

# Backpropagation and Jacobian Matrices

Lecture.1  
Why Backpropagation  
and Jacobians?

# Lecture.1 Why Backpropagation and Jacobians?

## - Trainable Models and Params

### Artificial Neurons

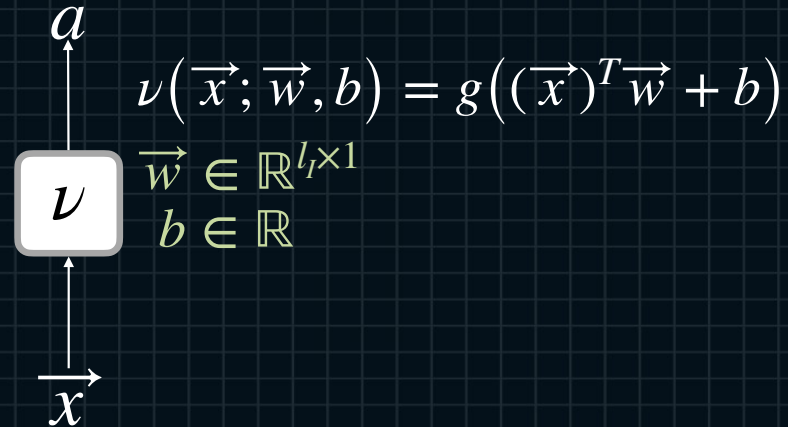
$$\hat{y} = xw + b$$

$$\hat{y} = \vec{x}^T \cdot \vec{w} + b$$

$\vec{x}$  is  $n \times 1$ ,  
 $\vec{w}$  is  $n \times 1$   
 $b = 1$

$$\hat{y} = g(\vec{x}^T \cdot \vec{w} + b)$$

$\downarrow$   
sigmoid...  
 $w: n$   
 $b: 1$

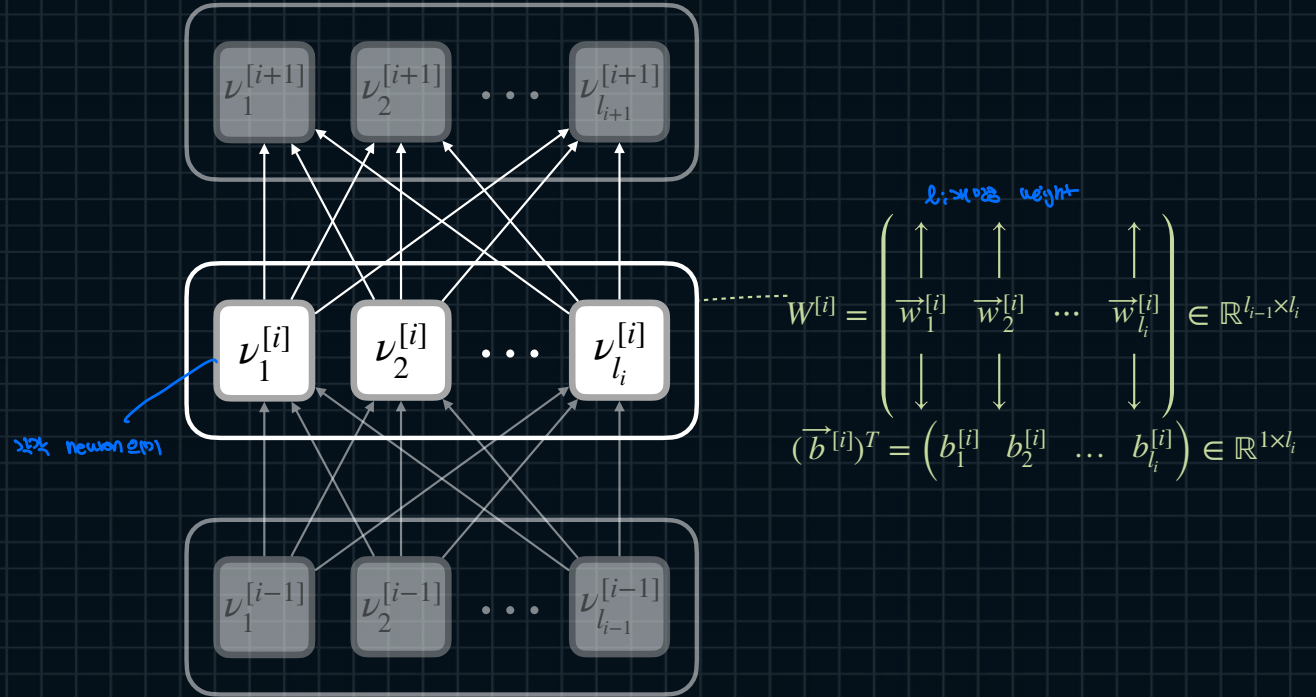


# Lecture.1

## Why Backpropagation and Jacobians?

### - Trainable Models and Params

#### Dense Layers



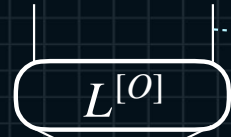
# Lecture.1

## Why Backpropagation and Jacobians?

### - Trainable Models and Params

Dense Layers

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$$W^{[0]} = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \bar{w}_1^{[0]} & \bar{w}_2^{[0]} & \dots & \bar{w}_{l_0}^{[0]} \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \in \mathbb{R}^{l_{o-1} \times l_o} \quad (A^{[0]})^T = (A^{[o-1]})^T W^{[0]} + (\vec{b}^{[0]})^T$$

$\mathbb{R}^{N \times l_o} \quad \mathbb{R}^{N \times l_{o-1}} \quad \mathbb{R}^{l_{o-1} \times l_o} \quad \mathbb{R}^{1 \times l_o}$

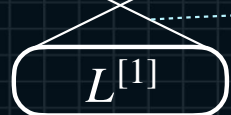
$$(\vec{b}^{[0]})^T = (b_1^{[0]} \quad b_2^{[0]} \quad \dots \quad b_{l_0}^{[0]}) \in \mathbb{R}^{1 \times l_o}$$



$$W^{[2]} = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \bar{w}_1^{[2]} & \bar{w}_2^{[2]} & \dots & \bar{w}_{l_2}^{[2]} \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \in \mathbb{R}^{l_1 \times l_2} \quad (A^{[2]})^T = (A^{[1]})^T W^{[2]} + (\vec{b}^{[2]})^T$$

$\mathbb{R}^{N \times l_2} \quad \mathbb{R}^{N \times l_1} \quad \mathbb{R}^{l_1 \times l_2} \quad \mathbb{R}^{1 \times l_2}$

$$(\vec{b}^{[2]})^T = (b_1^{[2]} \quad b_2^{[2]} \quad \dots \quad b_{l_2}^{[2]}) \in \mathbb{R}^{1 \times l_2}$$



$$(A^{[1]})^T = X^T W^{[1]} + (\vec{b}^{[1]})^T$$

$\mathbb{R}^{N \times l_1} \quad \mathbb{R}^{N \times l_1} \quad \mathbb{R}^{l_1 \times l_1} \quad \mathbb{R}^{1 \times l_1}$

$$W^{[1]} = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \bar{w}_1^{[1]} & \bar{w}_2^{[1]} & \dots & \bar{w}_{l_1}^{[1]} \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \in \mathbb{R}^{l_1 \times l_1}$$

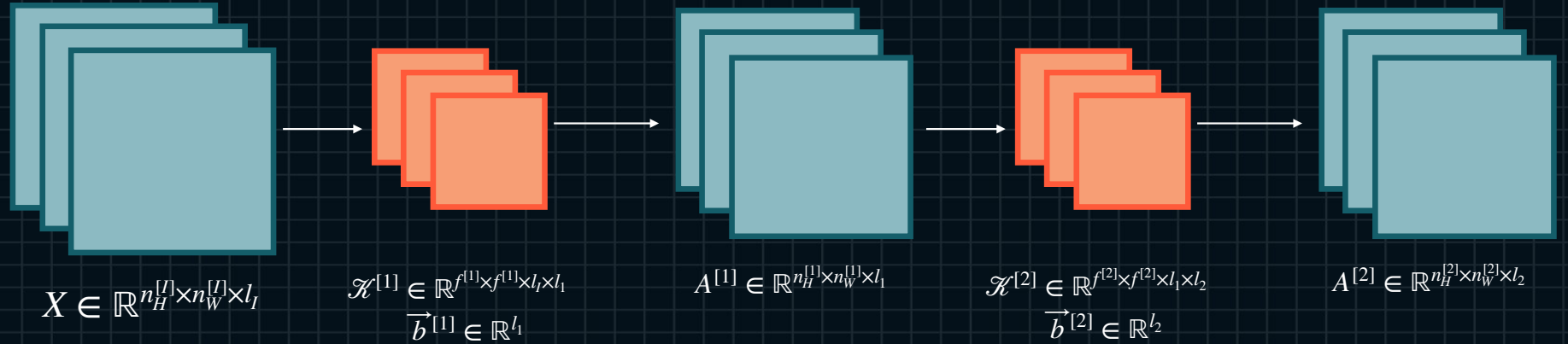
$$(\vec{b}^{[1]})^T = (b_1^{[1]} \quad b_2^{[1]} \quad \dots \quad b_{l_1}^{[1]}) \in \mathbb{R}^{1 \times l_1}$$

24. Layerwise weight matrix

# Lecture.1 Why Backpropagation and Jacobians?

## - Trainable Models and Params

### Conv Layers



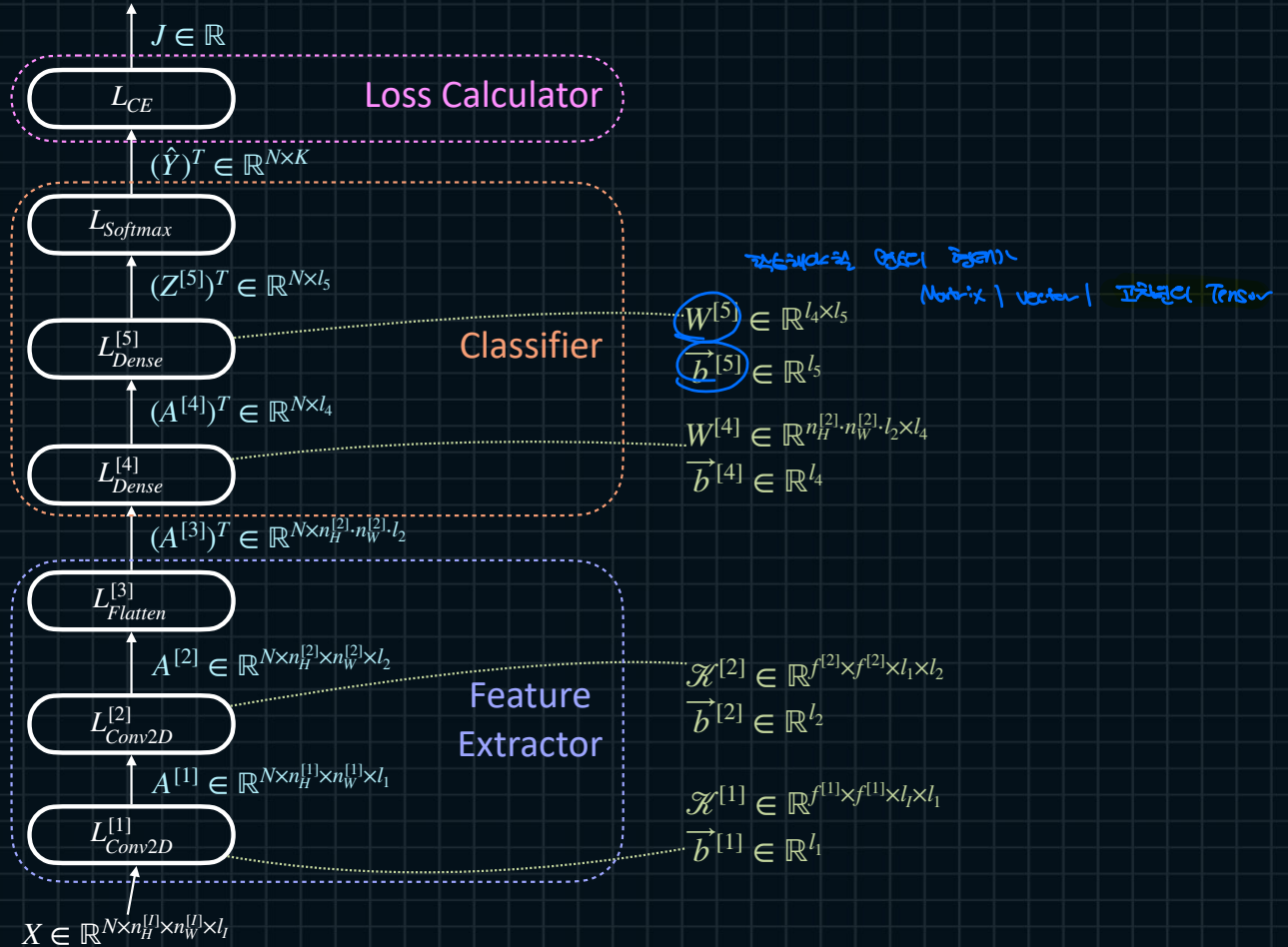
Output the filter size and color

Kernel (Tensor) of size and bias vec. size

# Lecture.1 Why Backpropagation and Jacobians?

CNNs

## - Trainable Models and Params



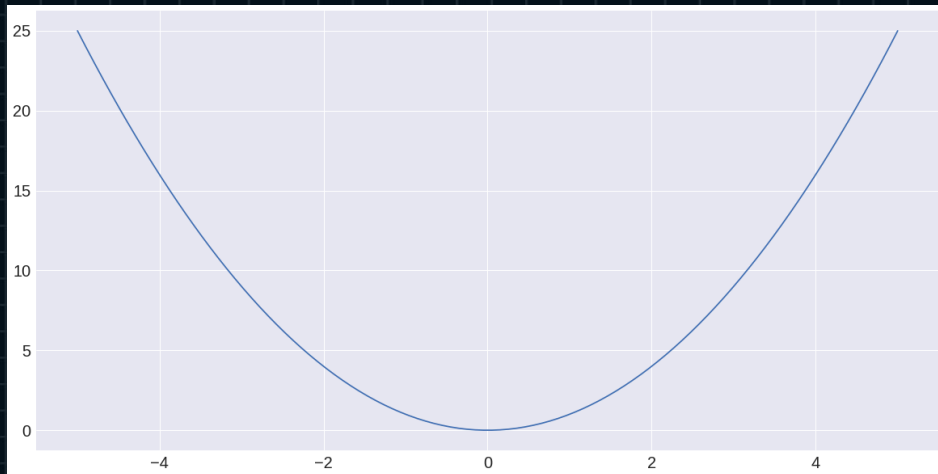


# Lecture.1

## Why Backpropagation and Jacobians?

## - Gradient-based Learning

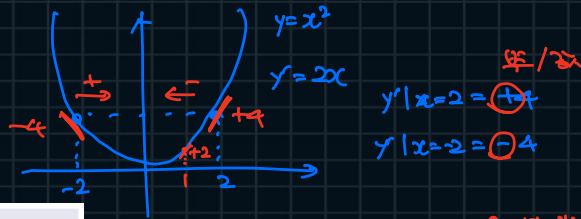
### Differential Coefficient in DL



$$y' = 2x$$

$$y'|_{x=2} = 2 \cdot 2 = +4$$

$$y'|_{x=-2} = 2 \cdot (-2) = -4$$



이 값은 왜 나오는지?  
 $y'$  (미분계수) =  $\frac{dy}{dx}$   
 $x$  값의 증가에 따른  $y$ 의 변화량

$$y' = 2x$$

$$y'|_{x=1} = 2 \cdot 1 = +2$$

$$y'|_{x=2} = 2 \cdot 2 = +4$$

# Lecture.1

## Why Backpropagation and Jacobians?

### - Gradient-based Learning

Update Notation

$$\underline{x := x + a}$$

update e.g.  $x = x + 2$

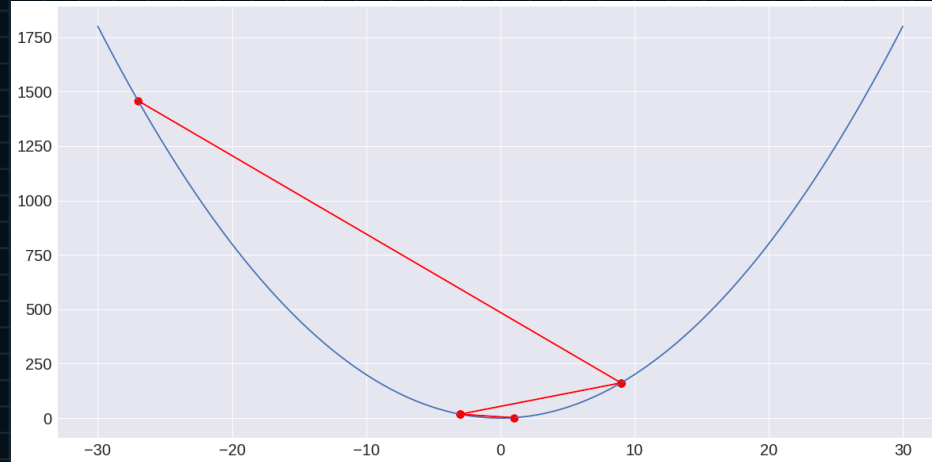


# Lecture.1

## Why Backpropagation and Jacobians?

## - Gradient-based Learning

### Effectiveness of Gradients



$$\begin{aligned}x &:= x - \frac{f'(x)}{\frac{dy}{dx}} \\&= x - 4x \\&= -3x\end{aligned}$$

$$\begin{aligned}x &:= -3 \cdot 1 = -3 \\x &:= -3 \cdot (-3) = 9 \\x &:= -3 \cdot 9 = -27\end{aligned}$$

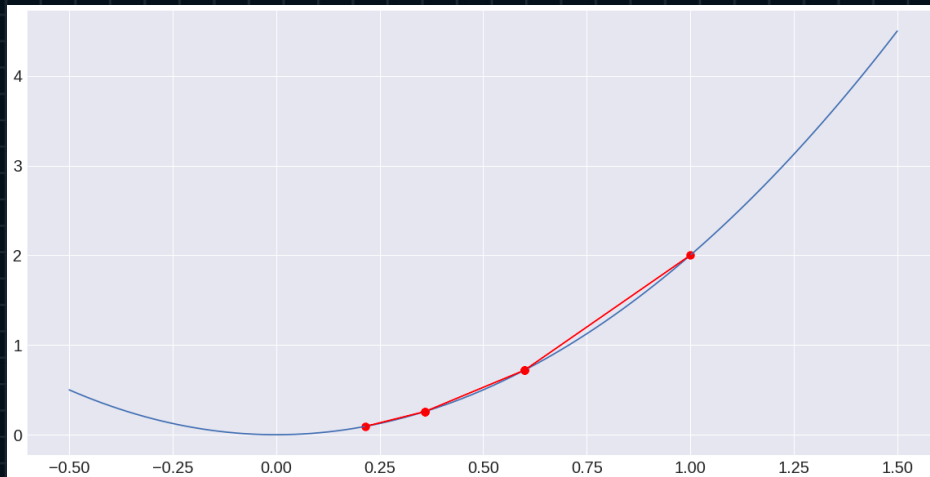
→ 오차의 방향을 따라 이동

이렇게 하는 문제?

## - Gradient-based Learning

→ like. 디렉터리

의상디자인의 한시곡조!



↳ 다른 계층을 완전히 버리는 역할 (suppressed)

$$\alpha = \text{צדית תיכונה (128 ז"ב)}$$

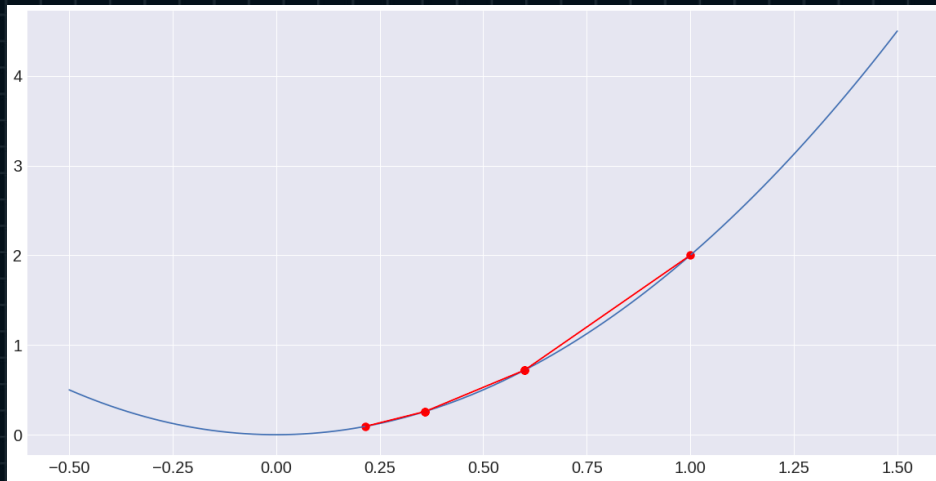
$$x := 0.6 \cdot 0.36 = 0.216$$

강이 죽어가는 과정에

# Lecture.1 Why Backpropagation and Jacobians?

## - Gradient-based Learning

### Descending Without a Map



이제 DL에서는  $y=x^2$  대신  
복잡한 함수를 씀

$$x := x - \alpha f'(x)$$

이걸을 뭐라고 하죠?

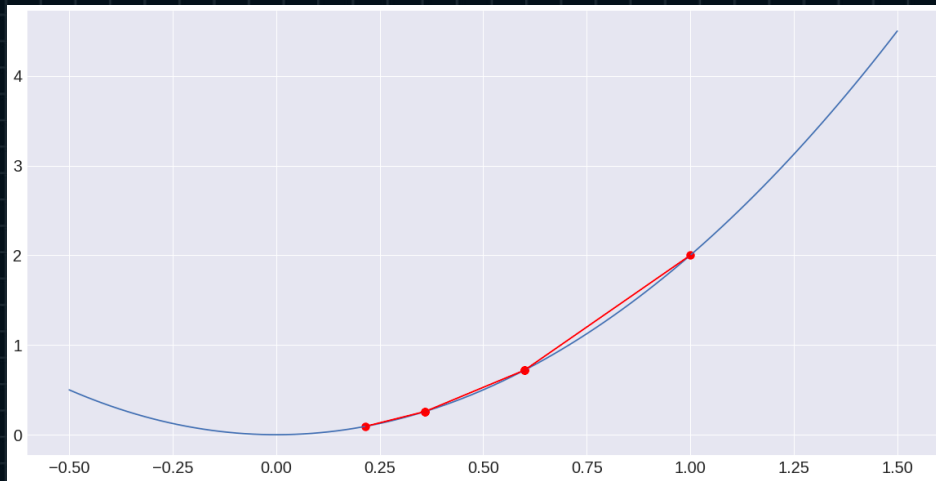
- 함수값을 최솟값으로 내리려 하는 방법
- 미분계수의 값에 양함수로 조정 필요

# Lecture.1

## Why Backpropagation and Jacobians?

### - Gradient-based Learning

#### Target of Gradient



왜 함수 값을 줄여야 하나?

→ Loss ↓

MSE (Mean Squared Error)

BCE (Binary Cross Entropy)

CCE (Categorical Cross Entropy)

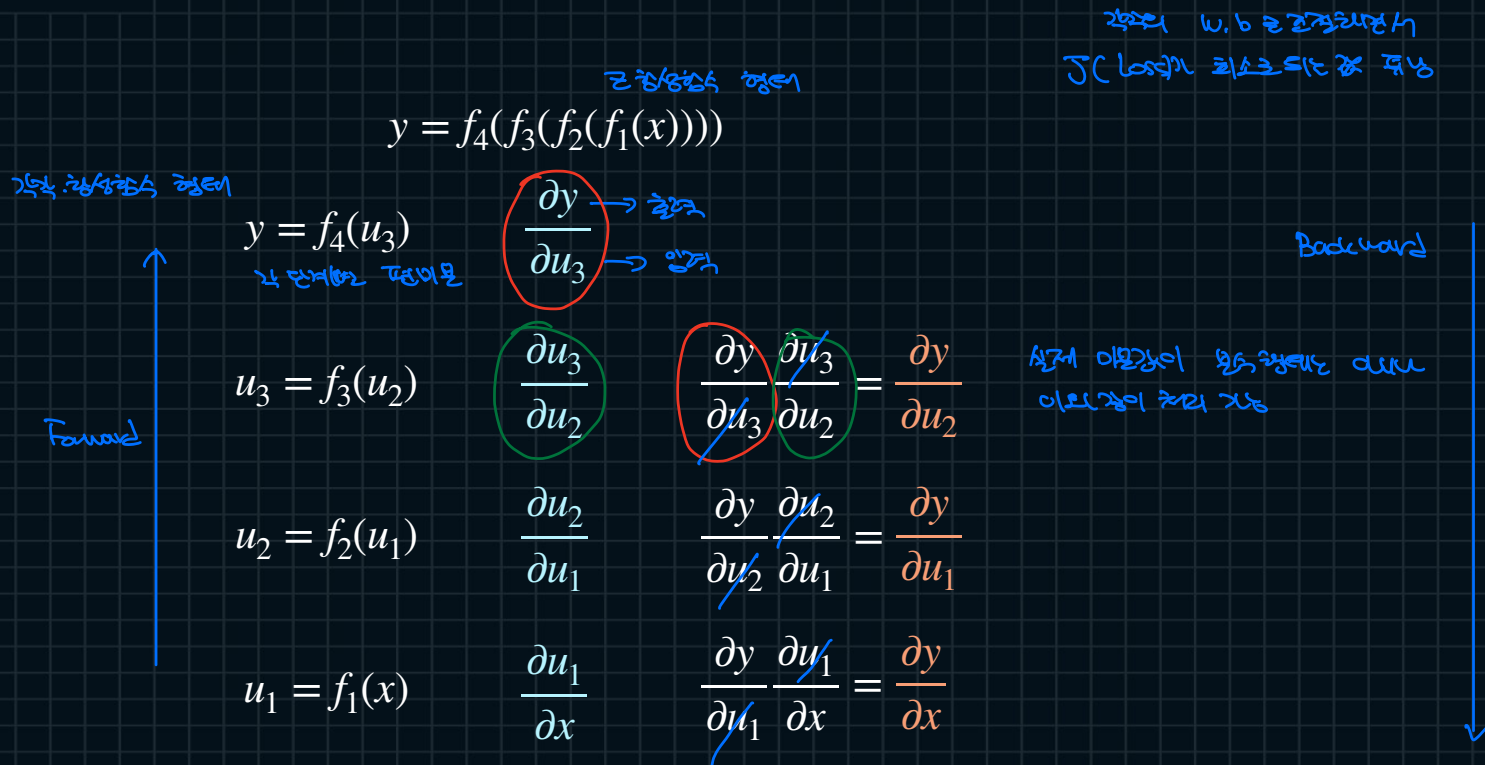
$$\underbrace{J}_{\text{Loss}} = \mathcal{L}(y, \hat{y}) = \frac{\partial J}{\partial x}$$
$$x := x - \alpha \mathcal{L}'(x)$$

Loss를 줄이기 위해 x 값을 어떻게 조정해야 하나?

# Lecture.1 Why Backpropagation and Jacobians?

## - Backpropagation

### Chain Rule



# Lecture.1

## Why Backpropagation and Jacobians?

### - Backpropagation

Chain Rule

$$y = f_4(f_3(f_2(f_1(x))))$$

$$y = f_4(u_3)$$

$$u_3 = f_3(u_2)$$

$$u_2 = f_2(u_1)$$

$$u_1 = f_1(x)$$

$$\frac{\partial y}{\partial x}$$

$$\frac{\partial y}{\partial u_1}$$

$$\frac{\partial y}{\partial u_2}$$

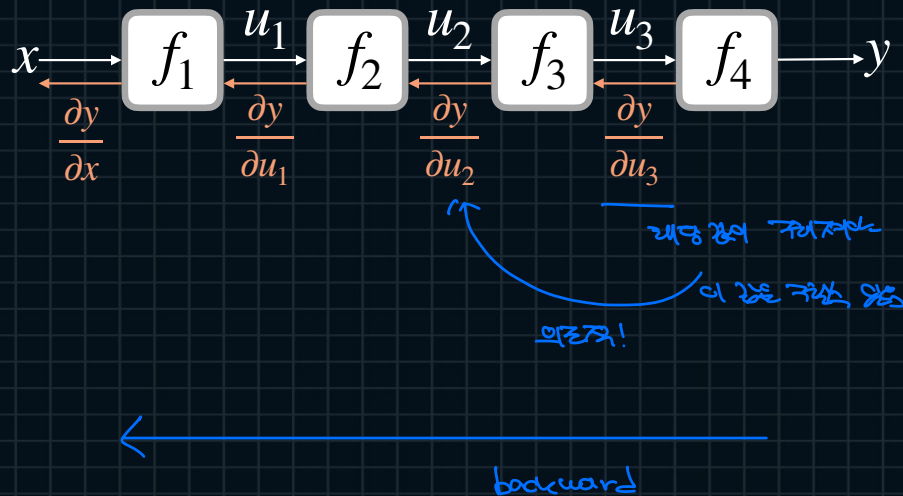
Forward

Backward

# Lecture.1 Why Backpropagation and Jacobians?

## - Backpropagation

Chain Rule

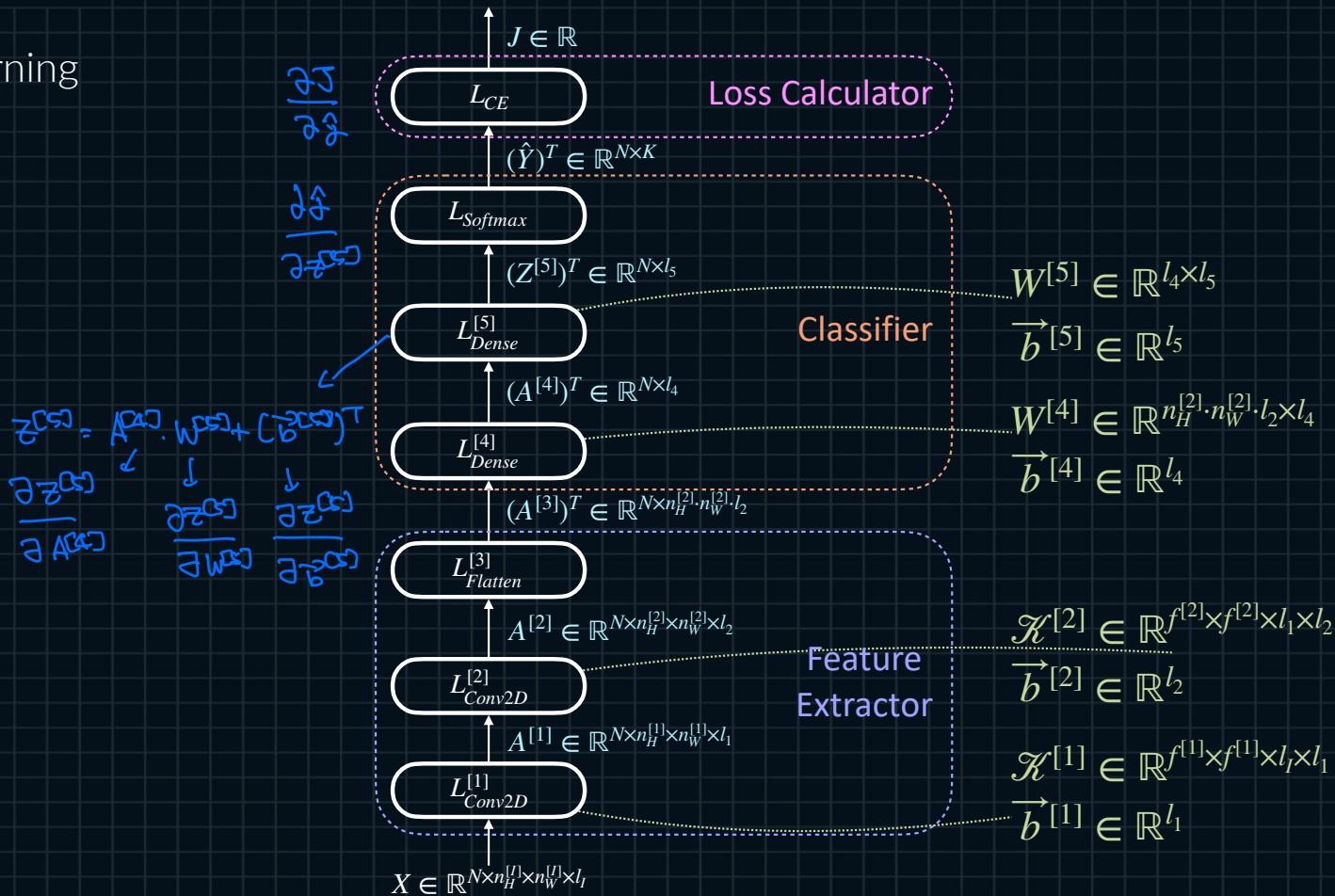




# Lecture.1 Why Backpropagation and Jacobians?

## Chain Rule in Deep Learning

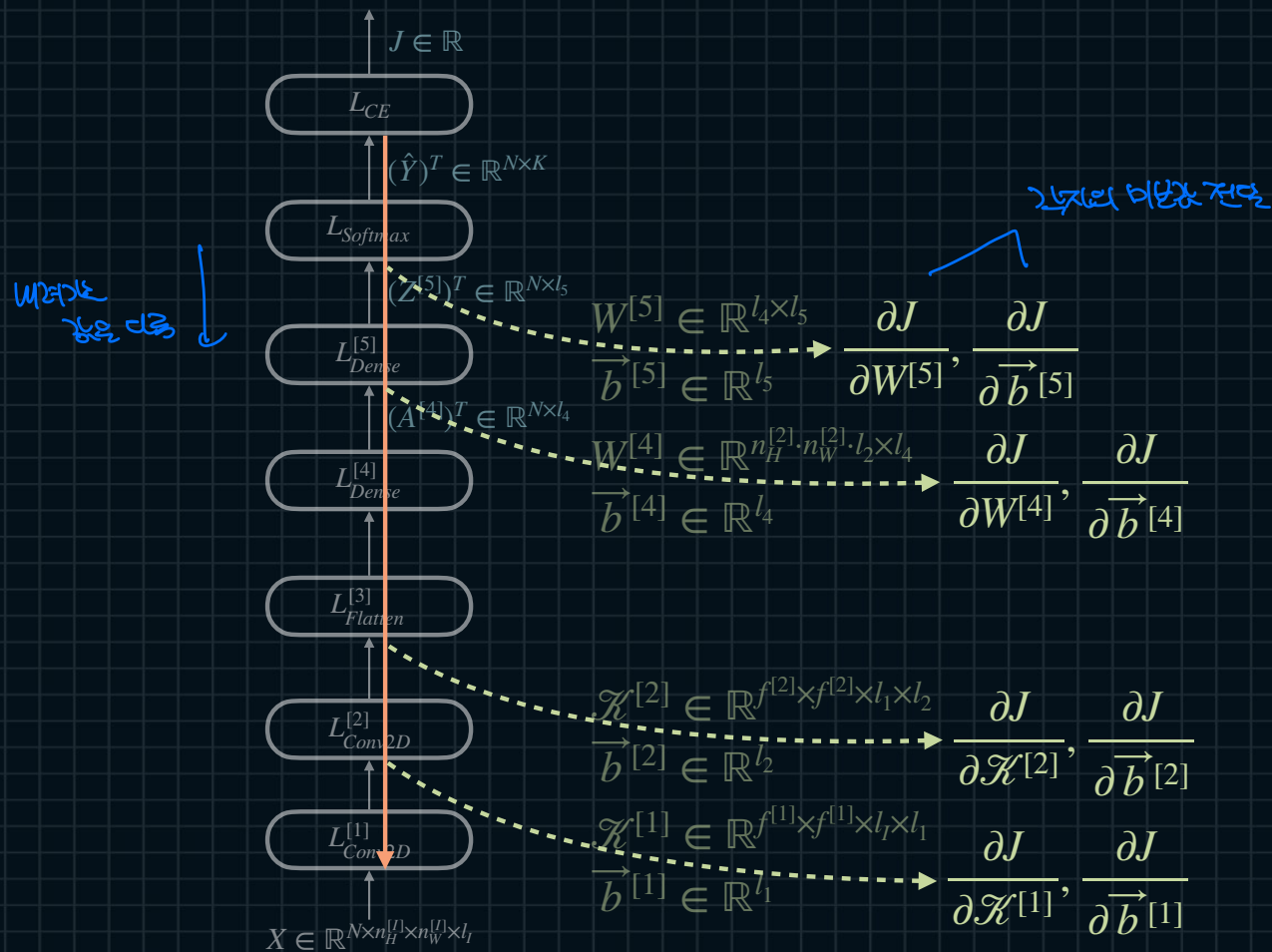
## - Backpropagation



# Lecture.1 Why Backpropagation and Jacobians?

## Backpropagation

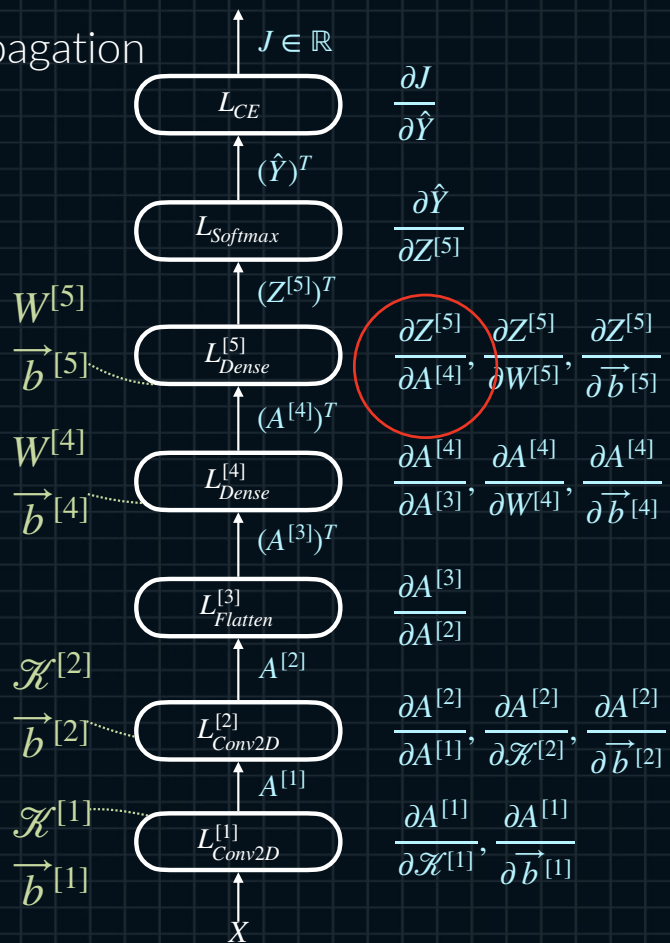
### - Backpropagation



# Lecture.1 Why Backpropagation and Jacobians?

## - Backpropagation

Backpropagation



$$\frac{\partial J}{\partial \hat{Y}} \frac{\partial \hat{Y}}{\partial Z^{[5]}} = \frac{\partial J}{\partial Z^{[5]}}$$

$$\frac{\partial J}{\partial Z^{[5]}} \frac{\partial Z^{[5]}}{\partial A^{[4]}} = \frac{\partial J}{\partial A^{[4]}}$$

$$\frac{\partial J}{\partial A^{[4]}} \frac{\partial A^{[4]}}{\partial A^{[3]}} = \frac{\partial J}{\partial A^{[3]}}$$

$$\frac{\partial J}{\partial A^{[3]}} \frac{\partial A^{[3]}}{\partial A^{[2]}} = \frac{\partial J}{\partial A^{[2]}}$$

$$\frac{\partial J}{\partial A^{[2]}} \frac{\partial A^{[2]}}{\partial A^{[1]}} = \frac{\partial J}{\partial A^{[1]}}$$

$$\frac{\partial J}{\partial Z^{[5]}} \frac{\partial Z^{[5]}}{\partial W^{[5]}} = \frac{\partial J}{\partial W^{[5]}}$$

$$\frac{\partial J}{\partial A^{[4]}} \frac{\partial A^{[4]}}{\partial W^{[4]}} = \frac{\partial J}{\partial W^{[4]}}$$

→ 이항미분 곱셈 법칙을 사용

$$\frac{\partial J}{\partial A^{[2]}} \frac{\partial A^{[2]}}{\partial \mathcal{K}^{[2]}} = \frac{\partial J}{\partial \mathcal{K}^{[2]}}$$

$$\frac{\partial J}{\partial A^{[1]}} \frac{\partial A^{[1]}}{\partial \mathcal{K}^{[1]}} = \frac{\partial J}{\partial \mathcal{K}^{[1]}}$$

$$\frac{\partial J}{\partial Z^{[5]}} \frac{\partial Z^{[5]}}{\partial \vec{b}^{[5]}} = \frac{\partial J}{\partial \vec{b}^{[5]}}$$

$$\frac{\partial J}{\partial A^{[4]}} \frac{\partial A^{[4]}}{\partial \vec{b}^{[4]}} = \frac{\partial J}{\partial \vec{b}^{[4]}}$$

$$\frac{\partial J}{\partial A^{[2]}} \frac{\partial A^{[2]}}{\partial \vec{b}^{[2]}} = \frac{\partial J}{\partial \vec{b}^{[2]}}$$

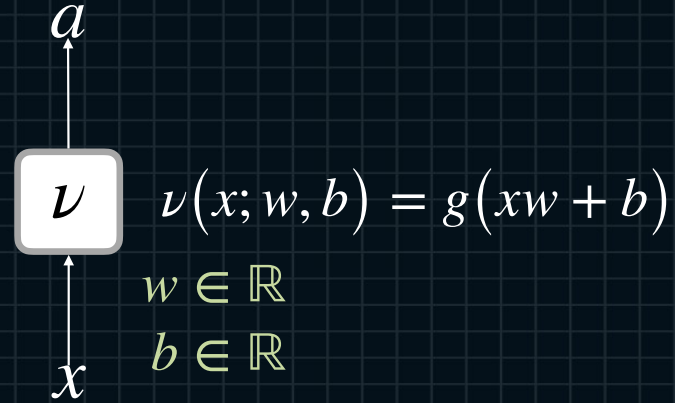
$$\frac{\partial J}{\partial A^{[1]}} \frac{\partial A^{[1]}}{\partial \vec{b}^{[1]}} = \frac{\partial J}{\partial \vec{b}^{[1]}}$$

# Lecture.1

## Why Backpropagation and Jacobians?

### - Why Jacobians?

Derivatives of Scalars




$$w := w - \alpha \frac{\partial J}{\partial w} \quad b := b - \alpha \frac{\partial J}{\partial b}$$

# Lecture.1

## Why Backpropagation and Jacobians?

### - Why Jacobians?

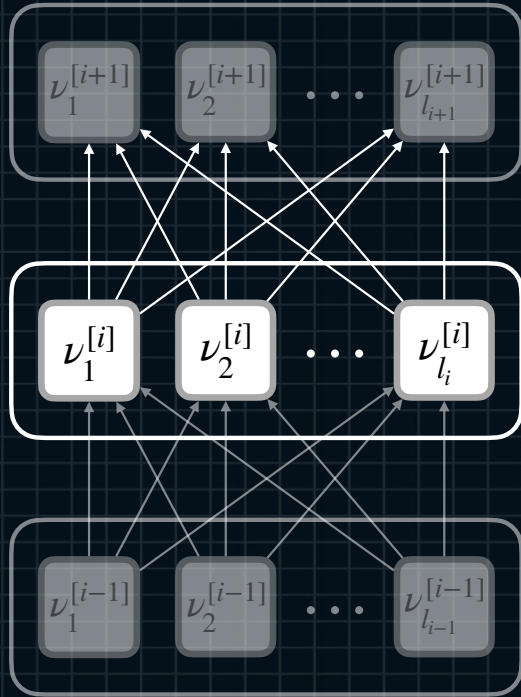
#### Derivatives of Vectors


$$\nu(\vec{x}; \vec{w}, b) = g((\vec{x})^T \vec{w} + b)$$
$$\vec{w} \in \mathbb{R}^{l \times 1}$$
$$b \in \mathbb{R}$$
$$\vec{w} := \vec{w} - \alpha \frac{\partial J}{\partial \vec{w}} \quad b := b - \alpha \frac{\partial J}{\partial b}$$

# Lecture.1 Why Backpropagation and Jacobians?

## - Why Jacobians?

### Derivatives of Matrices



$$W^{[i]} = \begin{pmatrix} \uparrow & \uparrow & \dots & \uparrow \\ \overrightarrow{w}_1^{[i]} & \overrightarrow{w}_2^{[i]} & \dots & \overrightarrow{w}_{l_i}^{[i]} \\ \downarrow & \downarrow & \dots & \downarrow \end{pmatrix} \in \mathbb{R}^{l_{i-1} \times l_i}$$

$$(\overrightarrow{b}^{[i]})^T = \begin{pmatrix} b_1^{[i]} & b_2^{[i]} & \dots & b_{l_i}^{[i]} \end{pmatrix} \in \mathbb{R}^{1 \times l_i}$$

$$W^{[i]} := W^{[i]} - \alpha \frac{\partial J}{\partial W^{[i]}}$$

$$\overrightarrow{b}^{[i]} := \overrightarrow{b}^{[i]} - \alpha \frac{\partial J}{\partial \overrightarrow{b}^{[i]}}$$

# Lecture.1 Why Backpropagation and Jacobians?

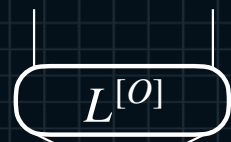
## - Why Jacobians?

Derivatives of Matrices

$$W^{[O]} = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{w}_1^{[O]} & \vec{w}_2^{[O]} & \dots & \vec{w}_{l_o}^{[O]} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \in \mathbb{R}^{l_{o-1} \times l_o}$$

$$W^{[O]} := W^{[O]} - \alpha \frac{\partial J}{\partial W^{[O]}}$$

$$\vec{b}^{[O]} := \vec{b}^{[O]} - \alpha \frac{\partial J}{\partial \vec{b}^{[O]}}$$

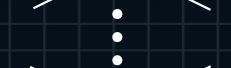


$$(\vec{b}^{[O]})^T = (b_1^{[O]} \quad b_2^{[O]} \quad \dots \quad b_{l_o}^{[O]}) \in \mathbb{R}^{1 \times l_o}$$

$$W^{[2]} = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{w}_1^{[2]} & \vec{w}_2^{[2]} & \dots & \vec{w}_{l_2}^{[2]} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \in \mathbb{R}^{l_1 \times l_2}$$

$$W^{[2]} := W^{[2]} - \alpha \frac{\partial J}{\partial W^{[2]}}$$

$$\vec{b}^{[2]} := \vec{b}^{[2]} - \alpha \frac{\partial J}{\partial \vec{b}^{[2]}}$$



$$(\vec{b}^{[2]})^T = (b_1^{[2]} \quad b_2^{[2]} \quad \dots \quad b_{l_2}^{[2]}) \in \mathbb{R}^{1 \times l_2}$$



$$W^{[1]} = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{w}_1^{[1]} & \vec{w}_2^{[1]} & \dots & \vec{w}_{l_1}^{[1]} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \in \mathbb{R}^{l \times l_1}$$

$$W^{[1]} := W^{[1]} - \alpha \frac{\partial J}{\partial W^{[1]}}$$

$$\vec{b}^{[1]} := \vec{b}^{[1]} - \alpha \frac{\partial J}{\partial \vec{b}^{[1]}}$$

$X^T$

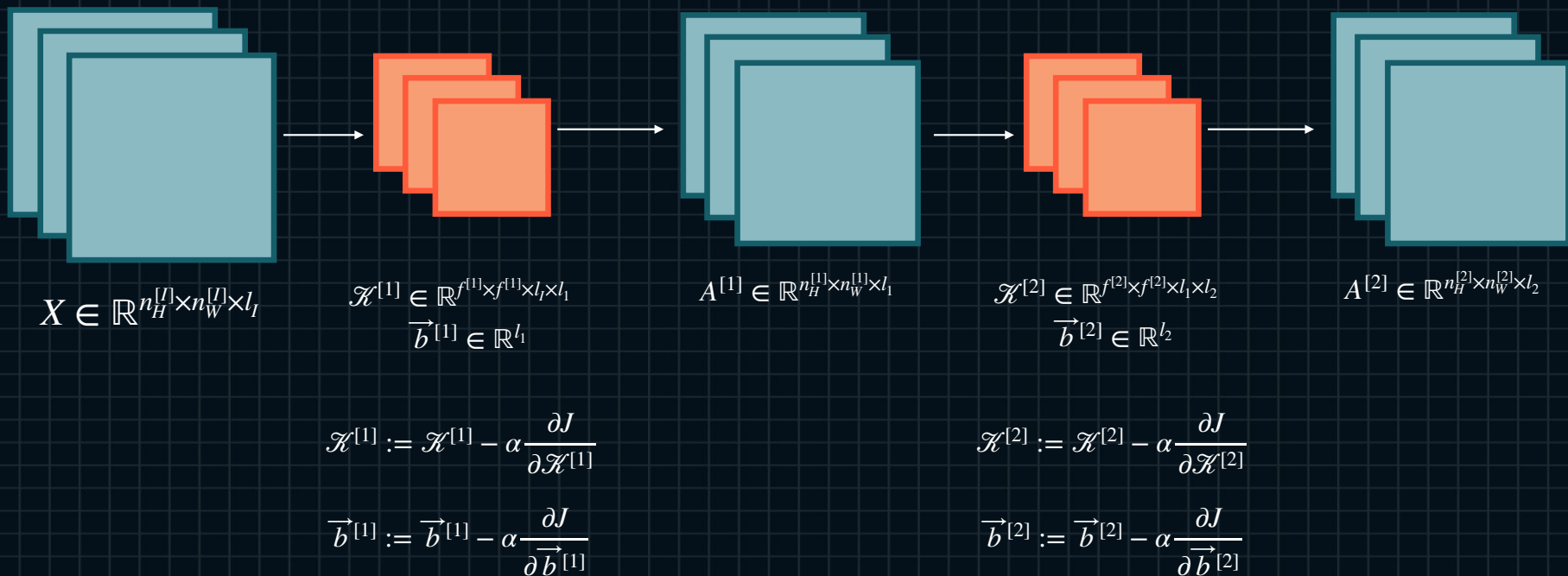
$$(\vec{b}^{[1]})^T = (b_1^{[1]} \quad b_2^{[1]} \quad \dots \quad b_{l_1}^{[1]}) \in \mathbb{R}^{1 \times l_1}$$



# Lecture.1 Why Backpropagation and Jacobians?

## - Why Jacobians?

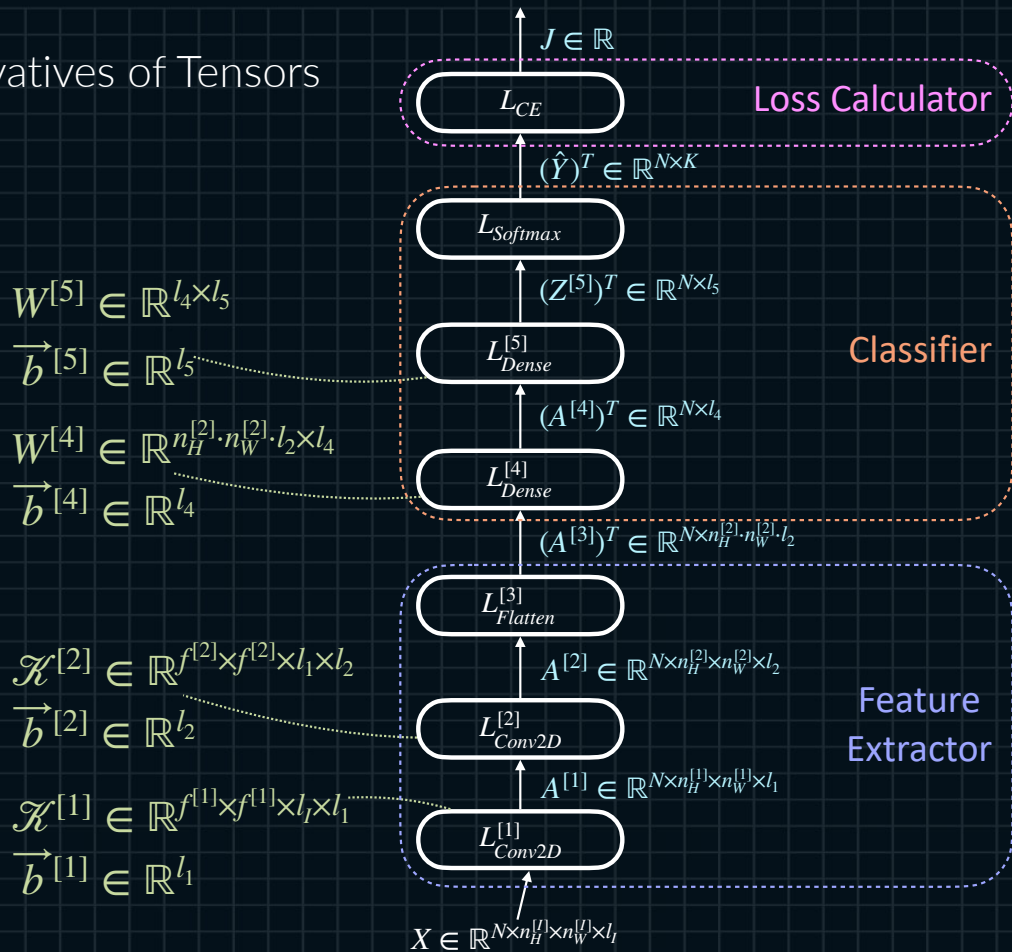
### Derivatives of Tensors



# Lecture.1 Why Backpropagation and Jacobians?

## - Why Jacobians?

Derivatives of Tensors



$$W^{[5]} := W^{[5]} - \alpha \frac{\partial J}{\partial W^{[5]}}$$

$$\vec{b}^{[5]} := \vec{b}^{[5]} - \alpha \frac{\partial J}{\partial \vec{b}^{[5]}}$$

$$W^{[4]} := W^{[4]} - \alpha \frac{\partial J}{\partial W^{[4]}}$$

$$\vec{b}^{[4]} := \vec{b}^{[4]} - \alpha \frac{\partial J}{\partial \vec{b}^{[4]}}$$

$$\mathcal{K}^{[2]} := \mathcal{K}^{[2]} - \alpha \frac{\partial J}{\partial \mathcal{K}^{[2]}}$$

$$\vec{b}^{[2]} := \vec{b}^{[2]} - \alpha \frac{\partial J}{\partial \vec{b}^{[2]}}$$

$$\mathcal{K}^{[1]} := \mathcal{K}^{[1]} - \alpha \frac{\partial J}{\partial \mathcal{K}^{[1]}}$$

$$\vec{b}^{[1]} := \vec{b}^{[1]} - \alpha \frac{\partial J}{\partial \vec{b}^{[1]}}$$

Lecture.1  
Why Backpropagation  
and Jacobians?

- Theoretical and Practical Jacobians

Upgrades of Derivatives(Jacobians)

$$\frac{\partial f}{\partial x} \longrightarrow \frac{\partial f}{\partial \vec{x}} \longrightarrow \frac{\partial \vec{f}}{\partial x} \longrightarrow \frac{\partial \vec{f}}{\partial \vec{x}} \longrightarrow \frac{\partial f}{\partial X}$$

Lecture.1  
Why Backpropagation  
and Jacobians?

- Theoretical and Practical Jacobians

Theoretical Jacobians

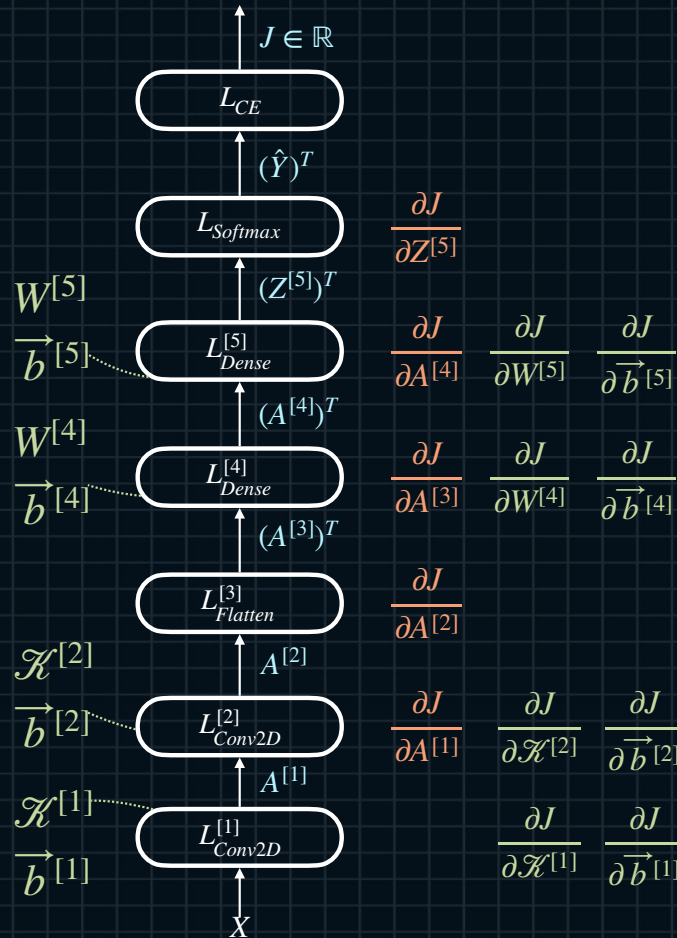
	$x$	$\vec{x}$
$f$	$\frac{\partial f}{\partial x}$	$\frac{\partial f}{\partial \vec{x}}$
$\vec{f}$	$\frac{\partial \vec{f}}{\partial x}$	$\frac{\partial \vec{f}}{\partial \vec{x}}$

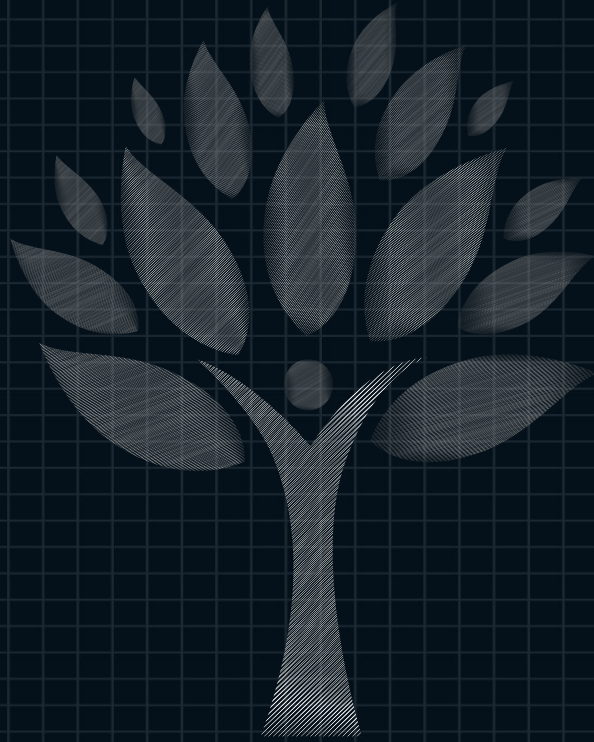
# Lecture.1

## Why Backpropagation and Jacobians?

## - Theoretical and Practical Jacobians

### Practical Jacobians





# Backpropagation and Jacobian Matrices

Lecture.1

Why Backpropagation  
and Jacobians?