Numerical Analysis — FMN011 — 140602

The exam lasts 4 hours and has 14 questions. A minimum of 35 points out of the total 70 are required to get a passing grade. These points will be added to those you obtained in your two home assignments, and the final grade is based on your total score.

Justify all your answers and write down all important steps. Unsupported answers will be disregarded.

During the exam you are allowed a pocket calculator, but no textbook, lecture notes or any other electronic or written material.

1. I have designed a new iterative method for solving non-linear equations, and I want to test it by solving the equation

$$4x^3 + x - 1 = 0,$$

whose exact solution is $x = 0.5$. When I apply the method with initial guess $x_0 = 1$, I get the following sequence: $\{1, 0.1464, 0.2897, 0.4036, 0.4701, 0.4948\}$.

   (a) **(2p)** Calculate the absolute error at each iteration step

   (b) **(2p)** Linear convergence is defined by $\|e_{k+1}\| \leq c \cdot \|e_k\|$ and $0 < c < 1$. Does the method converge linearly?

2. **(6p)** Construct a fixed point iteration that will solve $x^4 + 0.5x^2 - x + 0.25 = 0$. Can it converge to the solution $x = 0.3053$? Can it converge to the solution $x = 0.6630$? Explain.

3. Construct an iterative method for solving $Ax = b$, $x_{k+1} = Bx_k + c$, by partitioning A as $A = M - N$ and letting $M = I$.

   (a) **(3p)** What is the iteration matrix, B? What is c?

   (b) **(2p)** Give a good reason why we can solve the system

$$\begin{aligned} 0.5x - 0.4y &= 1 \\ -0.1x + 1.3y &= 2 \end{aligned}$$

   with this method.

4. **(5p)** I read on the web that the following statements are true, but they are actually false. Give a good argument to show their falseness.

   (a) Every matrix has an LU factorization, $A = LU$.

   (b) If a matrix A is not invertible, and $A = LU$, then L is not invertible.

   (c) After solving $Ax = b$, a small residual implies a small error.

   (d) Solving a linear system by means of LU factorization is numerically more stable than using Gauss elimination.

   (e) Partial pivoting may change the 1-norm condition number of a matrix.

5. **(5p)** Consider

$$A = \begin{pmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

   (a) Show that the columns of A are orthonormal vectors.
   (b) How can (a) be used to solve the least squares problem $Ax = b$ efficiently?
   (c) Calculate the least squares solution.
   (d) Calculate the residual.
   (e) Is there a vector v such that $\|Av - b\|_2 < 2\sqrt{2}/3$?

6. **(5p)** A friend tells me that

$$A = \begin{pmatrix} 7 & -4 & 4 \\ -4 & 5 & 0 \\ 4 & 0 & 9 \end{pmatrix}$$

   has three different eigenvalues, and that its corresponding eigenvectors are $(2/3, 2/3, -1/3)^{\mathrm{T}}, (-1/3, 2/3, 2/3)^{\mathrm{T}}$ and $(1, 0, 0)^{\mathrm{T}}$. Without calculating the eigenvectors, I tell her that she is wrong: it looks like one of the eigenvectors was incorrectly calculated. Can you reason how I concluded that?

7. **(5p)** When certain exact data,

x	1	2.15	3.2	5.1	5.4
y	0.100000	0.9938375	3.2768	13.2651	15.7464

   is plotted on a loglog scale, (i.e., the values of $\log(x_i)$ and $\log(y_i)$ are plotted), the outcome looks like a straight line. What is the type of relation between x and y? Write the equation of the model. What would the parameters of the model be? Describe how to find them; you do not need to do the computations.

8. **(5p)** A quadratic Bézier curve that starts at (0,0) with a slope of 1/2, and ends at (1,1) with a slope of -2/3. Make a sketch of the curve, give all of its control points, and show the control polygon.

9. **(5p)** Explain why: if an $n \times n$ matrix A has eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$, the power method will converge (for any appropriate choice of initial vector).

10. **(5p)** The DFT of a real vector is

```
-0.3536
-0.3536 + 3.5990i
-0.3536 - 0.9239i
-0.3536 - 0.2294i
-0.3536
-0.3536 + 0.2294i
-0.3536 + 0.9239i
-0.3536 - 3.5990i
```

Given that the DFT trigonometric interpolation polynomial is

$$P_n(t) = \frac{a_0}{\sqrt{n}} + \frac{2}{\sqrt{n}} \sum_{k=1}^{n/2-1} (a_k \cos(2\pi kt) - b_k \sin(2\pi kt)) + \frac{a_{n/2}}{\sqrt{n}} \cos(n\pi t),$$

construct a low-pass filter that keeps all frequencies up to $2\pi t$.

11. Consider the message

SKÅNSKT KNÄCKE

(a) **(1p)** Construct a Huffman tree for this message.

(b) **(2p)** Construct a table with the binary code for each symbol.

(c) **(1p)** How many bits are required to code the message?

(d) **(1p)** What is the average number of bits/symbol used?

12. **(5p)** If the QR factorization of a matrix A is given as

```
>> [Q,R]=qr(A)

Q =
   -0.2582    0.7115    0.6383    0.0468   -0.1321
   -0.5164   -0.5534    0.5003   -0.1403    0.3964
   -0.7746    0.1581   -0.5175   -0.2417   -0.2207
   -0.2582   -0.0791   -0.0863    0.9590    0.0016
         0    0.3953   -0.2588    0.0079    0.8813

R =
   -3.8730   -0.7746   -1.2910
         0    2.5298    1.5811
         0         0    2.4152
         0         0         0
         0         0         0
```

write down the system that must be solved to get the least squares solution to $Ax = b$ with $b = [1, 0, 0, 0, 1]^{\mathrm{T}}$. How many operations are (approximately) needed to solve the system? Do not solve.

13. **(5p)** When I calculate the eigenvalues and the singular values of a certain 5×5 matrix, I obtain the following results:

```
>> eigenvalues = eig(A)

eigenvalues =
   -1.2836
   -0.6084
   -0.0100
    0.1024
    4.1611
```

```
>> singular_values = svd(A)

singular_values =
    4.1611
    1.2836
    0.6084
    0.1024
    0.0100
```

(a) Is the matrix invertible? Why?

(b) What kind of structure (of A) does this result suggest?

(c) Why does this result guarantee a basis of $\mathbb{R}^5$ consisting of eigenvectors of A?

(d) What eigenvalue would I get if I applied the inverse power method?

(e) If I had a matrix B with eigenvalues $\{-2.2836, -0.6084, -0.0100, 0.1024, 4.1611\}$, for which of the two matrices would the power method converge faster to 4.1611? Why?

14. **(5p)** Given that

```
    >> [U,S,V]=svd(A)

U =
   -0.5544   -0.7279    0.3310   -0.2305   -0.0000
    0.7496   -0.6132   -0.1982   -0.1509    0.0000
   -0.3504   -0.1358   -0.8951   -0.0136   -0.2397
   -0.0425    0.1808   -0.1734   -0.7178    0.6482
   -0.0781   -0.2072   -0.1412    0.6392    0.7228

S =
    4.1611         0         0         0         0
         0    1.2836         0         0         0
         0         0    0.6084         0         0
         0         0         0    0.1024         0
         0         0         0         0    0.0000

V =
   -0.5544    0.7279   -0.3310   -0.2305         0
    0.7496    0.6132    0.1982   -0.1509   -0.0000
   -0.3504    0.1358    0.8951   -0.0136    0.2397
   -0.0425   -0.1808    0.1734   -0.7178   -0.6482
   -0.0781    0.2072    0.1412    0.6392   -0.7228
```

How can you achieve a lossy compression of A of at least 50%?

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