Numerical Analysis — FMN011 — Practice2016

The exam lasts 5 hours and has 14 questions. A minimum of 35 points out of the total 70 are required to get a passing grade. These points will be added to those you obtained in your two home assignments, and the final grade is based on your total score.

Justify all your answers and write down all important steps. Unsupported answers will be disregarded.

During the exam you are allowed a pocket calculator, but no textbook, lecture notes or any other electronic or written material.

1. To solve the equation
$$4x^3 + x - 1 = 0$$
I applied a fixed point iteration ($x_{n+1} = g(x_n)$) to $g(x) = 1 - 4x^3$ with initial guess $x_0 = 1$.

   (a) **(2p)** Explain why this will not solve my problem.

   Then I applied the same method to $g(x) = \sqrt[3]{\dfrac{1-x}{4}}$ and found a solution with relative error less than 0.005.

   (b) **(3p)** How many iterations did I have to perform and what was the result I got?

2. **(5p)** When solving the equation $4x^5 - 16x^4 + 17x^3 - 19x^2 + 13x - 3$ using the Newton-Raphson method ($x_{n+1} = x_n - f_n/f'_n$) with precision of 10^{-4}, I get very different rates of convergence, depending on where I start. It takes 11 iterations to solve the problem for the solution $x = 0.5$, starting at $x_0 = 0$, but only 4 iterations to solve for the solution $x = 3$, starting at $x_0 = 3.5$. Give a good explanation of why this happens.

3. Under certain conditions, the iterative method $x_{k+1} = (I - A)x_k + b$ will solve the linear system $Ax = b$.

   (a) **(2p)** State a condition that will ensure convergence.

   (b) **(2p)** Will the method be convergent for $A = \begin{pmatrix} 2/3 & 1/12 \\ 0 & 1/4 \end{pmatrix}$? Why?

4. **(5p)** I read on the web that the following statements are true, but they are actually false. Give a good argument to show their falseness.

   (a) Only invertible matrices have an LU factorization.

   (b) If a matrix A has an LU factorization only after row permutations are made, that is, $PA = LU$, then U can only be guarantied to be upper triangular after its rows have been permuted.

   (c) The residual cannot always be calculated after solving $Ax = b$.

(d) Solving a linear system $Ax = b$ by means of LU factorization is numerically unstable if the condition number of A is greater than 1.

(e) Row and column permutations of A will not change the result of solving $Ax = b$ if b is permuted accordingly.

5. **(5p)** Consider

$$A = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & -2/\sqrt{6} \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

(a) Show that the columns of A are orthonormal vectors.

(b) Show that finding the least squares solution of $Ax = b$ requires no actual arithmetic operations.

(c) Calculate the least squares solution.

(d) Calculate the residual.

(e) Is there a vector v such that $\|Av - b\|_2 < 1/\sqrt{2}$?

6. **(6p)** Let A be an invertible $n \times n$ real matrix.

(a) Prove that the eigenvalues of A and A^T are the same. (Hint: we know that $\det(A) = \det(A^T)$.)

(b) Prove that if λ is an eigenvalue of A then $1/\lambda$ is an eigenvalue of A^{-1}.

(c) Prove that if A is an orthogonal matrix, its real eigenvalues are ± 1.

7. **(5p)** Study the following Matlab code. What does it do and what can we use it for?

```
function val = whatisit(p,t)
%
% INPUT:   p   nx2 Matrix
%          t   mx1 Vector
%
% OUTPUT: val   mx2 Matrix

n = size(p,1);
m = length(t);
val = zeros(m,2);
X(:,1) = p(:,1);
Y(:,1) = p(:,2);

for j = 1:m
    for i = 2:n
        X(i:n,i) = (1-t(j))*X(i-1:n-1,i-1) + t(j)*X(i:n,i-1);
        Y(i:n,i) = (1-t(j))*Y(i-1:n-1,i-1) + t(j)*Y(i:n,i-1);
    end
    val(j,1) = X(n,n);
    val(j,2) = Y(n,n);
end
```

```
plot(val(:,1),val(:,2))
```

8. **(5p)** A quadratic Bézier curve starts at (1,1) with a slope of 1/2, and ends at (0,0) with a slope of -1. Make a sketch of the curve, give all of its control points, and show the control polygon.

9. **(5p)** When the built-in Matlab function `eig` is used on a 3×3 matrix C, we get the following results:

```
[V,D]=eig(C)
V =
   -0.8666    0.5345   -0.5345
    0.4333   -0.2673    0.2673
    0.2476   -0.8018    0.8018
D =
    8.5     0         0
    0      17         0
    0       0        17
```

   Explain what are V and D and how they relate to each other.

10. **(5p)** The DFT of a real vector is

```
-0.3536
-0.3536 + 3.5990i
-0.3536 - 0.9239i
-0.3536 - 0.2294i
-0.3536
-0.3536 + 0.2294i
-0.3536 + 0.9239i
-0.3536 - 3.5990i
```

   Given that the DFT trigonometric interpolation polynomial is

$$P_n(t) = \frac{a_0}{\sqrt{n}} + \frac{2}{\sqrt{n}} \sum_{k=1}^{n/2-1} \left(a_k \cos(2\pi k t) - b_k \sin(2\pi k t)\right) + \frac{a_{n/2}}{\sqrt{n}} \cos(n\pi t),$$

   show how to construct a low-pass filter that keeps all frequencies up to $4\pi t$.

11. Consider the message

   ERROR CONTROL

   (a) **(1p)** Construct a Huffman tree for this message.
   (b) **(2p)** Construct a table with the binary code for each symbol.
   (c) **(1p)** How many bits are required to code the message?
   (d) **(1p)** What is the average number of bits/symbol used?

12. **(5p)** Explain what the Gram-Schmidt procedure does, and illustrate it with the vectors $(1, 2)$ and $(4, 1)$, either algebraically or by drawing a clear picture.
13. **(5p)** Suppose you can get the singular value decomposition of a matrix, $A = USV^T$. How can you use it to determine its rank?
14. **(5p)** If A is a full 20×20 matrix, describe a way to achieve a lossy compression of at least 75% without losing information unnecessarily.

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