FMN011 — Seminar 3 — Interpolation, Splines and Bézier Curves

1. Name some common polynomial representations. What are their basis functions?

2. True or false:

   (a) There are arbitrarily many different mathematical functions that interpolate a given set of data points.

   (b) If the polynomial interpolating a given function is unique, then so is the representation of that polynomial.

   (c) Obtaining an expression of the polynomial passing through n data points requires the solution of a system of $n+1$ equations.

   (d) If a function is interpolated at n points $\{x_1, x_2, \dots, x_n\}$, the error of the interpolation at a point $x \neq x_j$ depends on the basis chosen for the interpolation.

   (e) To define a quadratic spline with n data points requires the use of $3(n-1)$ conditions.

   (f) To define a piecewise cubic polynomial with n knots, $4n$ parameters are required.

   (g) With linear splines, the function and its first derivative are continuous at the knots.

   (h) If the value at a single knot is changed, only the corresponding cubic spline piece needs to be modified.

3. Is it possible to construct two different polynomials that interpolate the same n data points? If so, under what conditions, and if not, why?

4. Consider the following polynomial bases:

   - Monomial basis
   - Lagrange basis

   (a) Describe the pattern of nonzero entries in the basis matrix A used to determine the coefficients for polynomial interpolation using each of the two bases.

   (b) Rank the two methods for polynomial interpolation according to the cost of determining the coefficients of the interpolating polynomial.

   (c) Which of the two methods has the best-conditioned basis matrix?

5. For Lagrange polynomial interpolation of n data points, what is the degree of each polynomial function $L_{m,j}(x)$ in the basis?

6. Construct a polynomial interpolation for $f(x) = \tan(x)$ in the interval $[-1.5, 1.5]$

(a) with 5 equally spaced nodes;

(b) with 5 Chebyshev nodes.

Plot the error for this interpolation in the given interval.

7. Why is piecewise polynomial interpolation more appropriate than interpolation by a single polynomial when fitting a large number of data points?

8. What is the degree of each polynomial function $B_m^n(x)$ in the Bernstein basis?

9. What type of matrix is involved in the construction of a cubic spline?

10. How is the structure of the matrix affected by the end conditions of the spline?

11. The not-a-knot conditions for cubic splines, $S_1'''(x_2) = S_2'''(x_2)$ and $S_{n-1}'''(x_{n-1}) = S_n'''(x_{n-1})$ imply that x_2 and x_{n-1} are not knots. Why?

12. Why are cubic splines favored over other type of splines?

13. Determine if there are any values of a and b that make the function

$$f(x) = \begin{cases} x^3 + 1 & x \in [0, 2] \\ a(x-2)^2 + (x-2) + b & x \in [2, 3] \end{cases}$$

a cubic spline.

14. Determine whether the following are quadratic splines

(a)

$$S(x) = \begin{cases} 0.1x^2 & x \in [0, 1] \\ 9.3x^2 - 18.4x + 9.2 & x \in [1, 1.3] \end{cases}$$

(b)

$$S(x) = \begin{cases} x & -\infty < x \leq 1 \\ x^2 & 1 \leq x \leq 2 \\ 4 & 2 \leq x < \infty \end{cases}$$

15. Consider the following data

x	3.0	4.5	7.0	9.0
f(x)	2.5	1.0	2.5	0.5

(a) Interpolate the data with a first-degree spline and evaluate the function at $x = 5$.

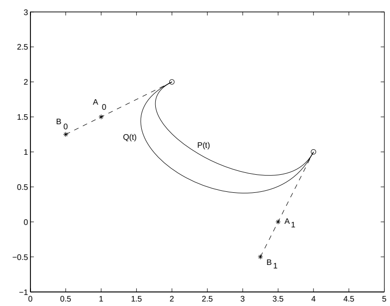
(b) Interpolate with a quadratic spline imposing $S''(3) = 0$, and evaluate at $x = 5$.

(c) Interpolate with a clamped cubic spline ($S'(3) = S'(9) = 0$) and evaluate at $x = 5$.

(d) Interpolate with a not-a-knot cubic spline and evaluate at $x = 5$.

16. Build a Bézier spline that forms a square with sides of length 2.

17. Here you see two Bézier curves with the same endpoints but different control points, (A_0, A_1) and (B_0, B_1). Say which curve, P or Q, has the control points A_0, A_1, and why.



18. Is it possible to have a Bézier curve with repeated control points?

19. Consider the Bézier curve B_1 with control points (0,0), (2,3), (5,1), (3,3). Construct a Bézier curve that starts at (3,3), ends at (0,0), and such that the resulting piecewise Bézier curve is $\mathcal{C}^1$.

20. Make a sketch of the de Casteljau algorithm for the construction of a quadratic Bézier curve. The sketch should show how to find a point for a few values of t, say, $t = 0.2, 0.4, 0.5, 0.75$.