Numerical Analysis, FMN011

Carmen Arévalo

Lund University

carmen@maths.lth.se

Bézier Curves

BERNSTEIN POLYNOMIALS

Binomial expansion of 1:

$$1 = (t + (1-t))^n = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i}$$

The **Bernstein polynomials** of degree n are

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad i = 0, 1, \ldots, n$$

COEFFICIENTS OF THE BERNSTEIN POLYNOMIALS

The coefficients can be obtained from Pascal's Triangle:

					1							$n = 0$
				1		1						$n = 1$
			1		2		1					$n = 2$
		1		3		3		1				$n = 3$
	1		4		6		4		1			$n = 4$
1		5		10		10		5		1		$n = 5$

PROPERTIES OF BERNSTEIN POLYNOMIALS

- **Recursion formula**: $B_i^n(t) = (1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t)$
- **Partition of unity**: $\sum_{i=0}^{n} B_i^n(t) \equiv 1$
- **Linearly independent**
- **Symmetric**: $B_i^n(t) = B_{n-i}^n(1-t)$
- **Roots** only at 0 and 1: $B_i^n(0) = 0 \quad (i = 1, \ldots, n)$, $B_i^n(1) = 0 \quad (i = 0, \ldots, n-1)$
- **Positive**: $B_i^n(t) > 0$ for $0 < t < 1$

BÉZIER CURVES

The $n+1$ Bernstein polynomials form a basis of all polynomials of degree $\leq n$.

Every polynomial curve of degree n can be written as

$$\mathbf{b}(u(t)) = \sum_{i=0}^{n} \mathbf{b_i} B_i^n(t)$$

where $u = (1-t)c + td$ and $t \in [0,1]$.

The coefficients $\mathbf{b}_i$ are points and are called **Bézier points** or **control points**. They are the vertices of the **Bézier polygon** of $\mathbf{b}(t)$ over the interval $[c,d]$.

SOME PROPERTIES OF BÉZIER CURVES

- **Convex hull property**: A Bézier curve is entirely contained inside the convex hull of its control points.
- **Endpoints**: $b(c) = b_0$ and $b(d) = b_n$
- **Symmetry**: If the control points are given in reverse order, the shape of the Bézier curve is the same, but the curve is transversed in the opposite direction.
- **Variation diminishing property**: The Bézier curve will not wiggle more than its control polygon.

CALCULATION OF A POINT ON A BÉZIER CURVE

$$\mathbf{b}(u) = \sum_{i=0}^{n} \mathbf{b_i^0 B_i^n}(t), \qquad \text{with} \qquad u = (1-t)c + td$$

Use the recurrence relation and collect terms:

$$\mathbf{b}(u) = \sum_{i=0}^{n} \mathbf{b_i^0} B_i^n(t) = \sum_{i=0}^{n-1} \mathbf{b_i^1} B_i^{n-1}(t) = \cdots = \sum_{i=0}^{0} \mathbf{b_i^n} B_i^0(t) = \mathbf{b_0^n}$$

where

$$\mathbf{b_i^{k+1}} = (1-t)\mathbf{b_i^k} + t\mathbf{b_{i+1}^k}$$

THE DE CASTELJAU ALGORITHM

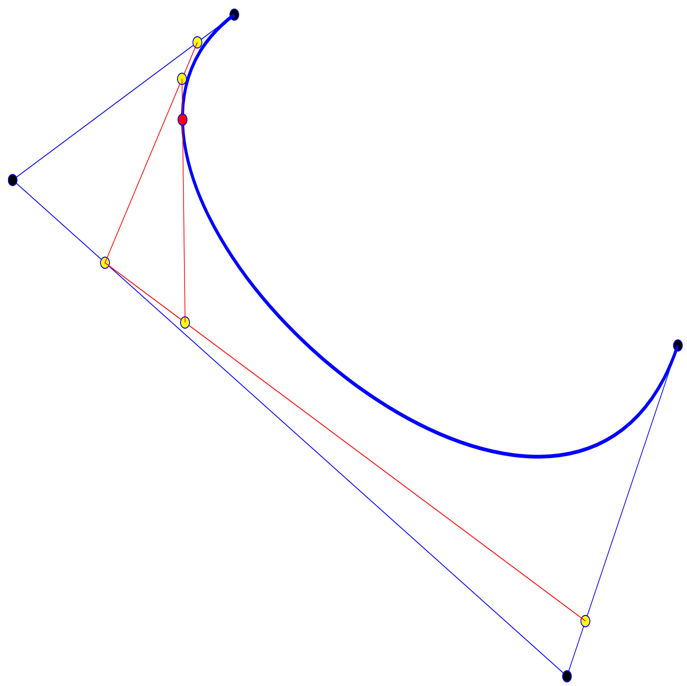
To compute a point for $u = u(t_0)$, on the Bézier curve $\mathbf{b}(u)$, with control points $\mathbf{b_0}, \mathbf{b_1}, \ldots, \mathbf{b_n}$:

1. Define $\mathbf{b_i^{(0)}} = \mathbf{b_i}$ for $i = 0, \ldots, n$.

2. for j=1:n
   for i=0:n-j
   $$\mathbf{b_i^j} = (1 - t_0)\mathbf{b_i^{j-1}} + t_0 \mathbf{b_{i+1}^{j-1}}$$

3. $\mathbf{b}(t_0) = \mathbf{b_0^n}$

ILLUSTRATION OF DE CASTELJAU's ALGORITHM



Evaluation of a single point on the Bézier curve. In this case, for t=1/6.

THE DE CASTELJAU ARRAY

$$
\begin{array}{ccccccc}
\mathbf{b_0^0} & & & & & & \\
\mathbf{b_1^0} & \mathbf{b_0^1} & & & & & \\
\vdots & & \ddots & & & & \\
\mathbf{b_n^0} & \mathbf{b_{n-1}^1} & \cdots & \mathbf{b_0^n} & = & \mathbf{b}(t_0) &
\end{array}
$$