Homework 1: Dynamic Programming

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Problem 1 (Asset market equilibrium) Each agent h will maximize their utility

$$\max_{\{c_0,\dots,c_4\}} U^h(c) = \max_c \left[\frac{c_0^{1-\gamma}}{1-\gamma} + \frac{1}{4}\sum_{s=1}^{4} \frac{c_s^{1-\gamma}}{1-\gamma} \right] \tag{1}$$

over two periods subject to the budget constraint

$$c_0^h = e_0^h - q_1\theta_1^h - q_2\theta_2^h \qquad \text{at } t=0 \tag{2}$$

and

$$c_s^h = e_s^h + \theta_1^h A_s^1 + \theta_2^h A_s^2 \qquad \text{otherwise} \tag{3}$$

where q_i the initial price of asset i, e_s^h is agent h his endowment (note, $e_0 = 1$ for both agents), θ_i^h his holdings of asset i and A_s^i the pay out from asset i in state s. Moreover, the following market clearing conditions have to hold:

$$\theta_1^1 + \theta_1^2 = 0 \tag{4}$$

$$\theta_2^1 + \theta_2^2 = 0 \tag{5}$$

We can plug the budget constraints and market clearing conditions directly into (1). Consequently, the First Order Conditions (FOCs) are as follows:

$$\frac{\partial U^1}{\partial \theta_1^1} = -q_1(e_0^1 - q_1\theta_1^1 - q_2\theta_2^1)^{-\gamma} + \frac{1}{4}\sum_{s=1}^{4} A_s^1(e_s^1 + \theta_1^1 A_s^1 + \theta_2^1 A_s^2)^{-\gamma} = 0 \tag{6}$$

$$\frac{\partial U^1}{\partial \theta_2^1} = -q_2(e_0^1 - q_1\theta_1^1 - q_2\theta_2^1)^{-\gamma} + \frac{1}{4}\sum_{s=1}^{4} A_s^2(e_s^1 + \theta_1^1 A_s^1 + \theta_2^1 A_s^2)^{-\gamma} = 0 \tag{7}$$

$$\begin{aligned}\frac{\partial U^2}{\partial \theta_1^2} &= -q_1(e_0^2 - q_1\theta_1^2 - q_2\theta_2^2)^{-\gamma} + \frac{1}{4}\sum_{s=1}^{4} A_s^1(e_s^2 + \theta_1^2 A_s^1 + \theta_2^2 A_s^2)^{-\gamma} \\ &= -q_1(e_0^2 + q_1\theta_1^1 + q_2\theta_2^1)^{-\gamma} + \frac{1}{4}\sum_{s=1}^{4} A_s^1(e_s^2 - \theta_1^1 A_s^1 - \theta_2^1 A_s^2)^{-\gamma} = 0 \end{aligned} \tag{8}$$

$$\begin{aligned}\frac{\partial U^2}{\partial \theta_2^2} &= -q_2(e_0^2 - q_1\theta_1^2 - q_2\theta_2^2)^{-\gamma} + \frac{1}{4}\sum_{s=1}^{4} A_s^2(e_s^2 + \theta_1^2 A_s^1 + \theta_2^2 A_s^2)^{-\gamma} \\ &= -q_2(e_0^2 + q_1\theta_1^1 + q_2\theta_2^1)^{-\gamma} + \frac{1}{4}\sum_{s=1}^{4} A_s^2(e_s^2 - \theta_1^1 A_s^1 - \theta_2^1 A_s^2)^{-\gamma} = 0 \end{aligned} \tag{9}$$

Hence, resulting in a system of 4 equations in 4 unknowns (q_1, q_2, θ_1^1, θ_2^1)