AR + MA processes

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Overview

1. Moving-average process
2. Autoregressive process
3. Stationarity constraints for AR and MA processes
4. ARMA process
5. Stationarisation procedure for time series
6. ARIMA process

Moving-average process

- A stochastic process (X_t) is called a **moving-average process of order** q, denoted as **MA[q]**, if it is represented as:

$$X_t = \varepsilon_t - \beta_1 \varepsilon_{t-1} - \beta_2 \varepsilon_{t-2} - \cdots - \beta_q \varepsilon_{t-q},$$

  where:
  - ε_t is a **white noise process**, meaning it is a series of random shocks (**uncorrelated**) with:

$$\mathbb{E}[\varepsilon_t] = 0 \quad \text{and} \quad \mathsf{Var}(\varepsilon_t) = \sigma^2.$$

  - The coefficients $\beta_1, \beta_2, \ldots, \beta_q$ determine how past shocks affect the current observation X_t.
- If $q = \infty$, the process is called an **infinite moving-average process**, denoted as **MA[∞]**.
- **Interpretation**:
  - At time t, a random shock ε_t occurs. This shock is **independent** of other shocks at different time points.
  - The observation X_t is a **weighted average** of the current shock ε_t and past shocks $\varepsilon_{t-1}, \varepsilon_{t-2}, \ldots, \varepsilon_{t-q}$.

Properties of MA Processes

The following applies to $MA[q]$ processes:

- **Stationarity**: An $MA[q]$ process is **stationary** because it is a linear combination of white noise processes, which are stationary by definition. This means that its statistical properties (mean, variance, and autocovariance) do not depend on time.
- **Mean of the Process**: The mean of the $MA[q]$ process is given by: $\mu_x = E(X_t) = \mu_\varepsilon \sum_{u=0}^{q}(-\beta_u)$. Here, μ_ε is the mean of the white noise process ε_t. If ε_t has zero mean (a typical assumption), then $\mu_x = 0$.
- **Variance of the Process**: The variance of the $MA[q]$ process is: $\sigma_x^2 = \left[\sum_{u=0}^{q} \beta_u^2\right] \sigma_\varepsilon^2$. This indicates that the variance of X_t depends on the coefficients β_u and the variance of the white noise process σ_ε^2. Each β_u determines how much a particular lagged shock contributes to the overall variance.

Properties of MA Processes

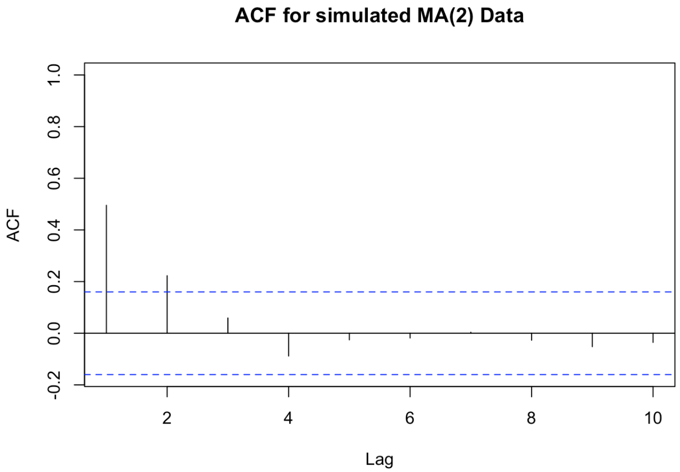
Furthermore, it applies:

- **Autocorrelation Function (ACF)**: The autocorrelation function (ACF) of an $MA[q]$ process vanishes for lags $\tau > q$.
- **Stationarity of $MA[\infty]$ Processes**: $MA[\infty]$ processes are stationary if the sequence of weights β_u satisfies the summability condition:

$$\sum_{u=0}^{\infty} |\beta_u| < \infty$$

This condition ensures that the effect of shocks diminishes as we go further into the past. Intuitively, the absolute contributions of past shocks must decrease to maintain finite variance and stationarity.

Example



Autoregressive Process

- **Definition:** A stochastic process (X_t) is called an **autoregressive (AR) process of order** p, denoted as $AR[p]$, if it can be represented as:

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \cdots + \alpha_p X_{t-p} + \varepsilon_t$$

where:

- $\alpha_1, \alpha_2, \ldots, \alpha_p$ are the coefficients that determine the influence of past values.
- ε_t is a white noise process, characterized by:
  - Mean of zero: $\mathbb{E}[\varepsilon_t] = 0$.
  - No autocorrelation: $\text{Cov}(\varepsilon_t, \varepsilon_{t-k}) = 0$ for $k \neq 0$.
  - Constant variance: $\text{Var}(\varepsilon_t) = \sigma^2$.

Autoregressive Process

- **Infinite Order:** If $p = \infty$, the process is called an **infinite autoregressive process** (**AR[∞]**), which theoretically depends on all past values.

Interpretation

- The process X_t is a **linear combination** of its past values $(X_{t-1}, X_{t-2}, \ldots, X_{t-p})$ and a random shock (ε_t).
- The model can be viewed as a **(multiple) linear regression** where the explanatory variables are the past values of X_t.
- Practical application example:
  - In an AR[2] model for stock prices:

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \varepsilon_t$$

Example of an Autoregressive Process

Multiplier-Accelerator Model by Paul A. Samuelson (1939):

(i) $C_t = \alpha Y_{t-1}$

(ii) $I_t = \beta(C_t - C_{t-1}) + c$

(iii) $Y_t = C_t + I_t$

Where:

- Y_t: Income in period t.
- C_t: Consumption in period t.
- I_t: Investment in period t.

Derivation:

- Substitute (i) and (ii) into (iii) and replace constant c by ε_t (random noise):

$$Y_t = \alpha(1+\beta)Y_{t-1} - \alpha\beta Y_{t-2} + \varepsilon_t$$

- This is an $AR[2]$ process because it depends on Y_{t-1} and Y_{t-2}.

Stationarity Constraints for AR Processes

- An AR[1] process is stationary if:

  $$|\alpha| < 1$$

- An AR[2] process is stationary if:
  - $|\alpha_2| < 1$
  - $\alpha_2 + \alpha_1 < 1$
  - $\alpha_2 - \alpha_1 < 1$
- An AR[p] process is stationary if all roots of its **characteristic polynomial** lie outside the unit circle:

  $$P(z) = 1 - \alpha_1 z - \alpha_2 z^2 - \cdots - \alpha_p z^p$$

- **Note:** A stationary AR[p] process can be represented as a moving average process of infinite order (MA[∞]).

ARMA Processes

- A stochastic process (X_t) is called an **autoregressive moving-average process of order** $[p, q]$, denoted as $ARMA[p, q]$, if it can be represented as:

$$X_t = \alpha_1 X_{t-1} + \cdots + \alpha_p X_{t-p} + \varepsilon_t - \beta_1 \varepsilon_{t-1} - \cdots - \beta_q \varepsilon_{t-q}$$

  where (ε_t) is a white noise process.
- **Components of ARMA[p, q]:**
  - **AR part:** Dependency on past values $X_{t-1}, \ldots, X_{t-p}$.
  - **MA part:** Dependency on past shocks $\varepsilon_{t-1}, \ldots, \varepsilon_{t-q}$.
- $ARMA[p, q]$ processes are autoregressive processes with **correlated residuals**.
- An $ARMA[p, q]$ process is **stationary** if and only if its AR part is stationary.

Testing for Stationarity

- In practice, time series are often **non-stationary**.
- Examples of non-stationary time series include those with:
  - **Trend**: Systematic increases or decreases over time.
  - **Seasonal Variations**: Repeating patterns over specific periods.
- **Statistical Tests** are used to determine whether an observed time series is stationary or non-stationary.
- The **null hypothesis** (H_0) for these tests is:

H_0 : The generating stochastic process is **not** stationary w.r.t. the mean.

- Common statistical tests for stationarity:
  - **Dickey-Fuller Test (DF Test)**: Tests for a unit root.
  - **Augmented Dickey-Fuller Test (ADF Test)**: Accounts for higher-order correlations.
  - **Phillips-Perron Test**: Adjusts for serial correlation and heteroskedasticity.
- A simple way to convert non-stationary time series into stationary ones is through a **filtering process**:
  - Differencing, seasonal differencing, etc..

Difference and Seasonal Filtering

- By **filtering**, non-stationary time series can be converted into stationary ones.
- For time series **with trend component**: Difference filter Δ^D
  - **First-order difference:**

    $$\Delta X_t = X_t - X_{t-1}$$

    Example sequence: $\Delta X_2, \Delta X_3, \Delta X_4, \ldots, \Delta X_T$.
  - **Second-order difference:**

    $$\Delta^2 X_t = \Delta X_t - \Delta X_{t-1}$$

    Example sequence: $\Delta^2 X_2, \Delta^2 X_3, \Delta^2 X_4, \ldots, \Delta^2 X_T$.
- For time series **with seasonal component**: Seasonal filter Δ_S
- For example: Quarterly data with seasonal length $S = 4$:

  $$\Delta_4 X_t = X_t - X_{t-4}$$

  Example sequence: $\Delta_4 X_5, \Delta_4 X_6, \Delta_4 X_7, \ldots, \Delta_4 X_T$.

ARIMA Processes

- A stochastic process that results from d-times integration (differencing) of an $ARMA[p, q]$ process is **called an autoregressive integrated moving-average process** or $ARIMA[p, d, q]$ process.
- **Key Characteristics:**
  - p: Order of the autoregressive (AR) part.
  - d: Number of times the series is differenced to achieve stationarity.
  - q: Order of the moving average (MA) part.
- $ARIMA[p, 1, q]$ models represent $ARMA[p, q]$ models applied to the **first-order differences** of the time series:

$$\Delta X_t = X_t - X_{t-1}$$

- **Steps in ARIMA Modeling:**
  - **Step 1:** Differencing (d) to remove trends or seasonality and achieve stationarity.
  - **Step 2:** Fit an ARMA$[p, q]$ model to the differenced series.
  - **Step 3:** Integrate to obtain forecasts for the original series.

Summary of the Results

- Idea: A (stationary) stochastic process can be represented as:
  1. A **weighted sum of lagged values** (**AR process**),
  2. A **weighted sum of present or past shocks** (**MA process**), or
  3. A **combination of both processes** (**ARMA process**).
- Precondition: The time series must be **stationary**!
- Stationarity can often be achieved using proper **difference** or **seasonal filter methods**:

  ARIMA Process.

- Advantage: ARIMA models are a **very useful category** of forecasting models:

  Box-Jenkins approach!