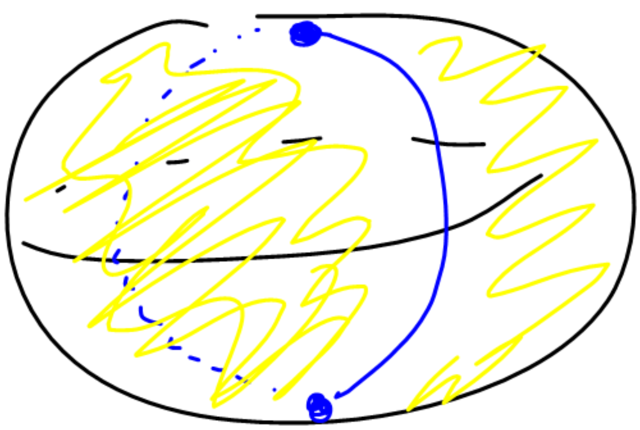
4.

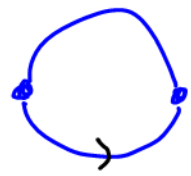
$\mathbb{R}P^2 =$  $/ \; x \sim -x$

$S^2 =$  $=$ 0-cell $\cup$ 2-cell

$\pi_1(S^2) = 1$.

S^2



$X_1 =$  S^1

$\pi_1(X_1) = \langle \gamma \rangle$

Gluing maps for two 2-cells are

$\gamma \, , \, \gamma^{-1}$

$\mathbb{RP}^2$

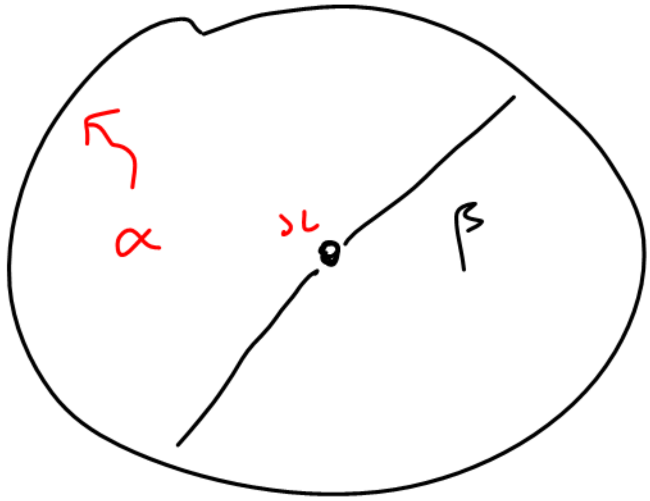
1 0-cell
1 1-cell
1 2-cell

$X_1 = S^1$

$\pi_1(X_1) = \langle \beta \rangle$

Gluing map for 2-cell is $\gamma = \beta^2$

$$\pi_1(\mathbb{RP}^2) = \langle \beta \mid \beta^2 \rangle = \mathbb{Z}/2$$

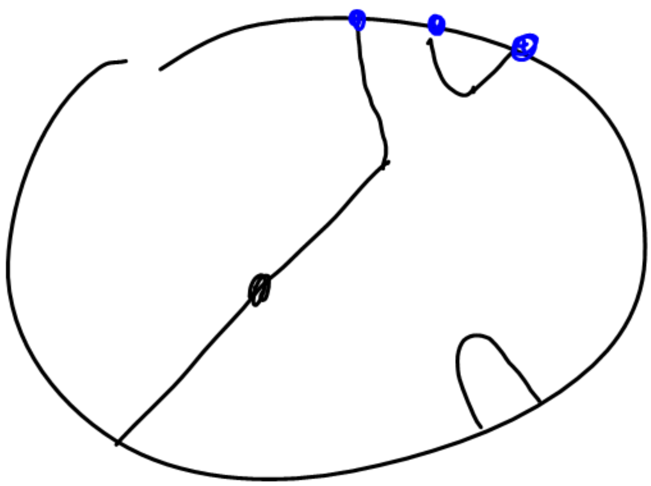


Prove β can't be contracted

$\#(\alpha \cap \beta) = \underline{1}$

Can show that # intersections between α and loop based at x cannot change mod 2 by homotoping the loop

(If intersections are transverse)



7. $\mathbb{Z}/_m \otimes \mathbb{Z}/_n$

Classification of f.g. abelian gps:

$$A \cong \mathbb{Z}/_{d_1} \oplus \cdots \oplus \mathbb{Z}/_{d_k} \oplus \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{m}$$

$$1 < d_1 \mid d_2 \mid \cdots \mid d_k$$

$$A \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Z}^m \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^m$$

$\therefore$ # $\mathbb{Z}$'s in A is $\dim_{\mathbb{Q}} A \otimes_{\mathbb{Z}} \mathbb{Q}$

$A \otimes_{\mathbb{Z}} \mathbb{Z}/_{(p)}$ is a $\mathbb{Z}/_{(p)}$-vector space

<u>Q</u>: How to recover the d_i's from this kind of computation?

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9. Surjectivity ✓

$$M \xrightarrow{f} \operatorname{Im} f \subset N$$

$$\begin{tikzcd} \mathrm{Im} f \times P \arrow[r] & M \otimes P \\ \mathrm{Im} f \otimes P \arrow[ur] & \end{tikzcd}$$

$f(m) \otimes p$ vs $f(m') \otimes p'$

These could be equal in $N \otimes P$

but <u>not</u> in $\mathrm{Im} f \otimes P$

eg $P = \mathbb{Z}/(n)$ $\quad M = \mathbb{Z} \longrightarrow \mathbb{Q} = N$

$$\begin{array}{c} \mathrm{Im} f \otimes P \cong \mathbb{Z}/(n) \\ \cong \\ \mathbb{Z} \end{array}$$

$N \otimes P = 0$

$\mathbb{Z} \hookrightarrow \mathbb{Q}$ injective

$\otimes \mathbb{Z}/2 \rightsquigarrow$ not injective

$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$ works too

<u>Defn</u>: An R-module P is <u>projective</u>

if

$$\begin{tikzcd} P \arrow[d, dotted, "h"'] \arrow[dr, "f"] & \\ M \arrow[r, two heads, "g"'] & N \end{tikzcd}$$

given $f : P \to N$
$g : M \twoheadrightarrow N$ surj
$\exists\, h : P \to M$
such that

$$f = g \circ h$$

<u>Q</u>: • Show free R-modules are projective

• Show that if P is projective then $\otimes P$ preserves injectivity of maps

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Algebras

R ring.

<u>Defn</u>: An R-<u>algebra</u> is a ring S with a ring hom $f: R \to S$

<u>Note</u>: S is an R-module automatically

<u>Ex</u>: • A $\mathbb{Z}$-algebra is a ring

• $\mathbb{C}[x, y, z]$ is a $\mathbb{C}$-algebra

$\mathbb{C}[x]$-algebra.

M R-module, S R-algebra.

Can make $S \otimes_R M$ into an S-module

$$s \cdot (s' \otimes m) := (ss') \otimes m$$

<u>Extension of scalars</u> (from R to S)

Can go the other way. Given an S-module N, can turn it into an R module N_R

$$\begin{array}{ccc} r \cdot n & := & f(r)n \\ \rotatebox{90}{$\in$} \quad \rotatebox{90}{$\in$} & & \\ R \quad N & & \end{array}$$

<u>Restriction of scalars</u>

<u>Q</u>: Show that

$$\mathrm{Hom}_{S\text{-mod}}(S \otimes_R M, N)$$

$$= \mathrm{Hom}_{R\text{-mod}}(M, N_R)$$

<u>Tensor algebra</u> Ring R

R-module M.

$$TM := R \oplus M \oplus M \otimes M \oplus M^{\otimes 3} \oplus \dots$$

some big R-module

<u>Claim</u>: This is a non-commutative R-algebra.

$$m_1 \cdot m_1' := m_1 \otimes m_1'$$

$$(m_1 \otimes \cdots \otimes m_k) \cdot (m_1' \otimes \cdots \otimes m_\ell')$$

$$:= m_1 \otimes \cdots \otimes m_k \otimes m_1' \otimes \cdots \otimes m_\ell'$$

<u>eg</u> $M = R^2$ basis x, y

Then $M \otimes M \cong R^4$ $\quad x \otimes x,\ x \otimes y,\ y \otimes x,\ y \otimes y$

$M^{\otimes 3} \cong R^8$ $\quad x^3,\ x^2 y,\ xyx,\ \ldots$

TM = "non-commutative polynomial ring in x, y"

<u>Symmetric algebra</u>

$$SM := TM \big/ (m_1 \otimes m_2 - m_2 \otimes m_1)$$

two-sided ideal

[Left ideal I in R: $\forall i \in I, \forall r \in R$ have $ri \in I$

Right ideal ______ $ir \in I$.

Two-sided ______ both]

eg If M is free then SM is poly ring.

<u>Exterior algebra</u>

$$\Lambda M := TM \big/ (M \otimes M)$$

<u>Check</u>: $m_1 \otimes m_2 = -m_2 \otimes m_1$

Why <u>the converse fails</u> in char 2.

<u>Q.</u> $\Lambda^i M$ = image of $M^{\otimes i}$ in ΛM

Assume M is free of rank n.
What is $\Lambda^i M$?

Given $f: M \to M$ of R-mods,
write down two interesting maps
$$\Lambda^i M \longrightarrow \Lambda^i M$$ induced by f.

Interpret these maps when $i=n$.