习题六

1. 解：$y=a_0+a_1x$

正则方程组为：

$$\begin{pmatrix} 8 & \sum_{i=1}^{8}x_i \\ \sum_{i=1}^{8}x_i & \sum_{i=1}^{8}x_i^2 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}=\begin{pmatrix} \sum_{i=1}^{8}y_i \\ \sum_{i=1}^{8}y_ix_i \end{pmatrix}$$

即

$$\begin{pmatrix} 8 & 15.26 \\ 15.26 & 30.1556 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}=\begin{pmatrix} 145.227 \\ 284.8363 \end{pmatrix}$$

$\Rightarrow a_0=3.9160 \qquad a_1=7.4639$

所以 $y=3.9160+7.4639x$

2. 解：$y=a_0+a_1x+a_2x^2$

正则方程组为：

$$\begin{pmatrix} 9 & \sum_{i=1}^{9}x_i & \sum_{i=1}^{9}x_i^2 \\ \sum_{i=1}^{9}x_i & \sum_{i=1}^{9}x_i^2 & \sum_{i=1}^{9}x_i^3 \\ \sum_{i=1}^{9}x_i^2 & \sum_{i=1}^{9}x_i^3 & \sum_{i=1}^{9}x_i^4 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}=\begin{pmatrix} \sum_{i=1}^{9}y_i \\ \sum_{i=1}^{9}y_ix_i \\ \sum_{i=1}^{9}y_ix_i^2 \end{pmatrix}$$

即

$$\begin{pmatrix} 9 & 53 & 281 \\ 53 & 381 & 3017 \\ 381 & 3017 & 25317 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}=\begin{pmatrix} 32 \\ 147 \\ 1025 \end{pmatrix}$$

$\Rightarrow a_0=13.4597 \qquad a_1=-3.6053 \qquad a_2=0.2676$

所以 $y=13.4597-3.6053x+0.2676x^2$

3、解：观察数据近似一次线性关系，

则用最小二乘一次拟合多项式 $y=a_0+a_1x$.

正则方程组为：$\begin{bmatrix} 5 & 700 \\ 700 & 99376 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}=\begin{bmatrix} 758 \\ 108146 \end{bmatrix}$

$\Rightarrow a_0=-54.536 \qquad a_1=1.4724$

则 $y=1.4724x-54.536$

即近似为钢液碳含量与冶炼时间的关系函数.

4. 解：考虑 y 与 x^2 的关系：

x_i^2	361	625	961	1444	1936
y_i	19.0	32.3	49.0	73.3	97.8

用最小二乘一次拟合：$y=a_0+a_1x^2$

正则方程组为：

$$\begin{bmatrix} 5 & 5327 \\ 5327 & 7277699 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}=\begin{bmatrix} 271.4 \\ 369321.5 \end{bmatrix}$$

$\Rightarrow a_0=0.9726 \quad a_1=0.0500$

则 $y=0.9726+0.05x^2$

5. 解：考虑 $\varphi_0=x \quad \varphi_1=x^3$，线性无关. 在各节点处的值：

$\varphi_0(x)=[-3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3]$

$\varphi_1(x)=[-27 \quad -8 \quad -1 \quad 0 \quad 1 \quad 8 \quad 27]$

计算得正则方程组：

$$\begin{bmatrix} 28 & 196 \\ 196 & 1588 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}=\begin{bmatrix} 209.73 \\ 1683.63 \end{bmatrix} \quad \Rightarrow a_0=0.5059 \quad a_1=0.9978$$

则 $y=0.5059+0.9978x^3$

6. 解: $y=ae^{bx} \Rightarrow \ln y=\ln a+bx$. 数据变为:

x_i	1	2	3	4	5	6	7	8
$\ln y_i$	2.7279	3.0204	3.3105	3.6000	3.8939	4.1836	4.4759	4.7673

得正则方程组:

$$\begin{cases} 8a_0+36a_1=29.9795 \\ 36a_0+204a_1=147.1406 \end{cases}$$

$\Rightarrow a_0=2.4367 \quad a_1=0.2913$

则 $a=e^{a_0}=11.4352 \quad b=0.2913$

则经验公式为 $y=11.4352e^{0.2913x}$.

7. 解: $y=\frac{t}{at+b} \Rightarrow \frac{1}{y}=a+b\frac{1}{t}$. 数据变为下图, 用最小二乘法拟合:

$\frac{1}{t_i}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{8}$	$\frac{1}{10}$	$\frac{1}{12}$	$\frac{1}{14}$	$\frac{1}{16}$
$\frac{1}{y_i}$	0.25	0.156	0.1248	0.1138	0.1049	0.1014	0.0968	0.096	0.095	0.0943

将数据代入正则方程组得:

$$\begin{bmatrix} 10 & 2.6923 \\ 2.6923 & 1.4930 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1.2330 \\ 0.4586 \end{bmatrix}$$

$\Rightarrow a_0=0.0789 \qquad a_1=0.1649$

则 $\frac{1}{y}=0.0789+0.1649\frac{1}{t}$

$\Rightarrow y=\frac{t}{0.0789t+0.1649}$

8. 解: $y=ae^{\frac{b}{t}} \Rightarrow \ln y=\ln a+b\frac{1}{t}$. 数据变为:

$\frac{1}{t_i}$	1	0.5	0.333	0.25	0.1667	0.125	0.1	0.833	0.0714	0.0625
$\ln y_i$	1.3863	1.8579	2.0807	2.1736	2.2544	2.2885	2.3351	2.3437	2.3542	2.3618

用最小二乘法拟合：

则正则方程组为：

$$\begin{bmatrix} 10 & 2.6923 \\ 2.6923 & 1.4930 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 21.4362 \\ 4.9586 \end{bmatrix}$$

$\Rightarrow a_0 = 2.4284 \quad a_1 = -1.0579$

则 $\ln y = 2.4284 + (-1.0579)\frac{1}{t}$

$a = e^{2.4284} = 11.3407 \quad b = -1.0579$

则 $y = 11.3407 e^{\frac{-1.0579}{t}}$

对于题7的残差平方和为：$\sum_{i=1}^{10}\left(y_i - \frac{t_i}{0.018t_i + 0.1649}\right)^2 = 1.7030$

题8的残差平方和为 $\sum_{i=1}^{10}\left(y_i - 11.3407e^{\frac{-1.0579}{t_i}}\right)^2 = 0.1109 < 1.7030$

则 $y = 11.3407e^{\frac{-1.0579}{t}}$ 更好。

9：解：$k=0$ 时，$\varphi_0(x) = 1$，则 $\int_0^1 x\,dx = \frac{1}{2}$.

$k \neq 0$ 时：$\alpha_1 = \frac{(x\varphi_0, \varphi_0)}{(\varphi_0, \varphi_0)} = \frac{\int_0^1 x \cdot x \cdot 1^2 dx}{\int_0^1 x \cdot 1^2 dx} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$

则 $\varphi_1(x) = x - \frac{2}{3}$

则 $\int_0^1 x\varphi_1(x) = \int_0^1 (x^2 - \frac{2}{3}x)dx = 0$

$$\alpha_2 = \frac{\int_0^1 x \cdot x \cdot (x-\frac{2}{3})dx}{\int_0^1 x(x-\frac{2}{3})dx} = \frac{8}{15},$$

$$\beta_2 = \frac{\int_0^1 x \cdot (x-\frac{2}{3})^2 dx}{\int_0^1 x \cdot 1^2 dx} = \frac{7}{30}$$

$\therefore \varphi_2(x) = x^2 - \frac{6}{5}x + \frac{3}{10}$

根据定理6.3，有 $\int_0^1 x\varphi_k(x)dx$ 在 $k \geq 1$ 时，为0

则综上：$\int_0^1 x\varphi_k(x)dx = \begin{cases} \frac{1}{2} & k=0 \\ 0 & k \neq 0 \end{cases}$

$\varphi_2(x) = x^2 - \frac{6}{5}x + \frac{3}{10}$

另解：

由 $\{\varphi_k(x)\}$ 的正交性，

当 $k>0$ 时，$\int_0^1 x\varphi_k(x)dx = \int_0^1 \varphi_0(x)\varphi_k(x)w(x)dx = 0$

当 $k=0$ 时，$\int_0^1 x\varphi_k(x)dx = \int_0^1 x\,dx = \frac{1}{2}$.

设 $\varphi_2(x) = x^2 + ax + b$，$a, b$ 待定。

由 $\{\varphi_k(x)\}_{k=0}^n$ 的正交性。

$$\begin{cases} \int_0^1 \varphi_2(x) x \cdot x\,dx = 0 \\ \int_0^1 \varphi_2(x) x\,dx = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{1}{5} + \frac{1}{4}a + \frac{1}{3}b = 0 \\ \frac{1}{4} + \frac{1}{3}a + \frac{1}{2}b = 0 \end{cases}$$

$\Rightarrow a = -\frac{6}{5},\ b = \frac{3}{10}$

则 $\varphi_2(x) = x^2 - \frac{6}{5}x + \frac{3}{10}$

10. 证：我们只要在实内积空间讨论。由内积定义知 $(\varphi_k,\varphi_j)=(\varphi_j,\varphi_k)$，故 G 对应的矩阵是对称矩阵，考虑以 $a_0,a_1\cdots a_n$ 为未知元的线性方程组

$$\sum_{k=0}^{n} a_k(\varphi_j,\varphi_k)=0 \quad (j=0,1\cdots n). \quad (*)$$

其系数行列式为 G，由线性代数知识，(*) 式仅有零解 $a_k=0\ (k=0,1,\cdots n)$ 的充要条件是 $G\neq 0$.

充分性：设 $G\neq 0$ 要证 $\{\varphi_k(x)\}_{k=0}^{n}$ 线性无关，作线性组合 $\sum_{k=0}^{n} a_k\varphi_k=0$，

显然有 $(\sum_{k=0}^{n} a_k\varphi_k,\varphi_j)=\sum_{k=0}^{n} a_k(\varphi_k,\varphi_j)=\sum_{k=0}^{n} a_k(\varphi_j,\varphi_k)=0.\ (j=0,1\cdots n)$

这表明 $a_k\ (k=0,1\cdots n)$ 满足 (*) 式，又因 $G\neq 0$. 故有 $a_k=0\ (k=0,1\cdots n)$. 按线性无关的定义知 $\{\varphi_k(x)\}_{k=0}^{n}$ 线性无关.

必要性：设 $\{\varphi_k(x)\}_{k=0}^{n}$ 线性无关，要证 $G\neq 0$

设 $a_k\ (k=0,1\cdots n)$ 满足 (*) 式，即 $\sum_{k=0}^{n} a_k(\varphi_j,\varphi_k)=0\ (j=0,1\cdots n)$

则有 $\sum_{k=0}^{n} a_k(\varphi_k,\varphi_j)=(\sum_{k=0}^{n} a_k\varphi_k,\varphi_j)=0 \quad (j=0,1\cdots n)$

从而有 $(\sum_{k=0}^{n} a_k\varphi_k,\sum_{k=0}^{n} a_k\varphi_k)=0$.

则 $\sum_{k=0}^{n} a_k\varphi_k=0$，由于 $\{\varphi_k(x)\}_{k=0}^{n}$ 线性无关，则有 $a_k=0\ (k=0,1\cdots n)$

即齐次方程组 (*) 仅有零解. 故 $G\neq 0$

证毕

11. (1) 解：$x=[0,0.2,0.4,0.6,0.8,1]$

此题有误!!!

$y=[0,0.1987,0.3894,0.5646,0.7174,0.8415]$

则 $\varphi_0(x)=1$

$$\alpha_1=\frac{(x\varphi_0,\varphi_0)}{(\varphi_0,\varphi_0)}=\frac{\sum_{k=1}^{6}1\cdot\varphi_0^2(x_k)\cdot x_k}{\sum_{k=1}^{6}1\cdot\varphi_0^2(x_k)}=\frac{3}{6}=0.5$$

$$\varphi_1(x)=x-\frac{3}{6}=x-0.5$$

$$\alpha_2=\frac{(x\varphi_1,\varphi_1)}{(\varphi_1,\varphi_1)}=\frac{\sum_{k=1}^{6}1\cdot\varphi_1^2(x_k)\cdot x_k}{\sum_{k=1}^{6}1\cdot\varphi_1^2(x_k)}=\frac{0.12}{0.5288}=0.2269$$

$$\beta_2=\frac{(\varphi_1,\varphi_1)}{(\varphi_0,\varphi_0)}=\frac{\sum_{k=1}^{6}\varphi_1^2(x_k)}{\sum_{k=1}^{6}1\,\varphi_0^2(x_k)}=\frac{0.5288}{6}=0.0881$$

$$\varphi_2(x)=(x-0.2269)(x-0.5)-0.0881=x^2-0.7269x+0.0253$$

$$\alpha_3=\frac{(x\varphi_2,\varphi_2)}{(\varphi_2,\varphi_2)}=\frac{\sum_{k=1}^{6}1\cdot x_k\cdot\varphi_2^2(x_k)}{\sum_{k=1}^{6}1\cdot\varphi_2^2(x_k)}=\frac{0.033}{0.4785}=0.069$$

$$\beta_3=\frac{(\varphi_2,\varphi_2)}{(\varphi_1,\varphi_1)}=\frac{\sum_{k=1}^{6}1\cdot\varphi_2^2(x_k)}{\sum_{k=1}^{6}1\cdot\varphi_1^2(x_k)}=\frac{0.4785}{0.5288}=0.9049$$

$$\varphi_3(x)=(x-0.069)(x^2-0.7269x+0.0253)-0.9049(x-0.5)$$

$$=x^3-0.7959x^2-0.8294x+0.4542$$

$$a_0=\frac{(y\cdot\varphi_0)}{(\varphi_0\ \varphi_0)}=\frac{\sum_{k=1}^{6}1\cdot y_k\cdot 1}{\sum_{k=1}^{6}1\cdot 1^2}=\frac{2.7116}{6}=0.4519$$

$$a_1=\frac{(y,\varphi_1)}{(\varphi_1\ \varphi_1)}=\frac{\sum_{k=1}^{6}1\cdot y_k\cdot\varphi_1(x_k)}{\sum_{k=1}^{6}1\cdot\varphi_1^2(x_k)}=\frac{0.5939}{0.5288}=1.1231$$

$$a_2=\frac{(y\cdot\varphi_2)}{(\varphi_2\ \varphi_2)}=\frac{\sum_{k=1}^{6}1\cdot y_k\varphi_2(x_k)}{\sum_{k=1}^{6}1\cdot\varphi_2^2(x_k)}=\frac{0.2255}{0.4785}=0.4713$$

$$a_3=\frac{(y\cdot\varphi_3)}{(\varphi_3\ \varphi_3)}=\frac{\sum_{k=1}^{6}1\cdot y_k\varphi_3(x_k)}{\sum_{k=1}^{6}1\cdot\varphi_3^2(x_k)}=\frac{-0.2810}{0.3647}=-0.7705$$

则得最小二乘三次拟合多项式为:

$$\varphi(x)=a_0\varphi_0(x)+a_1\varphi_1(x)+a_2\varphi_2(x)+a_3\varphi_3(x)$$

$$=0.4519+a_1(x-0.5)+a_2(x^2-0.7269x+0.0253)+a_3(x^3-0.7959x^2-0.8294x+0.4542)$$

$=$

(2) 解: $x=[0,0.2,0.4\quad 0.6\quad 0.8\quad 1]$

$y=[0,0.1823,0.3365,\ 0.4700,\ 0.5878\quad 0.6931]$

设 $\begin{cases}\varphi_0(x)=1\\ \varphi_1(x)=x+a\\ \varphi_2(x)=x^2+bx+c\\ \varphi_3(x)=x^3+dx^2+ex+f\end{cases}$

其中, a,b,c,d,e,f 待定.

由正交性：

$$(\varphi_1,1)=\sum_{k=1}^{6}1\cdot\varphi_1(x_k)=3+6a=0 \quad \Rightarrow a=-0.5. \quad 则\ \varphi_1(x)=x-0.5.$$

$$\begin{cases}(\varphi_2,1)=\sum_{k=1}^{6}\varphi_2(x_k)\cdot 1=2.2+3b+6c=0\\(\varphi_2,x)=\sum_{k=1}^{6}\varphi_2(x_k)x_k=1.8+2.2b+3c=0\end{cases}\Rightarrow\begin{cases}b=-1.0\\c=0.1333\end{cases}$$

$$则\ \varphi_2(x)=x^2-1.0x+0.1333$$

$$\begin{cases}(\varphi_3,1)=\sum_{k=1}^{6}\varphi_3(x_k)=1.8+2.2d+3e+6f=0\\(\varphi_3,x)=\sum_{k=1}^{6}\varphi_3(x_k)\cdot x_k=1.5664+1.8d+2.2e+3f=0\\(\varphi_3,x^2)=\sum_{k=1}^{6}\varphi_3(x_k)x_k^2=1.416+1.5664d+1.8e+2.2f=0\end{cases}\Rightarrow\begin{cases}d=-1.5\\e=0.548\\f=-0.024\end{cases}$$

$$则\ \varphi_3(x)=x^3-1.5x^2+0.548x-0.024$$

$$a_0=\frac{(y\cdot\varphi_0)}{(\varphi_0\ \varphi_0)}=\frac{2.2697}{6}=0.3783$$

$$a_1=\frac{(y\cdot\varphi_1)}{(\varphi_1\ \varphi_1)}=\frac{0.4816}{0.7}=0.6880$$

$$a_2=\frac{(y\cdot\varphi_2)}{(\varphi_2\ \varphi_2)}=\frac{-0.0142}{0.0597}=0.2379$$

$$a_3=\frac{(y\cdot\varphi_3)}{(\varphi_3\ \varphi_3)}=\frac{0.0004464}{0.0041}=0.1089$$

$$则\ y=a_0\varphi_0+a_1\varphi_1+a_2\varphi_2+a_3\varphi_3$$

$$=0.3783+0.6880(x-0.5)+0.2379(x^2-x+0.1333)+0.1089(x^3-1.5x^2+0.548x-0.024)$$

$$=0.1089x^3+0.0746x^2+0.5098x+0.0634$$

(3) $x=[0\ \ 0.2\ \ 0.4\ \ 0.6\ \ 0.8\ \ 1]$

$y=[1\ \ 0.8187\ \ 0.6703\ \ 0.5488\ \ 0.4493\ \ 0.3679]$

设 $\begin{cases} \varphi_0(x)=1 \\ \varphi_1(x)=x+a \\ \varphi_2(x)=x^2+bx+c \\ \varphi_3(x)=x^3+dx^2+ex+f \end{cases}$

由 a.b.c.d.e.f 为待定系数. 由正交性:

$(\varphi_1, 1)=0 \Rightarrow a=$

$(\varphi_2\ 1)=0\ \ (\varphi_2, x)=0 \Rightarrow b= \quad c=$

$(\varphi_3, 1)=0\ \ (\varphi_3, x)=0\ \ (\varphi_3, x^2)=0 \Rightarrow d= \quad e= \quad f=$

则所求最小二乘三次拟合多项式为 $\varphi(x)=\sum_{k=0}^{3} a_k \varphi_k(x)$.

其中 $a_k=\frac{(y\ \varphi_k)}{(\varphi_k\ \varphi_k)}$. $k=0,1,2,3$.

12. 解: 令 $\varphi_0=1$. $\varphi_1=x^2$ $\varphi_2=x^4$.

$(\varphi_0\ \varphi_0)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1dx=\pi \quad (\varphi_0\ \varphi_1)=\frac{\pi^3}{12} \quad (\varphi_0\ \varphi_2)=\frac{\pi^5}{80}$ ~~叫~~

$(\varphi_1\ \varphi_1)=\frac{\pi^5}{80} \quad (\varphi_1\ \varphi_2)=\frac{\pi^7}{448} \quad (\varphi_2\ \varphi_2)=\frac{\pi^9}{2304}$

$(f, \varphi_0)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\, dx=2 \quad (f, \varphi_1)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2\cos x\, dx=\frac{\pi^2}{2}-4 \quad (f, \varphi_2)=\frac{\pi^4}{8}-6\pi^2+48$

设最佳平方逼近多项式为 $\varphi(x)=a_0\varphi_0(x)+a_1\varphi_1(x)+a_2\varphi_2(x)$

则 $\begin{bmatrix} (\varphi_0\ \varphi_0) & (\varphi_0\ \varphi_1) & (\varphi_0\ \varphi_2) \\ (\varphi_1\varphi_0) & (\varphi_1\varphi_1) & (\varphi_1\varphi_2) \\ (\varphi_2\varphi_0) & (\varphi_2\varphi_1) & (\varphi_2\varphi_2) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} (f,\varphi_0) \\ (f,\varphi_1) \\ (f,\varphi_2) \end{bmatrix}$

代入数据 $\begin{bmatrix} \pi & \frac{\pi^3}{12} & \frac{\pi^5}{80} \\ \frac{\pi^3}{12} & \frac{\pi^5}{80} & \frac{\pi^7}{448} \\ \frac{\pi^5}{80} & \frac{\pi^7}{448} & \frac{\pi^9}{2304} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{\pi^2}{2}-4 \\ \frac{\pi^4}{8}-6\pi^2+48 \end{bmatrix}$

$\Rightarrow a_0 \approx 0.999580 \quad a_1=-0.496392 \quad a_2=0.037209$

则最佳平方逼近多项式为 $\varphi(x)=0.999580-0.496392x^2+0.037209x^4$

13. (1) 解：先作变量替换 $x=\frac{t+1}{2}$，则 $f(x)=\sqrt{\frac{(1+t)}{2}}\equiv g(t)$，$t\in[-1,1]$，用 Legendre 多项式计算.

下面计算 $g(t)$ 在 $[-1,1]$ 上的一次最佳平方逼近多项式 $q_1(t)$.

$$a_0^*=\frac{(g,P_0)}{2}=\frac{1}{2}\int_{-1}^{1}\sqrt{\frac{1+t}{2}}\,dt=\frac{2}{3}$$

$$a_1^*=\frac{3(g,P_1)}{2}=\frac{3}{2}\int_{-1}^{1}\sqrt{\frac{1+t}{2}}\times t\,dt=\frac{2}{5}.$$

则 $q_1(t)=\frac{2}{3}P_0+\frac{2}{5}P_1=\frac{2}{3}+\frac{2}{5}(t)$

把 $t=2x-1$ 代入得：$f(x)$ 在 $[0,1]$ 上一次最佳平方逼近多项式为：

$$S_1^*(x)=\frac{4}{15}+\frac{4}{5}x.$$

(2) 法一：取 $\varphi_0(x)=1$　$\varphi_1(x)=x$. 则相应正则方程组为：

$$\begin{bmatrix}(\varphi_0,\varphi_0) & (\varphi_0,\varphi_1)\\(\varphi_1,\varphi_0) & (\varphi_1,\varphi_1)\end{bmatrix}\begin{bmatrix}a_0\\a_1\end{bmatrix}=\begin{bmatrix}(f,\varphi_0)\\(f,\varphi_1)\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}1 & \frac{1}{2}\\\frac{1}{2} & \frac{1}{3}\end{bmatrix}\begin{bmatrix}a_0\\a_1\end{bmatrix}=\begin{bmatrix}e-1\\1\end{bmatrix}$$

$$\Rightarrow a_0=4e-10=0.8731\qquad a_1=18-6e=1.6903$$

则最佳平方逼近多项式为 $y=0.8731+1.6903x$

法二：取正交多项式 $\varphi_0(x)=1$　$\varphi_1(x)=2x-1$

$(\varphi_0,\varphi_0)=\int_0^1 1\cdot 1dx=1$　$(\varphi_1,\varphi_1)=\int_0^1(2x-1)^2dx=\frac{1}{3}$

$(f,\varphi_0)=\int_0^1 e^x\cdot 1dx=e-1$　$(f,\varphi_1)=\int_0^1 e^x(2x-1)\,dx=3-e$

$\Rightarrow a_0=\frac{e-1}{1}=e-1$　$a_1=\frac{3-e}{\frac{1}{3}}=9-3e$

此最佳平方逼近一次多项式为：

$$y=a_0\varphi_0+a_1\varphi_1=(e-1)+3(3-e)(2x-1)=0.8731+1.6903x$$

(3) Legendre 多项式在[0,1]上正交多项式 $\hat{P}_n(x)=P_n(2x-1)$，取.

$\varphi_0(x)=\hat{P}_0(x)=1 \qquad \varphi_1(x)=\hat{P}_1(x)=2x-1$

$(\varphi_0,\varphi_0)=\int_0^1 1\cdot 1dx=1 \qquad (\varphi_1\ \varphi_1)=\int_0^1(2x-1)^2dx=\frac{1}{3}$

$(f,\varphi_0)=\int_0^1\sin\frac{\pi}{2}x\cdot 1\,dx=\frac{2}{\pi} \qquad (f,\varphi_1)=\int_0^1\sin\frac{\pi}{2}x\cdot(2x-1)dx=\frac{2(4-\pi)}{\pi^2}$

$\Rightarrow a_0=\frac{\frac{2}{\pi}}{1}=\frac{2}{\pi} \qquad a_1=\frac{8-2\pi}{\pi^2}/\frac{1}{3}=\frac{24-6\pi}{\pi^2}$

则最佳平方一次逼近多项式为：

$y=a_0\varphi_0(x)+a_1\varphi_1(x)$

$=\frac{2}{\pi}+\frac{24-6\pi}{\pi^2}(2x-1)=0.6369+0.5233(2x-1)=1.0467x+0.1136$.

14. Legendre 多项式在[0,1]上的正交多项式 $\hat{P}_n(x)=P_n(2x-1)$.取：

$\varphi_0(x)=1 \qquad \varphi_1(x)=2x-1 \qquad \varphi_2(x)=6x^2-6x+1$

则 $(\varphi_0\ \varphi_0)=1 \qquad (\varphi_1\ \varphi_1)=\frac{1}{3} \qquad (\varphi_2\ \varphi_2)=\frac{1}{5}$

① 对于 $f(x)=\sqrt{x}$，有 $(f,\varphi_0)=\int_0^1\sqrt{x}\cdot 1\,dx=\frac{2}{3} \qquad (f\ \varphi_1)=\int_0^1\sqrt{x}(2x-1)dx=\frac{2}{15}$

$(f,\varphi_2)=\int_0^1\sqrt{x}(6x^2-6x+1)dx=\frac{-2}{105}$

则 $f(x)=\sqrt{x}$ 在[0,1]上的二次逼近多项式为：

$\varphi(x)=\frac{\frac{2}{3}}{1}\cdot\times 1+\frac{\frac{2}{15}}{\frac{1}{3}}\times(2x-1)+\frac{\frac{-2}{105}}{\frac{1}{5}}\times(6x^2-6x+1)$

$=-0.5714x^2+1.3714x+0.1714$

② 对 $f(x)=e^x$. 有 $(f,\varphi_0)=\int_0^1 e^x dx=e-1=1.71823$

$(f,\varphi_1)=\int_0^1 e^x(2x-1)dx=3-e=0.2817$

$(f,\varphi_2)=\int_0^1 e^x(6x^2-6x+1)dx=7e-19=0.02797$

则 $\varphi(x)=1.7183\varphi_0(x)+0.8451\varphi_1(x)+0.1399\varphi_2(x)$

$=0.8394x^2+0.8508x+1.0131$

③ 对 $f(x)=\sin\frac{\pi}{2}x$，有 $(f,\varphi_0)=\frac{2}{\pi}$. $(f,\varphi_1)=\frac{24-6\pi}{\pi^2} \qquad (f,\varphi_2)=\frac{24\pi+2\pi^2-96}{\pi^3}$

则 $\varphi(x)=\frac{\frac{2}{\pi}}{1}\varphi_0(x)+\frac{24-6\pi}{\pi^2}\varphi_1(x)+\frac{24\pi+2\pi^2-96}{\pi^3}\varphi_2(x)$

$=1.0467x+0.1136+(-0.0297)(6x^2-6x+1)$

$=-0.1785x^2+1.2249x+0.0839$

15. 先求Legendre多项式在$[-\frac{\pi}{2}\ \frac{\pi}{2}]$上的形式：

$$\hat{P}_n(x)=P_n\left(\frac{2x-(-\frac{\pi}{2}+\frac{\pi}{2})}{\frac{\pi}{2}+\frac{\pi}{2}}\right)=P_n\left(\frac{2x}{\pi}\right)$$

$$\varphi_0=\hat{P}_0(x)=P_0\left(\frac{2x}{\pi}\right)=1$$

$$\varphi_1=\hat{P}_1(x)=P_1\left(\frac{2x}{\pi}\right)=\frac{2x}{\pi}.$$

$$\varphi_2=\hat{P}_2(x)=P_2\left(\frac{2x}{\pi}\right)=\frac{1}{2}\left(3\left(\frac{2x}{\pi}\right)^2-1\right)=\frac{6x^2}{\pi^2}-\frac{1}{2}$$

$$\varphi_3=\hat{P}_3(x)=P_3\left(\frac{2x}{\pi}\right)=\frac{1}{2}\left(5\left(\frac{2x}{\pi}\right)^3-3\times\frac{2x}{\pi}\right)=\frac{20x^3}{\pi^3}-\frac{3x}{\pi}.$$

则 $(\varphi_0\ \varphi_0)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}1\times1\,dx=\pi=3.14$ $(\varphi_1,\varphi_1)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{2x}{\pi}\right)^2dx=\frac{\pi}{3}=1.0467$

$(\varphi_2,\varphi_2)=0.6280$ $(\varphi_3,\varphi_3)=0.4472$

$(f\ \varphi_0)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin x\cdot1\,dx=0$ $(f,\varphi_1)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin x\cdot\frac{2x}{\pi}dx=\frac{4}{\pi}=1.2739$

$(f,\varphi_2)=0$ $(f,\varphi_3)=-0.1116$

则 $P_3(x)=\frac{0}{3.14}\varphi_0+\frac{1.2739}{1.0467}\varphi_1+\frac{0}{0.6280}\varphi_2+\frac{-0.1116}{0.4472}\varphi_3$

$$=1.2171\cdot\left(\frac{2x}{\pi}\right)-0.2496\left(\frac{20x^3}{\pi^3}-\frac{3x}{\pi}\right)$$

$$=-0.1612x^3+1.0137x$$

则均方误差 $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\sin x-(-0.1612x^3+1.0137x)\right]dx=0.$