FMNF05 - Computer Project 2

Designing a car using tension splines

See course web page for instructions on how to write your report.

To be handed in before **Monday, February 26, 2018 at 12 noon.**

February 10, 2018

1 Description of Tension Splines

Splines may be used to interpolate functions requiring particular smoothness conditions. In some data-interpolation problems, it is useful to have a parameter $\tau \geq 0$, called the *tension*. When τ is large, the curve passing through the given data points will have "high tension." It is also interesting, because one can construct splines that are not polynomials, but instead connect other types of functions into a piecewise function of prescribed smoothness. Here we are going to use such splines to draw the shape of a 2D car.

Given the knots $x_0, x_1, \ldots, x_n$ and the corresponding values $y_0, y_1, \ldots, y_n$, the *tension spline* is defined as a function T such that

1. $T \in \mathcal{C}^2[x_0, x_n]$
2. $T(x_i) = y_i, \ i = 0, 1, 2, \ldots, n$
3. $T''''(x) - \tau^2 \cdot T''(x) = 0$ for each $x \in (x_{i-1}, x_i)$.

2 Determination of T

To determine T, we set $z_i = T''(x_i)$. The conditions T must satisfy on $[x_i, x_{i+1}]$ are

$$
\begin{aligned}
T''''(x) - \tau^2 \cdot T''(x) &= 0 && (2.1)\\
T(x_i) &= y_i && (2.2)\\
T''(x_i) &= z_i && (2.3)\\
T(x_{i+1}) &= y_{i+1} && (2.4)\\
T''(x_{i+1}) &= z_{i+1} && (2.5)
\end{aligned}
$$

This is a differential equation – a 4th order two-point boundary-value problem – whose solution[1] is

$$
\begin{aligned}
T(x) = & \frac{z_i \sinh(\tau(x_{i+1} - x)) + z_{i+1} \sinh(\tau(x - x_i))}{\tau^2 \sinh(\tau h_i)} \\
& + \frac{(y_i - z_i/\tau^2)(x_{i+1} - x) + (y_{i+1} - z_{i+1}/\tau^2)(x - x_i)}{h_i} \qquad (2.6)
\end{aligned}
$$

[1]The hyperbolic sine is defined as $\sinh = (\mathrm{e}^x - \mathrm{e}^{-x})/2$

for $x \in [x_i, x_{i+1}]$. Once the coefficients z_i have been determined, this equation can be used to calculate values of T on the interval $[x_i, x_{i+1}]$.

3 Tasks

Task 1. Using a given silhouette of a car (or your own preferred silhouette), generate coordinate data by placing the profile on a grid, making a table of data choosing an adequate number of points, placing knots where needed. Try out standard cubic splines (not tension splines) to generate the upper part of the profile curve. Plot your result.

Task 2. Study the third property of a tension spline and describe a tension spline in the limiting cases, when $\tau = 0$ and when $\tau \to \infty$.

Task 3. By imposing the conditions

$$\lim_{x \downarrow x_i} T'(x) = \lim_{x \uparrow x_i} T'(x), \quad i = 1, 2, \ldots, n-1 \tag{3.7}$$

we find the system of equations for the unknowns $z_0, z_1, \ldots, z_n$. These calculations are similar to those for the cubic spline. Setting $h_i = x_{i+1} - x_i$, with

$$\begin{aligned}
\alpha_i &= \frac{1}{h_i} - \frac{\tau}{\sinh(\tau h_i)} \\
\beta_i &= \frac{\tau \cosh(\tau h_i)}{\sinh(\tau h_i)} - \frac{1}{h_i} \\
\gamma_i &= \frac{\tau^2 (y_{i+1} - y_i)}{h_i}
\end{aligned}$$

we arrive at

$$\alpha_{i-1} z_{i-1} + (\beta_{i-1} + \beta_i) z_i + \alpha_i z_{i+1} = \gamma_i - \gamma_{i-1}$$

for $i = 1, 2, \ldots, n-1$. This defines a linear system in the variables $z_i, i = 0, \ldots, n$.

Construct the matrix-vector system with the additional conditions $z_0 = z_n = 0$. Can you be certain that the system has a unique solution? Motivate your answer.

Task 4. Construct the matrix-vector system with the additional conditions $T'(x_0) = y'_0$ and $T'(x_n) = y'_n$, where y'_0 and y'_n are given.

Task 5. Program the algorithm and test it for the function $f(x) = \cos(x)$, $x \in [0, \pi]$, with a variety of values of τ (for instance, $\tau = 0, 0.3, 1, 3, 10$) and for different additional conditions, with 10 data points generated from the given function. Show the results of your tests.

Task 6. Let us now return to the car silhouette. Construct a tension spline representing the upper part of the profile. You may want to use (at least) four different tension splines, and choose the starting and end points of each tension spline according to appropriate initial conditions (compare Tasks 4 and 5). Generate and plot the tension splines that best suit the curve and compare to the original figure.

Optional task (+3p): Add the lower part of the profile, and some other features (e.g. the overall outline of the windows of the car).

G. Söderlind, C. Arévalo
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