

Backpropagation and Jacobian Matrices

Lecture.1
Why Backpropagation
and Jacobians?

Lecture.1 Why Backpropagation and Jacobians?

- Trainable Models and Params

Artificial Neurons

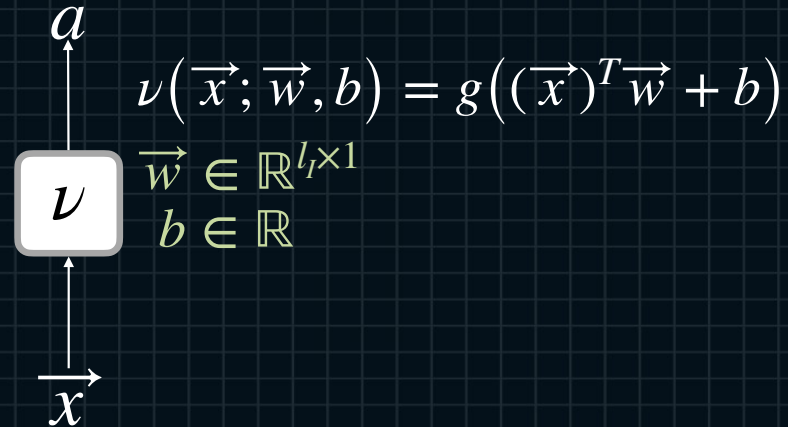
$$\hat{y} = xw + b$$

$$\hat{y} = \vec{x}^T \cdot \vec{w} + b$$

\vec{x} is $n \times 1$,
 \vec{w} is $n \times 1$
 $b = 1$

$$\hat{y} = g(\vec{x}^T \cdot \vec{w} + b)$$

\downarrow
sigmoid...
 $w: n$
 $b: 1$

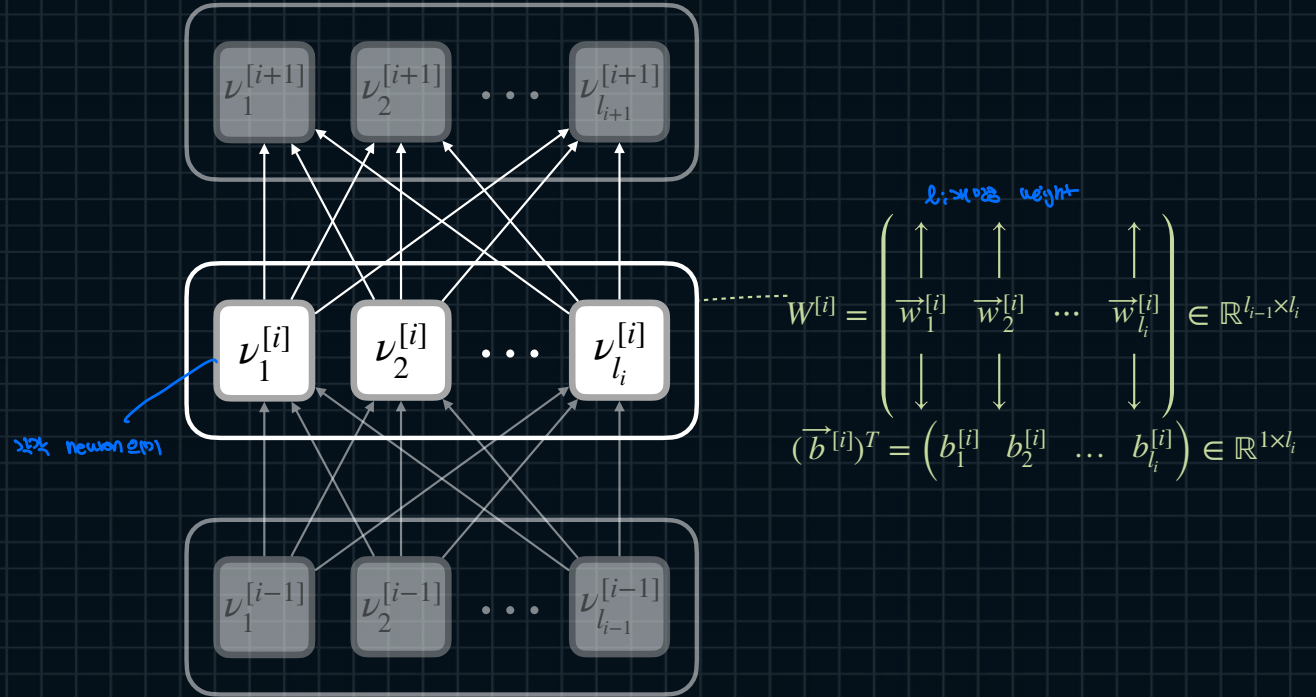


Lecture.1

Why Backpropagation and Jacobians?

- Trainable Models and Params

Dense Layers

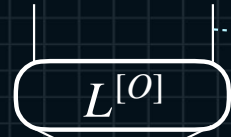


Lecture.1 Why Backpropagation and Jacobians?

- Trainable Models and Params

Dense Layers

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$$W^{[0]} = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \bar{w}_1^{[0]} & \bar{w}_2^{[0]} & \dots & \bar{w}_{l_0}^{[0]} \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \in \mathbb{R}^{l_{o-1} \times l_o} \quad (A^{[0]})^T = (A^{[o-1]})^T W^{[0]} + (\vec{b}^{[0]})^T$$

$\mathbb{R}^{N \times l_o} \quad \mathbb{R}^{N \times l_{o-1}} \quad \mathbb{R}^{l_{o-1} \times l_o} \quad \mathbb{R}^{1 \times l_o}$

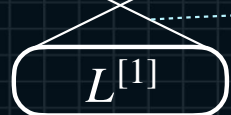
$$(\vec{b}^{[0]})^T = (b_1^{[0]} \quad b_2^{[0]} \quad \dots \quad b_{l_0}^{[0]}) \in \mathbb{R}^{1 \times l_o}$$



$$W^{[2]} = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \bar{w}_1^{[2]} & \bar{w}_2^{[2]} & \dots & \bar{w}_{l_2}^{[2]} \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \in \mathbb{R}^{l_1 \times l_2} \quad (A^{[2]})^T = (A^{[1]})^T W^{[2]} + (\vec{b}^{[2]})^T$$

$\mathbb{R}^{N \times l_2} \quad \mathbb{R}^{N \times l_1} \quad \mathbb{R}^{l_1 \times l_2} \quad \mathbb{R}^{1 \times l_2}$

$$(\vec{b}^{[2]})^T = (b_1^{[2]} \quad b_2^{[2]} \quad \dots \quad b_{l_2}^{[2]}) \in \mathbb{R}^{1 \times l_2}$$



$$(A^{[1]})^T = X^T W^{[1]} + (\vec{b}^{[1]})^T$$

$\mathbb{R}^{N \times l_1} \quad \mathbb{R}^{N \times l_1} \quad \mathbb{R}^{l_1 \times l_1} \quad \mathbb{R}^{1 \times l_1}$

$$W^{[1]} = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \bar{w}_1^{[1]} & \bar{w}_2^{[1]} & \dots & \bar{w}_{l_1}^{[1]} \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \in \mathbb{R}^{l_1 \times l_1}$$

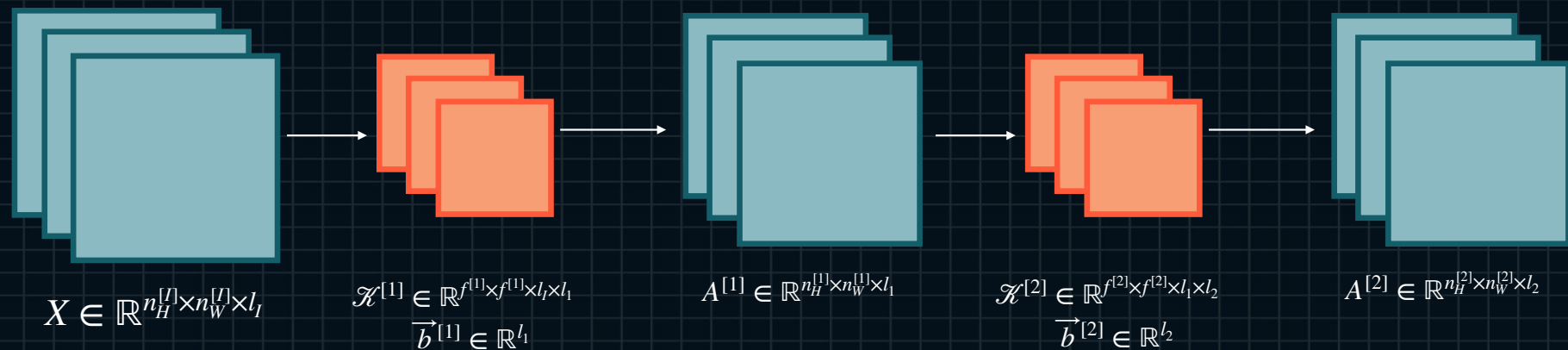
$$(\vec{b}^{[1]})^T = (b_1^{[1]} \quad b_2^{[1]} \quad \dots \quad b_{l_1}^{[1]}) \in \mathbb{R}^{1 \times l_1}$$

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Lecture.1 Why Backpropagation and Jacobians?

- Trainable Models and Params

Conv Layers



Output the filter output color

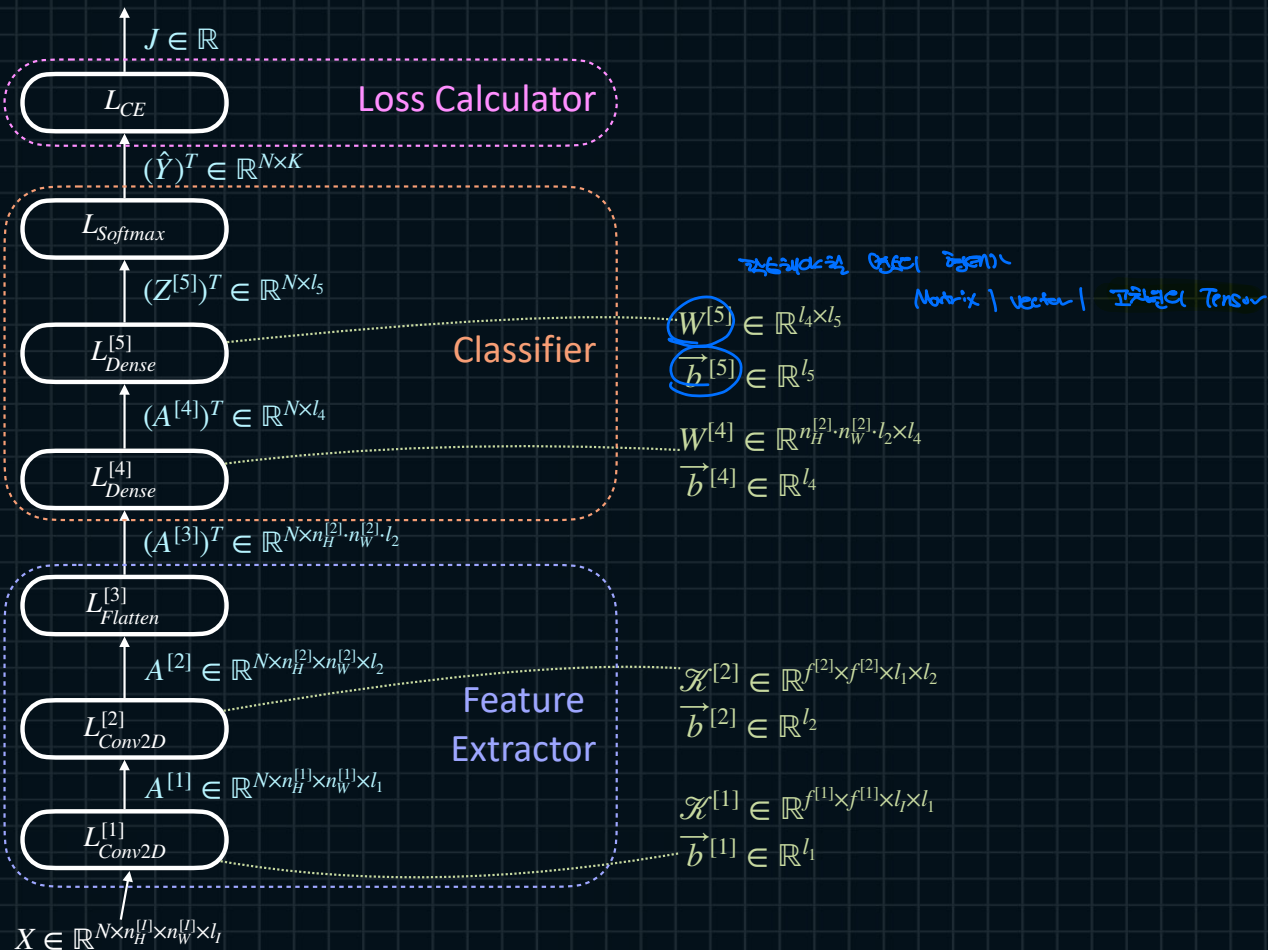
Kernel (Tensor) of input and bias vec. 2373

Lecture.1

Why Backpropagation and Jacobians?

CNNs

- Trainable Models and Params

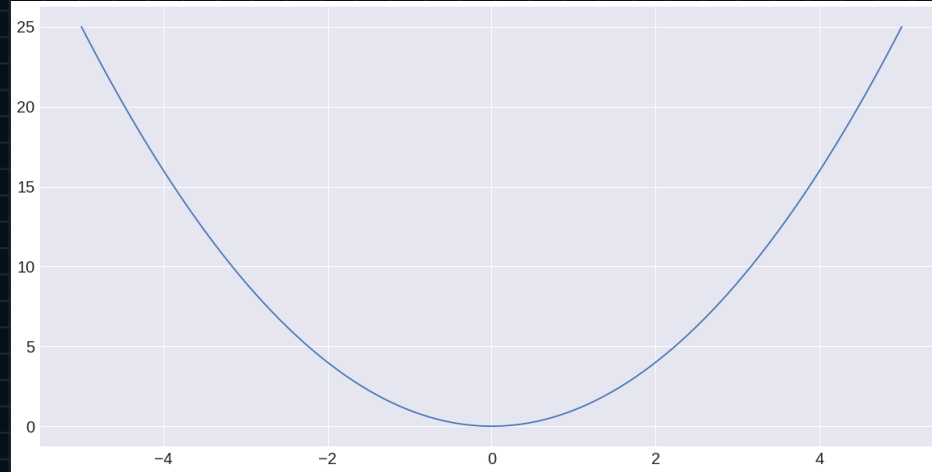


Lecture.1

Why Backpropagation and Jacobians?

- Gradient-based Learning

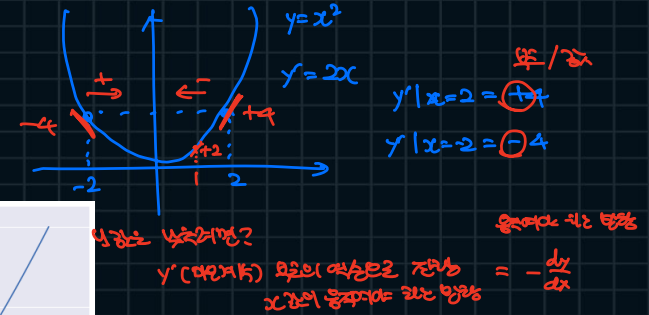
Differential Coefficient in DL



$$y' = 2x$$

$$y'|_{x=2} = 2 \cdot 2 = +4$$

$$y'|_{x=-2} = 2 \cdot (-2) = -4$$



$$y' = 2x$$

$$y'|_{x=1} = 2 \cdot 1 = +2$$

$$y'|_{x=2} = 2 \cdot 2 = +4$$

Lecture.1

Why Backpropagation and Jacobians?

- Gradient-based Learning

Update Notation

$$\underline{x := x + a}$$

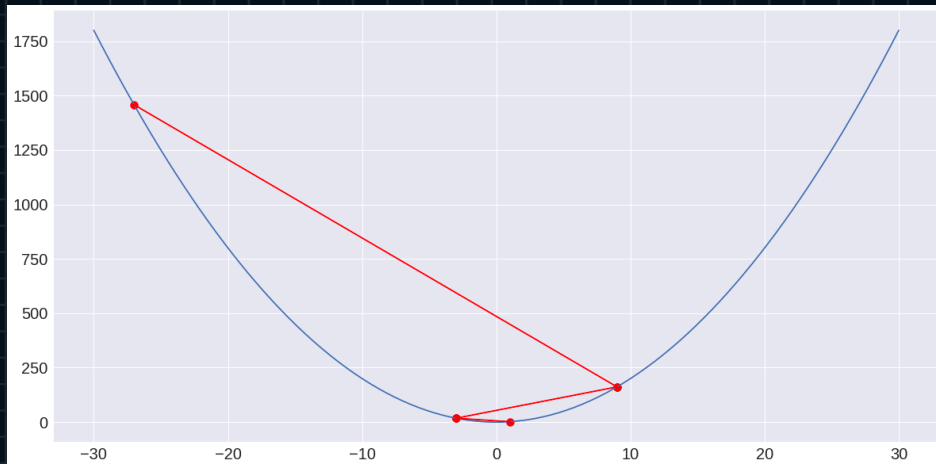
update e.g. $x = x + 2$

Lecture.1

Why Backpropagation and Jacobians?

- Gradient-based Learning

Effectiveness of Gradients



이렇게 하는 문제?

$$\begin{aligned}x &:= x - \frac{f'(x)}{f''(x)} = -\frac{dy}{dx} \\&= x - 4x \\&= -3x\end{aligned}$$

$$\begin{aligned}x &:= -3 \cdot 1 = -3 \\x &:= -3 \cdot (-3) = 9 \quad \rightarrow \text{이제 이 방향으로 움직일 것} \\x &:= -3 \cdot 9 = -27\end{aligned}$$

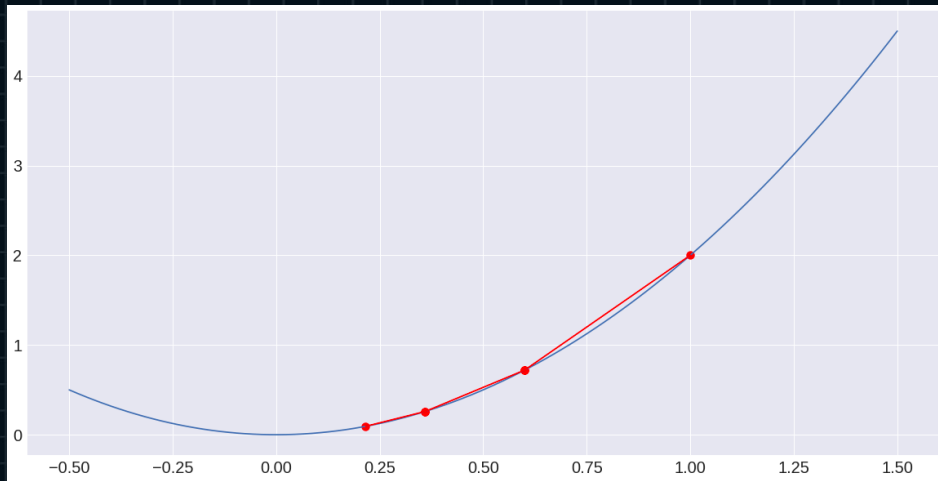
Lecture.1 Why Backpropagation and Jacobians?

- Gradient-based Learning

→ live. 다들 물어

의상되어서의 관계-구분!

Learning Rate and Gradient-based Learning



$$x := x - \alpha f'(x)$$

→ 다들 물어, 왜 양의 계수는 빼는 거지 (expressed)

$$x := x - 0.1 \cdot f'(x)$$

$$= x - 0.4x$$

$$= 0.6x$$

$\alpha = 0.1$ 이걸로 업데이트 (학습률 조정)

$$x := 0.6x$$

$$x := 0.6 \cdot 1 = 0.6$$

$$x := 0.6 \cdot 0.6 = 0.36$$

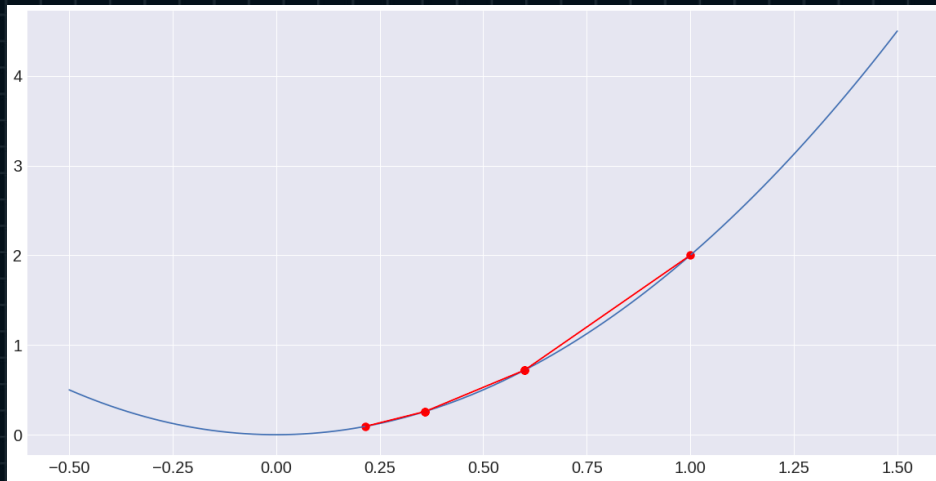
$$x := 0.6 \cdot 0.36 = 0.216$$

) 값이 작아지는 거임

Lecture.1 Why Backpropagation and Jacobians?

- Gradient-based Learning

Descending Without a Map



이제 DLO에서 $y=x^2$ 를
찾기 위해서는 x

$$x := x - \alpha f'(x)$$

이걸로 뭐가 바뀌?

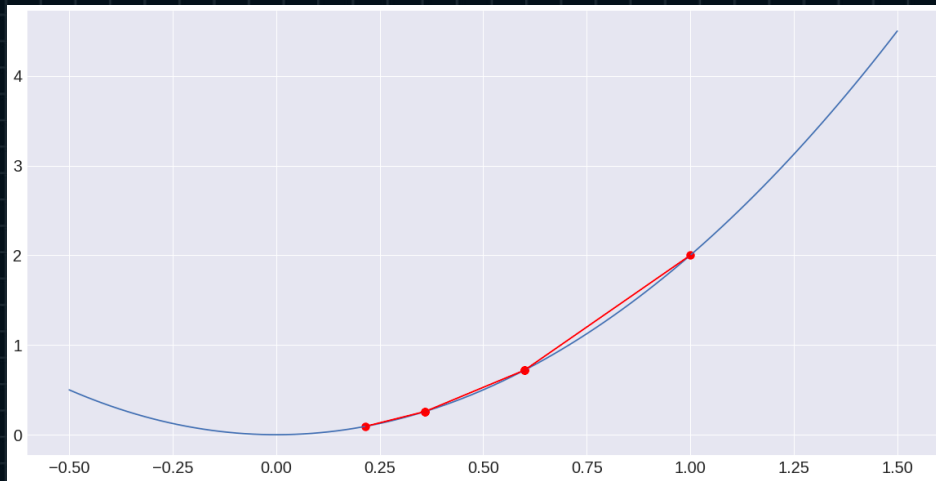
- 함수값은 최소로 근접, x값은 초기 추정
- 미분계수의 값에 양함수로 조정 필요

Lecture.1

Why Backpropagation and Jacobians?

- Gradient-based Learning

Target of Gradient



왜 함수 값을 줄여야 하나?

→ Loss ↓

MSE (Mean Squared Error)

BCE (Binary Cross Entropy)

CCE (Categorical Cross Entropy)

$$\begin{aligned} J &= \mathcal{L}(y, \hat{y}) \\ \text{Loss} &= \frac{\partial J}{\partial x} \\ x &:= x - \alpha \mathcal{L}'(x) \end{aligned}$$

Loss를 줄이기 위해 x 값을 어떻게 조정해야 하나?

- Backpropagation

각각의 w, b 은 조정하면
 $J(\text{loss})$ 최소로 되는 것 구함

$$y = f_4(f_3(f_2(f_1(x))))$$
$$y = f_4(u_3)$$

각 단계(층) 편미분

$$\frac{\partial y}{\partial u_3} \rightarrow \text{이것}$$

Badenward

$$u_3 = f_3(u_2)$$

$$\frac{\partial u_3}{\partial u_2} = \frac{\partial y}{\partial u_2}$$

실제 이모양이 될 수 있는 두
이모양이 여러 개

$$u_2 = f_2(u_1) \quad \frac{\partial u_2}{\partial u_1} \cdot \frac{\partial y}{\partial u_2} \cdot \frac{\partial u_2}{\partial u_1} = \frac{\partial y}{\partial u_1}$$

$$u_1 = f_1(x) \quad \frac{\partial u_1}{\partial x} = \frac{\partial y}{\partial y_1} \frac{\partial y_1}{\partial x} = \frac{\partial y}{\partial x}$$

Lecture.1

Why Backpropagation and Jacobians?

- Backpropagation

Chain Rule

$$y = f_4(f_3(f_2(f_1(x))))$$

$$y = f_4(u_3)$$

$$u_3 = f_3(u_2)$$

$$u_2 = f_2(u_1)$$

$$u_1 = f_1(x)$$

$$\frac{\partial y}{\partial x}$$

$$\frac{\partial y}{\partial u_1}$$

$$\frac{\partial y}{\partial u_2}$$

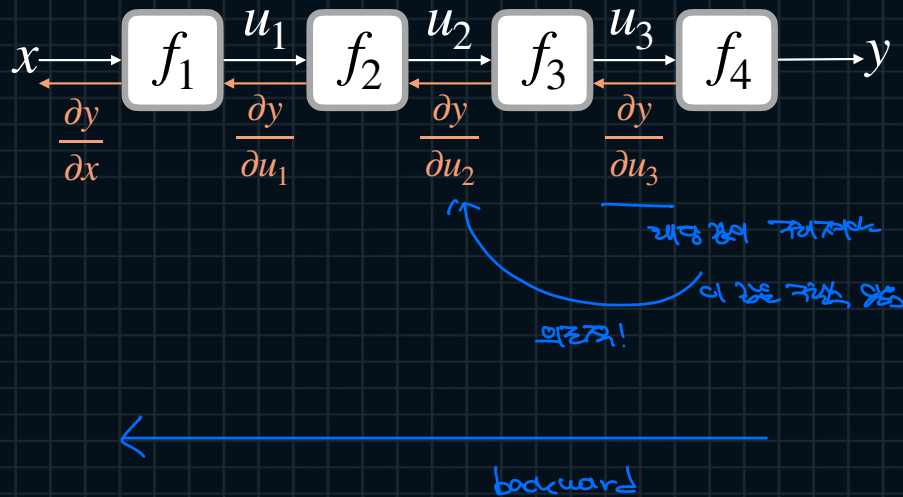
Forward

Backward

Lecture.1 Why Backpropagation and Jacobians?

- Backpropagation

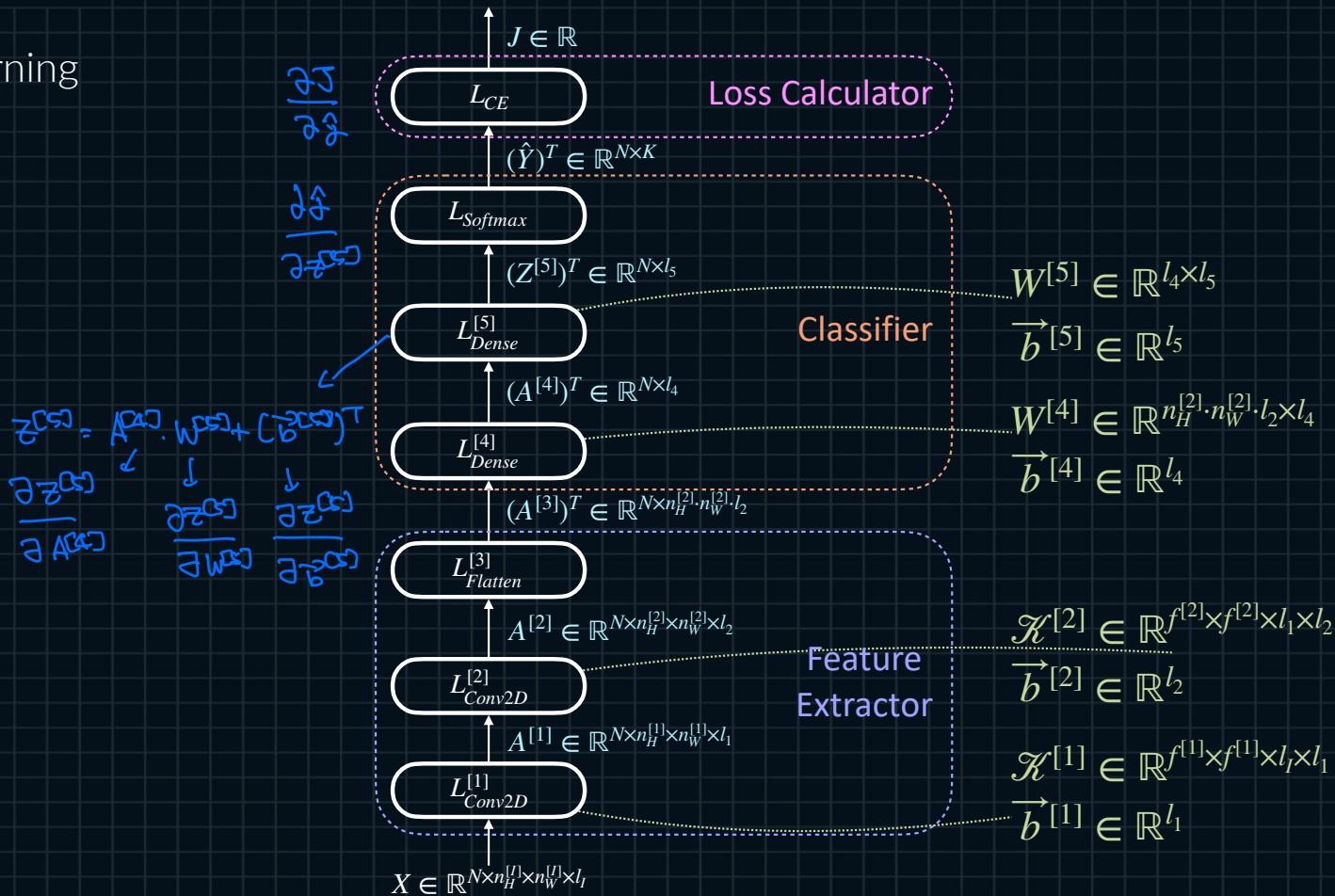
Chain Rule



Lecture.1 Why Backpropagation and Jacobians?

Chain Rule in Deep Learning

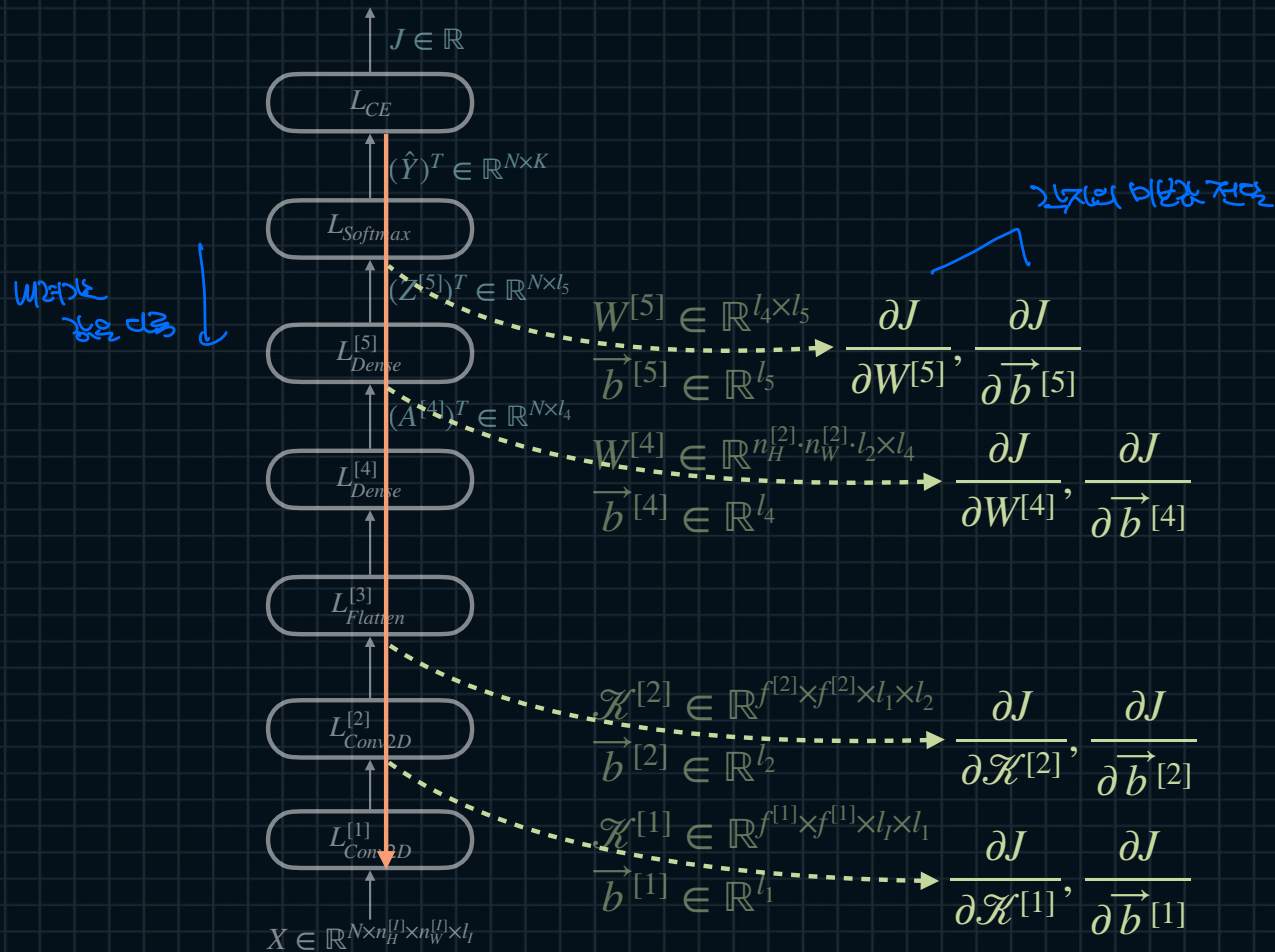
- Backpropagation



Lecture.1 Why Backpropagation and Jacobians?

Backpropagation

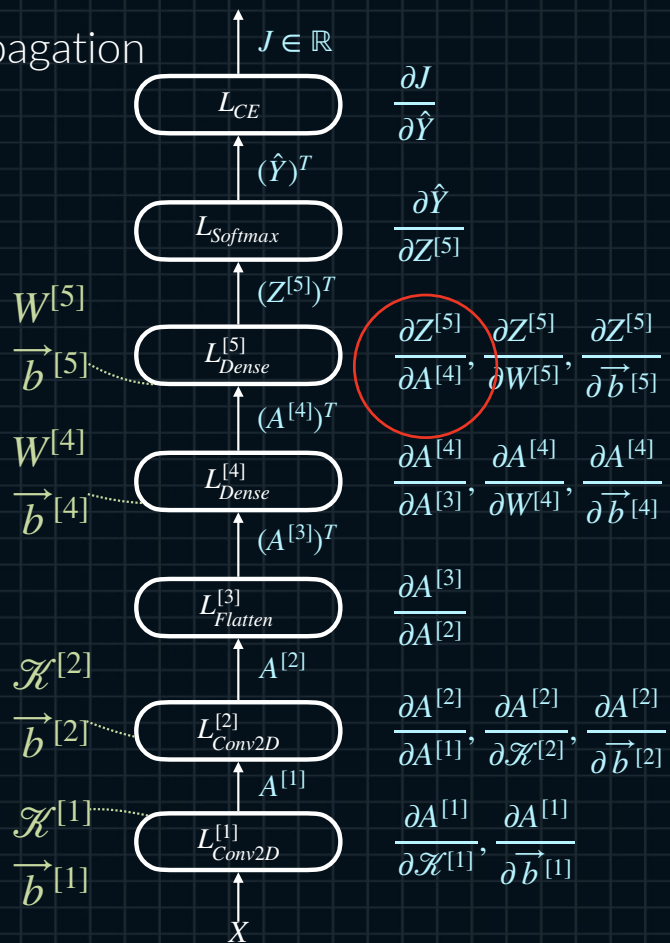
- Backpropagation



Lecture.1 Why Backpropagation and Jacobians?

- Backpropagation

Backpropagation



$$\frac{\partial J}{\partial \hat{Y}} \frac{\partial \hat{Y}}{\partial Z^{[5]}} = \frac{\partial J}{\partial Z^{[5]}}$$

$$\frac{\partial J}{\partial Z^{[5]}} \frac{\partial Z^{[5]}}{\partial A^{[4]}} = \frac{\partial J}{\partial A^{[4]}}$$

$$\frac{\partial J}{\partial A^{[4]}} \frac{\partial A^{[4]}}{\partial A^{[3]}} = \frac{\partial J}{\partial A^{[3]}}$$

$$\frac{\partial J}{\partial A^{[3]}} \frac{\partial A^{[3]}}{\partial A^{[2]}} = \frac{\partial J}{\partial A^{[2]}}$$

$$\frac{\partial J}{\partial A^{[2]}} \frac{\partial A^{[2]}}{\partial A^{[1]}} = \frac{\partial J}{\partial A^{[1]}}$$

$$\frac{\partial J}{\partial Z^{[5]}} \frac{\partial Z^{[5]}}{\partial W^{[5]}} = \frac{\partial J}{\partial W^{[5]}}$$

$$\frac{\partial J}{\partial A^{[4]}} \frac{\partial A^{[4]}}{\partial W^{[4]}} = \frac{\partial J}{\partial W^{[4]}}$$

→ 이항미분 곱셈법인 W, b에 대한 미분

$$\frac{\partial J}{\partial A^{[2]}} \frac{\partial A^{[2]}}{\partial \mathcal{K}^{[2]}} = \frac{\partial J}{\partial \mathcal{K}^{[2]}}$$

$$\frac{\partial J}{\partial A^{[1]}} \frac{\partial A^{[1]}}{\partial \mathcal{K}^{[1]}} = \frac{\partial J}{\partial \mathcal{K}^{[1]}}$$

$$\frac{\partial J}{\partial Z^{[5]}} \frac{\partial Z^{[5]}}{\partial \vec{b}^{[5]}} = \frac{\partial J}{\partial \vec{b}^{[5]}}$$

$$\frac{\partial J}{\partial A^{[4]}} \frac{\partial A^{[4]}}{\partial \vec{b}^{[4]}} = \frac{\partial J}{\partial \vec{b}^{[4]}}$$

$$\frac{\partial J}{\partial A^{[2]}} \frac{\partial A^{[2]}}{\partial \vec{b}^{[2]}} = \frac{\partial J}{\partial \vec{b}^{[2]}}$$

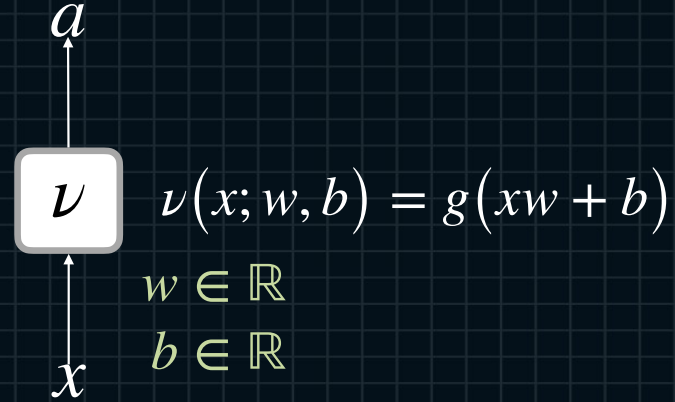
$$\frac{\partial J}{\partial A^{[1]}} \frac{\partial A^{[1]}}{\partial \vec{b}^{[1]}} = \frac{\partial J}{\partial \vec{b}^{[1]}}$$

Lecture.1

Why Backpropagation and Jacobians?

- Why Jacobians?

Derivatives of Scalars




$$w := w - \alpha \frac{\partial J}{\partial w} \quad b := b - \alpha \frac{\partial J}{\partial b}$$

Lecture.1 Why Backpropagation and Jacobians?

- Why Jacobians?

Derivatives of Vectors



$$\nu(\vec{x}; \vec{w}, b) = g((\vec{x})^T \vec{w} + b)$$

$\vec{w} \in \mathbb{R}^{l \times 1}$
 $b \in \mathbb{R}$

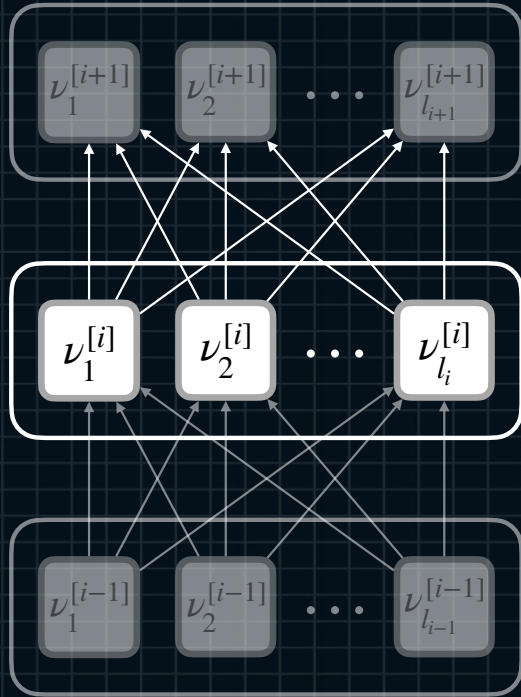
$\vec{w} := \vec{w} - \alpha \frac{\partial J}{\partial \vec{w}}$ *← w의 미분, w를 업데이트*
 $b := b - \alpha \frac{\partial J}{\partial b}$

Lecture.1

Why Backpropagation and Jacobians?

- Why Jacobians?

Derivatives of Matrices



$$W^{[i]} = \begin{pmatrix} \uparrow & \uparrow & \dots & \uparrow \\ \overrightarrow{w}_1^{[i]} & \overrightarrow{w}_2^{[i]} & \dots & \overrightarrow{w}_{l_i}^{[i]} \\ \downarrow & \downarrow & \dots & \downarrow \end{pmatrix} \in \mathbb{R}^{l_{i-1} \times l_i}$$

$$(\overrightarrow{b}^{[i]})^T = \begin{pmatrix} b_1^{[i]} & b_2^{[i]} & \dots & b_{l_i}^{[i]} \end{pmatrix} \in \mathbb{R}^{1 \times l_i}$$

$$W^{[i]} := W^{[i]} - \alpha \frac{\partial J}{\partial W^{[i]}}$$

$$\overrightarrow{b}^{[i]} := \overrightarrow{b}^{[i]} - \alpha \frac{\partial J}{\partial \overrightarrow{b}^{[i]}}$$

Lecture.1 Why Backpropagation and Jacobians?

- Why Jacobians?

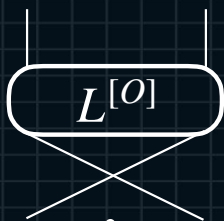
Derivatives of Matrices

$$W^{[O]} = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{w}_1^{[O]} & \vec{w}_2^{[O]} & \dots & \vec{w}_{l_o}^{[O]} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \in \mathbb{R}^{l_{o-1} \times l_o}$$

$$W^{[O]} := W^{[O]} - \alpha \frac{\partial J}{\partial W^{[O]}}$$

$$\vec{b}^{[O]} := \vec{b}^{[O]} - \alpha \frac{\partial J}{\partial \vec{b}^{[O]}}$$

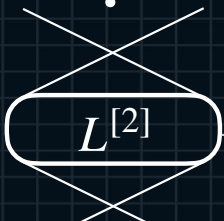
왜? 왜 Jacobian
이냐



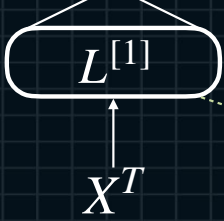
$$W^{[2]} = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{w}_1^{[2]} & \vec{w}_2^{[2]} & \dots & \vec{w}_{l_2}^{[2]} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \in \mathbb{R}^{l_1 \times l_2}$$

$$W^{[2]} := W^{[2]} - \alpha \frac{\partial J}{\partial W^{[2]}}$$

$$\vec{b}^{[2]} := \vec{b}^{[2]} - \alpha \frac{\partial J}{\partial \vec{b}^{[2]}}$$



$$(\vec{b}^{[2]})^T = (b_1^{[2]} \quad b_2^{[2]} \quad \dots \quad b_{l_2}^{[2]}) \in \mathbb{R}^{1 \times l_2}$$



$$W^{[1]} = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{w}_1^{[1]} & \vec{w}_2^{[1]} & \dots & \vec{w}_{l_1}^{[1]} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} \in \mathbb{R}^{l_0 \times l_1}$$

$$W^{[1]} := W^{[1]} - \alpha \frac{\partial J}{\partial W^{[1]}}$$

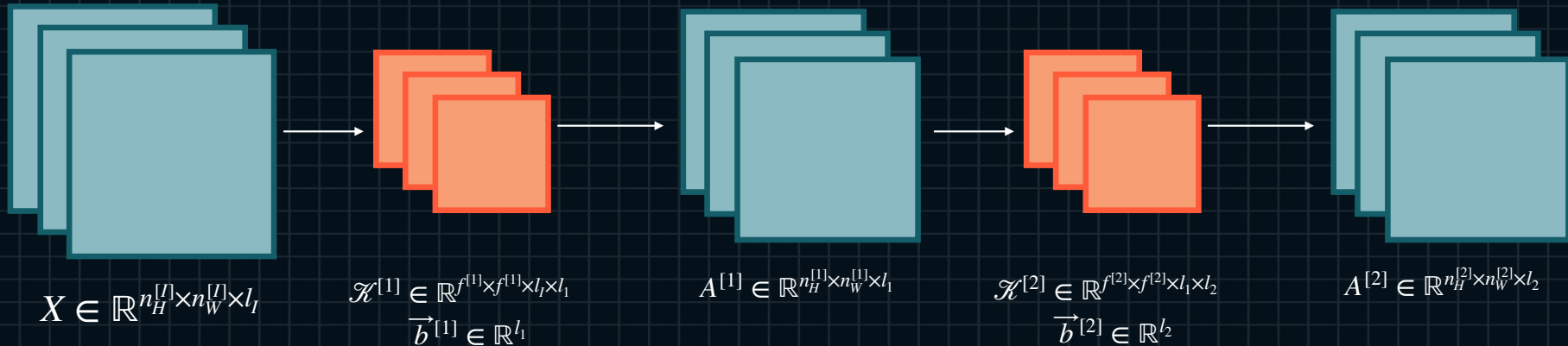
$$\vec{b}^{[1]} := \vec{b}^{[1]} - \alpha \frac{\partial J}{\partial \vec{b}^{[1]}}$$

$$(\vec{b}^{[1]})^T = (b_1^{[1]} \quad b_2^{[1]} \quad \dots \quad b_{l_1}^{[1]}) \in \mathbb{R}^{1 \times l_1}$$

Lecture.1 Why Backpropagation and Jacobians?

- Why Jacobians?

Derivatives of Tensors



$$\mathcal{K}^{[1]} := \mathcal{K}^{[1]} - \alpha \frac{\partial J}{\partial \mathcal{K}^{[1]}}$$

$$\vec{b}^{[1]} := \vec{b}^{[1]} - \alpha \frac{\partial J}{\partial \vec{b}^{[1]}}$$

Conv Layer에
기각되는 모든 텐서
Tensor에 대해 다함

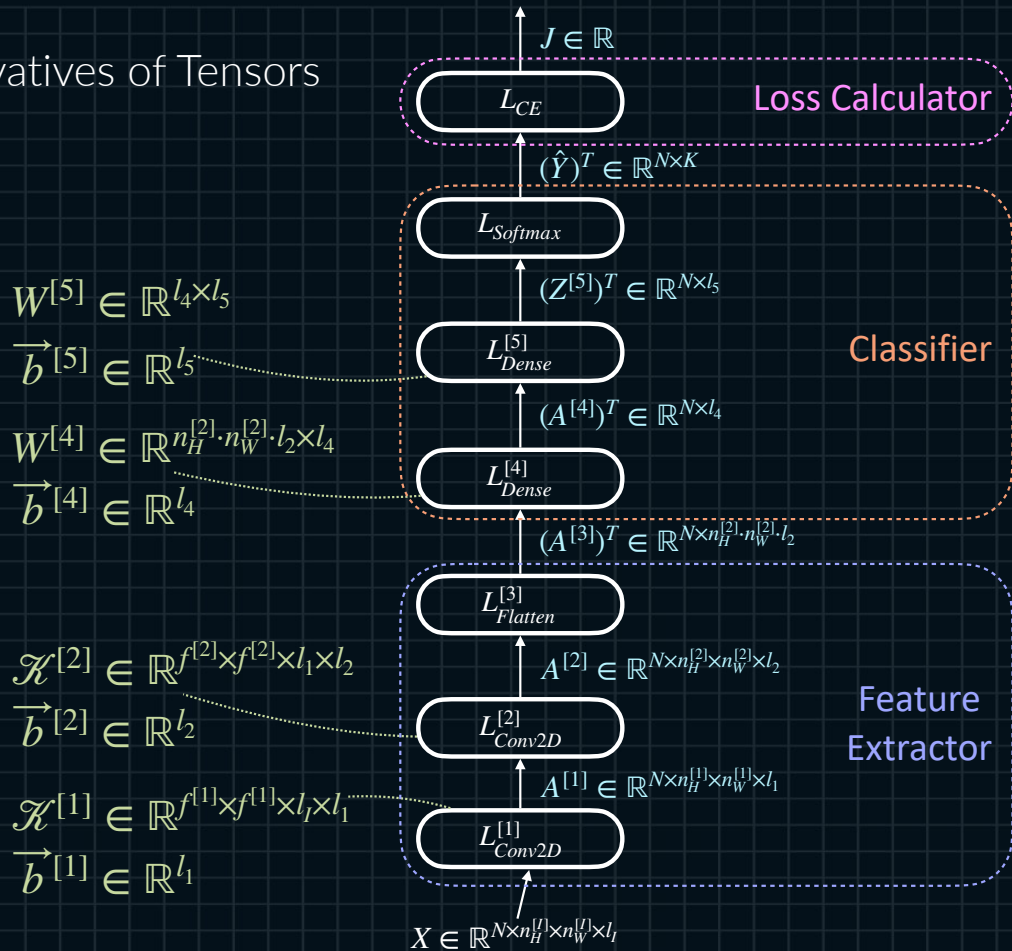
$$\mathcal{K}^{[2]} := \mathcal{K}^{[2]} - \alpha \frac{\partial J}{\partial \mathcal{K}^{[2]}}$$

$$\vec{b}^{[2]} := \vec{b}^{[2]} - \alpha \frac{\partial J}{\partial \vec{b}^{[2]}}$$

Lecture.1 Why Backpropagation and Jacobians?

- Why Jacobians?

Derivatives of Tensors



각각의 미분값을 구해 W, \vec{b} 를 업데이트 하는 과정

$$W^{[5]} := W^{[5]} - \alpha \frac{\partial J}{\partial W^{[5]}}$$

$$\vec{b}^{[5]} := \vec{b}^{[5]} - \alpha \frac{\partial J}{\partial \vec{b}^{[5]}}$$

$$W^{[4]} := W^{[4]} - \alpha \frac{\partial J}{\partial W^{[4]}}$$

$$\vec{b}^{[4]} := \vec{b}^{[4]} - \alpha \frac{\partial J}{\partial \vec{b}^{[4]}}$$

$$\mathcal{K}^{[2]} := \mathcal{K}^{[2]} - \alpha \frac{\partial J}{\partial \mathcal{K}^{[2]}}$$

$$\vec{b}^{[2]} := \vec{b}^{[2]} - \alpha \frac{\partial J}{\partial \vec{b}^{[2]}}$$

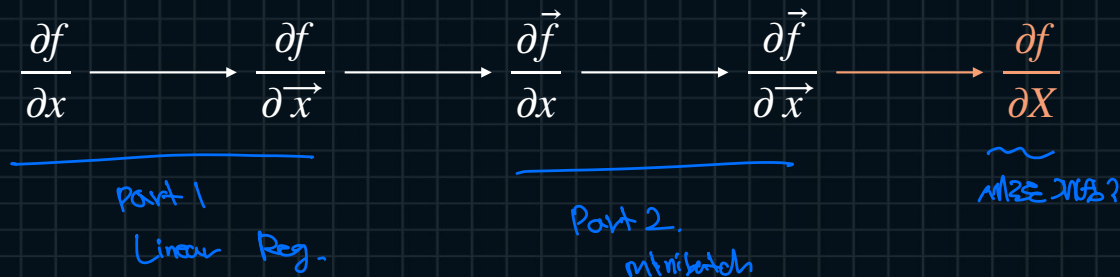
$$\mathcal{K}^{[1]} := \mathcal{K}^{[1]} - \alpha \frac{\partial J}{\partial \mathcal{K}^{[1]}}$$

$$\vec{b}^{[1]} := \vec{b}^{[1]} - \alpha \frac{\partial J}{\partial \vec{b}^{[1]}}$$

Lecture.1 Why Backpropagation and Jacobians?

- Theoretical and Practical Jacobians

Upgrades of Derivatives(Jacobians)



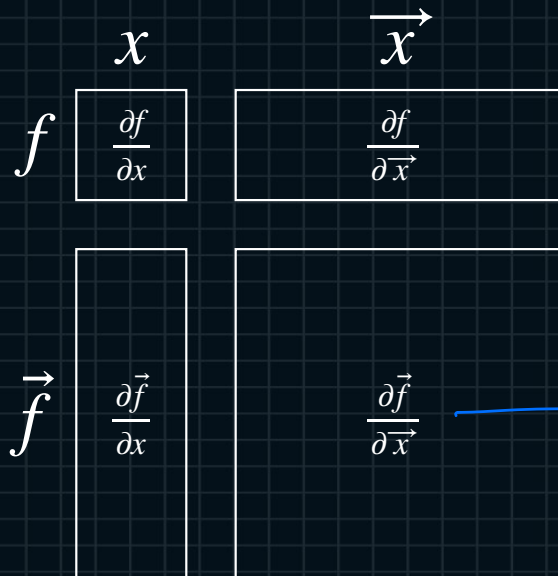
Lecture.1

Why Backpropagation and Jacobians?

- Theoretical and Practical Jacobians

Theoretical Jacobians

이론적 Jacobian / 함수에 대한 미분

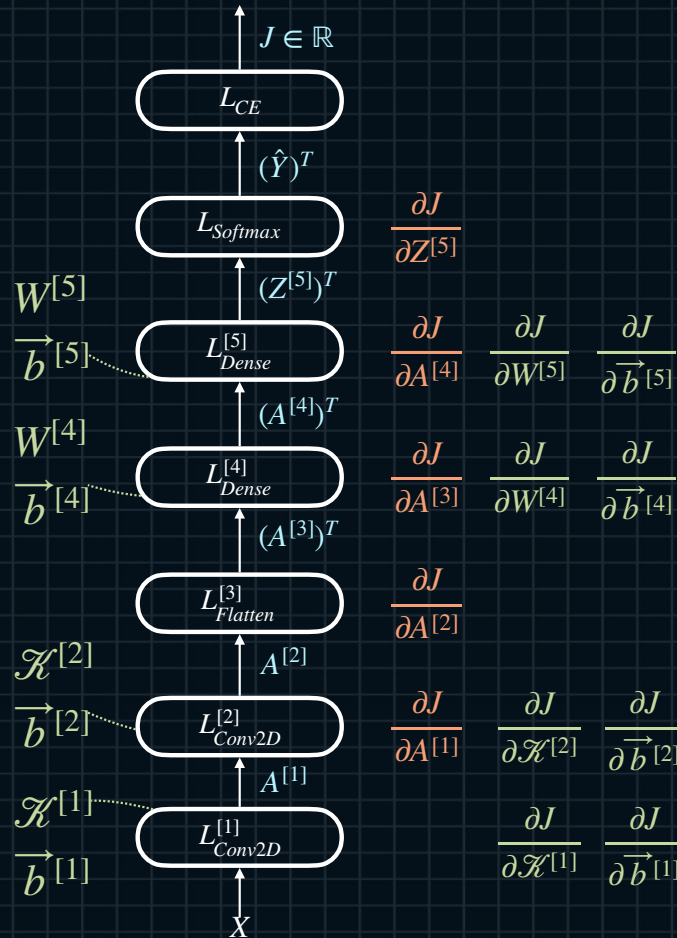


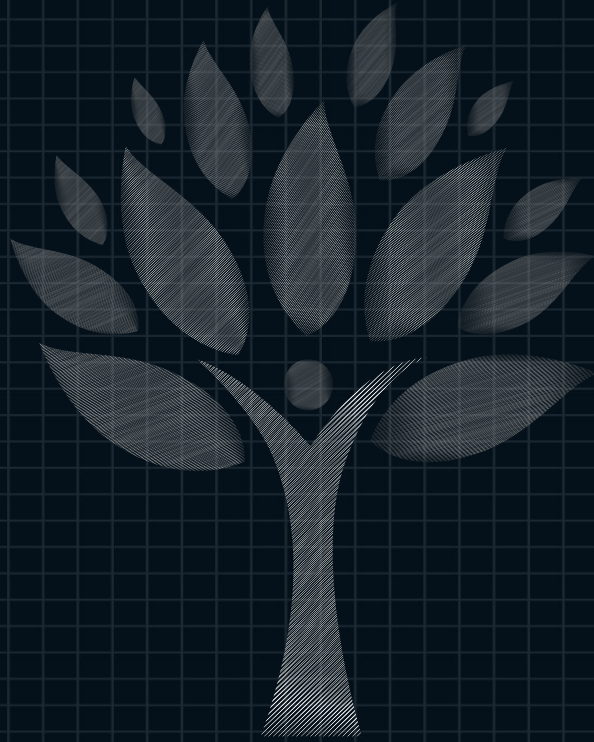
실용적 Jacobian / 벡터에 대한 미분

Lecture.1 Why Backpropagation and Jacobians?

- Theoretical and Practical Jacobians

Practical Jacobians





Backpropagation and Jacobian Matrices

Lecture.1

Why Backpropagation
and Jacobians?