Numerical Analysis
FMN011

Carmen Arévalo

Lund University

carmen@maths.lth.se

Eigenvalues and Power Iteration

Eigenvalues and Eigenvectors

If A is a square matrix such that

$$Au = \lambda u,$$

λ is an eigenvalue of A and
u is a corresponding eigenvector.

The characteristic polynomial of a matrix A is

$$p(\lambda) = \det(A - \lambda I)$$

The eigenvalues of A are the roots of this polynomial.

The eigenvectors are vectors in the null space of $A - \lambda I$.

Some Properties of Eigenvalues and Eigenvectors

- If u is an eigenvector, then ku is one too.
- The corresponding eigenvalue of u is the Rayleigh quotient, $\boxed{\lambda = \frac{u^T A u}{u^T u}}$
- λ eigenvalue of $A \Rightarrow \lambda^{-1}$ eigenvalue of A^{-1} (same eigenvector)
- λ eigenvalue of $A \Rightarrow \lambda - s$ eigenvalue of $A - sI$ (same eigenvector)
- $(\lambda - s)^{-1}$ eigenvalue of $(A - sI)^{-1}$ (same eigenvector)
- If $A = S^{-1}BS$, then A and B have the same eigenvalues (but not the same eigenvectors)

Computing Eigenvalues and Eigenvectors with the Characteristic Polynomial

$$A = \begin{bmatrix} -1 & 10 \\ 0 & -2 \end{bmatrix} \Rightarrow \det(\lambda I - A) = 0 \Rightarrow \lambda_1 = -1,\ \lambda_2 = -2$$

$$\begin{bmatrix} -1 & 10 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -1 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = c \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 10 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -2 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = c \begin{bmatrix} -10 \\ 1 \end{bmatrix}$$

The Power Method

Computing the dominant eigenvalue/eigenvector

Suppose:
- The eigenvectors of A form a basis
- A has unique λ_1 of maximum modulus

Start with x_0 and define the iteration method

$$\boxed{x_k = A x_{k-1}}$$

If $\{u_1, u_2, \dots, u_n\}$ is the basis of eigenvectors,

$$x_0 = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$$

and

$$x_k = Ax_{k-1} = A^2 x_{k-2} = \cdots = A^k x_0 = \sum_{i=1}^{n} A^k c_i u_i$$

$$\begin{aligned} x_k &= \sum_{i=1}^{n} A^k c_i u_i \\ &= \sum_{i=1}^{n} \lambda_i^k c_i u_i \\ &= \lambda_1^k (c_1 u_1 + \sum_{i=2}^{n} (\frac{\lambda_i}{\lambda_1})^k c_i u_i) \to \lambda_1^k c_1 u_1 \text{ as } k \to \infty \end{aligned}$$

$$u \cong \frac{x_k}{\|x_k\|}, \quad \lambda \cong \frac{u^T A u}{u^T u}$$

Normalization

Iterates could converge to zero or become unboundedly large, so we require the largest component of each iterate to have modulus 1.

Given the initial vector x_0. Take

$$
\begin{aligned}
y_{k-1} &= \frac{x_{k-1}}{\|x_{k-1}\|_2} \\
x_k &= A y_{k-1} \\
\lambda_k &= y_{k-1}^T x_k
\end{aligned}
$$

Speed of convergence is linear, and governed by $|\lambda_2/\lambda_1|$

Example: Power Method

From Scientific Computing by Heath

$$A = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix} \quad \text{and} \quad x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

i	1	2	3	4	...	14	15
y_i	0.3162 0.9487	0.5145 0.8575	0.6139 0.7894	0.6616 0.7498	$\ldots$ $\ldots$	0.7070 0.7072	0.7071 0.7071
$y_i^T A y_i$	1.50	1.80	1.941	1.985	...	1.999	2.000

$\lambda \approx 2.000$, $u \approx [0.7071, 0.7071]^T$

The inverse power method

The smallest eigenvalue (in magnitude) of A is the inverse of the dominant eigenvalue of A^{-1}.

$$\begin{aligned} Ax &= \lambda x \\ \lambda^{-1} x &= A^{-1} x \end{aligned}$$

Apply the power method to A^{-1},

$$x_{k+1} = A^{-1} x_k$$

and take $\lambda = 1/\eta$, where $\eta = \dfrac{x_j^T A^{-1} x_j}{x_j^T x_j} = \dfrac{x_j^T x_{j+1}}{x_j^T x_j}$.

To avoid calculation of the inverse, for each iteration solve the system

$$A x_{k+1} = x_k.$$

The Shifted Inverse Power Method

To find the eigenvalue nearest to s look for the smallest eigenvalue (in magnitude) of $A - sI$, because

$$(A - sI)x = (\lambda - s)x$$

$$\begin{aligned}
(A - sI)^{-1}x &= \frac{1}{\lambda - s}x \\
(A - sI)x_{k+1} &= \eta x_k \\
\lambda &= \frac{1}{\eta} + s
\end{aligned}$$

The Shifted Inverse Power Method algorithm

Start with x_0
Set $B = A - sI$
Set $y_{k-1} = x_{k-1}/\|x_{k-1}\|_2$
Solve $Bx_k = y_{k-1}$
Set $\eta_k = x_k^T y_{k-1}$

$$\lambda = \frac{1}{\eta} + s$$

Speed of convergence is linear.

Example: Inverse Power Method (s=0)

$$A = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 2.5 \end{bmatrix}, \qquad x_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

1	2	...	13	14
-0.9806 0.1961	-0.9558 0.2941	$\dots$ $\dots$	-0.9239 0.3826	-0.9239 0.3827
1.4000	1.3188	...	1.2929	1.2929

$\lambda \approx 1.2929$, $u \approx [-0.9239, 0.3827]^T$,

Residual vector: $Au - \lambda u = [-0.00001508, -0.00003640]^T$