Numerical Analysis
FMN011

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Lecture 3

Multiple roots

If $f(r) = 0$ and $f'(r) = f''(r) = \cdots = f^{(m-1)}(r) = 0$ but $f^{(m)}(r) \neq 0$, we say that f has a root of **multiplicity** m at r.

$f(x) = (x - 2/3)^3$ has a root of multiplicity 3 at $x = 2/3$

$$
\begin{aligned}
f(x) &= \sin(x) - x \\
f(0) &= 0 \\
f'(x) &= \cos(x) - 1 \Rightarrow f'(0) = 0 \\
f''(x) &= -\sin(x) \Rightarrow f''(0) = 0 \\
f'''(x) &= -\cos(x) \Rightarrow f'''(0) = -1
\end{aligned}
$$

f has a root of multiplicity 3 at 0.

Convergence Rate

Nonlinear equations must be solved by iterative methods

For $f(p) = 0$, let $e_k = p_k - p$.

The method converges if

$$\lim_{k \to \infty} \|e_k\| = 0$$

The convergence rate is:

- linear if $\boxed{\|e_{k+1}\| \leq c \cdot \|e_k\| \text{ and } 0 < c < 1}$
- superlinear if $\boxed{\|e_{k+1}\| \leq c \cdot \|e_k\|^r \text{ and } r > 1}$
- quadratic if $\boxed{\|e_{k+1}\| \leq c \cdot \|e_k\|^2}$

Convergence of methods for nonlinear equations depends on choice of p_0

Newton-Raphson Method

Taylor polynomial:

$$f(x) = f(p_0) + (x - p_0)f'(p_0) + (x - p_0)^2 f''(\zeta)/2$$

We take the linear approximation

$$f(x) \approx \hat{f}(x) = f(p_0) + (x - p_0)f'(p_0)$$

Note: $y = f'(p_0)(x - p_0) + f(p_0)$ is the tangent to f at $x = p_0$.

$$\hat{f}(x) = 0 \quad \text{iff} \quad x = p_0 - \frac{f(p_0)}{f'(p_0)}$$

The iteration for Newton's method is $\boxed{p_{k+1} = p_k - \frac{f(p_k)}{f'(p_k)}}$

Convergence of Newton's Method

Fixed point iteration with $\quad g(x) = x - \dfrac{f(x)}{f'(x)}$

$$g'(x) = 1 - \frac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} = \frac{f(x)f''(x)}{(f'(x))^2}$$

If $f(x) = 0$ and $f'(x) \neq 0$, then $g'(x) = 0$ and this means the rate of convergence is quadratic:

$$\boxed{e_{k+1} = \frac{g''(\zeta)}{2} e_k^2}$$

Example: Newton's Method

Find the largest root of $16x^2 - 32x + 15 = 0$

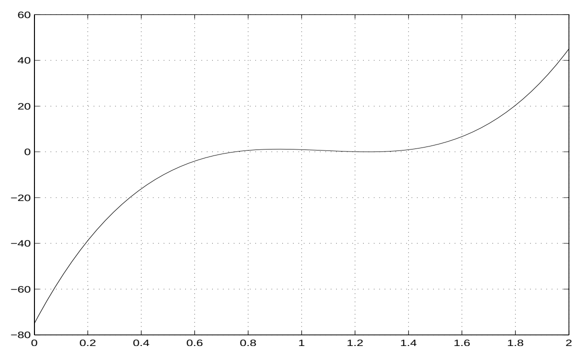
$$
\begin{aligned}
p_{k+1} &= p_k - \frac{16p_k^2 - 32p_k + 15}{32p_k - 32} \\
&= \frac{16p_k^2 - 15}{32p_k - 32}
\end{aligned}
$$

$p_0 = 1.2000$
$p_1 = 1.2563$
$p_2 = 1.2501$
$p_3 = 1.2500$

Example: Multiple Roots

Find the largest root of $64x^3 - 208x^2 + 220x - 75 = 0$

$$p_{k+1} = p_k - \frac{64p_k^3 - 208p_k^2 + 220p_k - 75}{192p_k^2 - 416p_k + 220}$$



$p_0 = 1.5000$
$p_1 = 1.3929$

$p_2 = 1.3286$
$p_3 = 1.2918$
$\vdots$
$p_{11} = 1.2502$
$p_{12} = 1.2501$
$p_{13} = 1.2500$ No quadratic convergence!
because 1.25 is a double root: $f'(1.25) = 0$

Remedy: Modified Newton's Method for a root of multiplicity m:

$$\boxed{p_{k+1} = p_k - m\frac{f(p_k)}{f'(p_k)}}$$

$p_0 = 1.5000$
$p_1 = 1.2857$
$p_2 = 1.2512$
$p_3 = 1.2500 \Rightarrow$ Quadratic convergence!

In MATLAB

The built-in function `fzero` (`fsolve`) computes the solution and the residual for a non-linear equation f(x)=0 with a starting guess x0:

```
>> f = @(x) 64*x.^3-208*x.^2+220*x-75;
>> x0 = 1;
>> [x,res] = fzero(f,x0)

x =
 0.749999999999999
res =
 0
```