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Question 1. Can you embed the following groups into F_2?

- F_3

  Same as F_∞, but restricted to first three generators.

- F_∞

  Let a, b be generators of F_2. Consider $x_i = b^{-i}ab^{-i}$ for $i = 1, 2, ...$ Let's prove that $\langle x_1, x_2, ...\rangle$ is free: Suppose that some reduced word $w = x_{y_1}^{z_1} x_{y_2}^{z_2}...$ is equal to identity. It is reduced, so $y_i \neq y_i + 1$ for any i

  $$b^{-y_1} a^{z_1} b^{y_1 - y_2} a^{z_2} b^{y_2 - y_3} ... = e$$

  All exponents in the above word are non-zero, so the above word is reduced and non-empty, so it is not equal to identity, contradiction. Hence $\langle x_1, x_2, ...\rangle \cong F_\infty$, so F_2 has a subgroup isomorphic to F_∞.

Question 2. How to recover d_i's in classification of finitely generated abelian groups?

Tensor product is distributive over finite direct sums, as proven in question 4. Also $\mathbb{Z}/a \underset{Z}{\otimes} Z \cong \mathbb{Z}/a$ and $\mathbb{Z}/a \underset{Z}{\otimes} \mathbb{Z}/b \cong \mathbb{Z}/\gcd(a, b)$. We can recover number of factors of $\mathbb{Z}$ using:

$$m = \dim_{\mathbb{Q}} A \underset{Z}{\otimes} Q$$

For any prime p we have:

$$Z/p \underset{Z}{\otimes} A = \mathbb{Z}/\gcd(d_1, p) \oplus \mathbb{Z}/\gcd(d_2, p) \cdots \oplus \mathbb{Z}/\gcd(d_k, p) \oplus \underbrace{\mathbb{Z}/p \oplus \cdots \oplus \mathbb{Z}/p}_{\text{m times}}$$

So $|Z/p \underset{Z}{\otimes} A| < p^{m+k}$ with equality if $p|d_i$ for all i. But if $p|d_1$ then $p|d_i$ for all i, and $d_1 > 1$, so there exists a prime p which divides all d_i. Hence:

$$k = \max_{p \text{ is prime}} \log_p(|Z/p \underset{Z}{\otimes} A|) - m$$

For a given prime p, define:

$$a_i = \log_p(|Z/p^i \underset{Z}{\otimes} A|) - mi$$

Then $k - (a_{i+1} - a_i)$ is precisely number of d_i with exponent of p in prime factorization equal to i. But sequence of these exponents in d_i has to be non-decreasing, so it is uniquely determined of a_i. This holds for any prime p, so d_i are uniquely determined by A.

Question 3. Show that free R-modules are projective.

Let $p_1, ...p_k$ be a basis for a free R-module P. Suppose $f : N \to M$ is a surjective homomorphism and $g : N \to M$ is a homomorphism. f is surjective, so let n_i be s.t. $f(n_i) = g(p_i)$. Now let $h : P \to M$ be defined by $h(r_i p_i) = r_i n_i$. This is well defined, because P is free, and it's also a homeomorphism. Moreover $f \circ h(r_i p_i) = f(r_i n_i) = r_i g(p_i) = g(r_i p_i)$, so $g = f \circ h$, so P is projective.

Question 4. Show that if P is projective then $\otimes P$ preserves injectivity of maps.

Let's first prove the following: For any R-modules M, N, Q we have

$$(M \oplus N) \otimes Q \cong (M \otimes Q) \oplus (N \otimes Q) .$$

Proof. Consider the bilinear map $f : (M \oplus N) \times Q \to (M \otimes Q) \oplus (N \otimes Q)$ given by $f\left((m, n), q\right) = m \otimes q + n \otimes q$. Then by the fundamental property of tensor products the corresponding map $\phi : (M \oplus N) \otimes Q \to (M \otimes Q) \oplus (N \otimes Q)$ that sends $(m, n) \otimes q$ to $m \otimes q + n \otimes q$ is a well-defined module homomorphism.
We would now like to find its inverse. Consider the "obvious" maps $\varphi_N, \varphi_M : M \otimes Q, N \otimes Q \to (M \oplus N) \otimes Q$. Then the map $\psi : (M \otimes Q) \oplus (N \otimes Q) \to (M \oplus N) \otimes Q$ given by $(m \otimes q_m, n \otimes q_n) \mapsto \varphi_M(m \otimes q_m) + \varphi_N (n \otimes q_n)$ is a well-defined homomorphism since it's a sum of two homomorphisms. Now it remains to check that ψ is an inverse of ϕ but this is a trivial check, so we are done $\square$

Now let's show that taking a tensor product with a free module preserves the injectivity of maps. I will denote a free R-module by R^n but everything that follows work just as well for modules of infinite rank. Let $f : M \to N$ be injective. Then we claim that $f \otimes id : M \otimes R^k \to N \otimes R^k$is injective. Suppose that $f(m) \otimes (r_1, \ldots, r_k) = f(m') \otimes (r'_1, \ldots, r'_k)$. We would like to show that this implies the equality of arguments. Since $M \otimes R^k \cong M^k$ we can instead assume $(r_1 f(m), \ldots, r_k f(m)) = (r'_1 f(m'), \ldots, r'_k f(m'))$. But since f is a homomorphism we can move the $r_i$and r'_i inside the argument and use the injectivity of f to establish $r_i m = r'_i m'$ which is exactly what we wanted, so we are done. $\square$
One more step in the proof (which, in case it isn't apparent, was created by reading a lot of simpler facts on wikipedia, trying to prove these first and seeing if these lead to anything) : For any projective module P there exists a module Q s.t. $P \oplus Q$ is free. Once again I will be assuming that P has a generating set of size n but $n = \infty$ works the same.
Let P be generated by $\{p_1, \ldots, p_n\}$. Consider $f : R^n \to P$ that sends the i-th basis element to p_i. This is clearly a surjective module homomorphism. Then by the defining property of projective modules, there exists a homomorphism $h : P \to R^n$ s.t. $f \circ h = id_P$

$$\begin{tikzcd}
R^n \arrow[dr, two heads, "f"] & \\
P \arrow[u, hook, dashed, "h"] \arrow[r, "id"] & P
\end{tikzcd}$$

Now clearly h is injective, because $f \circ h$ has trivial kernel, so $\mathrm{Im}(h) \cong P$. We claim that $R^n \cong \mathrm{im}\,(h) \oplus \ker\,(f)$ and the isomorphism between these is

$$\begin{aligned}(r_1, \ldots, r_n) \mapsto \left(h\left(\sum r_i p_i\right), h\left(\sum r_i p_i\right) - (r_1, \ldots, r_n)\right) = \\ = \left(h \circ f\,(r_i)\,, h \circ f\,(r_i) - (r_i)\right).\end{aligned}$$

First question : Why is the second term in the kernel of f?
Because applying f to both sides gives identity on first term and sth that cancels it on the second.

Bad question : Is this a well-defined homomorphism? Yes.
It is also a bijection, because $g : \mathrm{Im}(h) \oplus \ker(f) \to R^n$ given by $(a, b) \mapsto a - b$ is a homomorphism and is inverse to this map. $\square$
Finally, let's put it together. Given a projective module P we can find Q s.t. $P \oplus Q \cong R^n$. Then given an injective map $f : M \to N$ we get an injective map $M \otimes R^n \to N \otimes R^n$. Then using the distributivity of tensor product over direct sum (or the other way around) we see it's an injective homomorphism $(M \otimes P) \oplus (M \otimes Q) \to (N \otimes P) \oplus (N \otimes Q)$ and, importantly, the homomorphism had the $f \otimes id$ form on each of the summands. Now putting 0 in the $\otimes Q$ entry this gives an injective homomorphism $M \otimes P \to N \otimes P$, which is what we wanted to show.

Question 5. Show that $\mathrm{Hom}_{S-mod}(S \underset{R}{\otimes} M, N) = \mathrm{Hom}_{R-mod}(M, N_R)$.

Consider a map $\varphi : \mathrm{Hom}\left(S \otimes_R M, N\right) \to \mathrm{Hom}\left(M, N_R\right)$ defined by

$$\varphi\left(g\right)\left(m\right) = g\left(1 \otimes m\right)$$

It is clearly well defined. First let's note that it maps homomorphisms to homomorphisms since $1 \otimes m + 1 \otimes n = 1 \otimes (m + n)$ and $\varphi(g)(rm) = g(1 \otimes rm) = rg\left(1 \otimes m\right) = r\varphi(g)(m)$ where the r multiplication in the S-module N is defined by some homomorphism $R \to S$ but it doesn't seem to matter how exactly it is defined. The map φ is also a module homomorphism, since the sum of homomorphisms and a homomorphism multiplied by any s is still a homomorphism. It remains to find an inverse to φ in order to prove it's an isomorphism. Consider ϕ defined by $\phi(h)(s \otimes m) = sh(m)$. A similar check shows that it's well defined and a homomorphism. It also clearly is the inverse of φ so the two homomorphism sets are isomorphic as S-modules.

Question 6. Check why dividing by $\{m \otimes m : m \in M\}$ implies that $m_1 \otimes m_2 + m_2 \otimes m_1 = 0$ but the opposite doesn't hold in characteristic 2

Question 7. Assume M is free of rank n. What is $\Lambda^i M$?

We will show that it is a free module of rank $\binom{n}{k}$. First note that $M^{\otimes k} \cong R^{n^k}$ and it is spanned by the tensor products of free basis elements of each copy of M. So the k-th exterior power, by definition, is spanned by the images of these. Now note that any basis tensor with two equal basis elements at two different positions is mapped to 0 in the quotient (straightforward to check. We can use $a \otimes b = -b \otimes a$ to move two equal elements closer to each other until we get $m \otimes m = 0$).
Hence $\Lambda^k M$ is spanned by the $\binom{n}{k}$ basis elements not mapped to zero. So there is a surjective homomorphism $R^{\binom{n}{k}} \to \Lambda^k M$. We would like to show it's also injective. Let's assume a fact which we will show later, namely $\Lambda^n M$ is a free module of rank 1 spanned by $e_1 \otimes \ldots \otimes e_n$. Now assume that the kernel of our map is non-trivial i.e. there exist r_i s.t. $\sum r_i (e_{i_1} \otimes \ldots \otimes e_{i_k}) = 0$ in the exterior power. Then the image of the tensor product of this element with any tensor product of $n - k$ basis elements of M is zero in the n-th exterior power (by definition of an ideal). So we can pick products of basis elements such that

after the tensor multiplication we are only left with $\pm r_i\,(e_1 \otimes \ldots \otimes e_n)$. Then this has to be zero so r_i is zero for all i and so the kernel is trivial.
It remains to show that the $\Lambda^n M$ is indeed a rank 1 free module. It is spanned by $e_1 \otimes \ldots \otimes e_n$ (that is its image under the quotient map) by the earlier observation so just need to check that it is non-zero.
Consider $f : R^n \times \ldots \times R^n \to R$ that sends $(r_{11}, \ldots, r_{1n}), \ldots, (r_{n1}, \ldots, r_{nn})$ to $\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_i r_{i\sigma(i)}$. This is a multilinear antisymmetric map, so it factors through the quotient defining exterior power, and $f(e_1 \otimes \ldots \otimes e_n) = 1 \neq 0$ so $e_1 \otimes \ldots \otimes e_n$ is indeed non-zero in the exterior power.

Question 8. Given $f : M \to M$ a homomorphism of R-modules, write down two "interesting" maps

$$\Lambda^i M \to \Lambda^i M$$

induced by f. Interpret these maps when $i = n$.

Let $g_1 : \Lambda^i M \to \Lambda^i M$ be defined by $g_1(m_1 \otimes ... \otimes m_i) = f(m_1) \otimes ... f(m_i)$. Let $g_2 : \Lambda^i M \to \Lambda^i M$ be defined by $g_2(m_1 \otimes ... \otimes m_i) = f(m_1) \otimes ... \otimes m_i + m_1 \otimes f(m_2) ... \otimes m_i + ... + m_1 \otimes m_2 ... \otimes f(m_i)$. They are well defined because they factor through the quotient defining the exterior algebra. If $i = n$, let $e_1, e_2, ..., e_n$ be the basis of M, and let f_ij be the j-th component of $f(e_i)$ in this basis. The only basis vector for $\Lambda^n M$ is $e_1 \otimes e_2 ... \otimes e_n$:

$$g_1(Ce_1 \otimes e_2 ... \otimes e_n) = Cf(e_1) \otimes (e_2) ... \otimes f(e_n) = C(f_{1j}e_j) \otimes (f_{2j}e_j) ... \otimes (f_{nj}e_j) = C\det(f_{ij}) e_1 \otimes e_2 ... \otimes e_n$$

So g_1 corresponds to multiplication by $\det(f_{ij})$. Similarly:

$$g_2(C(e_1 \otimes e_2 ... \otimes e_n)) = C(f(e_1) \otimes ... \otimes e_n + e_1 \otimes f(e_2) ... \otimes e_i + ... + e_1 \otimes e_2 ... \otimes f(e_n))$$

$$= C(e_{11} + e_{22} + ... + e_{nn}) e_1 \otimes e_2 ... \otimes e_n = C\,\mathrm{Tr}(f_{ij}) e_1 \otimes e_2 ... \otimes e_n$$

So g_2 corresponds to multiplication by $\mathrm{Tr}(f_{ij})$.