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Question 1. Can you embed the following groups into F_2?

- F_3

  Same as F_∞, but restricted to first three generators.

- F_∞

  Let a, b be generators of F_2. Consider $x_i = b^{-i}ab^{-i}$ for $i = 1, 2, ...$ Let's prove that $\langle x_1, x_2, ...\rangle$ is free: Suppose that some reduced word $w = x_{y_1}^{z_1} x_{y_2}^{z_2} ...$ is equal to identity. It is reduced, so $y_i \neq y_i + 1$ for any i

  $$b^{-y_1}a^{z_1}b^{y_1-y_2}a^{z_2}b^{y_2-y_3}... = e$$

  All exponents in the above word are non-zero, so the above word is reduced and non-empty, so it is not equal to identity, contradiction. Hence $\langle x_1, x_2, ...\rangle \cong F_\infty$, so F_2 has a subgroup isomorphic to F_∞.

Question 2. How to recover d_i's in classification of finitely generated abelian groups?

Question 3. Show that free R-modules are projective.

Let $p_1, ...p_k$ be a basis for a free R-module P. Suppose $f : N \to M$ is a surjective homomorphism and $g : N \to M$ is a homomorphism. f is surjective, so let n_i be s.t. $f(n_i) = g(p_i)$. Now let $h : P \to M$ be defined by $h(r_ip_i) = r_in_i$. This is well defined, because P is free, and it's also a homeomorphism. Moreover $f \circ h(r_ip_i) = f(r_in_i) = r_ig(p_i) = g(r_ip_i)$, so $g = f \circ h$, so P is projective.

Question 4. Show that if P is projective then $\otimes P$ preserves injectivity of maps.

Let's first prove the following: For any R-modules M, N, Q we have

$$(M \oplus N) \otimes Q \cong (M \otimes Q) \oplus (N \otimes Q).$$

Proof. Consider the bilinear map $f : (M \oplus N) \times Q \to (M \otimes Q) \oplus (N \otimes Q)$ given by $f((m, n), q) = m \otimes q + n \otimes q$. Then by the fundamental property of tensor products the corresponding map $\phi : (M \oplus N) \otimes Q \to (M \otimes Q) \oplus (N \otimes Q)$ that sends $(m, n) \otimes q$ to $m \otimes q + n \otimes q$ is a well-defined module homomorphism.
We would now like to find its inverse. Consider the "obvious" maps $\varphi_N, \varphi_M : M \otimes Q, N \otimes Q \to (M \oplus N) \otimes Q$. Then the map $\psi : (M \otimes Q) \oplus (N \otimes Q) \to (M \oplus N) \otimes Q$ given by $(m \otimes q_m, n \otimes q_n) \mapsto \varphi_M(m \otimes q_m) + \varphi_N(n \otimes q_n)$ is a well-defined homomorphism since it's a sum of two homomorphisms. Now it remains to check that ψ is an inverse of ϕ but this is a trivial check, so we are done $\square$

Now let's show that free modules preserve the injectivity of maps. I will denote a free R-module by R^n but everything that follows work just as well for modules of infinite rank. Let $f : M \to N$ be injective. Then we claim that $f \otimes id : M \otimes R^k \to N \otimes R^k$is injective. Suppose that $f(m) \otimes (r_1, \ldots, r_k) = f(m') \otimes (r'_1, \ldots, r'_k)$. We would like to show that this implies the equality of arguments. Since $M \otimes R^k \cong$

M^k we can instead assume $(r_1 f(m), \dots, r_k f(m)) = (r'_1 f(m'), \dots, r'_k f(m'))$. But since f is a homomorphism we can move the $r_i$and r'_i inside the argument and use the injectivity of f to establish $r_i m = r'_i m'$ which is exactly what we wanted, so we are done. $\square$

One more step in the proof (which, in case it isn't apparent, was created by reading a lot of simpler facts on wikipedia, trying to prove these first and seeing if these lead to anything) : For any projective module P there exists a module Q s.t. $M \oplus Q$ is free. Once again I will be assuming that P has a generating set of size n but $n = \infty$ works the same.

Let P be generated by $\{p_1, \dots, p_n\}$. Consider $f : R^n \to P$ that sends the i-th basis element to p_i. This is clearly a surjective module homomorphism. Then by the defining property of projective modules, there exists a homomorphism $h : P \to R^n$ s.t. $f \circ h = id_P$

$$\begin{tikzcd}
R^n \arrow[dr, "f"] & \\
P \arrow[u, dashed, hook, "h"] \arrow[r, "id"] & P
\end{tikzcd}$$

Now clearly h is injective, because $f \circ h$ has trivial kernel, so $h(P) = P$. We claim that $R^n \cong \operatorname{im}(h) \oplus \ker(f)$ and the isomorphism between these is

$$(r_1, \dots, r_n) \mapsto \left(h\left(\sum r_i p_i\right), h\left(\sum r_i p_i\right) - (r_1, \dots, r_n)\right) =$$
$$= \left(h \circ f(r_i), h \circ f(r_i) - (r_i)\right).$$

First quetion : Why is the second term in the kernel of f?
Because applying f to both sides gives identity on first term and sth that cancels it on the second.
Bad question : Is this a well-defined homomorphism? Yes.
Second question: Is it surjective?
An arbitrary element of the image of h can be written as $h \circ f(r_i)$ since f surjective. Also let $(r'_i) \in \ker f$. We want to find a preimage (w_i) of this pair. Let $(w_i) = h \circ f(r_i) - (r'_i)$ (note we don't really have any other choice by definition of our map). Then applying $(h \circ f)$ and using the composition properties of h, f we immediately see that it works.
Third question : Is it injective?
Just need to show trivial kernel. Assume that (r_i) is sent to zero. Then by looking at the image entry we get that it's in the kernel of $h \circ f$ and h has trivial kernel so in the kernel of f. Then applying this to second term we get $(r_i) = 0$
Hence our map is an isomorphism and we are done. $\square$

Finally, let's put it together. Given a free module P we can find Q s.t. $P \oplus Q \cong R^n$. Then given an injective map $f : M \to N$ we get an injective map $M \otimes R^n \to N \otimes R^n$. Then using the distributivity of tensor product over direct sum (or the other way around) we see it's an injective homomorphism $(M \otimes P) \oplus (M \otimes Q) \to (N \otimes P) \oplus (N \otimes Q)$ and, importantly, the homomorphism had the $f \otimes id$ form on each of the summands. Now putting 0 in the $\otimes Q$ entry this gives an injective homomorphism $M \otimes P \to N \otimes P$, which is what we wanted to show.

Question 5. Show that $\mathrm{Hom}_{S-mod}(S \underset{R}{\otimes} M, N) = \mathrm{Hom}_{R-mod}(M, N_R)$.

Question 6. Check why dividing by $\{m \otimes m : m \in M\}$ implies that $m_1 \otimes m_2 + m_2 \otimes m_1 = 0$ but the opposite doesn't hold in characteristic 2

Question 7. Assume M is free of rank n. What is $\Lambda^i M$?

Question 8. Given $f : M \to M$ a homomorphism of R-modules, write down two "interesting" maps

$$\Lambda^i M \to \Lambda^i M$$

induced by f. Interpret these maps when $i = n$.

Let $g_1 : \Lambda^i M \to \Lambda^i M$ be defined by $g_1(m_1 \otimes ... \otimes m_i) = f(m_1) \otimes ... f(m_i)$. Let $g_2 : \Lambda^i M \to \Lambda^i M$ be defined by $g_2(m_1 \otimes ... \otimes m_i) = f(m_1) \otimes ... \otimes m_i + m_1 \otimes f(m_2) ... \otimes m_i + ... + m_1 \otimes m_2 ... \otimes f(m_i)$. They are well defined because they factor through the quotient defining the exterior algebra. If $i = n$, let $e_1, e_2, ..., e_n$ be the basis of M, and let $f_i j$ be the j-th component of $f(e_i)$ in this basis. The only basis vector for $\Lambda^n M$ is $e_1 \otimes e_2 ... \otimes e_n$:

$$g_1(Ce_1 \otimes e_2 ... \otimes e_n) = Cf(e_1) \otimes (e_2) ... \otimes f(e_n) = C(f_{1j}e_j) \otimes (f_{2j}e_j) ... \otimes (f_{nj}e_j) = C \det(f_{ij}) e_1 \otimes e_2 ... \otimes e_n$$

So g_1 corresponds to multiplication by $\det(f_{ij})$. Similarly:

$$g_2(C(e_1 \otimes e_2 ... \otimes e_n)) = C(f(e_1) \otimes ... \otimes e_n + e_1 \otimes f(e_2) ... \otimes e_i + ... + e_1 \otimes e_2 ... \otimes f(e_n))$$

$$= C(e_{11} + e_{22} + ... + e_{nn}) e_1 \otimes e_2 ... \otimes e_n = C \operatorname{Tr}(f_{ij}) e_1 \otimes e_2 ... \otimes e_n$$

So g_2 corresponds to multiplication by $\operatorname{Tr}(f_{ij})$.