Numerical Analysis FMN011

Carmen Arévalo

Lund University

carmen@maths.lth.se

Lecture 5

Triangular factorization

A has a triangular factorization or LU decomposition if it can be expressed as the product

$$A = LU$$

where L is a lower triangular matrix and U is upper triangular.

For instance,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

Solving a system

$$Ax = b$$

Suppose we have L and U such that $A = LU$. Then we can write

$$LUx = b$$

Let $Ux = y$, then

1. Solve triangular system $Ly = b$ (forward substitution) to get y

2. Use y in $Ux = y$ and solve this triangular system (back substitution) to get x

Elementary Matrices

How to transform $Ax = b$ to a system $Tx = c$ (T upper $\triangle$); or
How to get a column of zeros below an entry.

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ -a_2/a_1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots & -m_{k+1} & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & -m_n & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ a_{k+1} \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

multiplier: $m_i = a_i/a_k$, pivot: a_k, elementary matrix: M_k

Properties of M_k

- M_k is lower triangular and its diagonal elements are all 1, so it is nonsingular

- M_k is I with zeros below diagonal element in column k changed to $-m_{k+1}, \ldots, -m_n$

- M_k^{-1} is I with zeros below diagonal element in column k changed to $m_{k+1}, \ldots, m_n$

- If $j < k$ (OBS: order important!), $M_j M_k$ is I with $-m_{j+1}, \ldots, -m_n$ below diagonal in column j and $-m_{k+1}, \ldots, -m_n$ below diagonal in column k

LU factorization

$$Ax = b$$

Gaussian elimination

- With a_{11} as pivot, construct M_1 to annihilate entries in column 1 below a_{11}.
- Multiply to get $M_1 A x = M_1 b$
- Continue process until matrix

$$U = M_{n-1} \cdots M_1 A$$

is upper triangular.

$L = (M_{n-1} \cdots M_1)^{-1}$ is lower triangular, and $A = LU$

Finding LU

$$A = \begin{bmatrix} 4 & -8 & 1 \\ 6 & 5 & 7 \\ 0 & -10 & -3 \end{bmatrix}, \quad \begin{matrix} m_{21} = 6/4; \\ m_{31} = 0/4; \end{matrix} \quad M_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{6}{4} & 1 & 0 \\ -\frac{0}{4} & 0 & 1 \end{bmatrix},$$

$$M_1 A = \begin{bmatrix} 4 & -8 & 1 \\ 0 & 17 & \frac{22}{4} \\ 0 & -10 & -3 \end{bmatrix}, \quad m_{32} = -10/17; \quad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{10}{17} & 1 \end{bmatrix},$$

$$U = M_2 M_1 A = \begin{bmatrix} 4 & -8 & 1 \\ 0 & 17 & \frac{22}{4} \\ 0 & 0 & \frac{4}{17} \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & 1 & 0 \\ 0 & -\frac{10}{17} & 1 \end{bmatrix}, \quad A = LU$$

Computational complexity of LU factorization

Same as Gauss elimination.

For a set of k problems with the same $n \times n$ matrix,

$$
\begin{aligned}
Ax &= b_1 \\
Ax &= b_2 \\
&\vdots \\
Ax &= b_k
\end{aligned}
$$

Gauss requires $\approx kn^3/3$ operations
LU requires $\approx 2n^3/3 + kn^2$ operations

Pivoting

- A pivot cannot be 0
- A small pivot can introduce large numerical errors
- Multipliers should be less than 1 (in magnitude)

Partial pivoting chooses largest magnitude in column as pivot. Possible because rows may be interchanged.

Scaled partial pivoting chooses largest magnitude in column (**relative to the entries in its row**) as pivot.

$$s_r = \max\{|a_{rp}|, |a_{rp+1}|, \ldots, |a_{rN}|\} \quad \text{for} \qquad r = p : N$$

The pivotal row k is defined by

$$\frac{|a_{kp}|}{s_k} = \max\left\{\frac{|a_{pp}|}{s_p}, \frac{|a_{p+1p}|}{s_{p+1}}, \ldots, \frac{|a_{Np}|}{s_N}\right\}$$

Permutation matrices

A permutation matrix is a matrix whose rows are permutations of the rows of I. Example:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \longrightarrow \quad P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Rows of matrix PA are permuted in the same order:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

Note: $P^{-1} = P^T$

LU decomposition with pivoting

If A is nonsingular, there is a P such that $PA = LU$

$$Ax = b \quad \Rightarrow \quad LUx = Pb$$

1. Compute L, U and P
2. Compute Pb
3. Solve $Ly = Pb$ with forward substitution
4. Solve $Ux = y$ with backward substitution

Computing the inverse with LU factorization

As $AA^{-1} = I$, each column of A^{-1} is the solution of $Ax_i = e_i$, where e_i is the canonical vector containing only zeros except for a one in position i.

1. Compute the LU(P) factorization of A

2. Perform n back substitutions (one for each e_i)

3. Perform n forward substitutions (one for each e_i)

This has a computational complexity of the order of $2n^3/3 + n(n^2 + n^2) = 8n^3/3$. Taking advantage of the sparseness of the operations we can bring this count down to $2n^3$.