Numerical Analysis
FMN011

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Lecture 2

Solving nonlinear equations

The annuity-due equation is $\quad A = \dfrac{P}{I/12}\left((1+\dfrac{I}{12})^N - 1\right)$

P monthly deposit, I annual interest, A amount after N deposits

You save Kr. 300 per month; what interest rate would allow you to have Kr. 50.000 after 12 years ($N = 144$)?

$$A(I) = \frac{300}{I/12}\left((1+\frac{I}{12})^{144} - 1\right) = 50.000$$

$A(0.04) = 55.331$;
$A(0.03) = 51.922$;

$A(0.02) = 48.779$.

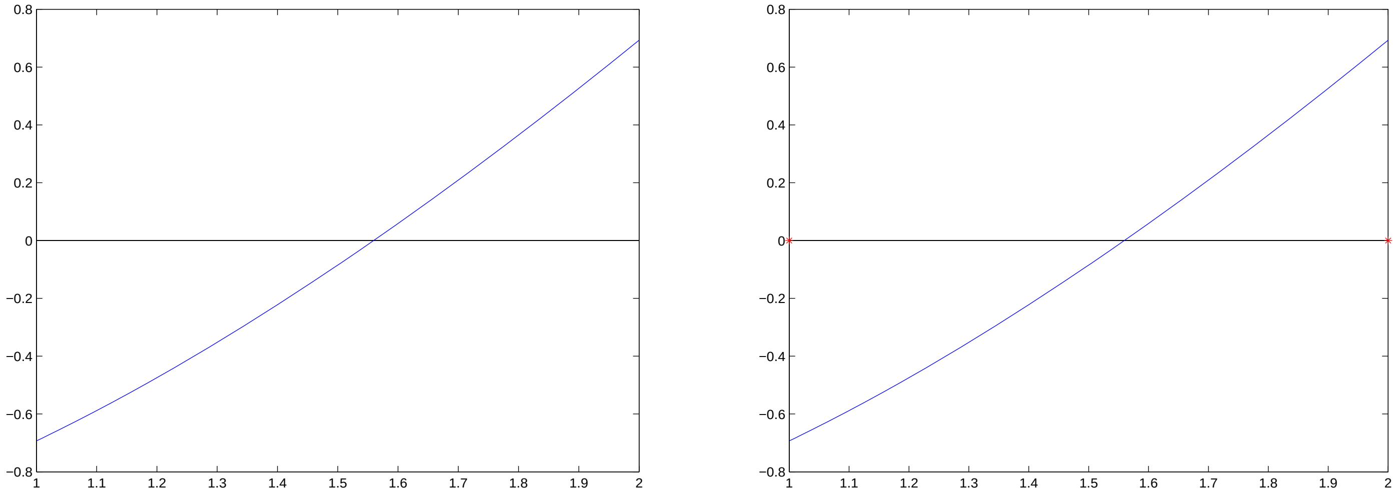
Answer lies in $[0.02, 0.03]$.

$A(0.025) = 50.319$, answer is in $[0.020, 0.025]$.

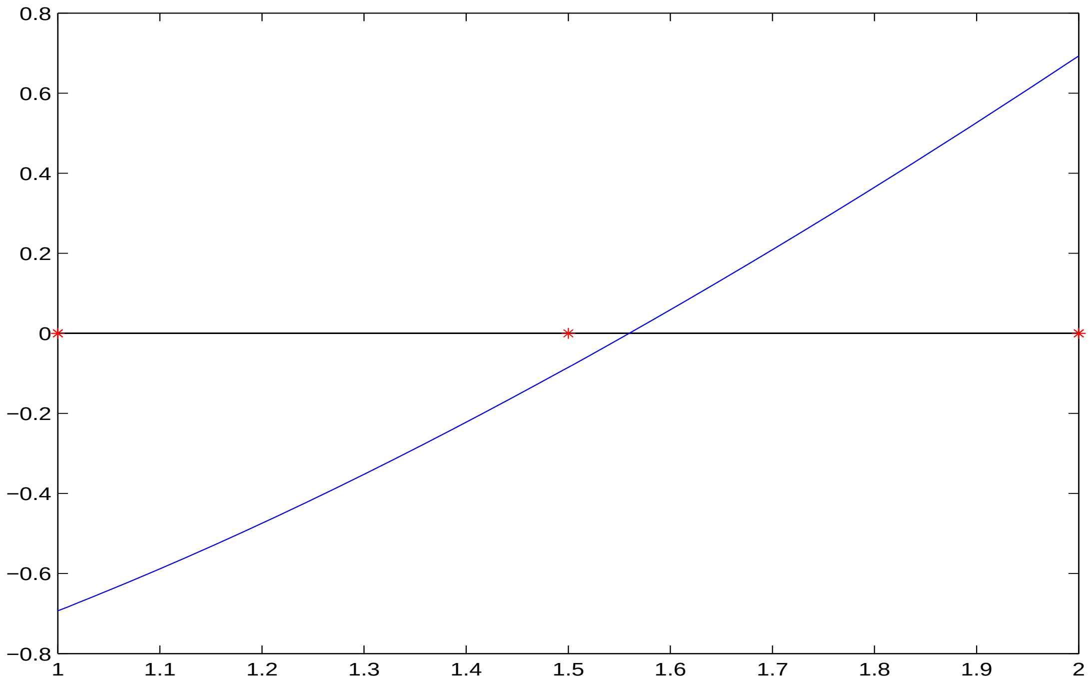
After a few more tries, $A(0.024) = 50.006$, so you must find a bank that will give you a yearly interest rate of 2.4%

Bisection method

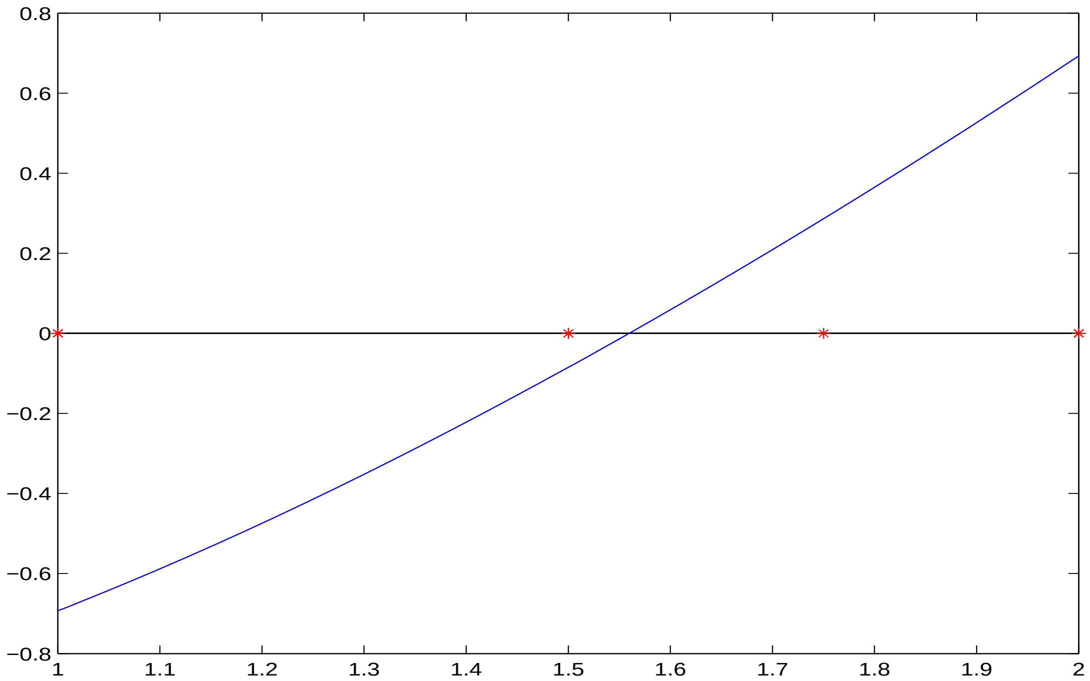
Problem: find a zero of a continuous $f(x)$



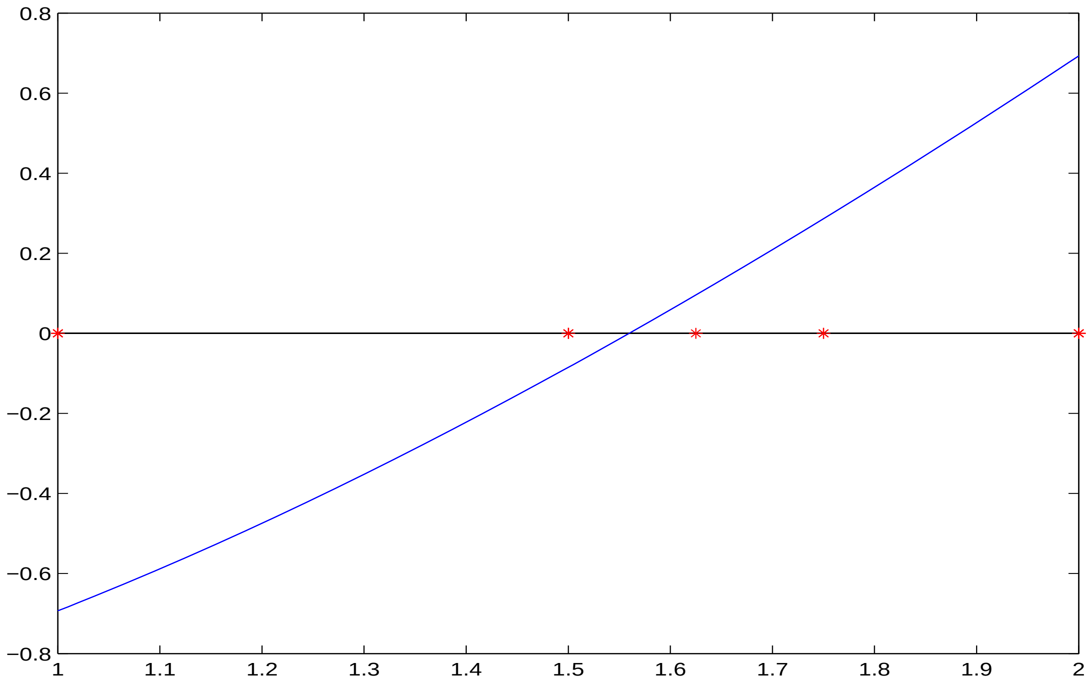
First bracket: $[1, 2]$, where $f(1) < 0$, $f(2) > 0$



First bracket: $[1, 2]$, where $f(1) < 0$, $f(2) > 0$
Midpoint: $x = 1.5$, $f(1.5) < 0 \Rightarrow [1.5, 2]$



First bracket: $[1, 2]$, where $f(1) < 0$, $f(2) > 0$
Midpoint: $x = 1.5$, $f(1.5) < 0 \Rightarrow [1.5, 2]$
Midpoint: $x = 1.75$, $f(1.75) > 0 \Rightarrow [1.5, 1.75]$



$[1, 2]$, where $f(1) < 0$, $f(2) > 0$

Midpoint: $x = 1.5$, $f(1.5) < 0 \Rightarrow [1.5, 2]$

Midpoint: $x = 1.75$, $f(1.75) > 0 \Rightarrow [1.5, 1.75]$

Midpoint: $x = 1.625$, $f(1.625) > 0 \Rightarrow [1.5, 1.625]$

Approximate solution: $x = 1.5625$

Bisection algorithm

```
function [c,possible_err,res] = bisection(f,a,b,tol)
%  f(a), f(b) must have opposite signs
% x is the approximate solution
while (b-a)/2>tol
    c = (a+b)/2; % midpoint
    if f(c)*f(a)>0
        a=c;
    elseif f(c)*f(b)>0
        b=c;
    else break
    end
end
possible_err = (b-a)/2;
res = f(c);
```

Bisection theorem

Suppose

- f is continuous in $[a, b]$
- $f(r) = 0$ for some $r \in [a, b]$
- $f(a)$ and $f(b)$ have opposite signs

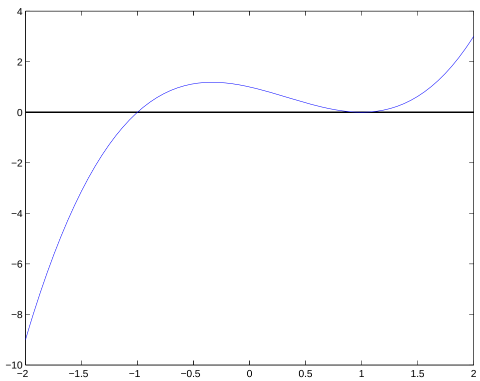
If $\{c_n\}$ is the sequence produced by the bisection method, then

$$\boxed{|r - c_n| \leq \frac{b_n - a_n}{2} = \frac{b-a}{2^{n+1}}}$$

so $\lim_{n \to \infty} c_n = r$

Initial approximation

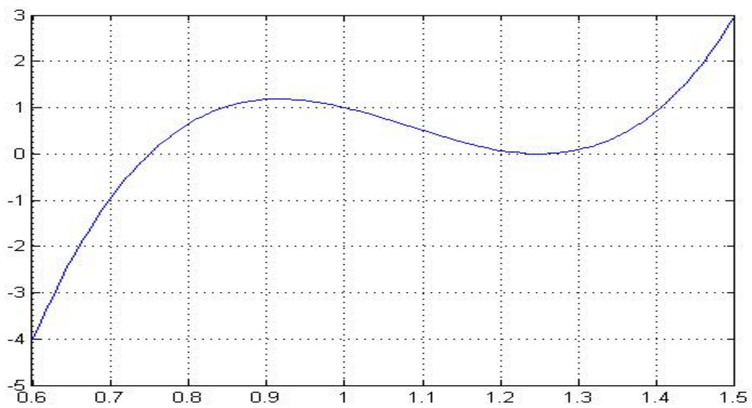
If $f(x)$ has several zeros in $[a, b]$ we must use a different starting interval for each root.



Finding the initial interval can be done graphically.

Example:

Find the smallest root of $p(x) = 64x^3 - 208x^2 + 220x - 75$



Bisection algorithm in $[0.7, 0.85]$ with $tol = 10^{-6}$ gives $p(0.750000381469726) = 6.1035 \times 10^{-6}$; $tol = 10^{-16}$ gives $p(0.75) = 0$.

Backward and Forward Errors

Suppose $f(r) = 0$ and $\hat{x}$ approximates r.

The backward error is $|f(\hat{x})|$ and the forward error is $|r - \hat{x}|$

Example, $f(x) = x^3 - 2x + 4/3\,x - 8/27$, $\hat{x} = 0.6666565$, $r = 2/3$.

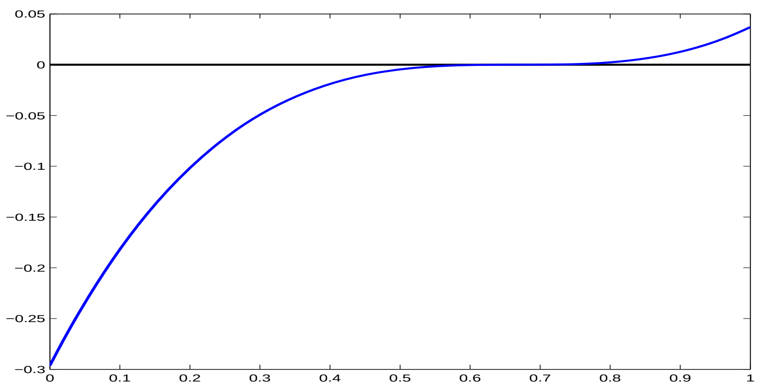
Backward error: $|f(\hat{x})| = 1.110223 \times 10^{-15}$ and
forward error: $|r-\hat{x}| = 1.01666666666 \times 10^{-5}$ (usually cannot be evaluated, as we do not know the exact solution r).

The backward error is the absolute value of the residual.

Backward error vs forward error

Desirable: small backward error $\Rightarrow$ small forward error

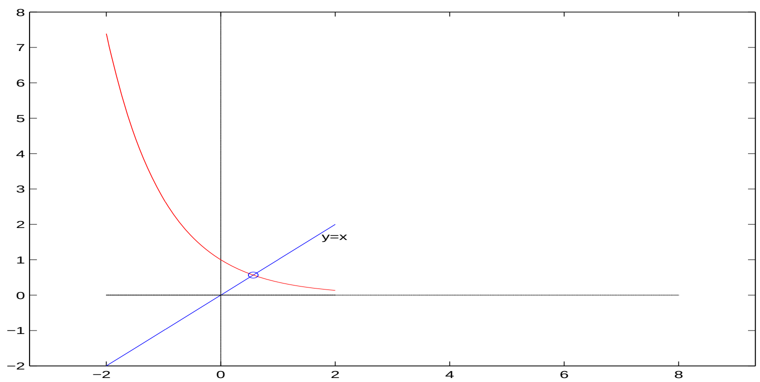
Actually, this is not always so. In this case the difficulty is due to $2/3$ being a multiple root.



Notice the graph is "flat" near the multiple root.

Fixed Points

$\mathbf{x}$ is a fixed point of the function $\mathbf{g}$ if $\mathbf{x} = \mathbf{g}(\mathbf{x})$



They are the points of intersection of curves $y = g(x)$ and $y = x$

Fixed Point Iteration

A fixed point iteration has the form $p_{k+1} = g(p_k)$

If g is continuous and $\lim_{n\to\infty} g(p_n) = P$, then P is a fixed point of g.

Theorem

If g and g' are continuous in $[a, b]$,
$g(x) \in [a, b]$ for all $x \in [a, b]$ and
$p_0 \in [a, b]$, then

- $|g'(P)| \leq K < 1 \Rightarrow \{p_n\} \longrightarrow P$
- $|g'(P)| > 1 \Rightarrow \{p_n\}$ will not converge to P

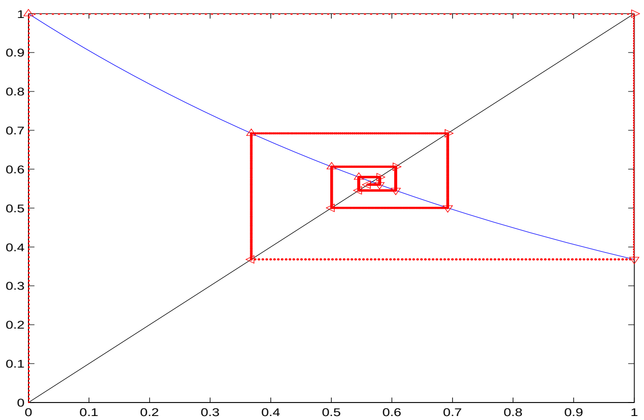
Convergent Iteration

$$x - e^{-x} = 0 \quad \Rightarrow \quad x_{k+1} = e^{-x_k},\ x_0 = 0$$

$$g(x) = e^{-x},$$

$$g : [0, 1] \to [0, 1]$$

$$|g'(x)| = e^{-x} < 1 \text{ for } x > 0$$



Example: Fixed Point Iteration

Find the largest root of $16x^2 - 32x + 15 = 0$ by fixed point iteration.

Some possibilities:

1. $x = 16x^2 - 31x + 15$

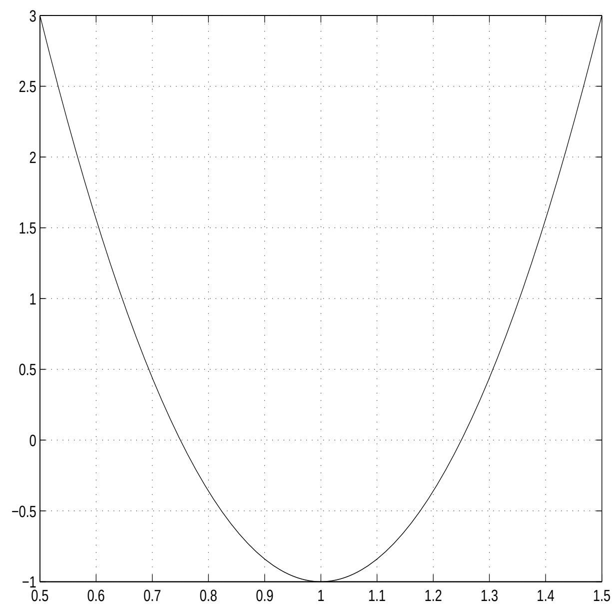
2. $x = \sqrt{32x - 15}/4$

3. $x = x^2/2 + 15/32$

4. $x = -15/16(x - 2)$

5. $x = 2 - 15/16x$

Plot of $f(x) = 16x^2 - 32x + 15$



Take $x \in [1.2, 1.3]$

Check hypothesis

1. $g(x) = 16x^2 - 31x + 15$
   $|g'(x)| = |32x - 31| > 1$

2. $g(x) = \sqrt{32x - 15}/4$
   $|g'(x)| = 4/\sqrt{32x - 15} < 1$ and $g : [1.2, 1.3] \to [1.2, 1.3]$

3. $g(x) = x^2/2 + 15/32$,
   $|g'(x)| = |x| > 1$

4. $g(x) = -15/16(x - 2)$,
   $g : [1.2, 1.3] \to [1.17, 1.34]$

5. $g(x) = 2 - 15/16x$
   $g : [1.2, 1.3] \to [1.21, 1.28]$ and $|g'(x)| = 15/16x^2 < 1$

Fixed point method for formulations 2 and 5

$g(x) = \sqrt{32x - 15}/4$	$g(x) = 2 - 15/16x$
$x_0 = 1.2000$	$x_0 = 1.2000$
$x_1 = 1.2093$	$x_1 = 1.2188$
$x_2 = 1.2170$	$x_2 = 1.2308$
$\vdots$	$\vdots$
$x_{12} = 1.2463$	$x_{12} = 1.2499$
$x_{13} = 1.2470$	$x_{13} = 1.2499$
$\vdots$	
$x_{27} = 1.2499$	
$x_{28} = 1.2499$	
$\lvert g'(1.25)\rvert = 0.8$	$\lvert g'(1.25)\rvert = 0.6$

Stopping criteria

We would like $f(p_n) \approx 0$ and $p_n \approx p_{n-1}$

The criteria can be

- For the ordinate: $|f(p_n)| < \epsilon$
- For the abscissa:
  - for the absolute error: $|p_n - p_{n-1}| < \delta$
  - for the relative error: $\dfrac{2|p_n - p_{n-1}|}{|p_n| + |p_{n-1}|} < \delta$