Box-Jenkins Approach

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Overview

1. Determination of model order
2. Estimation of parameters
3. Model fit
4. Model diagnosis

Determination of the model order (p, q)

- The model order (p, q) of an ARMA model is determined by graphical inspection of the autocorrelation function (ACF) and the partial autocorrelation function (PACF).
- The **autocorrelation coefficient** is computed by:

$$ACF_k = \frac{\sum_{t=1}^{T-k}(X_t - \bar{X})(X_{t-k} - \bar{X})}{\sum_{t=1}^{T}(X_t - \bar{X})^2}$$

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Determination of the model order (p, q)

- The **partial autocorrelation coefficient** measures the correlation between X_t and X_{t-k}, after eliminating the influence of the values between X_t and X_{t-k}.
- The partial autocorrelation function PACF shows correlations between X_t and X_{t-k}, *controlled* for $X_{t-1}, X_{t-2}, ..., X_{t-k+1}$.
- **Computation:** If time series X_t is predicted by the regression function of k previous time series, the partial autocorrelation is given by the regression coefficient b_k of the k-th (the last) predicting time series, thus

$$X_t = b_0 + b_1 X_{t-1} + b_2 X_{t-2} + b_k X_{t-k}$$

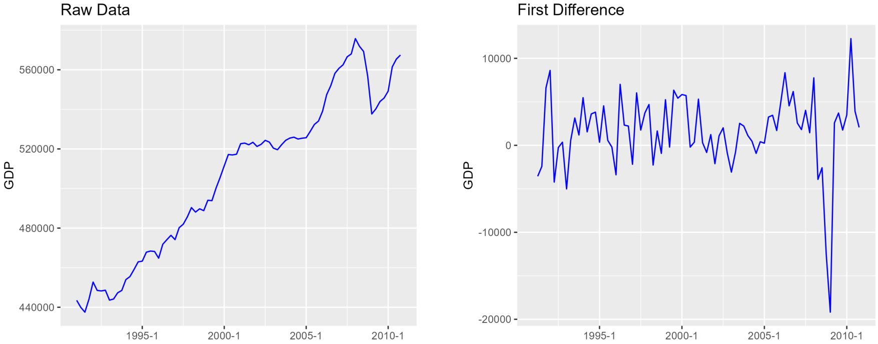
- Finally $PACF_k = b_k$ is the **partial autocorrelation coefficient** of lag k.

Determination of the model order (p, q)

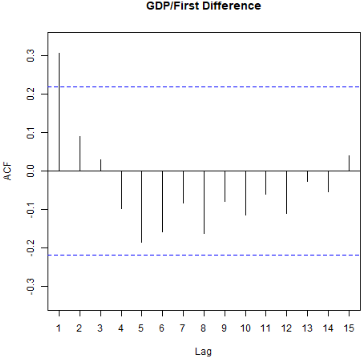
- The model order (p, q) of an ARMA process is identified by inspection of ACF and PACF.

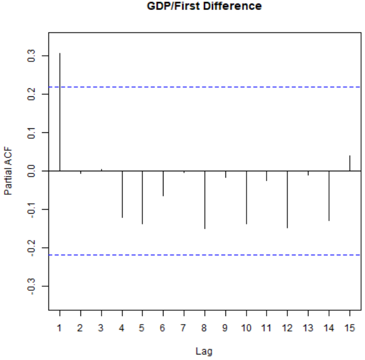
	ACF	PACF
$AR(p)$	• infinite • Exponential decay or damped or sine wave	• infinite • $\alpha_\tau^{(\tau)} = 0$ for $\tau > p$ Significant spikes at p lag but none beyond.
$MA(q)$	• infinite • $\rho_\tau = 0$ for $\tau > q$ Significant spikes at q lag but none beyond.	• infinite • Exponential decay or damped or Sine waves
$ARMA(p,q)$	• When $AR(p)$ $\tau > q$ Exponential decay	• When $MA(q)$ $\tau > p$ Exponential decay

GDP and First Difference



ACF and PACF Plot of GDP first difference





Determination of the model order (p, q)

- In practical application the determination of p and q by ACF and PACF often can be difficult.
- In many cases subjective decisions must be made.
- Alternative approach:
  - Choose upper limits p_{MAX} and q_{MAX} for p and q
  - Estimate every possible model with $0 \leq p \leq p_{MAX}$ and $0 \leq q \leq q_{MAX}$.
  - Choose the best model by using a proper measure for the goodness of fit!

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Parameter estimation

- ARMA(p, q) model:

$$X_t = c_0 + \phi_1 X_{t-1} + ... + \phi_p X_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + ... + \theta_q \varepsilon_{t-q}$$

- The estimation of the parameters of an ARMA(p, q) model is carried out by **iterative estimation procedures**. The following procedures are commonly used:
  - (Non linear) least squares method
  - Maximum likelihood estimation

- Iterative estimation procedures require the specification of (random) initial values for parameters c_0, ϕ_k and θ_l.

  Subsequently: Stepwise modification of the initial values until a predefined criterion (e.g. minimizing the error sum of squares of the model) is met.

The model fit

- In order to assess the model fit the residual variance $\hat{\sigma}^2(k)$ can be used:

$$\hat{\sigma}^2_{p,q} = \frac{1}{T} \sum_{t=1}^{T} \hat{\varepsilon}_t^2$$

  with T number of observations, p and q are parameters of the ARMA(p, q) model and $\hat{\varepsilon}_t$ is estimating the residual at time point t.

- Problem: the residual variance (the *error of the model*) is inversely proportional to the model size, i.e. models with a large number of parameters are favored (similar to the determination coefficient R^2 of the linear regression).

- In many cases the forecasting performance of such models is poor: they are **overfitted!**

Information criteria

An information criterion measures the model fit, considering a penalty term for the number of parameters p and q. Two well-known information criteria are:

1. AIC criterion (Akaike's information criterion)

$$AIC(p,q) = ln\ \hat{\sigma}^2_{p,q} + 2 \cdot \frac{(p+q)}{T}$$

2. BIC criterion (Bayesian information criterion)

$$BIC(p,q) = ln\ \hat{\sigma}^2_{p,q} + \frac{(p+q)ln\ T}{T}$$

Always choose the ARMA(p, q) model for which the criterion is **minimal**.

Model diagnosis

- Question: Is the selected model adequate?
- In an ideal case the ARMA(p, q) model reproduces the relevant characteristics of time series X_t. This means: the unexplained remainder (i.e. the residuals $\hat{\varepsilon}_t$) does not contain any information, structures or systematic $\rightarrow$ white noise!
- There are two different procedures for model diagnosis:
  1. Graphical:
     - plot of the residuals over time t: are there any systematic figures detectable?
     - ACF and PACF plot of the residuals: are there any **significant** correlations?
  2. Statistical: Box-Pierce test, Ljung-Box test

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Box-Pierce test

- Box-Pierce test is testing the hypothesis, that the first K auto-correlations of the residuals are equal to zero:

$$H_0 : \rho_{\hat{\varepsilon}}(1) = \rho_{\hat{\varepsilon}}(2) = ... = \rho_{\hat{\varepsilon}}(K) = 0$$

- The test statistic reads:

$$Q_{BP} = T \sum_{k=1}^{K} \rho_{\hat{\varepsilon}}^2(k)$$

Q_{BP} is approximately χ^2 distributed with $K - p - q$ degrees of freedom.

Ljung-Box test

- Ljung-Box test is also testing the hypothesis, that the first K autocorrelations of the residuals are equal to zero:

$$H_0 : \rho_{\hat{\varepsilon}}(1) = \rho_{\hat{\varepsilon}}(2) = ... = \rho_{\hat{\varepsilon}}(K) = 0$$

- The test statistic reads:

$$Q_{LB} = T(T+2)\sum_{k=1}^{K} \frac{\rho_{\hat{\varepsilon}}^2(k)}{T-k}$$

  Q_{LB} is approximately χ^2 distributed with $K-p-q$ degrees of freedom.

- For small sample sizes or more robust results, the Ljung-Box test is generally preferred.

GDP: ARIMA (0,1,0)

```
Series: df1$gdp
ARIMA(0,1,0) with drift

Coefficients:
          drift
      1568.9019
s.e.   487.6669

sigma^2 = 19031079:  log likelihood = -773.67
AIC=1551.34   AICc=1551.5   BIC=1556.08

Training set error measures:
                   ME     RMSE      MAE          MPE      MAPE      MASE      ACF1
Training set 5.524162 4307.587 2964.663 -0.002124409 0.5852234 0.8769136 0.3049348
```

GDP: ARIMA (1,1,0)

```
Series: df1$gdp
ARIMA(1,1,0) with drift

Coefficients:
         ar1      drift
      0.3082  1543.0566
s.e.  0.1071   666.7094

sigma^2 = 17434466:  log likelihood = -769.75
AIC=1545.5   AICc=1545.82   BIC=1552.61

Training set error measures:
                   ME     RMSE      MAE         MPE      MAPE      MASE         ACF1
Training set 28.28396 4096.422 2933.614 0.004053159 0.5811208 0.8677297 -0.002197978
```

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GDP: ARIMA (0,1,1)

```
Series: df1$gdp
ARIMA(0,1,1) with drift

Coefficients:
         ma1      drift
      0.2916  1557.5321
s.e.  0.1025   599.4028

sigma^2 = 17557870:  log likelihood = -770.02
AIC=1546.04   AICc=1546.36   BIC=1553.15

Training set error measures:
                   ME     RMSE     MAE         MPE      MAPE      MASE      ACF1
Training set 17.60785 4110.894 2924.23 0.001432736 0.5795691 0.8649541 0.0185144
```

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GDP: ARIMA (1,1,1)

```
Series: df1$gdp
ARIMA(1,1,1) with drift

Coefficients:
         ar1     ma1      drift
      0.3021  0.0061  1543.9456
s.e.  0.3498  0.3663   665.0868

sigma^2 = 17663908:  log likelihood = -769.75
AIC=1547.5   AICc=1548.04   BIC=1556.98

Training set error measures:
                   ME     RMSE      MAE         MPE      MAPE      MASE        ACF1
Training set 27.60567 4096.427 2932.842 0.003907974 0.5809815 0.8675012 -0.0021814
```

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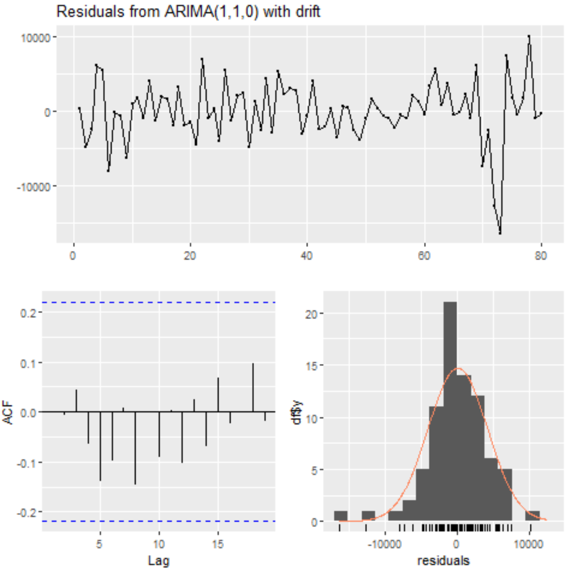
Model diagnosis: ARIMA (1,1,0)



Model diagnosis: ARIMA (1,1,0)

```
        Ljung-Box test

data:  Residuals from ARIMA(1,1,0) with drift
Q* = 5.8105, df = 9, p-value = 0.7587

Model df: 1.   Total lags used: 10
```