An Unscented Kalman Filter for Catheter Tracking

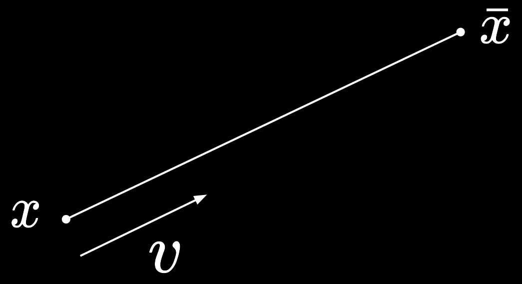
Sebastian Ferguson • June 18, 2019 • Realtime Meeting

A simple starting point

- Suppose we are tracking a point
- We have knowledge about how our point changes with time
- We have imperfect knowledge about the “state” of the point

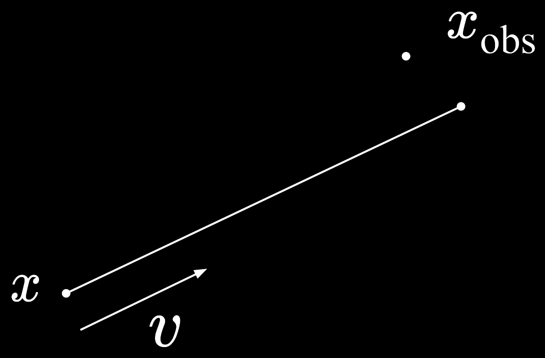
$$\text{state} = (x, v)$$

$$\bar{x} = x + \Delta t \, v$$
$$\bar{v} = v$$



A simple starting point

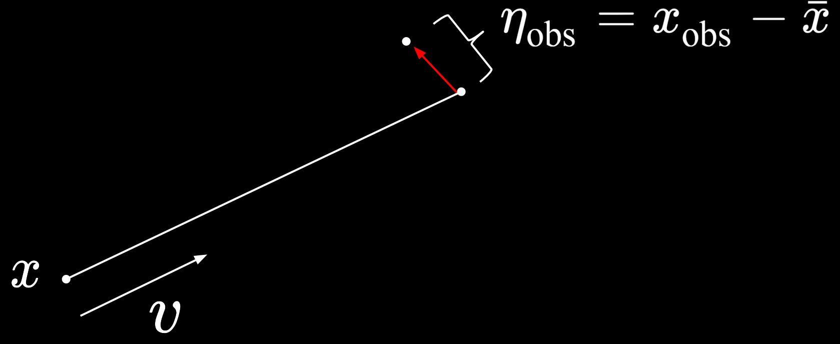
- After time passes we make an observation about the point
- It is not the same point we expected



A simple starting point

The difference could be due to:

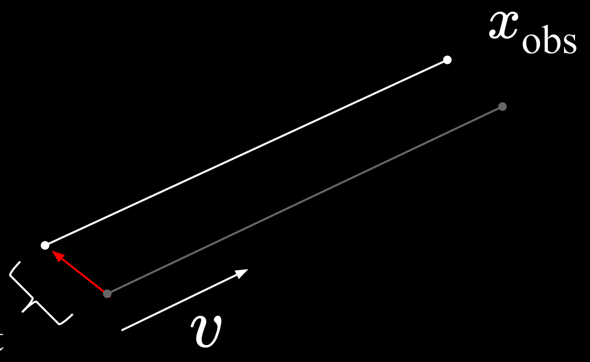
- Error in the measurement



A simple starting point

The difference could be due to:

- Error in the measurement
- Error in x

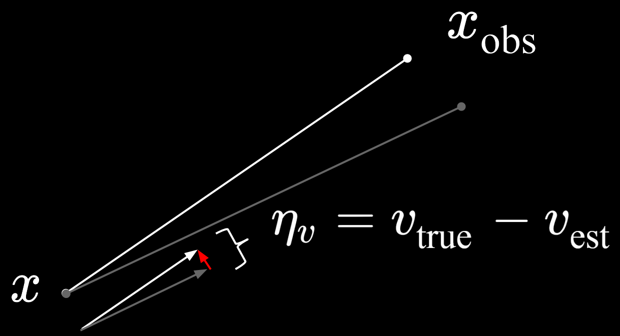


$$\eta_{\mathrm{x}} = x_{\mathrm{true}} - x_{\mathrm{est}}$$

A simple starting point

The difference could be due to:

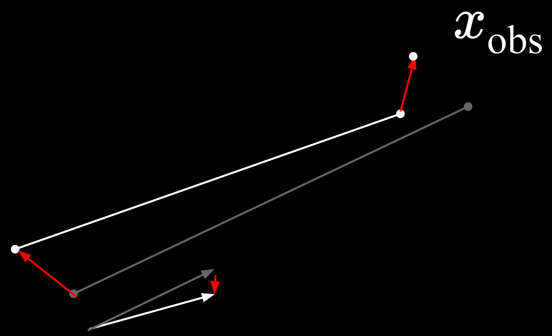
- Error in the measurement
- Error in x
- Error in v



A simple starting point

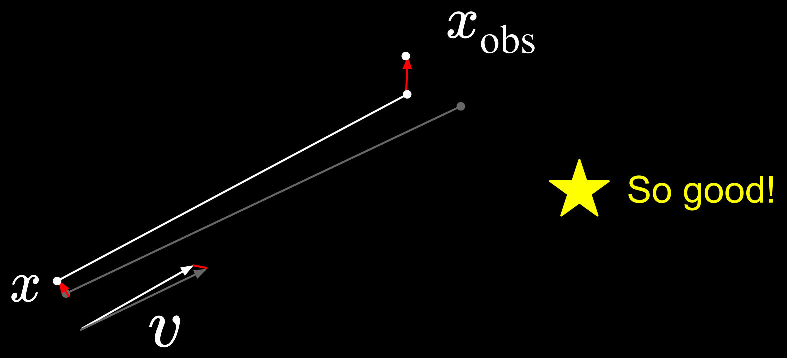
The difference could be due to:

- Error in the measurement
- Error in x
- Error in v
- All of the above



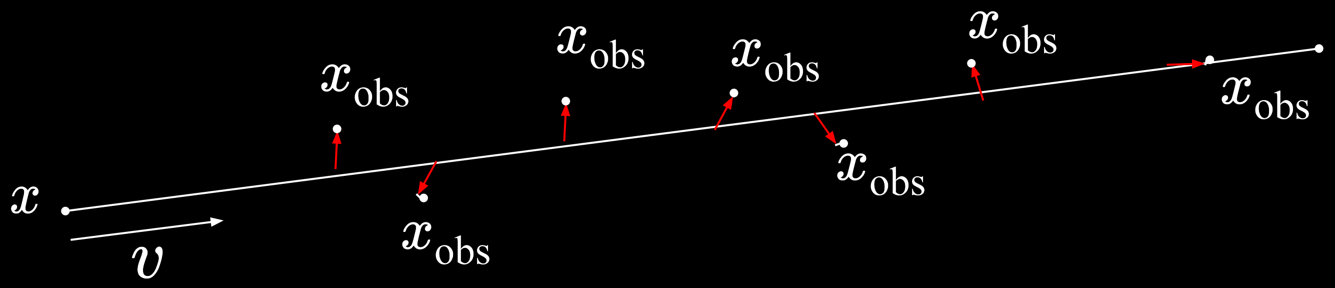
A simple starting point

If we want to use the new observation to improve our estimate of the state we will need a way to calculate the “best” distribution of error.



A simple starting point

If we keep track of all observations we can do (linear) regression.



Kalman filtering; what it will do for us

The challenges brought up in the previous slides can be overcome using a Kalman filter. It will:

- Update the state estimate using a single observation at a time.
- Give us a good (or even the best) estimate of our state.
- Use information about how well sequential predictions match observations to calculate “best” distribution of error.
- Let the state change in an unknown way (i.e., randomly).

There are multiple sets of hypotheses that lead to Kalman’s equations.

Kalman filtering; set up

State:

$$\bar{x} = Fx + \mathcal{N}(0, Q)$$

Observation:

$$z = Hx + \mathcal{N}(0, R)$$

Kalman filtering; set up

State:

$$\bar{x} = Fx + \mathcal{N}(0, Q)$$

Observation:

$$z = Hx + \mathcal{N}(0, R)$$

Keep track of:

$$(x, P)$$

Kalman filtering; set up

For example: The state can be position and velocity like before. State transition can be the kinematic equation, and observation can be projection (i.e., only observe position).

$$\bar{x} = Fx + \mathcal{N}(0, Q)$$

$$z = Hx + \mathcal{N}(0, R)$$

$$x \text{ ``=''} \begin{pmatrix} x \\ v \end{pmatrix} \quad F = \begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

Kalman filtering; procedure

Model

$$\bar{x} = Fx + \mathcal{N}(0, Q)$$
$$z = Hx + \mathcal{N}(0, R)$$

$$(x, P)$$

Predict

$$\bar{x} = Fx$$
$$\bar{P} = FPF^{\mathrm{T}} + Q$$

Update

$$S = H\bar{P}H^{\mathrm{T}} + R$$
$$K = \bar{P}H^{\mathrm{T}}S^{-1}$$
$$x = \bar{x} + K(z - H\bar{x})$$
$$P = (1 - KH)\bar{P}$$

Kalman filtering; remarks

The essential equation:

$$x = \bar{x} + K(z - H\bar{x})$$

- The new estimate is a linear combination of the prediction and the observation error.
- The Kalman gain is doing the work of selecting how to distribute the error.
- This procedure gives a ML/MVUE for the state with the linear + gaussian assumptions.

- Dynamics and measurement must be linear :(

Unscented Transform

A tool to estimate the mean and covariance of a nonlinear function of a random variable.

Unscented Transform

A tool to estimate the mean and covariance of a nonlinear function of a random variable.

Given a random variable x with known mean and covariance and a nonlinear function f:

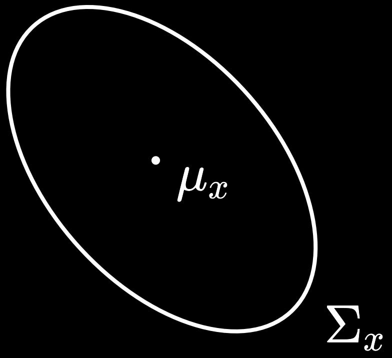
$$\mu_x = \mathbf{E}[x] \qquad \Sigma_x = \mathbf{E}[(x - \mu_x)(x - \mu_x)^{\mathrm{T}}] \qquad y = f(x)$$

Want to estimate,

$$\mu_y = \mathbf{E}[y] \qquad \Sigma_y = \mathbf{E}[(y - \mu_y)(y - \mu_y)^{\mathrm{T}}]$$

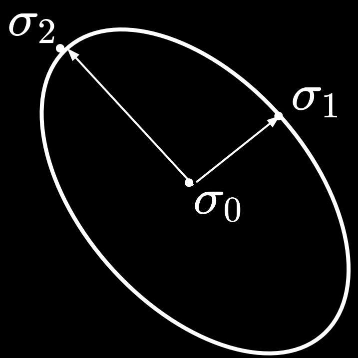
Unscented Transform

From the mean and covariance of the given rv ...



Unscented Transform

Generate a set of “sigma points” and weights, which capture those values ...

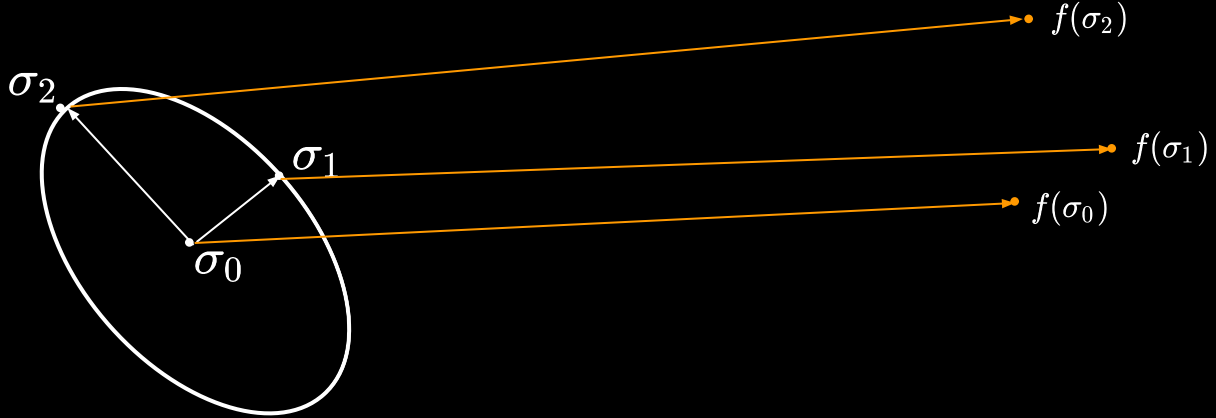


$$\mu_x = \Sigma_{i=0}^{n} w_i \sigma_i$$

$$\Sigma_x = \Sigma_{i=0}^{n} w_i (\sigma_i - \mu_x)(\sigma_i - \mu_x)^{\mathrm{T}}$$

Unscented Transform

Transform those points according to the nonlinear function ...

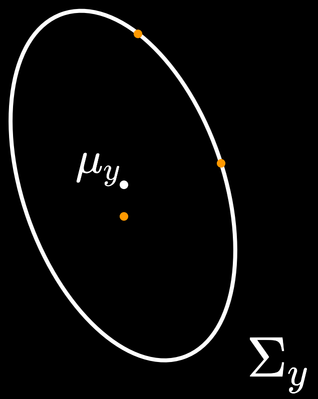


Unscented Transform

Estimate the mean and covariance from the transformed points.

$$\mu_y = \Sigma_{i=0}^{n} w_i f(\sigma_i)$$

$$\Sigma_y = \Sigma_{i=0}^{n} w_i (f(\sigma_i) - \mu_x)(f(\sigma_i) - \mu_x)^{\mathrm{T}}$$



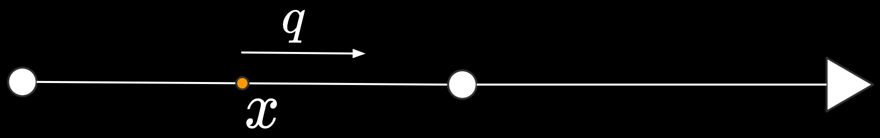
Unscented Kalman Filter

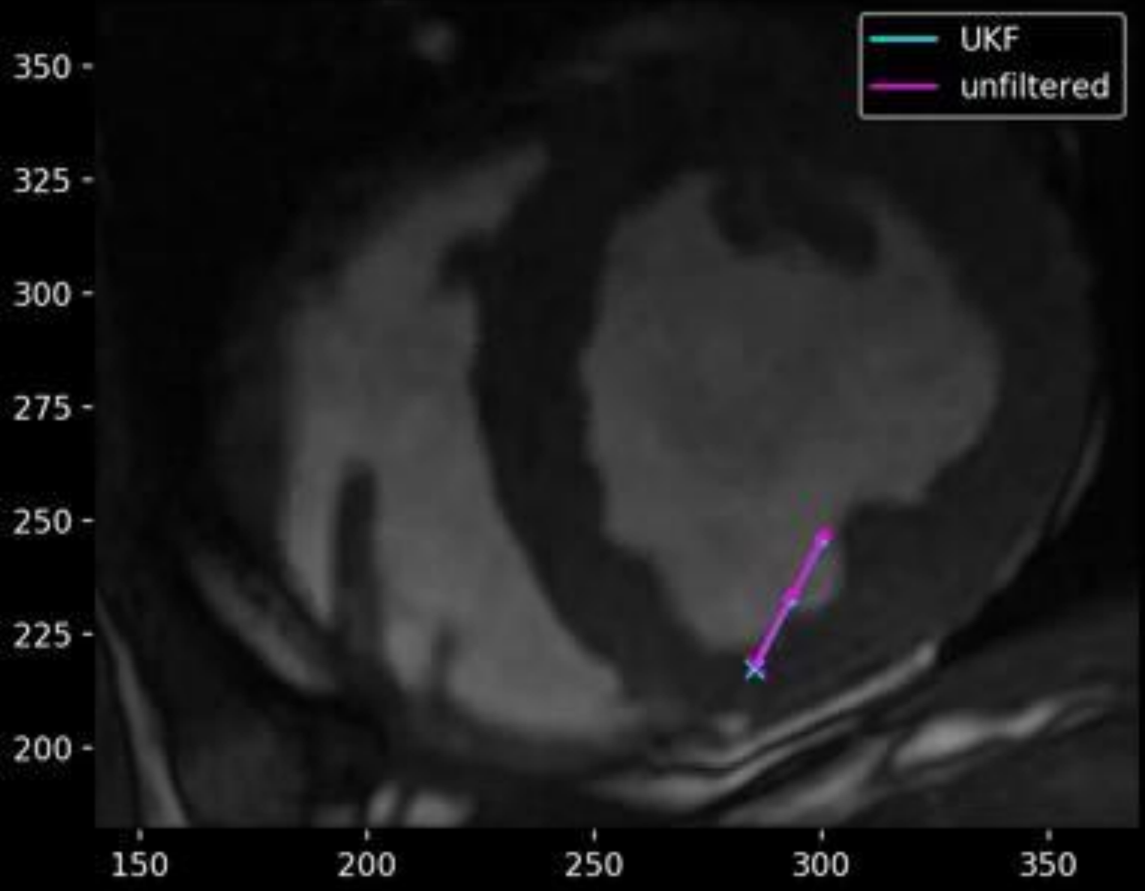
Uses nonlinear dynamics and measurement.

Approximates the covariances required for the predict/update equations using the unscented transform.

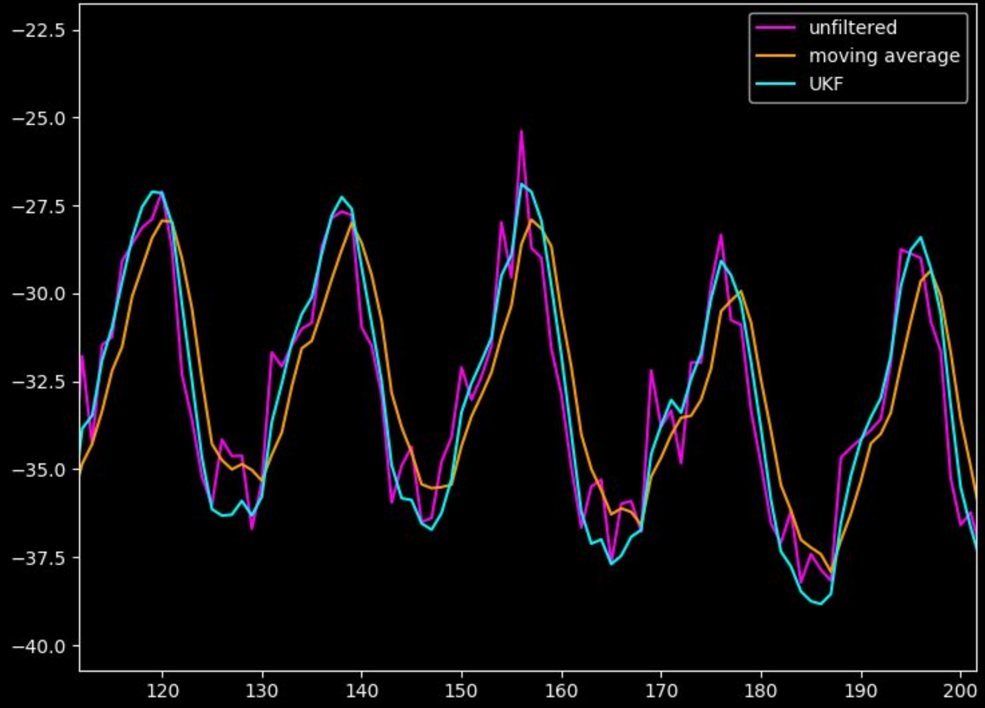
UKF applied to catheter tracking; model

x	The position of the midpoint
v_x	The velocity of the midpoint
a_x	The acceleration of the midpoint
q	The orientation (a unit vector)
v_q	The (2D) velocity of the orientation
a_q	The (2D) acceleration of the orientation




UKF
unfiltered
350
325
300
275
250
225
200
150
200
250
300
350

The effect of UKF and averaging



Some final thoughts

- KF can be used to get good model estimates from observations.
- KF can be used to combine multiple kinds of measurements.
  - Respiratory motion from multiple sources?
- I don’t think this is the end of KF in my projects.
  - This is a good self contained piece that could be implemented for experiments.
    - Less delay in visualization (if the code is fast enough …)
    - Make use of data between sets for smoothing