FMN011 — Seminar 1 — Errors and nonlinear equations

1. Consider the problem of evaluating the function $\sin(x)$, in particular the propagated data error, i.e., the error in the function value due to a perturbation h in the argument x, i.e., $\sin(x+h)$.
   (a) Estimate the absolute error in evaluating $\sin(x)$.
   (b) Estimate the relative error in evaluating $\sin(x)$.
   (c) For what values of the argument x is this problem highly sensitive?

2. Explain why a divergent infinite series, such as
$$\sum_{n=1}^{\infty} \frac{1}{n},$$
can have a finite sum in floating-point arithmetic. At what point will the partial sums cease to change?

3. What condition ensures that the bisection method will find a zero of a continuous nonlinear function in the interval $[a, b]$?

4. How many iterations are needed to guarantee an error no greater than 10^{-9} if we start the bisection method in the interval $[2, 2.5]$?

5. What is the convergence rate for Newton-Raphson's method for finding the root $x = 2$ of each of the following equations?
   (a) $f(x) = (x-1)(x-2)^2 = 0$
   (b) $f(x) = (x-1)^2(x-2) = 0$

   How can we restore the quadratic rate for multiple roots?

6. List one advantage and one disadvantage of the bisection method compared with Newton's method for solving a nonlinear equation in one dimension.

7. Write out Newton's iteration for solving the equation $x \sin x = 1$.

8. What methods does MATLAB's `fzero` function use? And `fsolve`?

9. Suppose you are using fixed-point iteration based on $x = g(x)$ to find a solution x^* to a nonlinear equation $f(x) = 0$. Which would be more favorable for the convergence rate: a horizontal tangent of g at x^* or a horizontal tangent of f at x^*?

10. Consider the function $f(x) = x + \ln x$.

(a) Plot the functions $y = x$ and $y = -\ln x$ to show f has a unique root P in $(0, \infty)$.

(b) Show that if $g(x) = -\ln x$ then $|g'(P)| > 1$.

(c) Can the root P be found using a fixed point iteration $x = -\ln x$?

(d) Can the root P be found using a fixed point iteration $x = g(x)$, with a different g?

11. Make graphs showing all basic patterns of convergence and divergence of fixed-point iteration.

12. Plot the following functions in [-5,5] and discuss the convergence of the Newton-Raphson method to find their roots:

    (a) $f(x) = \arctan(x)$. (In Matlab, `atan(x)`)

    (b) $f(x) = x^{1/3}$.
    (In Matlab, define the function as `sign(x)*abs(x)^(1/3)`.)

13. Implement your own Newton-Raphson method and solve the system

$$\begin{aligned} u^2 + v^2 &= 1 \\ (u-1)^2 + v^2 &= 1 \end{aligned}$$

Check that the convergence is quadratic.

14. Implement your own secant method and solve the system

$$\begin{aligned} u^2 + v^2 &= 1 \\ (u-1)^2 + v^2 &= 1 \end{aligned}$$

Can you determine what is the rate of convergence?

C. Arévalo