FMN011 — Seminar 4 — Power Iteration, QR algorithm and SVD

1. Define the following terms:
   (a) Eigenvalue
   (b) Eigenvector
   (c) Characteristic polynomial
   (d) Dominant eigenvalue
2. Name two methods for finding eigenvalues and eigenvectors of a matrix.
3. Name two methods for finding the rank of a matrix.
4. True or false:
   (a) The eigenvalues of a real symmetric matrix can be complex.
   (b) A matrix is singular if and only if all of its eigenvalues are equal to 0.
   (c) A singular matrix has at least one eigenvalue equal to 0.
   (d) The eigenvalues of a diagonal matrix are its diagonal elements.
   (e) The eigenvalues of a triangular matrix are its diagonal entries.
   (f) The eigenvectors of a matrix are unique.
   (g) An $n \times n$ matrix has n linearly independent eigenvectors.
   (h) An $n \times n$ symmetric matrix has n linearly independent eigenvectors.
   (i) The rank of a square matrix is the number of its nonzero eigenvalues.
   (j) The power method finds all eigenvalues/eigenvectors simultaneously.
   (k) The QR algorithm finds all eigenvalues/eigenvectors simultaneously.
   (l) If A is real symmetric matrix, the svd reduces to an eigenvalues/eigenvectors computation.
5. State the conditions under which the power method may be used.
6. If the eigenvalues of a 3 by 3 matrix A are 10, 20 and 40,
   (a) what are the eigenvalues of A^{-1}?
   (b) what are the eigenvalues of $P^T AP$, where P is an orthogonal matrix?
   (c) what are the eigenvalues of A^T?
   (d) what are the eigenvalues of $-A$?
7. For a given matrix A,
   (a) can the same eigenvalue correspond to two different eigenvectors?
   (b) can the same eigenvector correspond to two different eigenvalues?

8. Matrices A and B are similar if $A = T^{-1}BT$. True or false:

   (a) If A and B are similar, they have the same eigenvalues.

   (b) If A and B are similar, they have the same eigenvectors.

   (c) If A and B have the same eigenvalues, they are similar.

9. The eigenvalues of a matrix are the roots of its characteristic polynomial. Does this fact provide an effective numerical method for computing them?

10. A matrix can be reduced to triangular form by a QR factorization, and the eigenvalues of a triangular matrix are its diagonal entries. Does this procedure suffice to compute the eigenvalues of the original matrix?

11. Applied to a given matrix A, the QR iteration converges either to triangular or diagonal form. What property of A determines which of these two forms is obtained?

12. If a matrix has a simple dominant eigenvalue λ_1, what quantity determines the convergence rate of the power method for computing λ_1?

13. Given an approximate eigenvector x for a matrix A, how can we estimate the corresponding eigenvalue?

14. Given an approximate eigenvalue λ for a matrix A, how can we estimate a corresponding eigenvector?

15. To which eigenvalue does the inverse power iteration converge?

16. Given a general square matrix A, how would you compute the following?

   (a) The smallest eigenvalue of A

   (b) The largest eigenvalue of A

   (c) The eigenvalue of A closest to some specified scalar β

   (d) All of the eigenvalues of A

17. What are the eigenvalues of an orthogonal matrix? What are the singular values of an orthogonal matrix?

18. Show: In the SVD of matrix A, the columns of U are orthonormal eigenvectors of AA^T.

19. Show that the 2-norm condition number of an n by n matrix A is $\kappa(A) = s_1/s_n$.

20. Show that if $A = USV^T$, a least squares problem $Ax = b$ can be solved as $x = \displaystyle\sum_{i=1}^{r} \frac{u_i^T b}{s_i} v_i$.

21. Let $w \neq 0$ be a $n \times 1$ vector. What are its singular values?

22. Suppose A is a rank-1 matrix with given SVD, $A = USV^T$. Express the least squares solution of the system $Ax = b$ for $b = [1, 0, 0, \dots, 0]$.

23. Consider the following matrix and its SVD,

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Is the SVD unique? If it is not, can you find another one?

24. What is the compression rate if 3 terms are kept in the sum $A = \sum_{i=1}^{r} u_i s_i v_i^T$ for the 8×8 matrix A?

C. Arévalo