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Question 1. Can you embed the following groups into F_2 ?

- F_3

Same as F_∞ , but restricted to first three generators.

- F_∞

Let a, b be generators of F_2 . Consider $x_i = b^{-i}ab^{-i}$ for $i = 1, 2, \dots$. Let's prove that $\langle x_1, x_2, \dots \rangle$ is free: Suppose that some reduced word $w = x_{y_1}^{z_1} x_{y_2}^{z_2} \dots$ is equal to identity. It is reduced, so $y_i \neq y_i + 1$ for any i

$$b^{-y_1} a^{z_1} b^{y_1 - y_2} a^{z_2} b^{y_2 - y_3} \dots = e$$

All exponents in the above word are non-zero, so the above word is reduced and non-empty, so it is not equal to identity, contradiction. Hence $\langle x_1, x_2, \dots \rangle \cong F_\infty$, so F_2 has a subgroup isomorphic to F_∞ .

Question 2. How to recover d_i 's in classification of finitely generated abelian groups?

Tensor product is distributive over finite direct sums, as proven in question 4. Also $\mathbb{Z}/a \otimes \mathbb{Z} \cong \mathbb{Z}/a$ and $\mathbb{Z}/a \otimes \mathbb{Z}/b \cong \mathbb{Z}/\gcd(a, b)$. We can recover number of factors of \mathbb{Z} using:

$$m = \dim_{\mathbb{Q}} A \otimes_{\mathbb{Z}} \mathbb{Q}$$

For any prime p we have:

$$\mathbb{Z}/p \otimes_{\mathbb{Z}} A = \mathbb{Z}/\gcd(d_1, p) \oplus \mathbb{Z}/\gcd(d_2, p) \oplus \dots \oplus \mathbb{Z}/\gcd(d_k, p) \oplus \underbrace{\mathbb{Z}/p \oplus \dots \oplus \mathbb{Z}/p}_{m \text{ times}}$$

So $|\mathbb{Z}/p \otimes_{\mathbb{Z}} A| < p^{m+k}$ with equality if $p|d_i$ for all i . But if $p|d_1$ then $p|d_i$ for all i , and $d_1 > 1$, so there exists a prime p which divides all d_i . Hence:

$$k = \max_{p \text{ is prime}} \log_p(|\mathbb{Z}/p \otimes_{\mathbb{Z}} A|) - m$$

For a given prime p , define:

$$a_i = \log_p(|\mathbb{Z}/p^i \otimes_{\mathbb{Z}} A|) - mi$$

Then $k - (a_{i+1} - a_i)$ is precisely number of d_i with exponent of p in prime factorization equal to i . But sequence of these exponents in d_i has to be non-decreasing, so it is uniquely determined of a_i . This holds for any prime p , so d_i are uniquely determined by A .

Question 3. Show that free R-modules are projective.

Let p_1, \dots, p_k be a basis for a free R-module P . Suppose $f : N \rightarrow M$ is a surjective homomorphism and $g : N \rightarrow M$ is a homomorphism. f is surjective, so let n_i be s.t. $f(n_i) = g(p_i)$. Now let $h : P \rightarrow M$ be defined by $h(r_i p_i) = r_i n_i$. This is well defined, because P is free, and it's also a homomorphism. Moreover $f \circ h(r_i p_i) = f(r_i n_i) = r_i g(p_i) = g(r_i p_i)$, so $g = f \circ h$, so P is projective.

Question 4. Show that if P is projective then $\otimes P$ preserves injectivity of maps.

Let's first prove the following: For any R -modules M, N, Q we have

$$(M \oplus N) \otimes Q \cong (M \otimes Q) \oplus (N \otimes Q).$$

Proof. Consider the bilinear map $f : (M \oplus N) \times Q \rightarrow (M \otimes Q) \oplus (N \otimes Q)$ given by $f((m, n), q) = m \otimes q + n \otimes q$. Then by the fundamental property of tensor products the corresponding map $\phi : (M \oplus N) \otimes Q \rightarrow (M \otimes Q) \oplus (N \otimes Q)$ that sends $(m, n) \otimes q$ to $m \otimes q + n \otimes q$ is a well-defined module homomorphism.

We would now like to find its inverse. Consider the "obvious" maps $\varphi_N, \varphi_M : M \otimes Q, N \otimes Q \rightarrow (M \oplus N) \otimes Q$. Then the map $\psi : (M \otimes Q) \oplus (N \otimes Q) \rightarrow (M \oplus N) \otimes Q$ given by $(m \otimes q_m, n \otimes q_n) \mapsto \varphi_M(m \otimes q_m) + \varphi_N(n \otimes q_n)$ is a well-defined homomorphism since it's a sum of two homomorphisms. Now it remains to check that ψ is an inverse of ϕ but this is a trivial check, so we are done \square

Now let's show that taking a tensor product with a free module preserves the injectivity of maps. I will denote a free R -module by R^n but everything that follows work just as well for modules of infinite rank. Let $f : M \rightarrow N$ be injective. Then we claim that $f \otimes id : M \otimes R^k \rightarrow N \otimes R^k$ is injective. Suppose that $f(m) \otimes (r_1, \dots, r_k) = f(m') \otimes (r'_1, \dots, r'_k)$. We would like to show that this implies the equality of arguments. Since $M \otimes R^k \cong M^k$ we can instead assume $(r_1 f(m), \dots, r_k f(m)) = (r'_1 f(m'), \dots, r'_k f(m'))$. But since f is a homomorphism we can move the r_i and r'_i inside the argument and use the injectivity of f to establish $r_i m = r'_i m'$ which is exactly what we wanted, so we are done. \square

One more step in the proof (which, in case it isn't apparent, was created by reading a lot of simpler facts on wikipedia, trying to prove these first and seeing if these lead to anything) : For any projective module P there exists a module Q s.t. $P \oplus Q$ is free. Once again I will be assuming that P has a generating set of size n but $n = \infty$ works the same.

Let P be generated by $\{p_1, \dots, p_n\}$. Consider $f : R^n \rightarrow P$ that sends the i -th basis element to p_i . This is clearly a surjective module homomorphism. Then by the defining property of projective modules, there exists a homomorphism $h : P \rightarrow R^n$ s.t. $f \circ h = id_P$

$$\begin{array}{ccc} R^n & & \\ \uparrow h & \searrow f & \\ P & \xrightarrow{id} & P \end{array}$$

Now clearly h is injective, because $f \circ h$ has trivial kernel, so $\text{Im}(h) \cong P$. We claim that $R^n \cong \text{im}(h) \oplus \ker(f)$ and the isomorphism between these is

$$\begin{aligned} (r_1, \dots, r_n) &\mapsto \left(h \left(\sum r_i p_i \right), h \left(\sum r_i p_i \right) - (r_1, \dots, r_n) \right) = \\ &= (h \circ f(r_i), h \circ f(r_i) - (r_i)). \end{aligned}$$

First question : Why is the second term in the kernel of f ?

Because applying f to both sides gives identity on first term and sth that cancels it on the second.

Bad question : Is this a well-defined homomorphism? Yes.

It is also a bijection, because $g : \text{Im}(h) \oplus \ker(f) \rightarrow R^n$ given by $(a, b) \mapsto a - b$ is a homomorphism and is inverse to this map. \square

Finally, let's put it together. Given a projective module P we can find Q s.t. $P \oplus Q \cong R^n$. Then given an injective map $f : M \rightarrow N$ we get an injective map $M \otimes R^n \rightarrow N \otimes R^n$. Then using the distributivity of tensor product over direct sum (or the other way around) we see it's an injective homomorphism $(M \otimes P) \oplus (M \otimes Q) \rightarrow (N \otimes P) \oplus (N \otimes Q)$ and, importantly, the homomorphism had the $f \otimes \text{id}$ form on each of the summands. Now putting 0 in the $\otimes Q$ entry this gives an injective homomorphism $M \otimes P \rightarrow N \otimes P$, which is what we wanted to show.

Question 5. Show that $\text{Hom}_{S\text{-mod}}(S \otimes_R M, N) = \text{Hom}_{R\text{-mod}}(M, N_R)$.

Consider a map $\varphi : \text{Hom}(S \otimes_R M, N) \rightarrow \text{Hom}(M, N_R)$ defined by

$$\varphi(g)(m) = g(1 \otimes m)$$

It is clearly well defined. First let's note that it maps homomorphisms to homomorphisms since $1 \otimes m + 1 \otimes n = 1 \otimes (m + n)$ and $\varphi(g)(rm) = g(1 \otimes rm) = rg(1 \otimes m) = r\varphi(g)(m)$ where the r multiplication in the S -module N is defined by some homomorphism $R \rightarrow S$ but it doesn't seem to matter how exactly it is defined. The map φ is also a module homomorphism, since the sum of homomorphisms and a homomorphism multiplied by any s is still a homomorphism. It remains to find an inverse to φ in order to prove it's an isomorphism. Consider ϕ defined by $\phi(h)(s \otimes m) = sh(m)$. A similar check shows that it's well defined and a homomorphism. It also clearly is the inverse of φ so the two homomorphism sets are isomorphic as S -modules.

Question 6. Check why dividing by $\{m \otimes m : m \in M\}$ implies that $m_1 \otimes m_2 + m_2 \otimes m_1 = 0$ but the opposite doesn't hold in characteristic 2

1. Let $I = \langle m \otimes m : m \in M \rangle$, $J = \langle m_1 \otimes m_2 + m_2 \otimes m_1 : m_1, m_2 \in M \rangle$.

We have $m_1 \otimes m_2 + m_2 \otimes m_1 = (m_1 + m_2) \otimes (m_1 + m_2) - m_1 \otimes m_1 - m_2 \otimes m_2$, so $J \subseteq I$.

We also have $2m \otimes m \in J$, so if we can divide by 2, then $I = J$.

2. Consider $R = M = \mathbb{Z}/2$. Then $M \otimes_{\mathbb{Z}} M = \mathbb{Z}/2 \otimes_{\mathbb{Z}} \mathbb{Z}/2 = \mathbb{Z}/2$. We have $0 \otimes 0 = 0 \otimes 1 = 1 \otimes 0$ and $1 \otimes 1 \neq 0 \otimes 0$.

Hence $I = M$, and $J = \langle 0 \otimes 0, 1 \otimes 1 + 1 \otimes 1 \rangle = \{0 \otimes 0\}$. So J is strictly smaller than I , hence the opposite doesn't hold in characteristic 2.

Question 7. Assume M is free of rank n . What is $\Lambda^i M$?

We will show that it is a free module of rank $\binom{n}{i}$. First note that $M^{\otimes k} \cong R^{n^k}$ and it is spanned by the tensor products of free basis elements of each copy of M . So the k -th exterior power, by definition, is spanned by the images of these. Now note that any basis tensor with two equal basis elements at two different positions is mapped to 0 in the quotient (straightforward to check. We can use $a \otimes b = -b \otimes a$ to move two equal elements closer to each other until we get $m \otimes m = 0$).

Hence $\Lambda^k M$ is spanned by the $\binom{n}{k}$ basis elements not mapped to zero. So there is a surjective homomorphism $R^{\binom{n}{k}} \rightarrow \Lambda^k M$. We would like to show it's also injective. Let's assume a fact which we will show later, namely $\Lambda^n M$ is a free module of rank 1 spanned by $e_1 \otimes \dots \otimes e_n$. Now assume that the kernel of our map is non-trivial i.e. there exist r_i s.t. $\sum r_i(e_{i_1} \otimes \dots \otimes e_{i_k}) = 0$ in the exterior power. Then the image of the tensor product of this element with any tensor product of $n - k$ basis elements of M is zero in the n -th exterior power (by definition of an ideal). So we can pick products of basis elements such that after the tensor multiplication we are only left with $\pm r_i(e_1 \otimes \dots \otimes e_n)$. Then this has to be zero so r_i is zero for all i and so the kernel is trivial.

It remains to show that the $\Lambda^n M$ is indeed a rank 1 free module. It is spanned by $e_1 \otimes \dots \otimes e_n$ (that is its image under the quotient map) by the earlier observation so just need to check that it is non-zero.

Consider $f : R^n \times \dots \times R^n \rightarrow R$ that sends $(r_{11}, \dots, r_{1n}), \dots, (r_{n1}, \dots, r_{nn})$ to $\sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_i r_{i\sigma(i)}$. This is a multilinear antisymmetric map, so it factors through the quotient defining exterior power, and $f(e_1, \dots, e_n) = 1 \neq 0$ so $e_1 \otimes \dots \otimes e_n$ is indeed non-zero in the exterior power.

Question 8. Given $f : M \rightarrow M$ a homomorphism of R -modules, write down two "interesting" maps

$$\Lambda^i M \rightarrow \Lambda^i M$$

induced by f . Interpret these maps when $i = n$.

Let $g_1 : \Lambda^i M \rightarrow \Lambda^i M$ be defined by $g_1(m_1 \otimes \dots \otimes m_i) = f(m_1) \otimes \dots \otimes f(m_i)$. Let $g_2 : \Lambda^i M \rightarrow \Lambda^i M$ be defined by $g_2(m_1 \otimes \dots \otimes m_i) = f(m_1) \otimes \dots \otimes m_i + m_1 \otimes f(m_2) \otimes \dots \otimes m_i + \dots + m_1 \otimes m_2 \otimes \dots \otimes f(m_i)$. They are well defined because they factor through the quotient defining the exterior algebra. If $i = n$, let e_1, e_2, \dots, e_n be the basis of M , and let f_{ij} be the j -th component of $f(e_i)$ in this basis. The only basis vector for $\Lambda^n M$ is $e_1 \otimes e_2 \dots \otimes e_n$:

$$g_1(Ce_1 \otimes e_2 \dots \otimes e_n) = C f(e_1) \otimes (e_2) \dots \otimes f(e_n) = C(f_{1j}e_j) \otimes (f_{2j}e_j) \dots \otimes (f_{nj}e_j) = C \det(f_{ij}) e_1 \otimes e_2 \dots \otimes e_n$$

So g_1 corresponds to multiplication by $\det(f_{ij})$. Similarly:

$$\begin{aligned} g_2(C(e_1 \otimes e_2 \dots \otimes e_n)) &= C(f(e_1) \otimes \dots \otimes e_n + e_1 \otimes f(e_2) \dots \otimes e_i + \dots + e_1 \otimes e_2 \dots \otimes f(e_n)) \\ &= C(e_{11} + e_{22} + \dots + e_{nn}) e_1 \otimes e_2 \dots \otimes e_n = C \text{Tr}(f_{ij}) e_1 \otimes e_2 \dots \otimes e_n \end{aligned}$$

So g_2 corresponds to multiplication by $\text{Tr}(f_{ij})$.