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Question 1. Can you embed the following groups into F_2?

- F_3

  Same as F_∞, but restricted to first three generators.

- F_∞

  Let a, b be generators of F_2. Consider $x_i = b^{-i}ab^{-i}$ for $i = 1, 2, ...$ Let's prove that $\langle x_1, x_2, ...\rangle$ is free: Suppose that some reduced word $w = x_{y_1}^{z_1} x_{y_2}^{z_2}...$ is equal to identity. It is reduced, so $y_i \neq y_i + 1$ for any i

  $$b^{-y_1}a^{z_1}b^{y_1-y_2}a^{z_2}b^{y_2-y_3}... = e$$

  All exponents in the above word are non-zero, so the above word is reduced and non-empty, so it is not equal to identity, contradiciton. Hence $\langle x_1, x_2, ...\rangle \cong F_\infty$, so F_2 has a subgroup isomorphic to F_∞.

Question 2. How to recover d_i's in classification of finitely generated abelian groups?

Question 3. Show that free R-modules are projective.

Let $p_1, ...p_k$ be a basis for a free R-module P. Suppose $f : N \to M$ is a surjective homomorphism and $g : N \to M$ is a homomorphism. f is surjective, so let n_i be s.t. $f(n_i) = g(p_i)$. Now let $h : P \to M$ be defined by $h(r_ip_i) = r_in_i$. This is well defined, because P is free, and it's also a homeomorphism. Moreover $f \circ h(r_ip_i) = f(r_in_i) = r_ig(p_i) = g(r_ip_i)$, so $g = f \circ h$, so P is projective.

Question 4. Show that if P is projective then $\otimes P$ preserves injectivity of maps.

Question 5. Show that $\text{Hom}_{S-mod}(S \underset{R}{\otimes} M, N) = \text{Hom}_{R-mod}(M, N_R)$.

Question 6. Check why dividing by $\{m \otimes m : m \in M\}$ implies that $m_1 \otimes m_2 + m_2 \otimes m_1 = 0$ but the opposite doesn't hold in characteristic 2

Question 7. Assume M is free of rank n. What is $\Lambda^i M$?

Question 8. Given $f : M \to M$ a homomorphism of R-modules, write down two "interesting" maps

$$\Lambda^i M \to \Lambda^i M$$

induced by f. Interpret these maps when $i = n$.

Let $g_1 : \Lambda^i M \to \Lambda^i M$ be defined by $g_1(m_1 \otimes ... \otimes m_i) = f(m_1) \otimes ...f(m_i)$. Let $g_2 : \Lambda^i M \to \Lambda^i M$ be defined by $g_2(m_1 \otimes ... \otimes m_i) = f(m_1) \otimes ... \otimes m_i + m_1 \otimes f(m_2)...\otimes m_i + ... + m_1 \otimes m_2 ... \otimes f(m_i)$. They are well defined because they factor through the quotient defining the exterior algebra. If $i = n$, let $e_1, e_2, ..., e_n$ be the basis of M, and let f_ij be the j-th component of $f(e_i)$ in this basis. The only basis vector for $\Lambda^n M$ is $e_1 \otimes e_2 ... \otimes e_n$:

$$g_1(Ce_1 \otimes e_2 ... \otimes e_n) = Cf(e_1) \otimes (e_2) ... \otimes f(e_n) = C(f_{1j}e_j) \otimes (f_{2j}e_j) ... \otimes (f_{nj}e_j) = C \det(f_{ij}) e_1 \otimes e_2 ... \otimes e_n$$

So g_1 corresponds to multiplication by $\det(f_{ij})$. Similarly:

$$g_2(C(e_1 \otimes e_2 ... \otimes e_n)) = C(f(e_1) \otimes ... \otimes e_n + e_1 \otimes f(e_2) ... \otimes e_i + ... + e_1 \otimes e_2 ... \otimes f(e_n))$$

$$= C(e_{11} + e_{22} + ... + e_{nn}) e_1 \otimes e_2 ... \otimes e_n = C \,\mathrm{Tr}(f_{ij}) e_1 \otimes e_2 ... \otimes e_n$$

So g_2 corresponds to multiplication by $\mathrm{Tr}(f_{ij})$.