Area maximisation

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December 13, 2011

Abstract

We extremise the area under the synaptic memory curve

It will be convenient for us to define the diagonal elements of $\mathbf{T}$ to be zero (it is conventional to define them to be the recurrence times, $\mathbf{T}_{ii} = 1/\mathbf{p}^\infty_i$, but that would lead to a lot of extra δ_{ik}'s for us). Similarly, we define $N^k_{ij} = 0$ if $i = k$ or $j = k$.

1 Mixing times

sec:mixing

Define

$$\eta^\pm_i \equiv \sum_k \mathbf{T}_{ik} \mathbf{p}^\infty_k \left(\frac{1 \pm \mathbf{w}_k}{2} \right). \tag{1}$$ eq:mixingpm

It can be shown that [1][1]

$$\eta^+_i + \eta^-_i = \eta \qquad \text{(independent of } i\text{)}. \tag{2}$$ eq:mixing

It would be nice if we could prove (at least for the maximal area) that

$$\mathbf{w}_i = +1 \quad \text{and} \quad \mathbf{w}_j = -1 \qquad \Longrightarrow \qquad \eta^+_i < \eta^+_j \quad \Longleftrightarrow \quad \eta^-_i > \eta^-_j \tag{3}$$ eq:orderingh

then we could use these quantities to order the states.

The area under the memory curve is given by

$$\begin{aligned} A = \sqrt{N}(4f^+f^-) \sum_{i \neq j} q_{ij} \mathbf{p}^\infty_i (\eta^+_i - \eta^+_j) &= -\sqrt{N}(4f^+f^-) \sum_{i,j} q_{ij} \mathbf{p}^\infty_i \eta^+_j \\ = \sqrt{N}(4f^+f^-) \sum_{i \neq j} q_{ij} \mathbf{p}^\infty_i (\eta^-_j - \eta^-_i) &= \sqrt{N}(4f^+f^-) \sum_{i,j} q_{ij} \mathbf{p}^\infty_i \eta^-_j. \end{aligned} \tag{4}$$ eq:areamixin

[1]Not that their η is equal to our $\eta + 1$ due to the difference in diagonal elements of $\mathbf{T}$.

2 Shifting q_{ij}

sec:shiftq

Consider the following change

$$\mathbf{M}^+_{ij} \to \mathbf{M}^+_{ij} + f^- \epsilon_{ij}, \qquad \mathbf{M}^-_{ij} \to \mathbf{M}^-_{ij} - f^+ \epsilon_{ij}, \qquad \sum_j \epsilon_{ij} = 0. \tag{5}$$

eq:shiftq

This leaves $\mathbf{M}$ unchanged, and therefore $\mathbf{p}^\infty$, $\mathbf{T}$, η, N^j_{ik} and H^j_{ik} as well. This means

$$A \to A + \sqrt{N}(4f^+f^-) \sum_{i \neq j} \epsilon_{ij} \mathbf{p}^\infty_i (\eta^+_i - \eta^+_j). \tag{6}$$

eq:shiftqare

Suppose $\eta^+_i > \eta^+_j$. We can increase A by making $\epsilon_{ij} > 0$. The only thing that could stop us is if $\mathbf{M}^-_{ij}$ or $\mathbf{M}^+_{ii}$ hits zero (which also takes care of the possibility that $\mathbf{M}^+_{ij}$ hits unity). Similar considerations for $\eta^+_i < \eta^+_j$ imply that at the maximum:

$$\begin{aligned} \eta^+_i > \eta^+_j \qquad &\Longrightarrow \qquad \mathbf{M}^-_{ij} = 0 \quad \text{or} \quad \mathbf{M}^+_{ii} = 0, \\ \eta^+_i < \eta^+_j \qquad &\Longrightarrow \qquad \mathbf{M}^+_{ij} = 0 \quad \text{or} \quad \mathbf{M}^-_{ii} = 0. \end{aligned} \tag{7}$$

eq:shiftqmax

If $\eta^+_i = \eta^+_j$, ϵ_{ij} is a null direction, so we can impose either of the two conditions if we wish. If it weren't for the second possibility in each case, this would imply an upper/lower triangular structure for $\mathbf{M}^\pm$.

If we have a trio $\eta^+_i > \eta^+_j > \eta^+_k$, two of the transitions can help each other: decreasing $\mathbf{M}^-_{jk}$ increases $\mathbf{M}^-_{jj}$, which allows us to further decrease $\mathbf{M}^+_{ji}$. Decreasing $\mathbf{M}^+_{ji}$ increases $\mathbf{M}^+_{jj}$, which allows us to further decrease $\mathbf{M}^-_{jk}$. Successful triangulation requires:

$$\begin{aligned} \sum_{\{i|\eta^+_i > \eta^+_j\}} \mathbf{M}^+_{ji} \leq \mathbf{M}^-_{jj} + \sum_{\{i|\eta^+_i < \eta^+_j\}} \mathbf{M}^-_{ji} \qquad &\Longrightarrow \qquad \sum_{\{i|\eta^+_i > \eta^+_j\}} \left(\mathbf{M}^+_{ji} + \mathbf{M}^-_{ji}\right) \leq 1, \\ \sum_{\{i|\eta^+_i < \eta^+_j\}} \mathbf{M}^-_{ji} \leq \mathbf{M}^+_{jj} + \sum_{\{i|\eta^+_i > \eta^+_j\}} \mathbf{M}^+_{ji} \qquad &\Longrightarrow \qquad \sum_{\{i|\eta^+_i < \eta^+_j\}} \left(\mathbf{M}^+_{ji} + \mathbf{M}^-_{ji}\right) \leq 1. \end{aligned} \tag{8}$$

eq:triangles

The two left-hand-sides of the final inequalities sum to $(2 - \mathbf{M}^+_{jj} - \mathbf{M}^-_{jj})$, so they are not quite inconsistent.

3 Generalised fundamental matrix

undamental

Define the generalised fundamental matrix as

$$\mathbf{Z} = (\mathbf{I} - \mathbf{M} + \mathbf{c}\boldsymbol{\pi})^{-1}, \tag{9}$$

eq:funddef

where $\mathbf{c}$ is a row vector of 1's and $\boldsymbol{\pi}$ is any row vector such that $\boldsymbol{\pi}\mathbf{c} = 1$. The usual fundamental matrix has $\boldsymbol{\pi} = \mathbf{p}^\infty$ [2, §3.2], but it is more convenient to let it be independent of $\mathbf{M}$ [2, App.VIII], e.g. $\boldsymbol{\pi} = \mathbf{c}^{\mathrm{T}}/n$.

Then we have

$$
\begin{aligned}
\mathbf{p}^{\infty} &= \boldsymbol{\pi}\mathbf{Z}, \\
\mathbf{Z}\mathbf{c} &= \mathbf{c}, \\
\mathbf{T}_{ij} &= \frac{\mathbf{Z}_{jj} - \mathbf{Z}_{ij}}{\mathbf{p}^{\infty}_{j}}, \\
\eta_i^+ - \eta_j^+ &= \frac{1}{2}\sum_k (\mathbf{Z}_{jk} - \mathbf{Z}_{ik})\mathbf{w}_k.
\end{aligned} \tag{10}
$$

eq:fromfund

This allows us to write the area as

$$
A = \sqrt{N}(2f^+f^-)\sum_{ijkl} q_{ij}\boldsymbol{\pi}_l \mathbf{Z}_{li}(\mathbf{Z}_{jk} - \mathbf{Z}_{ik})\mathbf{w}_k. \tag{11}
$$

eq:arefund

4 Derivatives wrt. $\mathbf{M}^{\pm}_{ij}$

sec:derivs

We will regard the off-diagonal elements of $\mathbf{M}^{\pm}_{ij}$ to be the independent variables, with $\mathbf{M}^{\pm}_{ii} = 1 - \sum_{j\neq i} M^{\pm}_{ij}$. Thus,

$$
\frac{\partial \mathbf{M}_{ij}}{\partial \mathbf{M}^{\pm}_{gh}} = f^{\pm}\delta_{gi}(\delta_{hj} - \delta_{ij}), \qquad \frac{\partial q_{ij}}{\partial \mathbf{M}^{\pm}_{gh}} = \pm\delta_{gi}(\delta_{hj} - \delta_{ij}). \tag{12}
$$

eq:basicderi

The implicit $g \neq h$ that comes with all derivatives is unnecessary, as the derivatives above vanish when $g = h$.

Differentiating (9),

$$
\frac{\partial \mathbf{Z}_{ij}}{\partial \mathbf{M}_{gh}} = \mathbf{Z}_{ig}(\mathbf{Z}_{hj} - \mathbf{Z}_{gj}). \tag{13}
$$

eq:fundderiv

From [3], we have

$$
\frac{\partial \mathbf{p}^{\infty}_{k}}{\partial \mathbf{M}_{gh}} = \mathbf{p}^{\infty}_{k}\mathbf{p}^{\infty}_{g}(\mathbf{T}_{gk} - \mathbf{T}_{hk}). \tag{14}
$$

eq:pderiv

We can write [4]

$$
\mathbf{M}^{(k)}_{ij} = (1-\delta_{ik})(1-\delta_{jk})\mathbf{M}_{ij}, \qquad N^k_{ij} = (1-\delta_{jk})(I - \mathbf{M}^{(k)})^{-1}_{ij}, \qquad \mathbf{T}_{ik} = \sum_j N^k_{ij}, \tag{15}
$$

eq:NTexpr

with $i, j \neq k$. Differentiating,

$$
\begin{aligned}
\frac{\partial N^k_{ij}}{\partial \mathbf{M}_{gh}} &= -N^k_{ig}[N^k_{gj} - N^k_{hj}], \\
\frac{\partial \mathbf{T}_{ik}}{\partial \mathbf{M}_{gh}} &= -N^k_{ig}[\mathbf{T}_{gk} - \mathbf{T}_{hk}],
\end{aligned} \tag{16}
$$

eq:NTderiv

again with $i, j \neq k$.

5 Kuhn-Tucker conditions

Consider the Lagrangian

$$\mathcal{L} = \frac{A}{\sqrt{N}(2f^+f^-)} + \sum_{\pm}\sum_{ij} \mu_{ij}\mathbf{M}^{\pm}_{ij}. \tag{17}$$

`eq:lagrangia`

Necessary conditions for an extremum are

$$\frac{\partial \mathcal{L}}{\partial \mathbf{M}^{\pm}_{gh}} = 0, \qquad \mu^{\pm}_{ij} \geq 0, \quad \mathbf{M}^{\pm}_{ij} \geq 0, \quad \mu^{\pm}_{ij}\mathbf{M}^{\pm}_{ij} = 0. \tag{18}$$

`eq:extremum`

with $g \neq h$. We will make statements of the following form repeatedly:

$$A < B \qquad \Longrightarrow \qquad A < 0 \quad \text{or} \quad B > 0. \tag{19}$$

`eq:ineqstate`

Let $\lambda^{\pm}_{ij} = \mu^{\pm}_{ij} - \mu^{\pm}_{ii}$. Together with the bounds (18), applying (19) leads to

$$\begin{aligned} \lambda^{\pm}_{ij} > 0 \qquad &\Longrightarrow \qquad \mu^{\pm}_{ij} > 0 \qquad \Longrightarrow \qquad \mathbf{M}^{\pm}_{ij} = 0, \\ \lambda^{\pm}_{ij} < 0 \qquad &\Longrightarrow \qquad \mu^{\pm}_{ii} > 0 \qquad \Longrightarrow \qquad \mathbf{M}^{\pm}_{ii} = 0. \end{aligned} \tag{20}$$

`eq:lambdaine`

Suppose $\mathbf{M}^{+}_{ii} = 0$, then for some j, $\mathbf{M}^{+}_{ij} \neq 0$ and that $\lambda^{+}_{ij} < 0$. We can see that

$$\mathbf{M}^{\pm}_{ii} = 0 \quad \Longleftrightarrow \quad \min_j \lambda^{\pm}_{ij} < 0. \tag{21}$$

`eq:lambdadia`

Using the formulae of §4,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \mathbf{M}^{\pm}_{gh}} =& \lambda^{\pm}_{gh} \pm \sum_k \mathbf{p}^{\infty}_g \mathbf{p}^{\infty}_k \mathbf{w}_k (\mathbf{T}_{gk} - \mathbf{T}_{hk}) \\ &+ f^{\pm} \sum_k \sum_{i \neq j} q_{ij} \mathbf{p}^{\infty}_i \mathbf{p}^{\infty}_k \mathbf{w}_k \mathbf{p}^{\infty}_g (\mathbf{T}_{ik} - \mathbf{T}_{jk})(\mathbf{T}_{gi} - \mathbf{T}_{hi} + \mathbf{T}_{gk} - \mathbf{T}_{hk}) \\ &- f^{\pm} \sum_k \sum_{i \neq j} q_{ij} \mathbf{p}^{\infty}_i \mathbf{p}^{\infty}_k \mathbf{w}_k (N^k_{ig} - N^k_{jg})(\mathbf{T}_{gk} - \mathbf{T}_{hk}) \\ =& \lambda^{\pm}_{gh} \pm \sum_{kl} \mathbf{w}_k \boldsymbol{\pi}_l \mathbf{Z}_{lg} (\mathbf{Z}_{hk} - \mathbf{Z}_{gk}) \\ &+ f^{\pm} \sum_{ijkl} q_{ij} \mathbf{w}_k \boldsymbol{\pi}_l \left[\mathbf{Z}_{lg}(\mathbf{Z}_{hi} - \mathbf{Z}_{gi})(\mathbf{Z}_{jk} - \mathbf{Z}_{ik}) + \mathbf{Z}_{li}(\mathbf{Z}_{jg} - \mathbf{Z}_{ig})(\mathbf{Z}_{hk} - \mathbf{Z}_{gk}) \right] \\ =& \lambda^{\pm}_{gh} \pm 2\mathbf{p}^{\infty}_g (\eta^{+}_g - \eta^{+}_h) \\ &+ 2f^{\pm} \sum_{ij} q_{ij} \mathbf{p}^{\infty}_i \mathbf{p}^{\infty}_g \left[(\mathbf{T}_{gi} - \mathbf{T}_{hi})(\eta^{+}_i - \eta^{+}_j) + (\mathbf{T}_{ig} - \mathbf{T}_{jg})(\eta^{+}_g - \eta^{+}_h) \right] \\ =& 0. \end{aligned} \tag{22}$$

`eq:lagrangde`

Consider

$$f^{-}\frac{\partial \mathcal{L}}{\partial \mathbf{M}^{+}_{gh}} - f^{+}\frac{\partial \mathcal{L}}{\partial \mathbf{M}^{-}_{gh}} = (f^{-}\lambda^{+}_{gh} - f^{+}\lambda^{-}_{gh}) + 2\mathbf{p}^{\infty}_{g}(\eta^{+}_{g} - \eta^{+}_{h}) = 0, \tag{23}$$ eq:derivsshi

which corresponds to the shift considered in §2. We can see that

$$\begin{aligned}
\eta^{+}_{g} > \eta^{+}_{h} \quad &\implies \quad f^{+}\lambda^{-}_{gh} > f^{-}\lambda^{+}_{gh} &&\implies \quad \lambda^{-}_{gh} > 0 \quad \text{or} \quad \lambda^{+}_{gh} < 0 \\
& &&\implies \quad \mathbf{M}^{-}_{gh} = 0 \quad \text{or} \quad \mathbf{M}^{+}_{gg} = 0, \\
\eta^{+}_{g} < \eta^{+}_{h} \quad &\implies \quad f^{-}\lambda^{+}_{gh} > f^{+}\lambda^{-}_{gh} &&\implies \quad \lambda^{+}_{gh} > 0 \quad \text{or} \quad \lambda^{-}_{gh} < 0 \\
& &&\implies \quad \mathbf{M}^{+}_{gh} = 0 \quad \text{or} \quad \mathbf{M}^{-}_{gg} = 0,
\end{aligned} \tag{24}$$ eq:etalambda

which we've already seen in §2.

Consider

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \mathbf{M}^{+}_{gh}} + \frac{\partial \mathcal{L}}{\partial \mathbf{M}^{-}_{gh}} &= (\lambda^{+}_{gh} + \lambda^{-}_{gh}) + 2\sum_{ij} q_{ij}\mathbf{p}^{\infty}_{i}\mathbf{p}^{\infty}_{g}\left[(\mathbf{T}_{gi} - \mathbf{T}_{hi})(\eta^{+}_{i} - \eta^{+}_{j}) + (\mathbf{T}_{ig} - \mathbf{T}_{jg})(\eta^{+}_{g} - \eta^{+}_{h})\right] \\
&= 0.
\end{aligned} \tag{25}$$ eq:derivsove

If we could argue that $\lambda^{+}_{gh} + \lambda^{-}_{gh} > 0$, this would imply that $\mathbf{M}^{+}_{gh}\mathbf{M}^{-}_{gh} = 0$.

References

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