Homework 1: Measure Theory

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Exercise 1.3 (a) $\mathcal{G}_1$ is not an algebra (and hence not a σ-algebra). $\mathcal{G}_1$ is not closed under complements. Fix $a \in \mathbb{R}$ and let $A = (-\infty, a)$ be an open set in $\mathbb{R}$. Then $A^c = [a, \infty)$ is not an open set in $\mathbb{R}$. Therefore, we have that $A \in \mathcal{G}_1$ but $A^c \notin \mathcal{G}_1$, which violates the definition of an algebra.

(b) $\mathcal{G}_2$ is an algebra but not a σ-algebra. We first show that $\mathcal{G}_2$ is an algebra. Let $a = b \in \mathbb{R}$. Then $(a, b] = \emptyset \in \mathcal{G}_2$. $\mathcal{G}_2$ is also closed under complements. To see this, fix $a, b \in \mathbb{R}$. Then $(a, b]^c = (-\infty, a] \cup (b, \infty) \in \mathcal{G}_2$. Furthermore, $(-\infty, b]^c = (b, \infty) \in \mathcal{G}_2$, and $(a, \infty)^c = (-\infty, a] \in \mathcal{G}_2$. Finally, observe that $\mathcal{G}_2$ is closed under finite unions. A finite union of intervals of the form $(a, b], (-\infty, b]$, and (a, ∞) still results in a finite union of intervals of the same form. Therefore, $\mathcal{G}_2$ is an algebra. However, $\mathcal{G}_2$ is not a σ-algebra. An infinite (countable) union of intervals of the form $(a, b], (-\infty, b]$, and (a, ∞) does not belong to $\mathcal{G}_2$, by definition. Thus $\mathcal{G}_2$ is not a σ-algebra.

(c) $\mathcal{G}_3$ is an algebra and a σ-algebra. The proof that $\emptyset \in \mathcal{G}_3$ and $\mathcal{G}_3$ is closed under complements is the same as the proof for $\mathcal{G}_2$. However, $\mathcal{G}_3$ contains countable unions of intervals of the form $(a, b], (-\infty, b]$, and (a, ∞) by definition; therefore $\mathcal{G}_3$ is closed under countable unions and is a σ-algebra.

Exercise 1.7 Let X be a nonempty set and $\mathcal{A}$ be a σ-algebra of X. The definition of a σ-algebra requires $\emptyset \in \mathcal{A}$. Furthermore, a σ-algebra is closed under complements, which implies that $\emptyset^c = X \in \mathcal{A}$. Therefore, to satisfy the definition, any σ-algebra must contain both $\emptyset$ and X, so that the set $\{\emptyset, X\}$ is the smallest σ-algebra. The power set of X, $\mathcal{P}(X)$, is the largest possible σ-algebra because it contains all possible subset of X. Recall that a σ-algebra is a family of subsets of X. Therefore, the largest possible family of subsets of X contains all subsets of X, which is precisely $\mathcal{P}(X)$.

Exercise 1.10 Let $\{S_\alpha\}$ be a family of σ-algebras on X. We show that $\cap_\alpha S_\alpha$ is also a σ-algebra. Observe that $\emptyset \in S_\alpha$ for all α because $\{\mathcal{S}_\alpha\}$ is a family of σ-algebras. Therefore, $\emptyset \in \cap_\alpha S_\alpha$.

Moreover, let $A \in \cap_\alpha S_\alpha$. Therefore, $A \in S_\alpha$ for all α. Each S_α is a σ-algebra, which implies that $A^c \in S_\alpha$ for all α. Therefore, $A^c \in \cap_\alpha S_\alpha$. Thus, $A \in \cap_\alpha S_\alpha$ implies $A^c \in \cap_\alpha S_\alpha$, so that $\cap_\alpha S_\alpha$ is closed under complements.

Finally, let $A_1, A_2, \ldots \in \cap_\alpha S_\alpha$. Therefore, $A_1, A_2, \ldots \in S_\alpha$ for all α. Each S_α is a σ-algebra, which implies that $\cup_{n=1}^\infty A_n \in S_\alpha$ for all α. Thus, $\cup_{n=1}^\infty A_n \in \cap_\alpha S_\alpha$, which shows that $\cap_\alpha S_\alpha$ is closed under countable unions.

Therefore, we have showed that $\cap_\alpha S_\alpha$ is indeed a σ-algebra.

Exercise 1.22 Let $(X, \mathcal{S}, \mu)$ be a measure space.

(a) Observe that we can write $B = (B \cap A^c) \cup (B \cap A) = (B \cap A^c) \cup A$, where the last inequality follows because $A \subset B$. By the definition of a measure, we have that $\mu(B) = \mu(B \cap A^c) + \mu(A)$, as $(B \cap A^c)$ and $(B \cap A) = A$ are clearly disjoint. Finally, note that by definition, a measure is nonnegative, so $\mu(B \cap A^c) \geq 0$. Therefore, $\mu(B) \geq \mu(A)$.

(b) Let $A = \cup_{i=1}^{\infty} A_i$ and define $\{B_i\}_{i=1}^{\infty}$ as follows. Set $B_1 = A_1$, $B_2 = A_2 \cap A_1^c$, $B_3 = A_3 \cap (A_1 \cup A_2)^c$, and more generally, $B_i = A_i \cap (\cup_{n=1}^{i-1} A_i)^c$. Observe that $A = \cup_{i=1}^{\infty} B_i$. Also observe $B_i \subset A_i$, as we form B_i by intersecting A_i with other sets. Additionally, $B_i \cap B_j = \emptyset$ for all $i \neq j$, by construction. By monotonicity, proved above, we have that $\mu(B_i) \leq \mu(A_i)$. Therefore,

$$\mu(\cup_{i=1}^{\infty} A_i) = \mu(\cup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \mu(B_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

The second inequality follows by the disjointness of B_i and B_j for all $i \neq j$, and the fourth inequality follows by monotonicity.

Exercise 1.23 Let $(X, \mathcal{S}, \mu)$ be a measure space and $B \in \mathcal{S}$. Define $\lambda : \mathcal{S} \to [0, \infty]$ by $\lambda(A) = \mu(A \cap B)$. By the definition of λ, $\lambda(\emptyset) = \mu(\emptyset \cap B) = \mu(\emptyset) = 0$, because μ is a measure.

Next, fix $\{A_i\}_{i=1}^{\infty} \subset \mathcal{S}$ such that $A_i \cap A_j = \emptyset$ for all $i \neq j$. By the definition of λ,

$$\begin{aligned}
\lambda(\cup_{i=1}^{\infty} A_i) &= \mu((\cup_{i=1}^{\infty} A_i) \cap B) \\
&= \mu(\cup_{i=1}^{\infty} (A_i \cap B)) \\
&= \sum_{i=1}^{\infty} \mu(A_i \cap B) \\
&= \sum_{i=1}^{\infty} \lambda(A_i \cap B)
\end{aligned}$$

The third equality follows because $A_i \cap A_j = \emptyset$ for all $i \neq j$ implies that $(A_i \cap B) \cap (A_j \cap B) = \emptyset$ for all $i \neq j$. Thus, as μ is a measure, we may break of the disjoint into a sum of the individual measures. Finally, the last inequality follows by the definition of λ.

Exercise 1.26 We use the following lemma in our proof:

Lemma 1. If $A_n \subset A_1$, then $\mu(A_1) - \mu(A_n) = \mu(A_1 \setminus A_n)$

Proof. We can write $A_1 = (A_1 \setminus A_n) \cup A_n$, which is a disjoint union, because $A_n \subset A_1$. Therefore, $\mu(A_1) = \mu(A_1 \setminus A_n) + \mu(A_n)$. The lemma follows. □

Define $B_n = A_1 \setminus A_n$ for $n \in \mathbb{N}$ and let $B = \cup_{n=1}^{\infty} B_n$. Observe that $\{B_n\}_{n=1}^{\infty}$ forms an increasing sequence because $A_n \supset A_{n+1}$ for all n is a decreasing sequence of sets. Let $A = \cap_{n=1}^{\infty} A_n$. Then,

$$\begin{aligned}\mu(A_1)-\mu(A) &= \mu(A_1-A)\\ &= \mu(B)\\ &= \lim_{n\to\infty}\mu(B_n)\\ &= \lim_{n\to\infty}(\mu(A_1)-\mu(A_n))\end{aligned}$$

Therefore, because $\mu(A_1) < \infty$, we may subtract $\mu(A_1)$ from both sides of the above sequence of equations to find that,

$$\mu(A) = \lim_{n\to\infty}\mu(A_n)$$

Exercise 2.10 Recall that μ^* is countably subadditive. Write $B \subset X$ as $B = (B\cap E)\cup(B\cap E^c)$. By countable subadditivity, we have that $\mu^*(B) \leq \mu^*(B\cap E) + \mu^*(B\cap E^c)$. Therefore, if $\mu^*(B) = \mu^*(B\cap E)+\mu^*(B\cap E^c)$, then it follows that $\mu^*(B) \geq \mu^*(B\cap E)+\mu^*(B\cap E^c)$. Therefore, satisfying the equality condition implies the condition (*) in Theorem 2.8.

Exercise 2.14 Recall that $\sigma(\mathcal{O}) :=$ the smallest σ-algebra containing all open sets of X. We call $\sigma(\mathcal{O})$ the Borel σ-algebra of X written as $\mathcal{B}(X)$. Next, by the Carathéodory Construction, $\mathcal{M}$ is a σ-algebra. Additionally, $\mathcal{M}$ is the collection of all Lebesgue measurable sets. The collection of all Lebesgue measurable sets contains open sets. Therefore, because $\mathcal{B}(X)$ is defined as the intersection of all σ-algebra containing open sets, we must have that $\mathcal{B}(X) \subset \mathcal{M}$.

Exercise 3.1 Let $A = \{a_1, a_2, \ldots\}$ be a countable subset of the real line. Fix $\epsilon > 0$. We construct a sequence of intervals $\{I_n\}_{n=1}^{\infty}$ as follows. Let $I_1 = (a_1 - \frac{\epsilon}{2}, a_1 + \frac{\epsilon}{2})$, which has length ϵ. Similarly, let $I_2 = (a_2 - \frac{\epsilon}{4}, a_2 + \frac{\epsilon}{4})$, which has length $\frac{\epsilon}{2}$. For a general interval n, we write $I_n = (a_n - \frac{\epsilon}{2^n}, a_n + \frac{\epsilon}{2^n})$, which has length $\frac{\epsilon}{2^{n-1}}$. Now, the sum of the lengths of the intervals is,

$$\sum_{n=1}^{\infty}\frac{\epsilon}{2^{n-1}} = 2\epsilon$$

By definition, the Lebesgue (outer) measure of A is,

$$\mu(A) = \inf\left[\sum_{n=1}^{\infty}(d_n - c_n) : A \subset \bigcup_{n=1}^{\infty}(c_n, d_n]\right]$$

Above, we demonstrated an open cover of A such that the sum of the intervals is arbitrarily small. Therefore, the measure of this open cover is arbitrarily small (yet weakly positive) and hence 0. Thus, $\mu(A) = 0$.

Exercise 3.7

We show that the following conditions are equivalent:

1. $\{x \in X : f(x) < a\} \in \mathcal{M}$

2. $\{x \in X : f(x) \geq a\} \in \mathcal{M}$

   (1) $\implies$ (2): Suppose $\{x \in X : f(x) < a\} \in \mathcal{M}$. Observe that $f^{-1}([a, \infty)) = (f^{-1}(-\infty, a))^c$. $\mathcal{M}$ is closed under complements, therefore $f^{-1}([a, \infty)) \in \mathcal{M}$.

3. $\{x \in X : f(x) > a\} \in \mathcal{M}$

   (2) $\implies$ (3): Suppose $\{x \in X : f(x) \geq a\} \in \mathcal{M}$. Observe that $f^{-1}((a, \infty)) = \cap_{n=1}^{\infty} f^{-1}([a - \frac{1}{n}, \infty))$. By assumption, each of the sets in this intersection is in $\mathcal{M}$. $\mathcal{M}$ is closed under countable intersections. Therefore, $f^{-1}(a, \infty) \in \mathcal{M}$.

4. $\{x \in X : f(x) \leq a\} \in \mathcal{M}$

   (3) $\implies$ (4): Suppose $\{x \in X : f(x) > a\} \in \mathcal{M}$. Observe that $f^{-1}((-\infty, a]) = (f^{-1}(a, \infty))^c$. $\mathcal{M}$ is closed under complements, therefore $f^{-1}((-\infty, a]) \in \mathcal{M}$.

   (4) $\implies$ (1): Suppose $\{x \in X : f(x) \leq a\} \in \mathcal{M}$. Observe that $f^{-1}((-\infty, a)) = \cap_{n=1}^{\infty} f^{-1}((-\infty, a + \frac{1}{n}))$. By assumption, each of the sets in this intersection is in $\mathcal{M}$. $\mathcal{M}$ is closed under countable intersections. Therefore, $f^{-1}((a, \infty)) \in \mathcal{M}$.

Exercise 3.10 We prove $\min(f, g)$, $\max(f, g)$ and $|f|$ directly from the definition of measurable functions and use results from Exercise 3.7 to rewrite the condition for measurability in equivalent forms to make the proofs easier.

1. Consider $F(f(x) + g(x)) = f(x) + g(x)$. Then F is continuous and by part 4 of Theorem 3.9, measurable. Therefore, $f + g$ is measurable.

2. Consdier $F(f(x) + g(x)) = f(x)g(x)$. Then F is continuous and by part 4 of Theorem 3.9, measurable. Therefore, $f \cdot g$ is measurable.

3. Because f and g are measurable functions on $(X, \mathcal{M})$, we have that for all $a \in \mathbb{R}$, $\{x \in X : f(x) < a\} \in \mathcal{M}$ and $\{x \in X : g(x) < a\} \in \mathcal{M}$. Therefore, it follows that $\{x \in X : \max(f(x), g(x)) < a\} = \{x \in X : f(x) < a\} \cap \{x \in X : g(x) < a\}$. $\mathcal{M}$ is closed under countable intersections, therefore, $\{x \in X : \max(f(x), g(x)) < a\} \in \mathcal{M}$, so that $\max(f(x), g(x))$ is measurable.

4. The proof that $\min(f, g)$ is measurable is analogous to the proof of (3). The key observation here is that $\{x \in X : \min(f(x), g(x)) > a\} = \{x \in X : f(x) > a\} \cap \{x \in X : g(x) > a\}$. $\mathcal{M}$ is closed under countable intersections, therefore, $\{x \in X : \min(f(x), g(x)) > a\} \in \mathcal{M}$, so that $\min(f(x), g(x))$ is measurable.

5. Observe that $\{x \in X : |f(x)| > a\} = \{x \in X : f(x) < -a\} \cup \{x \in X : f(x) > a\}$. Both of these sets are in $\mathcal{M}$. $\mathcal{M}$ is closed under countable unions, therefore, $\{x \in X : |f(x)| > a\} \in \mathcal{M}$, so that $|f(x)|$ is measurable.

Exercise 3.17 Let f be bounded, and fix $\epsilon > 0$. Then, there exists an $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in X$. Therefore, $x \in E_i^M$ for some i and all $x \in X$. Observe that there is an $N \in \mathbb{R}$ and $N \geq M$ such that $\frac{1}{2^N} < \epsilon$. Therefore, for all $x \in X$ and $n \geq N$, $||s_n(x) - f(x)|| < \epsilon$. Therefore, the convergence in part (1) of Theorem 3.13 is uniform.

Exercise 4.13 To show that $f \in \mathscr{L}^1(\mu, E)$, we must show that both $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ are finite.

Recall that $||f|| = f^+ + f^-$. Also note that $0 \leq f^+$ and $0 \leq f^-$ by definition. Because $||f|| < M$ on E, then $0 \leq f^+ < M$ and $0 \leq f^- < M$ on E. Then, because $\mu(E) < \infty$, we have that,

$$\int_E f^+ d\mu < M\mu(E) < \infty$$
$$\int_E f^- d\mu < M\mu(E) < \infty$$

Therefore, both $\int_E f^+ d\mu$ and $\int_E f^- d\mu$ are finite. Hence, by definition, $f \in \mathscr{L}^1(\mu, E)$.

Exercise 4.14 We prove the contrapositive of this statement. To that end, suppose there exists a measurable set $\hat{E} \subset E$ such that f is infinite on $\hat{E}$. Here, we assume that f reaches positive infinity (without loss of generality, the proof for negative infinity or mixed between positive and negative infinity is analogous). It follows that,

$$\infty = \int_{\hat{E}} f d\mu \leq \int_E f d\mu \leq \int_E ||f|| d\mu$$

The first inequality is proved below. However, this implies that $f \notin \mathscr{L}^1(\mu, E)$.

Exercise 4.15 Let $f, g \in \mathscr{L}^1(\mu, E)$. Define the set of simple functions $B(f) = \{s : 0 \leq s \leq f, s \text{ simple, measurable}\}$. Let $f \leq g$. If follows that $f^+ \leq g^+$ and $f^- \geq g^-$. Then, we have that $B(f^+) \subset B(g^+)$ and $B(g^-) \subset B(f^-)$. These two relationships imply that $\int_E f^+ d\mu \leq \int_E g^+ d\mu$ and $\int_E f^- d\mu \geq \int_E g^- d\mu$. Then by the definition of the Lebesgue integral, we observe that,

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu \leq \int_E g^+ d\mu - \int_E g^- d\mu = \int_E g d\mu$$

Therefore,

$$\int_E f d\mu \leq \int_E g d\mu$$

Exercise 4.16 Fix a simple function $s(x) = \sum_{i=1}^N c_i \chi_{E_i}$, where $E_i \in \mathcal{M}$. Let $A \subset E \in \mathcal{M}$. Then, by the monotonicity of measures, we have that $\mu(A \cap E_i) \leq \mu(E \cap E_i)$ for all i. Therefore, combining this result with Definition 4.1, we have that,

$$\int_A s d\mu = \sum_{i=1}^N c_i \mu(A \cap E_i) \leq \sum_{i=1}^N c_i \mu(E \cap E_i) = \int_E s d\mu \tag{1}$$

Now, it follows that,

$$\int_A f d\mu = \sup \left[\int_A s d\mu : 0 \leq s \leq f, s \text{ simple, measurable} \right]$$

and

$$\int_E f d\mu = \sup \left[\int_E s d\mu : 0 \leq s \leq f, s \text{ simple, measurable} \right]$$

Now because our choice of s was arbitrary, we have by Equation (1) that,

$$\int_A f d\mu \leq \int_E f d\mu \tag{2}$$

Because $f \in \mathscr{L}^1(\mu, E)$, by definition we have that $\int_E ||f|| d\mu < \infty$. Therefore, $\int_E f d\mu < \infty$. Finally, it follows that $\int_A f d\mu < \infty$, which in turn implies $\int_A f^+ d\mu < \infty$ and $\int_A f^- d\mu < \infty$, so that $f \in \mathscr{L}^1(\mu, A)$.

Exercise 4.21 Let $A, B \in \mathcal{M}$, $B \subset A$, $\mu(A - B) = 0$, and $f \in \mathscr{L}^1$. Then, by Proposition 4.6. we have that,

$$\int_{A-B} f d\mu = 0.$$

Recall that f^+ and f^- are non-negative $\mathcal{M}$-measurable functions because $f \in \mathscr{L}^1$. By Theorem 4.19, we have that $\mu_1(A) = \int_A f^+ d\mu$ and $\mu_2(A) = \int_A f^- d\mu$ are measures on $\mathcal{M}$. Therefore, by the definition of the Lesbesgue integral,

$$\int_A f d\mu = \int_A f^+ d\mu - \int_A f^- d\mu = \mu_1(A) - \mu_2(A)$$

Now, consider the disjoint union $A = (A - B) \cup B$. Because both $\mu_1(A)$ and $\mu_2(A)$ are measures, we have that $\mu_i(A) = \mu_i(A - B) + \mu_i(B)$ for $i = 1, 2$, because measures are additively separable on disjoint sets. Therefore, we have that $\mu_i(A) = \mu_i(B)$ for $i = 1, 2$ because $\mu(A - B) = 0$. Therefore,

$$\int_A f d\mu = \mu_1(B) - \mu_2(B) = \int_B f d\mu$$

This result clearly implies that

$$\int_A f d\mu \leq \int_B f d\mu$$