A memory frontier for complex synapses

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Abstract

Blah blah blah.

1 Introduction

sec:intro

2 Synaptic models and their memory curves

sec:setup

In this section, we will describe the class of models of synaptic plasticity that we are studying and how we quantify their performance in memory storage. In the subsequent sections, we will find upper bounds on this performance.

We use a well established formalism for the study of learning and memory with complex synapses (see [1–3]). In this approach, potentiating and depressing plasticity events occur at random times, with all information about the neural activity and learning rules responsible for them absorbed into their rates. We assume that there are no spatial or temporal correlations in the pattern of potentiating and depressing events, and these events cause Markovian transitions between the internal states of the synapses. As a result of these assumptions, the states of different synapses will be independent, and the system can be fully described by the probability distribution across these states, which we will indicate with the row-vector $\mathbf{p}(t)$.

We also employ an "ideal observer" approach to the memory readout, where the synaptic weights are read directly. This provides an upper bound to the quality of any readout using neural activity. These synaptic weights will be restricted to two values, which we can shift and scale to ± 1.

For any single memory, there will be an ideal pattern of synaptic weights, the N-element vector $\vec{w}_{\text{ideal}}$, that is $+1$ at all synapses that were potentiated and -1 at all synapses that were depressed. The actual pattern of synaptic weights at some later time, t, will be $\vec{w}(t)$. We can use the overlap between these, $\vec{w}_{\text{ideal}} \cdot \vec{w}(t)$, as a measure of the quality of the memory. As $t \to \infty$, the system will return to its steady state distribution which will be uncorrelated with the memory in question. The probability distribution of the quantity $\vec{w}_{\text{ideal}} \cdot \vec{w}(\infty)$ can be used as a "null model" for comparison.

The extent to which the memory has been stored is described by a signal-to-noise ratio (SNR):

$$\mathrm{SNR}(t) = \frac{\langle \vec{w}_{\text{ideal}} \cdot \vec{w}(t) - \vec{w}_{\text{ideal}} \cdot \vec{w}(\infty) \rangle}{\sqrt{\mathrm{Var}(\vec{w}_{\text{ideal}} \cdot \vec{w}(\infty))}}. \tag{1}$$

eq:SNRdef

The noise is essentially $\sqrt{N}$. There is a correction when potentiation and depression are imbalanced, but this will not affect the upper bounds that we will discuss below and will be ignored in the subsequent formulae.

All of the preceding plasticity events will put the system in its steady-state distribution, $\mathbf{p}^\infty$. The memory we are tracking at $t = 0$ will change this to $\mathbf{p}^\infty \mathbf{M}^{\text{pot}}$ in those synapses that are potentiated and $\mathbf{p}^\infty \mathbf{M}^{\text{dep}}$ in those synapses that are depressed, where $\mathbf{M}^{\text{pot/dep}}$ are $M \times M$ matrices of transition probabilities for the Markov processes describing potentiation and depression. As the potentiating/depressing nature of the subsequent memories is independent of $\vec{w}_{\text{ideal}}$, we can average over all sequences, resulting in the evolution of the probability distribution:

$$\frac{\mathrm{d}\mathbf{p}(t)}{\mathrm{d}t} = r\mathbf{p}(t)\mathbf{W}^{\mathrm{F}}, \qquad \text{where} \quad \mathbf{W}^{\mathrm{F}} = f^{\text{pot}}\mathbf{M}^{\text{pot}} + f^{\text{dep}}\mathbf{M}^{\text{dep}} - \mathbf{I}. \tag{2}$$

eq:evol

Here, r is the rate of the Poisson process describing the timing of plasticity events, $f^{\text{pot/dep}}$ is the fraction of events that are potentiating/depressing and $\mathbf{I}$ is the identity matrix.

This results in the following SNR

$$\mathrm{SNR}(t) = \sqrt{N}\left(2f^{\text{pot}}f^{\text{dep}}\right)\mathbf{p}^{\infty}\left(\mathbf{M}^{\text{pot}} - \mathbf{M}^{\text{dep}}\right)\mathrm{e}^{rt\mathbf{W}^{\mathrm{F}}}\mathbf{w}, \tag{3}$$

eq:SNRcalc

where the weight of the synapse when it is in its i'th state is given by the corresponding element of the column vector $\mathbf{w}$. We will frequently refer to this function as the memory curve.

The parameters must satisfy the following constraints:

$$\begin{aligned}
\mathbf{M}^{\text{pot/dep}}_{ij} &\in [0,1], & f^{\text{pot/dep}} &\in [0,1], & \mathbf{p}^{\infty}\mathbf{W}^{\mathrm{F}} &= 0, & \mathbf{w}_i &= \pm 1, \\
\sum_j \mathbf{M}^{\text{pot/dep}}_{ij} &= 1, & f^{\text{pot}} + f^{\text{dep}} &= 1, & \sum_i \mathbf{p}^{\infty}_i &= 1.
\end{aligned} \tag{4}$$

eq:constr

The upper bounds on $\mathbf{M}^{\text{pot/dep}}_{ij}$ and $f^{\text{pot/dep}}$ follow automatically from the other constraints.

The question is: what do these constraints imply for the memory curve above? In practice, to make any statements about finite times, we need to got to the eigenmode description:

$$\mathbf{W}^{\mathrm{F}} = \sum_a -q_a\mathbf{u}^a\mathbf{v}^a, \quad \mathbf{v}^a\mathbf{u}^b = \delta_{ab}, \quad \mathbf{W}^{\mathrm{F}}\mathbf{u}^a = -q_a\mathbf{u}^a, \quad \mathbf{v}^a\mathbf{W}^{\mathrm{F}} = -q_a\mathbf{v}^a. \tag{5}$$

eq:eigendecomp

Where q_a are the negative of the eigenvalues, $\mathbf{u}^a$ are the right (column) eigenvectors and $\mathbf{v}^a$ are the left (row) eigenvectors. This allow us to write

$$\begin{aligned}
\mathrm{SNR}(t) = \sqrt{N}\sum_a \mathcal{I}_a\mathrm{e}^{-rt/\tau_a}, \qquad \text{where} \quad \mathcal{I}_a &= \left(2f^{\text{pot}}f^{\text{dep}}\right)\mathbf{p}^{\infty}(\mathbf{M}^{\text{pot}} - \mathbf{M}^{\text{dep}})\mathbf{u}^a\mathbf{v}^a\mathbf{w}, \\
\text{and} \quad \tau_a &= \frac{1}{q_a}.
\end{aligned} \tag{6}$$

eq:SNReigen

We can ask then ask the question: what are the constraints on these quantities implied by the constraints (4)? We will find some of these constraints in the next section.

3 Upper bounds

sec:bounds

In this section, we will find some upper bounds on certain properties of the memory curve discussed in the previous section, namely its initial value and the area under it. We will use these bounds to determine an upper bound on the SNR at finite times in the next section.

3.1 Initial SNR

sec:initial

Now we will discuss the SNR at $t = 0$:

$$\mathrm{SNR}(0) = \sqrt{N}\left(2f^{\text{pot}}f^{\text{dep}}\right)\mathbf{p}^{\infty}\left(\mathbf{M}^{\text{pot}} - \mathbf{M}^{\text{dep}}\right)\mathbf{w}. \tag{7}$$

eq:init

We will find an upper bound on this quantity for *all* possible models and also find the model that saturates this bound.

A useful quantity is the equilibrium probability flux between two disjoint sets of states, $\mathcal{A}$ and $\mathcal{B}$:

$$\mathbf{\Phi}_{\mathcal{A}\mathcal{B}} = \sum_{i\in\mathcal{A}}\sum_{j\in\mathcal{B}} r\mathbf{p}^{\infty}_i\mathbf{W}^{\mathrm{F}}_{ij}. \tag{8}$$

eq:flux

The initial SNR is closely related to the flux between the states with $\mathbf{w}_i = -1$ and those with $\mathbf{w}_j = +1$:

$$\mathrm{SNR}(0) \le \frac{4\sqrt{N}\mathbf{\Phi}_{-+}}{r}. \tag{9}$$

eq:initflux

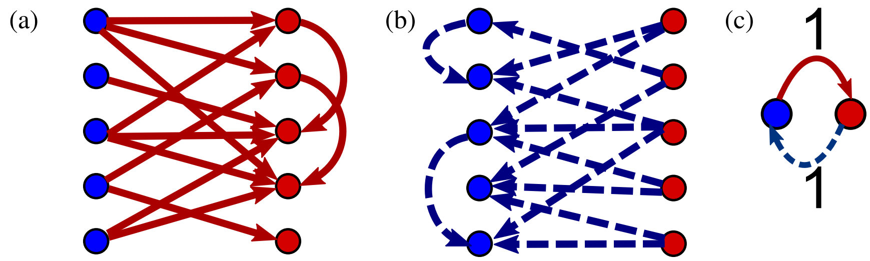

Figure 1: Synaptic models that maximize initial SNR. (a) For potentiation, all transitions starting from a weak state lead to a strong state, and the probabilities for all transitions leaving a given weak state sum to 1. (a) Depression is similar to potentiation, but with strong and weak interchanged. (c) The equivalent two state model, with transition probabilities under potentiation and depression equal to one.

This inequality becomes an equality if potentiation never decreases the synaptic weight and depression never increases it, which should be a property of any sensible model.

To maximise this flux, potentiation from a weak state must be guaranteed to end in a strong state, and depression must do the reverse. AN example of such a model is shown in Figure 1(a,b). These models have a property known as "lumpability" (see [4, §6.3] for the discrete time version and [5, 6] for continuous time). They are completely equivalent (i.e. have the same memory curve) as a two state model with transition probabilities equal to 1, as shown in Figure 1(c).

This two state model has the equilibrium distribution $\mathbf{p}^\infty = (f^{\text{dep}}, f^{\text{pot}})$ and its flux is given by $\mathbf{\Phi}_{-+} = r f^{\text{pot}} f^{\text{dep}}$. This is maximized when $f^{\text{pot}} = f^{\text{dep}} = \frac{1}{2}$, leading to the upper bound:

$$\mathrm{SNR}(0) \leq \sqrt{N}. \tag{10}$$

3.2 Area

Now consider the area under the memory curve:

$$A = \int_0^\infty \mathrm{d}t \; \mathrm{SNR}(t). \tag{11}$$

We will find an upper bound on this quantity as well as the model that saturates it.

4 Memory curve envelope

5 Discussion

References

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