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Question 1. Can you embed the following groups into F_2 ?

- F_3

Same as F_∞ , but restricted to first three generators.

- F_∞

Let a, b be generators of F_2 . Consider $x_i = b^{-i}ab^{-i}$ for $i = 1, 2, \dots$. Let's prove that $\langle x_1, x_2, \dots \rangle$ is free: Suppose that some reduced word $w = x_{y_1}^{z_1} x_{y_2}^{z_2} \dots$ is equal to identity. It is reduced, so $y_i \neq y_i + 1$ for any i

$$b^{-y_1} a^{z_1} b^{y_1 - y_2} a^{z_2} b^{y_2 - y_3} \dots = e$$

All exponents in the above word are non-zero, so the above word is reduced and non-empty, so it is not equal to identity, contradiction. Hence $\langle x_1, x_2, \dots \rangle \cong F_\infty$, so F_2 has a subgroup isomorphic to F_∞ .

Question 2. How to recover d_i 's in classification of finitely generated abelian groups?

Question 3. Show that free R -modules are projective.

Let p_1, \dots, p_k be a basis for a free R -module P . Suppose $f : N \rightarrow M$ is a surjective homomorphism and $g : N \rightarrow M$ is a homomorphism. f is surjective, so let n_i be s.t. $f(n_i) = g(p_i)$. Now let $h : P \rightarrow M$ be defined by $h(r_i p_i) = r_i n_i$. This is well defined, because P is free, and it's also a homomorphism. Moreover $f \circ h(r_i p_i) = f(r_i n_i) = r_i g(p_i) = g(r_i p_i)$, so $g = f \circ h$, so P is projective.

Question 4. Show that if P is projective then $\otimes P$ preserves injectivity of maps.

Let's first prove the following: For any R -modules M, N, Q we have

$$(M \oplus N) \otimes Q \cong (M \otimes Q) \oplus (N \otimes Q).$$

Proof. Consider the bilinear map $f : (M \oplus N) \times Q \rightarrow (M \otimes Q) \oplus (N \otimes Q)$ given by $f((m, n), q) = m \otimes q + n \otimes q$. Then by the fundamental property of tensor products the corresponding map $\phi : (M \oplus N) \otimes Q \rightarrow (M \otimes Q) \oplus (N \otimes Q)$ that sends $(m, n) \otimes q$ to $m \otimes q + n \otimes q$ is a well-defined module homomorphism.

We would now like to find its inverse. Consider the "obvious" maps $\varphi_N, \varphi_M : M \otimes Q, N \otimes Q \rightarrow (M \oplus N) \otimes Q$. Then the map $\psi : (M \otimes Q) \oplus (N \otimes Q) \rightarrow (M \oplus N) \otimes Q$ given by $(m \otimes q_m, n \otimes q_n) \mapsto \varphi_M(m \otimes q_m) + \varphi_N(n \otimes q_n)$ is a well-defined homomorphism since it's a sum of two homomorphisms. Now it remains to check that ψ is an inverse of ϕ but this is a trivial check, so we are done \square

Now let's show that taking a tensor product with a free module preserves the injectivity of maps. I will denote a free R -module by R^n but everything that follows work just as well for modules of infinite rank. Let $f : M \rightarrow N$ be injective. Then we claim that $f \otimes id : M \otimes R^k \rightarrow N \otimes R^k$ is injective. Suppose that $f(m) \otimes (r_1, \dots, r_k) = f(m') \otimes (r'_1, \dots, r'_k)$. We would like to show that this

implies the equality of arguments. Since $M \otimes R^k \cong M^k$ we can instead assume $(r_1 f(m), \dots, r_k f(m)) = (r'_1 f(m'), \dots, r'_k f(m'))$. But since f is a homomorphism we can move the r_i and r'_i inside the argument and use the injectivity of f to establish $r_i m = r'_i m'$ which is exactly what we wanted, so we are done. \square

One more step in the proof (which, in case it isn't apparent, was created by reading a lot of simpler facts on wikipedia, trying to prove these first and seeing if these lead to anything) : For any projective module P there exists a module Q s.t. $P \oplus Q$ is free. Once again I will be assuming that P has a generating set of size n but $n = \infty$ works the same.

Let P be generated by $\{p_1, \dots, p_n\}$. Consider $f : R^n \rightarrow P$ that sends the i -th basis element to p_i . This is clearly a surjective module homomorphism. Then by the defining property of projective modules, there exists a homomorphism $h : P \rightarrow R^n$ s.t. $f \circ h = id_P$

$$\begin{array}{ccc} R^n & & \\ \uparrow h & \searrow f & \\ P & \xrightarrow{id} & P \end{array}$$

Now clearly h is injective, because $f \circ h$ has trivial kernel, so $\text{Im}(h) \cong P$. We claim that $R^n \cong \text{im}(h) \oplus \ker(f)$ and the isomorphism between these is

$$\begin{aligned} (r_1, \dots, r_n) &\mapsto \left(h \left(\sum r_i p_i \right), h \left(\sum r_i p_i \right) - (r_1, \dots, r_n) \right) = \\ &= (h \circ f(r_i), h \circ f(r_i) - (r_i)). \end{aligned}$$

First question : Why is the second term in the kernel of f ?

Because applying f to both sides gives identity on first term and sth that cancels it on the second.

Bad question : Is this a well-defined homomorphism? Yes.

It is also a bijection, because $g : \text{Im}(h) \oplus \ker(f) \rightarrow R^n$ given by $(a, b) \mapsto a - b$ is a homomorphism and is inverse to this map. \square

Finally, let's put it together. Given a projective module P we can find Q s.t. $P \oplus Q \cong R^n$. Then given an injective map $f : M \rightarrow N$ we get an injective map $M \otimes R^n \rightarrow N \otimes R^n$. Then using the distributivity of tensor product over direct sum (or the other way around) we see it's an injective homomorphism $(M \otimes P) \oplus (M \otimes Q) \rightarrow (N \otimes P) \oplus (N \otimes Q)$ and, importantly, the homomorphism had the $f \otimes id$ form on each of the summands. Now putting 0 in the $\otimes Q$ entry this gives an injective homomorphism $M \otimes P \rightarrow N \otimes P$, which is what we wanted to show.

Question 5. Show that $\text{Hom}_{S\text{-mod}}(S \otimes_R M, N) = \text{Hom}_{R\text{-mod}}(M, N_R)$.

Question 6. Check why dividing by $\{m \otimes m : m \in M\}$ implies that $m_1 \otimes m_2 + m_2 \otimes m_1 = 0$ but the opposite doesn't hold in characteristic 2

Question 7. Assume M is free of rank n . What is $\Lambda^i M$?

Question 8. Given $f : M \rightarrow M$ a homomorphism of R -modules, write down two "interesting" maps

$$\Lambda^i M \rightarrow \Lambda^i M$$

induced by f . Interpret these maps when $i = n$.

Let $g_1 : \Lambda^i M \rightarrow \Lambda^i M$ be defined by $g_1(m_1 \otimes \dots \otimes m_i) = f(m_1) \otimes \dots \otimes f(m_i)$. Let $g_2 : \Lambda^i M \rightarrow \Lambda^i M$ be defined by $g_2(m_1 \otimes \dots \otimes m_i) = f(m_1) \otimes \dots \otimes m_i + m_1 \otimes f(m_2) \otimes \dots \otimes m_i + \dots + m_1 \otimes m_2 \otimes \dots \otimes f(m_i)$. They are well defined because they factor through the quotient defining the exterior algebra. If $i = n$, let e_1, e_2, \dots, e_n be the basis of M , and let f_{ij} be the j -th component of $f(e_i)$ in this basis. The only basis vector for $\Lambda^n M$ is $e_1 \otimes e_2 \dots \otimes e_n$:

$$g_1(Ce_1 \otimes e_2 \dots \otimes e_n) = Cf(e_1) \otimes (e_2) \dots \otimes f(e_n) = C(f_{1j}e_j) \otimes (f_{2j}e_j) \dots \otimes (f_{nj}e_j) = C \det(f_{ij}) e_1 \otimes e_2 \dots \otimes e_n$$

So g_1 corresponds to multiplication by $\det(f_{ij})$. Similarly:

$$\begin{aligned} g_2(C(e_1 \otimes e_2 \dots \otimes e_n)) &= C(f(e_1) \otimes \dots \otimes e_n + e_1 \otimes f(e_2) \dots \otimes e_i + \dots + e_1 \otimes e_2 \dots \otimes f(e_n)) \\ &= C(e_{11} + e_{22} + \dots + e_{nn}) e_1 \otimes e_2 \dots \otimes e_n = C \operatorname{Tr}(f_{ij}) e_1 \otimes e_2 \dots \otimes e_n \end{aligned}$$

So g_2 corresponds to multiplication by $\operatorname{Tr}(f_{ij})$.