Homework 1: DSGE Modelling

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Exercise 1 As a guess, it is suggested that the policy function for the Brock and Mirman model takes the following form: $K_{t+1} = Ae^{z_t}K_t^\alpha$. Moreover, the Euler equation is given. To verify the guess, we substitute the suggested policy function into each side of the Euler equation. Then, we try to solve for a value for A that results the left hand side (lhs) to be equal to the right hand side (rhs) of the Euler equation.

Starting with the lhs, we observe that,

$$\frac{1}{e^{z_t}K_t^\alpha - K_{t+1}} = \frac{1}{e^{z_t}K_t^\alpha - Ae^{z_t}K_t^\alpha} = \frac{1}{(1-A)e^{z_t}K_t^\alpha}$$

For the rhs, we observe that,

$$\begin{aligned}\beta E_t\left[\frac{\alpha e^{z_{t+1}}K_{t+1}^{\alpha-1}}{e^{z_{t+1}}K_{t+1}^\alpha - K_{t+2}}\right] &= \beta E_t\left[\frac{\alpha e^{z_{t+1}}(Ae^{z_t}K_t^\alpha)^{\alpha-1}}{e^{z_{t+1}}(Ae^{z_t}K_t^\alpha)^\alpha - Ae^{z_{t+1}}(Ae^{z_t}K_t^\alpha)^\alpha}\right] \\ &= \frac{\alpha\beta}{(1-A)Ae^{z_t}K_t^\alpha}\end{aligned}$$

Consequently, equating the lhs and rhs,

$$\frac{1}{(1-A)e^{z_t}K_t^\alpha} = \frac{\alpha\beta}{(1-A)Ae^{z_t}K_t^\alpha}$$

it follows that $A = \alpha\beta$. Therefore, the policy function is given by $k_{t+1} = \alpha\beta e^{z_t}k_t^\alpha$.

Exercise 2 Consider the following functional forms,

$$\begin{aligned}u\left(c_t, \ell_t\right) &= \ln c_t + a\ln\left(1-\ell_t\right) \\ F\left(K_t, L_t, z_t\right) &= e^{z_t}K_t^\alpha L_t^{1-\alpha}\end{aligned}$$

and, regardless of functional form, we have the following market clearing conditions $\ell_t = \text{Ł}_t$, $k_t = K_t$, $w_t = W_t$, and $r_t = R_t$. Then the seven equations characterizing equations and seven unknowns: $\{c_t, k_t, \ell_t, w_t, r_t, T_t, z_t\}$ for the model are as follows:

$$\begin{aligned}c_t &= (1-\tau)[w_t\ell_t + (r_t-\delta)k_t] + k_t + T_t - k_{t+1} \\ \frac{1}{c_t} &= \beta E_t\left[\frac{1}{c_{t+1}}[(r_{t+1}-\delta)(1-\tau)+1]\right] \\ \frac{a}{1-\ell_t} &= \frac{1}{c_t}w_t(1-\tau) \\ r_t &= \alpha e^{z_t}k_t^{\alpha-1}\ell_t^{1-\alpha} = \alpha e^{z_t}\left(\frac{\ell_t}{k_t}\right)^{1-\alpha} \\ w_t &= (1-\alpha)e^{z_t}k_t^\alpha\ell_t^{-\alpha} = (1-\alpha)e^{z_t}\left(\frac{k_t}{\ell_t}\right)^\alpha \\ T_t &= \tau[w_t\ell_t + (r_t-\delta)k_t] \\ z_t &= (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim iid.\ \mathcal{N}(0,\sigma_z^2)\end{aligned}$$

The trick from Exercise 1 cannot be applied here since households optimize over both their leisure and consumption decisions, hence e^{z_t} will not drop out.

Exercise 3 Consider the following functional forms,

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a\ln(1-\ell_t)$$
$$F(K_t, L_t, z_t) = e^{z_t} K_t^{\alpha} L_t^{1-\alpha}$$

and, regardless of functional form, we have the following market clearing conditions $\ell_t = Ł_t$, $k_t = K_t$, $w_t = W_t$, and $r_t = R_t$. Then the seven equations characterizing equations and seven unknowns: $\{c_t, k_t, \ell_t, w_t, r_t, T_t, z_t\}$ for the model are as follows:

$$\begin{aligned}
c_t &= (1-\tau)[w_t\ell_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1} \\
\frac{1}{c_t^{\gamma}} &= \beta E_t\left[\frac{1}{c_{t+1}^{\gamma}}[(r_{t+1}-\delta)(1-\tau)+1]\right] \\
\frac{a}{1-\ell_t} &= \frac{1}{c_t^{\gamma}} w_t(1-\tau) \\
r_t &= \alpha e^{z_t} k_t^{\alpha-1}\ell_t^{1-\alpha} = \alpha e^{z_t}\left(\frac{\ell_t}{k_t}\right)^{1-\alpha} \\
w_t &= (1-\alpha)e^{z_t}k_t^{\alpha}\ell_t^{-\alpha} = (1-\alpha)e^{z_t}\left(\frac{k_t}{\ell_t}\right)^{\alpha} \\
T_t &= \tau[w_t\ell_t + (r_t-\delta)k_t] \\
z_t &= (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim iid.\ \mathcal{N}(0,\sigma_z^2)
\end{aligned}$$

Exercise 4 Consider the following function forms:

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a\frac{(1-\ell_t)^{1-\xi} - 1}{1-\xi}$$
$$F(K_t, L_t, z_t) = e^{z_t}\left[\alpha K_t^{\eta} + (1-\alpha)L_t^{\eta}\right]^{\frac{1}{\eta}}$$

and, regardless of functional form, we have the following market clearing conditions $\ell_t = Ł_t$, $k_t = K_t$, $w_t = W_t$, and $r_t = R_t$. Then, the seven equations characterizing equations and seven unknowns: $\{c_t, k_t, \ell_t, w_t, r_t, T_t, z_t\}$ for the model are as follows:

$$\begin{aligned}
c_t &= (1-\tau)[w_t\ell_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1} \\
\frac{1}{c_t^{\gamma}} &= \beta E_t\left[\frac{1}{c_{t+1}^{\gamma}}[(r_{t+1}-\delta)(1-\tau)+1]\right] \\
\frac{a}{(1-\ell_t)^{\xi}} &= \frac{1}{c_t^{\gamma}} w_t(1-\tau) \\
r_t &= \alpha e^{z_t} k_t^{\eta-1}[\alpha k_t^{\eta} + (1-\alpha)\ell_t^{\eta}]^{\frac{1-\eta}{\eta}} \\
w_t &= (1-\alpha)e^{z_t}\ell_t^{\eta-1}[\alpha k_t^{\eta} + (1-\alpha)\ell_t^{\eta}]^{\frac{1-\eta}{\eta}} \\
T_t &= \tau[w_t\ell_t + (r_t-\delta)k_t] \\
z_t &= (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim iid.\ \mathcal{N}(0,\sigma_z^2)
\end{aligned}$$

Exercise 5 Consider the following functional forms:

$$u(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma}$$
$$F(K_t, L_t, z_t) = K_t^{\alpha}(L_t e_t^{z_t})^{1-\alpha}$$

and, regardless of functional form, we have the following market clearing conditions $\ell_t = \text{Ł}_t$, $k_t = K_t$, $w_t = W_t$, and $r_t = R_t$. Moreover, assume $\ell_t = 1$. Then, using the labour market clearing condition, we know that $L_t = \ell_t = 1$. Additionally, the following six equations characterize the model:

$$\begin{aligned}
c_t &= (1-\tau)[w_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1} \\
\frac{1}{c_t^{\gamma}} &= \beta E_t\left[\frac{1}{c_{t+1}^{\gamma}}[(r_{t+1} - \delta)(1-\tau) + 1]\right] \\
r_t &= \alpha k_t^{\alpha-1}(\ell_t e_t^{z_t})^{1-\alpha} = \alpha\left(\frac{e^{z_t}}{k_t}\right)^{1-\alpha} \\
w_t &= (1-\alpha)k_t^{\alpha}(\ell_t e_t^{z_t})^{-\alpha} e_t^{z_t} = (1-\alpha)k_t^{\alpha}(e_t^{z_t})^{1-\alpha} \\
T_t &= \tau[w_t + (r_t - \delta)k_t] \\
z_t &= (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim iid.\ \mathcal{N}(0, \sigma_z^2)
\end{aligned}$$

The equivalents of these equations at the steady-state are

$$\begin{aligned}
\bar{c} &= (1-\tau)[\bar{w} + (\bar{r} - \delta)\bar{k}] + \bar{T} \\
\frac{1}{\bar{c}^{\gamma}} &= \beta E_t\left[\frac{1}{\bar{c}^{\gamma}}[(\bar{r} - \delta)(1-\tau) + 1]\right] \\
\bar{r} &= \alpha\left(\frac{e^{\bar{z}}}{\bar{k}}\right)^{1-\alpha} \\
\bar{w} &= (1-\alpha)\bar{k}^{\alpha}(e^{\bar{z}})^{1-\alpha} \\
\bar{T} &= \tau[\bar{w} + (\bar{r} - \delta)\bar{k}] \\
\bar{z} &= (1-\rho_z)\bar{z} + \rho_z \bar{z} + \epsilon_t^z; \quad \epsilon_t^z \sim iid.\ \mathcal{N}(0, \sigma_z^2)
\end{aligned}$$

yielding the following analytical solution,

$$\begin{aligned}
\bar{r} &= \frac{1-\beta}{\beta(1-\tau)} + \delta \\
\bar{k} &= \left(\frac{\bar{r}}{\alpha}\right)^{\frac{1}{\alpha-1}} \\
\bar{w} &= (1-\alpha)\bar{k}^{\alpha} \\
\bar{c} &= \bar{w} + (\bar{r} - \delta)\bar{k} \\
\bar{T} &= \tau[\bar{w} + (\bar{r} - \delta)\bar{k}]
\end{aligned}$$

Plugging in the suggested parameters $\gamma = 2.5, \beta = .98, \alpha = .40, \delta = .10, \bar{z} = 0$ and $\tau = .05$, results in the following steady state values: $\bar{r} = 0.122, \bar{k} = 7.288, \bar{w} = 1.328, \bar{c} = 1.485, \bar{T} = 0.074$. This matches the solution in the Jupyter notebook.

Exercise 6 Consider the following functional forms:

$$
\begin{aligned}
u(c_t, \ell_t) &= \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a\frac{(1-\ell_t)^{1-\xi} - 1}{1-\xi} \\
F(K_t, \ell_t, z_t) &= K_t^{\alpha}(\ell_t e^{z_t})^{1-\alpha}
\end{aligned}
$$

then the seven equations characterizing equations and seven unknowns: $\{c_t, k_t, \ell_t, w_t, r_t, T_t, z_t\}$ for the model are as follows:

$$
\begin{aligned}
c_t &= (1-\tau)[w_t\ell_t + (r_t - \delta)k_t] + k_t + T_t - k_{t+1} \\
\frac{1}{c_t^{\gamma}} &= \beta E_t\left[\frac{1}{c_{t+1}^{\gamma}}[(r_{t+1} - \delta)(1-\tau) + 1]\right] \\
\frac{a}{(1-\ell_t)^{\xi}} &= c_t^{-\gamma} w_t(1-\tau) \\
r_t &= \alpha\left(\frac{\ell_t e^{z_t}}{k_t}\right)^{1-\alpha} \\
w_t &= (1-\alpha)e^{z_t}\left(\frac{k_t}{\ell_t e^{z_t}}\right)^{\alpha} \\
T_t &= \tau[w_t\ell_t + (r_t - \delta)k_t] \\
z_t &= (1-\rho_z)\bar{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim iid.\ \mathcal{N}(0, \sigma_z^2)
\end{aligned}
$$

The equivalents of these equations at the steady-state are

$$
\begin{aligned}
\bar{c} &= (1-\tau)[\bar{w}\bar{\ell} + (\bar{r} - \delta)\bar{k}] + \bar{T} \\
\frac{1}{\bar{c}^{\gamma}} &= \beta E_t\left[\frac{1}{\bar{c}^{\gamma}}[(\bar{r} - \delta)(1-\tau) + 1]\right] \\
\frac{a}{(1-\bar{\ell})^{\xi}} &= \frac{\bar{w}(1-\tau)}{\bar{c}^{\gamma}} \\
\bar{r} &= \alpha\left(\frac{\bar{\ell}e^{\bar{z}}}{\bar{k}}\right)^{1-\alpha} \\
\bar{w} &= (1-\alpha)e^{\bar{z}}\left(\frac{\bar{k}}{\bar{\ell}e^{\bar{z}}}\right)^{\alpha} \\
\bar{T} &= \tau[\bar{w}\bar{\ell} + (\bar{r} - \delta)\bar{k}] \\
\bar{z} &= (1-\rho_z)\bar{z} + \rho_z\bar{z} + \epsilon^{\bar{z}}; \quad \epsilon_t^z \sim iid.\ \mathcal{N}(0, \sigma_z^2)
\end{aligned}
$$

The numerical solution of this system is included in the accompanying Jupyter notebook.

PART II - Exercise 3

Given,

$$E_t\left[F\tilde{X}_{t+1} + G\tilde{X}_t + H\tilde{X}_{t-1} + L\tilde{Z}_{t+1} + M\tilde{Z}_t\right] = 0$$

the law of motion $\tilde{Z}_t = N\tilde{Z}_{t-1} + \varepsilon_t$, the hypothetical transition $\tilde{X}_t = P\tilde{X}_{t-1} + Q\tilde{Z}_t$, and the expected mean of the error term $E\left[\varepsilon_t\right] = 0$, we can show that

$$\begin{aligned}
0 &= E_t\left[F\tilde{X}_{t+1} + G\tilde{X}_t + H\tilde{X}_{t-1} + L\tilde{Z}_{t+1} + M\tilde{Z}_t\right] \\
&= E_t\left[F[P\tilde{X}_t + Q\tilde{Z}_{t+1}] + G[P\tilde{X}_{t-1} + Q\tilde{Z}_t] + H\tilde{X}_{t-1} + LN\tilde{Z}_t + M\tilde{Z}_t\right] \\
&= E_t\left[F[P(P\tilde{X}_{t-1} + Q\tilde{Z}_t) + Q(N\tilde{Z}_t + \varepsilon_{t+1})] + G[P\tilde{X}_{t-1} + Q\tilde{Z}_t] + H\tilde{X}_{t-1} + L(N\tilde{Z}_{t-1} + \varepsilon_t) + M\tilde{Z}_t\right] \\
&= FP^2\tilde{X}_{t-1} + FPQ\tilde{Z}_t + FQN\tilde{Z}_t + GP\tilde{X}_{t-1} + GQ\tilde{Z}_t + H\tilde{X}_{t-1} + LN\tilde{Z}_t + M\tilde{Z}_t \\
&= [(FP+G)P + H]\tilde{X}_{t-1} + [(FQ+L)N + (FP+G)Q + M]\tilde{Z}_t
\end{aligned}$$

PART II - Exercise 1

From the notes we know that the second-order representation is,

$$\begin{aligned}
&F_{xx}\{x(u),u\}x_u(u)x_u(u) + F_{xu}\{x(u),u\}x_u(u) \\
&+ F_x\{x(u),u\}x_{uu}(u) + F_{xu}\{x(u),u\}x_u(u) \\
&+ F_{uu}\{x(u),u\} = 0
\end{aligned}$$

To find the third-order terms we differentiate the again with respect to u. We get

$$\begin{aligned}
0 &= F_{xxx}\{x(u),u\}x_u(u)^3 + F_{xxu}\{x(u),u\}x_u(u)^2 + 2F_{xx}\{x(u),u\}x_u(u)x_{uu}(u) \\
&+ 2F_{xux}\{x(u),u\}x_u(u)^2 + 2F_{xuu}\{x(u),u\}x_u(u) + 2F_{xu}\{x(u),u\}x_{uu}(u) \\
&+ F_{xx}\{x(u),u\}x_{uu}(u)x_u(u) + F_{xu}\{x(u),u\}x_{uu}(u) + F_x\{x(u),u\}x_{uuu}(u) \\
&+ F_{uux}\{x(u),u\}x_u(u) + F_{uuu}\{x(u),u\}
\end{aligned}$$

therefore, evaluating the third derivative with respect to u around u_0, we find that,

$$x_{uuu} = -\frac{F_{xxx}x_u^3 + 3\left(F_{xx}x_u x_{uu} + F_{xux}x_u^2 + F_{xuu}x_u + F_{xu}x_{uu}\right) + F_{uuu}}{F_x}$$