Assignment 2

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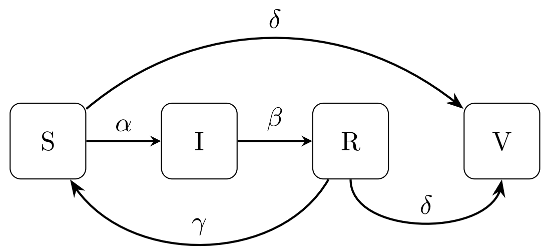
1 Model

A SIR model can let us predict the how infectious a virus is, but it is far from perfect. As seen from COVID-19, the virus can mutate and people can get re-infected. There are also concerns on how vaccination can help with the virus. Hence, by considering a SIRV model where the recovered can return to being susceptible again, with vaccination being considered as full immunity. We can then study how vaccination affect us. Below is a flowchart of the model considered.



1.1 Variables

t - time in weeks
S - Number of people Susceptible
I - Number of people Infected
R - Number of people Recovered
V - Number of people Vaccinated
B - Number of people Born
t_V - Vaccination start time
α/alpha - rate of infection
β/beta - rate of recovery
γ/gamma - rate of re-susceptibility
δ/delta - rate of vaccination

1.2 Assumptions

- The time t represents number of weeks. I.e. t=1 refers to one week.
- Infection and Vaccination is assumed to only take 1 week.
- Recovery and re-susceptibility period depends on the rate of recovery respectively. E.g. If $\beta = 1$, it means everyone recovers in 1 week on average.
- Birth is constant per week. Death not considered.
- Effective infection rate is an infective rate (α) multiplied by number of susceptibles multiplied by infected. If $\alpha = 1$, it means one person can spread to everyone susceptible.
- δ reflects the effective rate of the vaccine multiplied by the percentage of number of people who get vaccinated. Only people who are under S and R to be vaccinated.
- Vaccination only starts after certain time has passed, denoted by t_V, and t_n denotes total number of weeks

2 Discrete Time Analysis

2.1 Discrete Equations

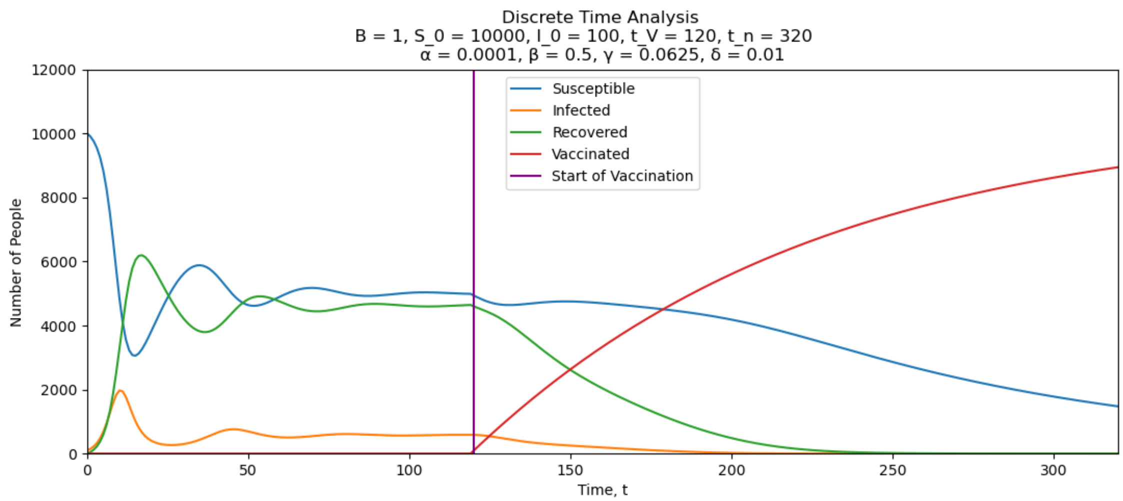
$$
\begin{aligned}
S_{t+1} &= S_t + B - \alpha S_t I_t + \gamma R_t - \delta S_t \\
I_{t+1} &= I_t + \alpha S_t I_t - \beta I_t \\
R_{t+1} &= R_t + \beta I_t - \gamma R_t - \delta R_t \\
V_{t+1} &= V_t + \delta S_t + \delta R_t
\end{aligned}
$$

$\alpha S_t I_t$ - refers to the number of people infected with the disease. In code, it is set to $\min(\alpha S_t I_t,\, S_t)$ so that it does not go over the current population
βI_t - refers to the number of people recovered from the disease.
γR_t - refers to the number of people who are susceptible to the disease again after recovering from it
$\delta S_t,\ \delta R_t$ - refers to the number of people who are vaccinated from the category S and R respectively.

With that, we have the following equations,
S at time $t+1 = S$ at time t + Birth $-$ Infected + Re-susceptible $-$ Vaccinated
I at time $t+1 = I$ at time t + Infected $-$ Recovered
R at time $t+1 = R$ at time t + Recovered $-$ Re-susceptible $-$ Vaccinated
V at time $t+1 = V$ at time t + Vaccinated from S + Vaccinated from R

2.2 Discrete Plots

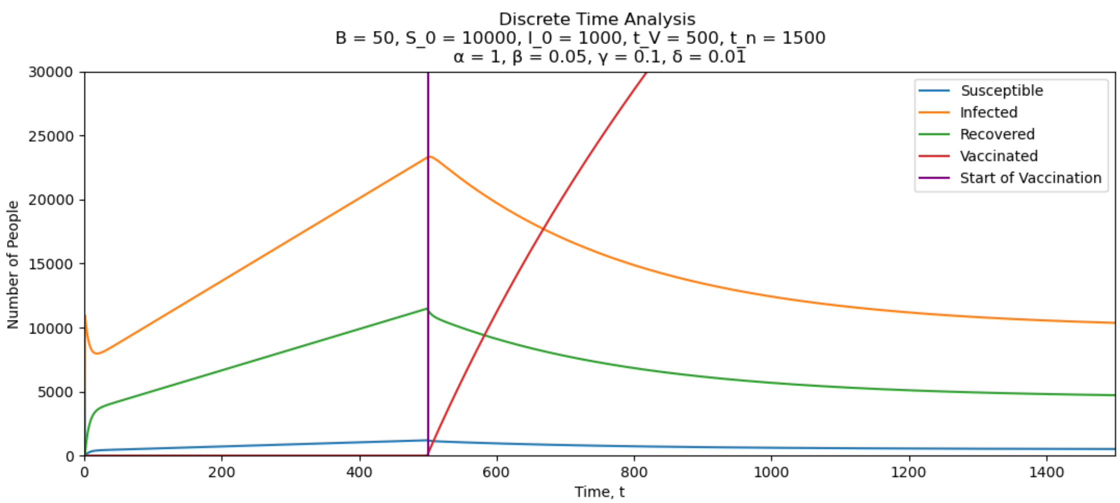
2.2.1 Plot 1



This plot uses 'realistic' variables:
- Population infected: around 1%
- Birth rate: around 5.0
- Infection rate: approximately 1 person to 1 person
- Recovering period: around 2 weeks
- Re-susceptibility period: around 4 months
- Vaccination rate: around 0.3 effective $\times$ 0.3 population

Before the vaccination, we can see the plot converging into a steady state. S decreased a lot at the start, due to exponential increase of people getting infected, as most of the population are susceptible. It start however starts to stabilise as soon as we see people recovering from the disease, reducing the number of people susceptible to the disease. I increased a lot at the start, but remains low due to the short recovering period and long re-susceptibility period. As long as many people remain in R, I will remain low. R increases to about the level of S, eventually converging to slightly less than S. This is due to balancing similar to Lokta-Volterra in the form of SIR, as every one of the three variables interact with each other.
After the vaccination however, we see a I drops to near zero, due to the reducing population of S due to vaccination, which results in I dying off. We also see S drops slowly compared to R, due to lesser people going to I, and the fact that people in R can go to S, therefore R drops faster than S. V reacts as expected despite low δ, as people cannot removed from V. We also see all other three variables converging to 0, a reassuring sign.

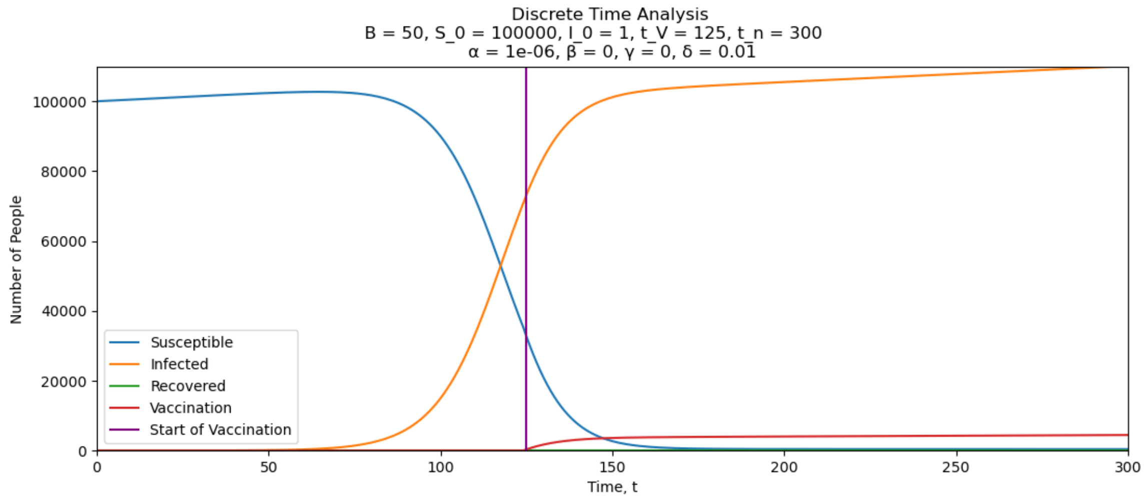
2.2.2 Plot 2



The next set of variables assume an extremely high infection rate disease, with longer recovery period, as well as human breeding like rabbits. The re-susceptibility rate is also set to be higher than recovery period, meaning people stay on I longer than R
The reduction of S is so fast that we couldn't even see in this graph, dropping to 0, if not for the constant birth rate. As for I and R increasing, it is to be expected, with I being twice of R as γ is twice of β, meaning they will converge to twice of the other.
After the vaccination however, I can be seen to be reduced immediately along with R. As we extend the t_n, it seems that I and R converge to a steady state to the value we see similar to what we see at $t = 1500$ here. This model shows that no matter how high the infection rate, as long as there is vaccination going on, things will be

remain stable (like COVID-19), but the disease will still be there.

2.2.3 Plot 3



Imagine you are watching a movie about the zombie apocalypse, where people get infected and don't recover, nor do they die. The infection rate is the same as the first plot, 1 to 1, but we now have a way higher population but only one zombie. And just about when the situation gets really bad, a vaccine is successfully made just in time (as usual). However, getting the vaccine is still very hard as people fight for it or gets bitten by a zombie (as usual), hence the relatively low vaccine rate. The birth rate remains slightly high, due to movies always include bed scenes in such situation (as usual).
Nothing special here as the number of zombies increase exponentially, along with number of susceptible decreasing. Despite the human race being almost wiped out at time t_V, there is still hope for humanity as long as we vaccine ourselves and procreate enough.

3 Continuous Time Analysis

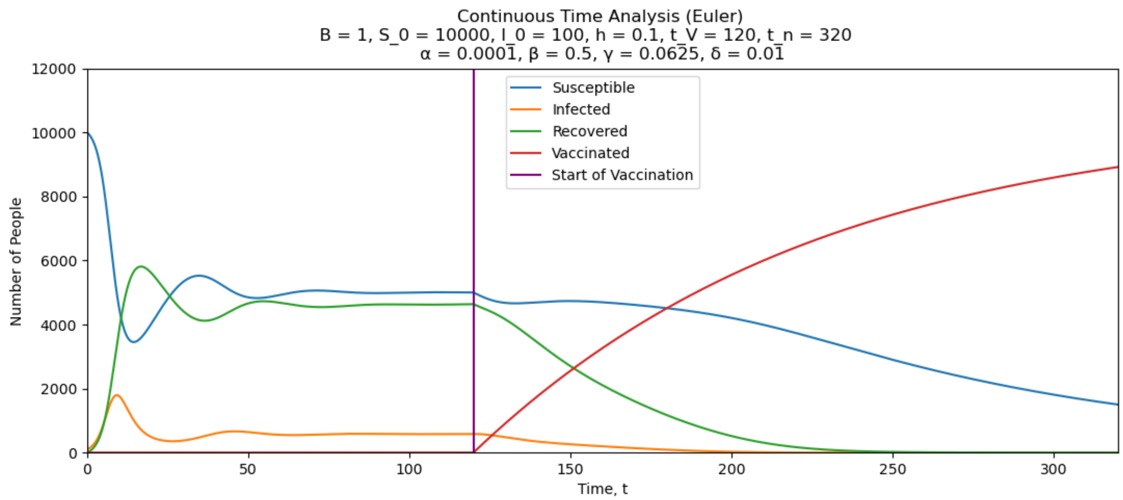
3.1 Continuous Equations

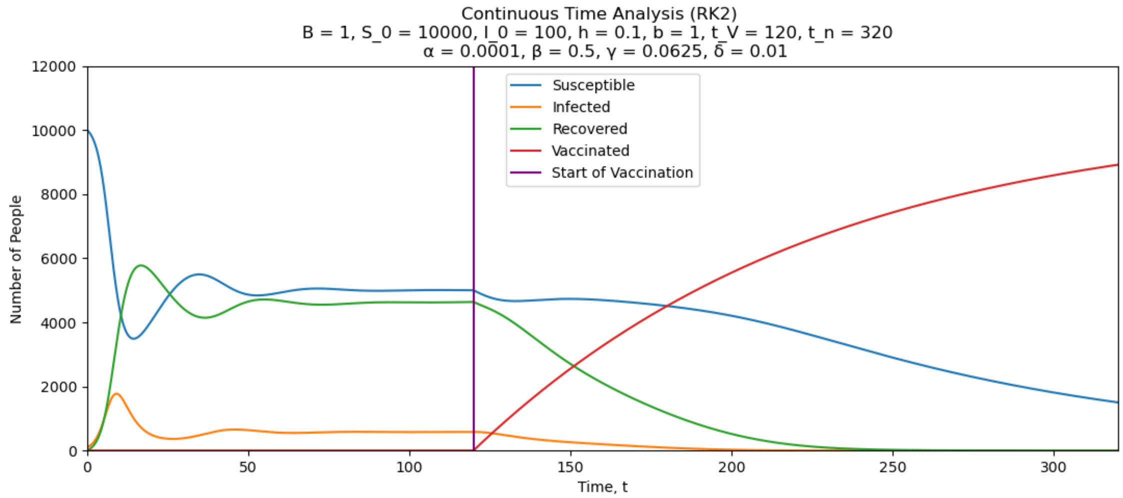
$$\frac{dS}{dt} = B - \alpha S_t I_t + \gamma R_t - \delta S_t$$
$$\frac{dI}{dt} = \alpha S_t I_t - \beta I_t$$
$$\frac{dR}{dt} = \beta I_t - \gamma R_t - \delta R_t$$
$$\frac{dV}{dt} = \delta S_t + \delta R_t$$

Not much changed from the discrete function, except the fact that $\frac{dS}{dt} = S_{t+1} - St$ and the rest respectively. This allow us to get a continuous function with we can use the RK method to get our plots.

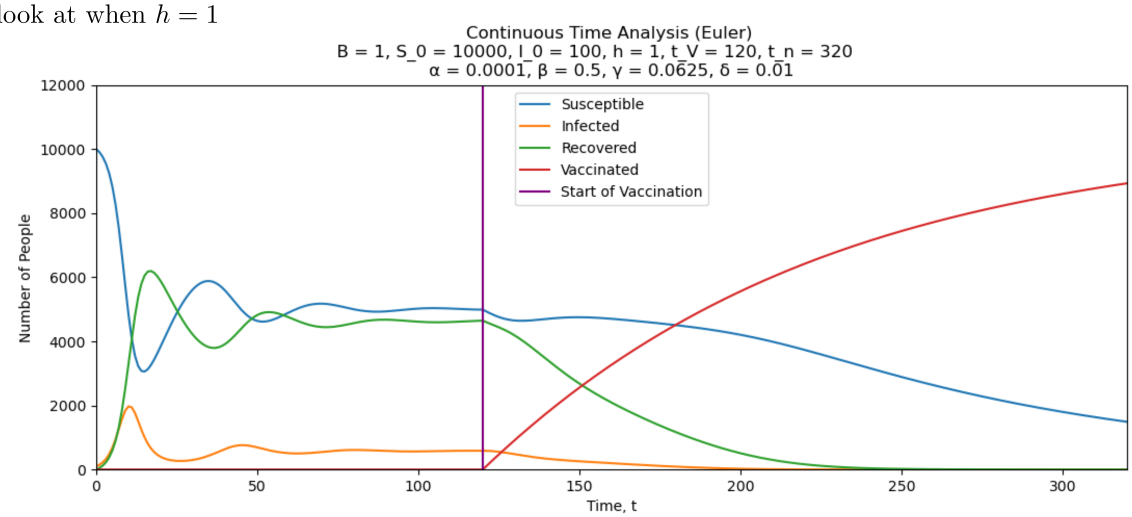
3.2 Continuous Plot

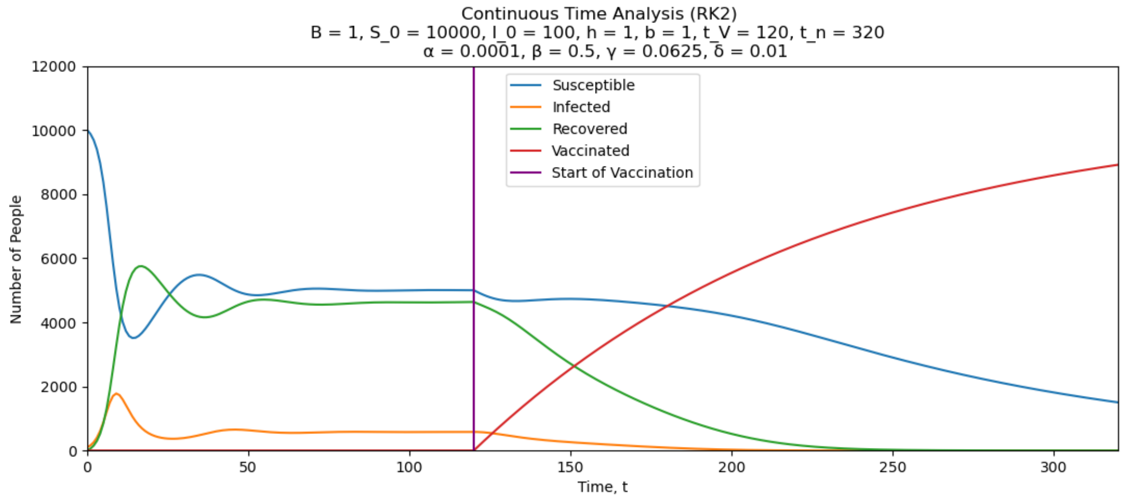
3.2.1 Plot 1





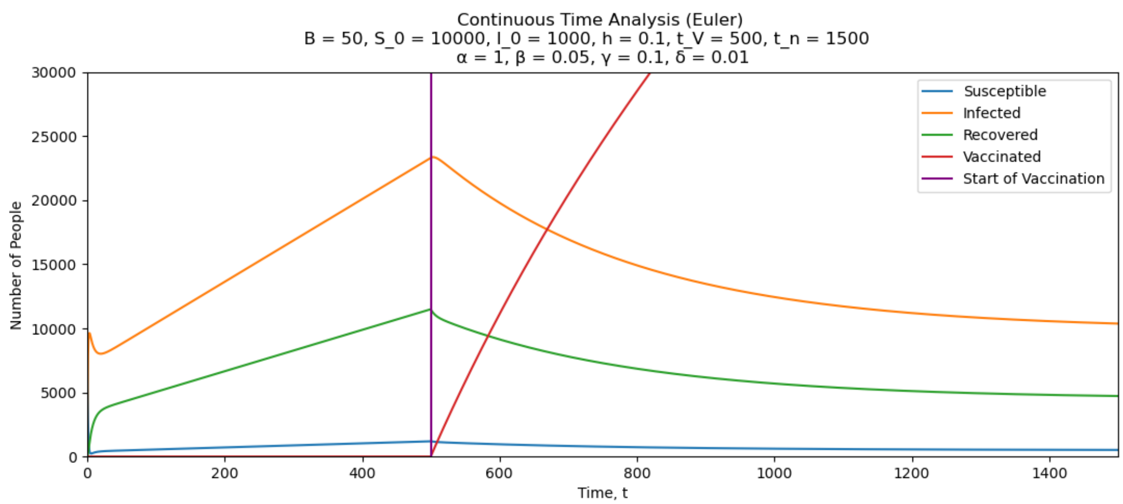
The variables remain the same as per the discrete plots, with the Euler and RK2 version identical. However, comparing them to the discrete version, there is a slight difference around the $t = 50$ part. If you were to look even more closely, you would realise that the amplitude of the wave is smaller than the plot in discrete analysis. This is kind of expected in continuous analysis as the curves does not oscillate greatly due to a sudden jump in numbers and tends to be smoother. This is for when $h = 0.1$.
Let's look at when $h = 1$

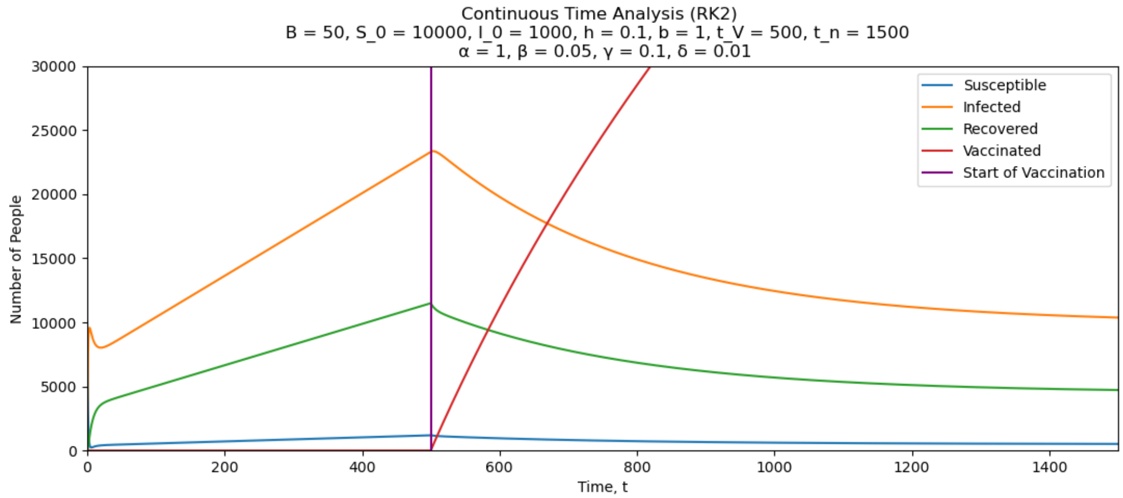




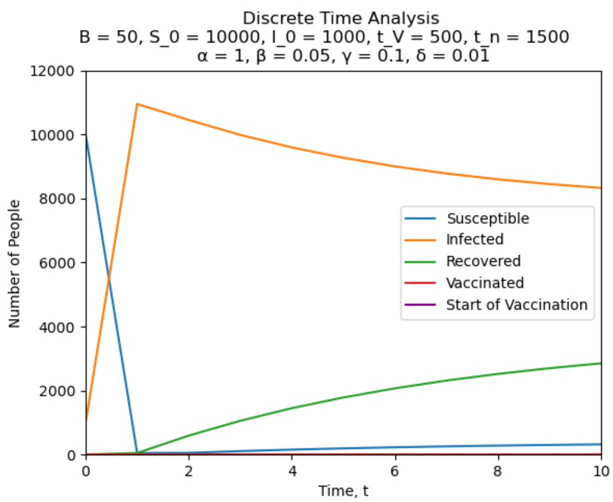
Since h is similar to t in this case, the Euler function is essentially the discrete function, as the step is 1. However, the RK2 function remains the same, due to taking more approximations of the taylor series. What is surprising is that when $h = 0.1$, both Euler and RK2 is the same. This is probably due to a simple graph where further terms in the sequence is small enough.

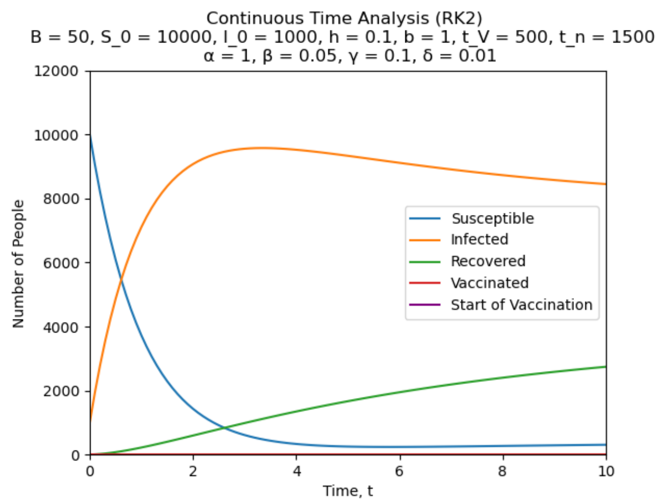
3.3 Plot 2





With $h = 0.1$, the 2 graphs are the same, nothing much to say. However, the first few t seems to be different from the discrete one. Let's zoom in.





Similar to reasons stated before, for the discrete plot, there is a sudden change in susceptible to infected from time $t = 0$ to $t = 1$, hence the sudden increase or decrease of population, resulting in the graph on the left. RK2 smoothens out the curve well to show gradual change over time more accurately. Euler's graph is the same. Also worth nothing that changing the b does not change how the graph look overall for RK2.

4 Steady State Analysis

For both discrete and continuous time, the equations for the steady state are the same. It is when $S_{t+1} - S_t = \frac{dS}{dt} = 0$. As a result, we have the following equations:

$$\begin{aligned}
&B = \alpha SI - \gamma R + \delta S \\
&\alpha SI = \beta I \\
&\beta I = \gamma R + \delta R \\
&\delta S + \delta R = 0
\end{aligned}$$

4.1 Plot 1

Let's do the steady state of pre-vaccination where $\delta = 0$. Inputting the variables we have, we get the following equations:

$$\begin{aligned}
&0.0001SI - 0.0625R = 1 \\
&0.0001SI = 0.5I \\
&0.5I = 0.0625R
\end{aligned}$$

Simplifying the equations, we get $S = 5000,\ 8I = R,\ 8I - R = 1$. We cannot simplify the equation further due to a conflict with the equations. We can see from the plots that R is indeed close to $8I$, and S is converging around 5000. These are the values taken at $t_n = 10000$, which fits the conclusion. $S = 5000.661686343056,\ I = 1679.2845731920756,\ R = 13420.053740464777$
Let's now look how vaccination changes the steady state.

$$\begin{aligned}
&0.0001SI - 0.0625R + 0.01S = 1 \\
&0.0001SI = 0.5I \\
&0.5I = 0.0625R + 0.01R \\
&0.01S + 0.01R = 0
\end{aligned}$$

We might be tempted to get started with the second equation first, but clearly from the plots, I goes to 0, and we cannot divide by 0. The last equation is false as well, as S is 1 at minimum. When we simplify the first three equations, we get $0.01S + 0.01R = 1$. Looking at $t_n = 10000$ again, we get $S = 100.00000000000071,\ I = 5e - 324,\ R = 3.58039e - 318,\ V = 19999.999999999927$. While we know V goes to constantly increasing, I and R goes to 0, it is surprising that S goes to 100. This is means that the last equation is slightly inaccurate, probably due to its assumption of the existence of $\frac{dV}{dt} = 0$. The reason S goes to 100, is probably due to $\delta = 0.01$, which means at 1 person gets vaccinated is being replaced by the 1 person who is born.

The results of plot 1 isn't shocking at all, seeing how everyone in S, I and R will eventually go to V, with only 1 birth per week to contribute to S.

4.2 Plot 2

For pre-vaccination:

$$\begin{aligned}
&SI - 0.1R = 50 \\
&SI = 0.05I \\
&0.05I = 0.1R
\end{aligned}$$

This is kinda tricky, because our number of infections αSI is not really αSI. Remember it is $\min(\alpha SI,\ S)$ as stated in the time analysis. Since $\alpha = 1$, it means that everyone in S gets infected, which makes the equation $S = 0.05I$. Then we have $I = 2R$ from the third equation, as noted in the discrete time analysis. All this checks at $t_n = 10000, S = 16520.291363163356,\ I = 329760.66597294447,\ R = 164719.04266389163$. As there are no bound, all three variables will continue to increase.

Post-vaccination:

$$
\begin{aligned}
&SI - 0.1R + 0.01S = 50 \\
&SI = 0.05I \\
&0.05I = 0.1R + 0.01R \\
&0.01S + 0.01R = 0
\end{aligned}
$$

Similar to pre-vaccination, we have $S = 0.05I$, but I changes slightly to $I = 2.2R$ due to the third equation. As usual, the last equation makes no sense. So, we substitute the other three and we get $0.01R + 0.01S = 50$, $S + R = 1000$. Since $S = 0.05 \times 2.2R = 0.11R$, we get $S = 495.495495...,\ I = 9909.90990...,\ R = 4504.50450...$ which is extremely close to the values at $t_n = 10000$.

Things certainly didn't go well for the non-vaccinated ones, as the situation keeps getting worse, with I more than S and R combined. While I is still high for post-vaccination, what is not noted here is the number of V, which is way higher than the steady states of SIR.

4.3 Plot 3

Pre-vaccination:

$$
\begin{aligned}
&10^{-6}SI - 0R = 50 \\
&10^{-6}SI = 0I \\
&0I = 0R
\end{aligned}
$$

There isn't much to work with, since many terms go to 0. We know that the number of infection eventually goes to S, just like in the previous plot. Therefore the only steady state is I goes to infinity while $S = B = 50$.

Post-vaccination:

$$
\begin{aligned}
&10^{-6}SI - 0R + 0.01S = 50 \\
&10^{-6}SI = 0I \\
&0I = 0R + 0.01R \\
&0.01S + 0.01R = 0
\end{aligned}
$$

Similarly to previous plot, only the first equation is workable, giving us $S + 0.01S = 50$, which leaves us with $S = 49.504950...$, while infected and vaccinated go to infinity. There doesn't seem like any pattern with I and V, as like previously, as the steady state equations do not depend on V.

Obviously, this is quite an extreme case, with pre-vaccination meaning human extinction (I don't think zombies can make babies). As for post-vaccination, as long as there are people vaccinated, and continue to vaccinate their offsprings, things should be still fine.