# FMN011 — Seminar 2 — Solution of Linear Systems

1. True or false:

   (a) If $A$ is nonsingular, then the number of solutions to $Ax = b$ depends on the particular choice of vector $b$.

   (b) For a symmetric matrix $S$, it is always the case that $\|S\|_1 = \|S\|_\infty$.

   (c) If a triangular matrix has a zero entry on its main diagonal, then the matrix is necessarily singular.

   (d) The product of two upper triangular matrices is also upper triangular.

   (e) If the rows of a square matrix are linearly dependent, then the columns of the matrix are also linearly dependent.

   (f) If $A$ is any $n \times n$ matrix and $P$ is any $n \times n$ permutation matrix, then $PA = AP$.

   (g) For $x \in \mathcal{R}^n$, $\|x\|_1 \geq \|x\|_\infty$.

   (h) If $\det(A) = 0$ , then $\|A\| = 0$.

   (i) The product of two symmetric matrices is also symmetric.

   (j) $\kappa_p(A) = \kappa_p(A^{-1})$.

   (k) A system $Ax = b$ can have exactly two distinct solutions.

   (l) Every nonsingular matrix $A$ can be written as $A = LU$, where $L$ is lower triangular and $U$ is upper triangular.

   (m) A singular matrix does not have an LU(P) factorization.

2. Given $Ax = b$, what effect on the solution vector $x$ results from

   (a) Permuting the rows of $[A \quad b]$?

   (b) Permuting the columns of $A$?

   (c) Multiplying both sides of the equation from the left by a nonsingular matrix $M$?

3. Consider the matrix

$$A = \begin{bmatrix} 4 & -8 & 1 & 2 \\ 6 & 5 & 7 & 3 \\ 0 & -10 & -3 & 5 \\ 5 & -1 & 1 & 0 \end{bmatrix}$$

   What will the initial pivot in Gaussian elimination be if

   (a) No pivoting is used?

   (b) Partial pivoting is used?

   (c) Scaled partial pivoting is used?

4. Given $n \times n$ matrices $A$ and $B$, what is the best way to compute $A^{-1}B$?

5. If $x$ is a column vector and $A$ is a matrix, which of the following computations require less work?

   (a) $y = (xx^T)A$

   (b) $y = x(x^T A)$

6. What is the inverse of a permutation matrix $P$?

7. Assume you have already computed the LU factorization, $PA = LU$. How would you use it to solve the system $A^T x = b$?

8. Classify each matrix as well conditioned or ill conditioned:

   (a) $\begin{pmatrix} 10^{10} & 0 \\ 0 & 10^{-10} \end{pmatrix}$

   (b) $\begin{pmatrix} 10^{10} & 0 \\ 0 & 10^{10} \end{pmatrix}$

   (c) $\begin{pmatrix} 10^{-10} & 0 \\ 0 & 10^{-10} \end{pmatrix}$

   (d) $\begin{pmatrix} 1.0000001 & 2 \\ 2 & 4 \end{pmatrix}$

9. In solving a linear system $Ax = b$, what is meant by the *residual* of an approximate solution $\hat{x}$? Does a small relative residual always imply that the solution is accurate?

10. Rank the following methods according to the amount of work required for solving most systems:

    (a) Gaussian elimination with partial pivoting

    (b) $LU$ factorization followed by forward- and back-substitutions

    (c) Explicit matrix inversion followed by matrix-vector multiplication

11. What quantity is minimized when using least squares to solve an overdetermined system $Ax \cong b$?

12. True or false:

    (a) A linear least squares problem always has a solution.

    (b) At the solution to a least squares problem $Ax \cong b$, the residual vector $r = b - Ax$ is orthogonal to the space generated by $A$ (i.e. $Ax$ for all $x$).

13. Let $A$ be an $m \times n$ matrix. Under what conditions on the matrix $A$ is the matrix $A^T A$ nonsingular?

14. In an overdetermined linear least squares problem $Ax \cong b$, where $A$ is an $m \times n$, if $\text{rank}(A) < n$, then which of the following situations are possible?

    (a) There is no solution

    (b) There is a unique solution

    (c) There is a solution, but it is not unique

15. In solving an overdetermined least squares problem $Ax \cong b$, which would be a more serious difficulty: that the rows of $A$ are linearly dependent, or that the columns of $A$ are linearly dependent?

16. In fitting a straight line $y = x_0 + x_1 t$ to the three data points $(0, 0), (1, 0), (1, 1)$, is the least squares solution unique? Why?

17. What is the Euclidean norm of the minimum residual vector for the following linear least-squares problem?

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \approx \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

    What is the solution vector for this problem?

C. Arévalo