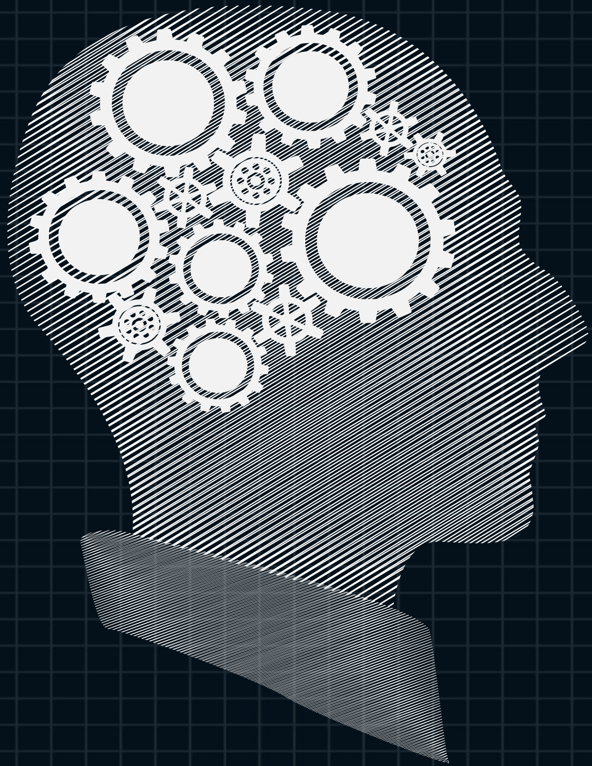


Backpropagation and Jacobian Matrices

Lecture.2

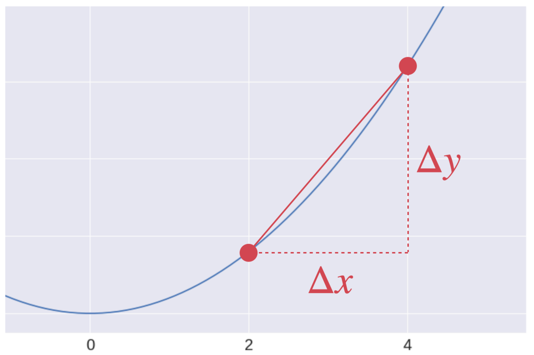
Basic Differentiation

Lecture.2
Basic Differentiation

- Rate of Changes

Average Rate of Change

$$y = x^2$$

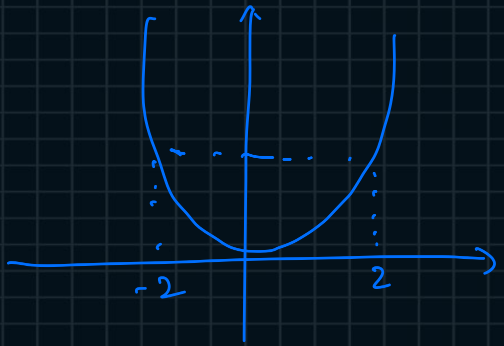


$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

$$\frac{y\text{의 변화량}}{x\text{의 변화량}} = \text{평균 변화율}$$

$$\frac{\Delta y}{\Delta x} = \frac{f(4) - f(2)}{4 - 2}$$

$$= \frac{16 - 4}{4 - 2} = 6$$



$$\rightarrow \frac{\Delta y}{\Delta x} = 0$$

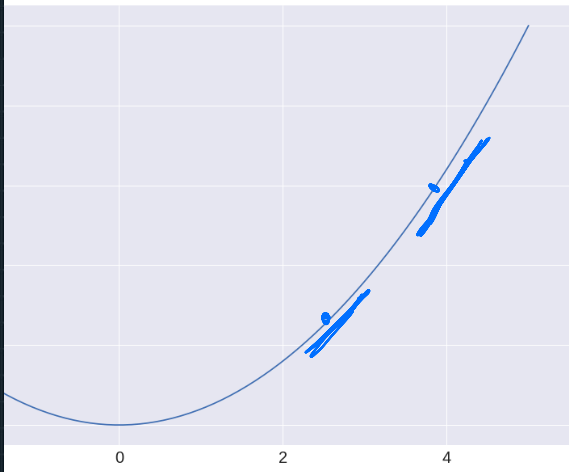
Lecture.2
Basic Differentiation

- Rate of Changes

Instantaneous Rate of Change

= 접선의 기울기

Δx 거리를 극한으로 줄인값



$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

미분계수 = 순간변화율

$$= f'(a) = \frac{dy}{dx}\bigg|_{x=a}$$

$$f'(2) = \lim_{h \to 0} \frac{(2+h)^2 - 2^2}{h} = \lim_{h \to 0} \frac{\cancel{4} + 4h + h^2 - \cancel{4}}{h}$$

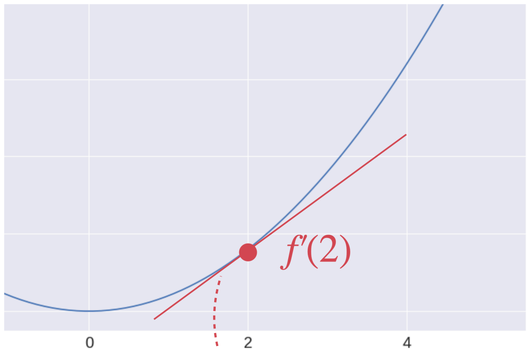
$$= \lim_{h \to 0} [4 + h] = 4$$

← $\lim_{h \to 0} \frac{\cancel{h}(4+h)}{\cancel{h}}$

Lecture.2
Basic Differentiation

- Rate of Changes

Instantaneous Rate of Change

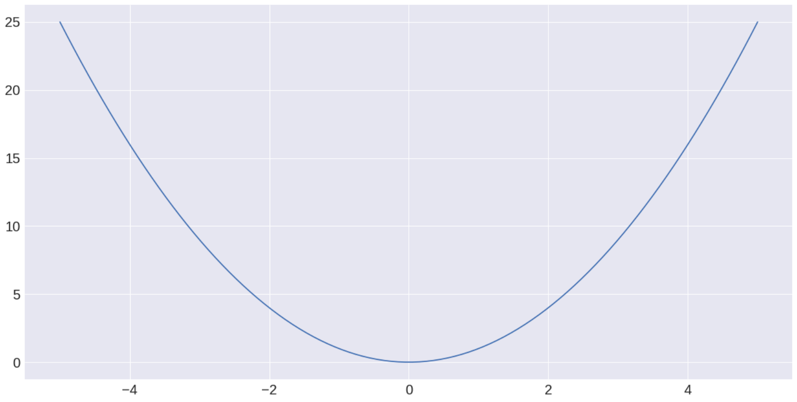


$$y - f(2) = f'(2)(x - 2)$$

Lecture.2
Basic Differentiation - Derivatives and Differentiation

Derivatives

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



$$y = x^2$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \left[2x + h\right] = 2x$$

$$f'(-2) = 2 \cdot (-2) = -4$$

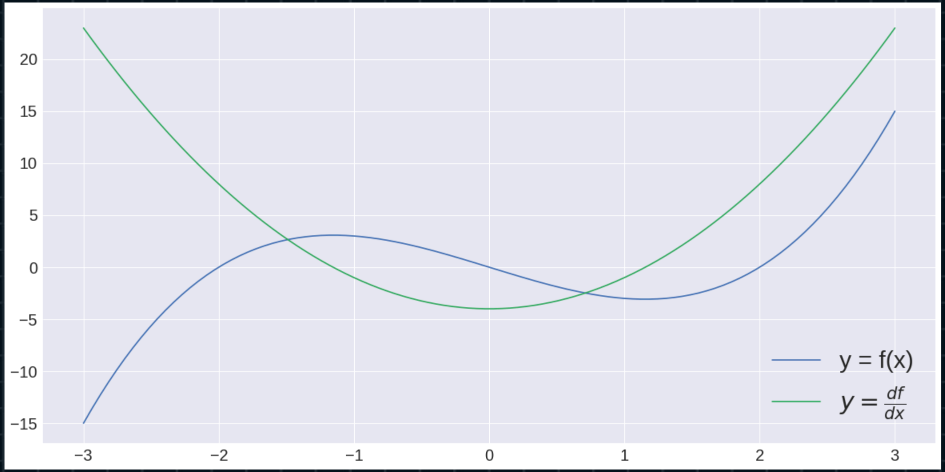
$$f'(0) = 2 \cdot 0 = 0$$

$$f'(2) = 2 \cdot 2 = 4$$

Lecture.2
Basic Differentiation

- Derivatives and Differentiation

Derivatives



$$f(x) = x(x+2)(x-2)$$

도함수 → 순간변화율을 계산한 함수

derivative

도함수를 구하는 과정 → 미분

$f(4) = 8$

x가 4일때 미분계수 8

Lecture.2
Basic Differentiation

- Diff. of Basic Functions

Constant Functions

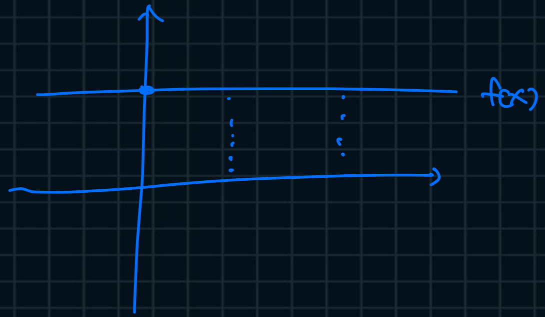
$$f(x) = c$$

상함수

x는 움직여도 값이 동일

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = 0$$

$$f(x) = c \implies \boxed{f'(x) = 0}$$



$f(x) = 100$

$f(x) = e^2 - ln(30)$

Lecture.2
Basic Differentiation

- Diff. of Basic Functions

Power Functions

$$f(x) = x^c$$ c, 정수

$$f'(x) = c \cdot x^{c-1}$$

$f(x) = x^2$ → $2x$

$f(x) = x^{10}$ → $10x^9$

$$f(x) = \frac{1}{x} \dashrightarrow f(x) = x^{-1}$$

$$f(x) = \frac{1}{x^2} \dashrightarrow f(x) = x^{-2}$$

$$f(x) = \sqrt{x} \dashrightarrow f(x) = x^{\frac{1}{2}}$$

$$f(x) = \frac{1}{x \cdot \sqrt[3]{x}} \dashrightarrow f(x) = x^{-\frac{4}{3}}$$

Lecture.2
Basic Differentiation

- Diff. of Basic Functions

Power Functions

$$f(x) = x^c \quad \Longrightarrow \quad f'(x) = c \cdot x^{c-1}$$

$f(x) = x$ $\quad 1$

$f(x) = x^2$ $\quad 2x$

$f(x) = \frac{1}{x} = x^{-1}$ $\quad -1 \cdot x^{-2} = -\frac{1}{x^2}$

$f(x) = \frac{-2}{x^3}$ $\quad -2 \cdot x^{-3} = -2 \cdot (-3) x^{-2} = \frac{6}{x^2}$

$f(x) = \sqrt{x}$ $\quad x^{\frac{1}{2}} = \frac{1}{2} \cdot x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$f(x) = \frac{1}{\sqrt[3]{x}}$ $\quad x^{-\frac{1}{3}} = -\frac{1}{3} x^{-\frac{4}{3}} = -\frac{1}{3} \cdot \frac{1}{\sqrt[3]{x^4}} = -\frac{1}{3\sqrt[3]{x^4}}$

Lecture.2
Basic Differentiation

- Diff. of Basic Functions

Logarithmic Functions

$$f(x) = log_a(x) \implies f'(x) = \frac{1}{x \cdot ln(a)}$$

$f(x) = log_2(x)$

$f(x) = log_e(x) = ln(x)$

‖

$\log x$ (default, $\log_e x$)

$f(x) = \log_2 x$

$f'(x) = \frac{1}{x \cdot \ln 2} = \frac{1}{\ln 2} \cdot \frac{1}{x}$

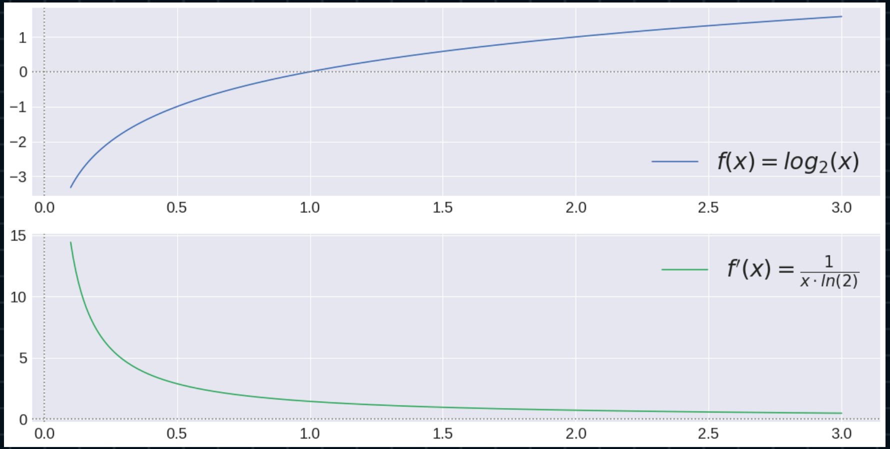
$f(x) = \ln x = \log_e x$

$f'(x) = \frac{1}{x \cdot \underset{1}{\cancel{\ln e}}} = \frac{1}{x}$

Lecture.2
Basic Differentiation

- Diff. of Basic Functions

Logarithmic Functions



도함수의 y값이 모두 >0 이라면.
단순증가 함수로 의미

log 함수는 단순 증가 함수

$a < b \rightarrow f(a) < f(b)$

Monotonically Increasing Functions → 단순증가함수

$$x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$$

Lecture.2
Basic Differentiation

- Diff. of Basic Functions

Inverse Function of Exponential

$$f(x) = a^x$$

$$log(y) = log(a^x) = x \cdot log(a)$$

$$x = \frac{log(y)}{log(a)} = log_a(y)$$

$$f(x) = log_a(x)$$

역함수 → x와 y의 위치 바꿈

$y = \log_a x \Leftrightarrow x = \log_a y$

$a^x = y$

$y = a^x \xleftrightarrow[\text{역함수}]{} y = \log_a x$

Lecture.2
Basic Differentiation

- Diff. of Basic Functions

Exponential Functions

$$f(x) = a^x \implies f'(x) = ln(a) \cdot a^x$$

$f(x) = 2^x$ $f'(x) = \ln 2 \cdot 2^x$

$f(x) = ln^x(2) = (\log_e 2)^x = c^x$ $f'(x) = c^x \cdot \ln(c)$

$= \ln^x(2) \cdot \ln(\ln(2))$

$f(x) = e^x$

$f'(x) = e^x \cdot \underset{1}{\cancel{\ln e}}$

$= e^x$

자연함수 는 도함수 = 원함수

$f(x) = a^x$

$f'(x) = \ln(a) \cdot a^x$

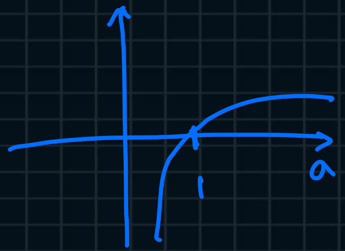
$y = a^x$

$y' = a^x \cdot \ln a$ → 단조 증가/감소 함수

$f'(x) = a^x \cdot \ln(a)$

지수함수의 도함수 $= f(x) \cdot \underline{\ln(a)}$

상수배 만큼 커짐



$a > 1,\ \ln(a) > 0$

$0 < a < 1,\ \ln(a) < 0$

- Diff. of Basic Functions

Trigonometric Equalities

$$\lim_{x\to 0}\frac{sin(x)}{x} = 1,\ \lim_{x\to 0}\frac{tan(x)}{x} = 1$$

$$sin(A) + sin(B) = 2sin\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right)$$

$$sin(A) - sin(B) = 2cos\left(\frac{A+B}{2}\right)sin\left(\frac{A-B}{2}\right)$$

$$cos(A) + cos(B) = 2cos\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right)$$

$$cos(A) - cos(B) = -2sin\left(\frac{A+B}{2}\right)sin\left(\frac{A-B}{2}\right)$$

Lecture.2
Basic Differentiation

- Diff. of Basic Functions

Diff. of Sin Function

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{sin(x+h)-sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{2cos\left(x+\frac{h}{2}\right)sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h\to 0}\frac{cos\left(x+\frac{h}{2}\right)sin\left(\frac{h}{2}\right)}{\frac{h}{2}} = cos(x)$$

$f(x) = sin(x)$

$f'(x) = cos(x)$

Lecture.2
Basic Differentiation

- Diff. of Basic Functions

Trigonometric Functions

$$f(x) = sin(x) \implies f'(x) = cos(x)$$

$$f(x) = cos(x) \implies f'(x) = -\,sin(x)$$

$$f(x) = tan(x) \implies f'(x) = sec^2(x) = \frac{1}{cos^2(x)}$$

$$\sec(x) = \frac{1}{\cos(x)}$$ (역함수 관계)

$f(x) = \cos(x)$

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \quad , \quad \frac{\cos(x+h)-\cos(x)}{h} = \frac{-2\sin\left(\frac{x+h+x}{2}\right)\cdot\sin\left(\frac{x+h-x}{2}\right)}{h}$$

$$= \frac{-\cancel{2}\sin\left(\frac{2x+h}{2}\right)\cdot\cancel{\sin\left(\frac{h}{2}\right)}}{\cancel{\frac{h}{2}}}$$

$$= -\sin\left(\frac{2x+h}{2}\right) = -\sin(x)$$

Lecture.2
Basic Differentiation

- Diff. of Basic Functions

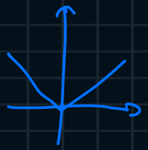
Piecewise-defined Functions

← 구간별로 정의된 함수

$$f(x) = \begin{cases} f_1(x), & x \geq \alpha \\ f_2(x), & x < \alpha \end{cases} \Longrightarrow f'(x) = \begin{cases} f_1'(x), & x \geq \alpha \\ f_2'(x), & x < \alpha \end{cases}$$

↓ 구간별로 미분하면 됨

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



$$f'(x) \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

$$f(x) = ReLU(x) = max(0, x)$$

0과 비교해서 큰값

$$f'(x) \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Lecture.2
Basic Differentiation - Differentiation Rules

Constant Multiple Rule

상수가 곱해져있을 때의 미분

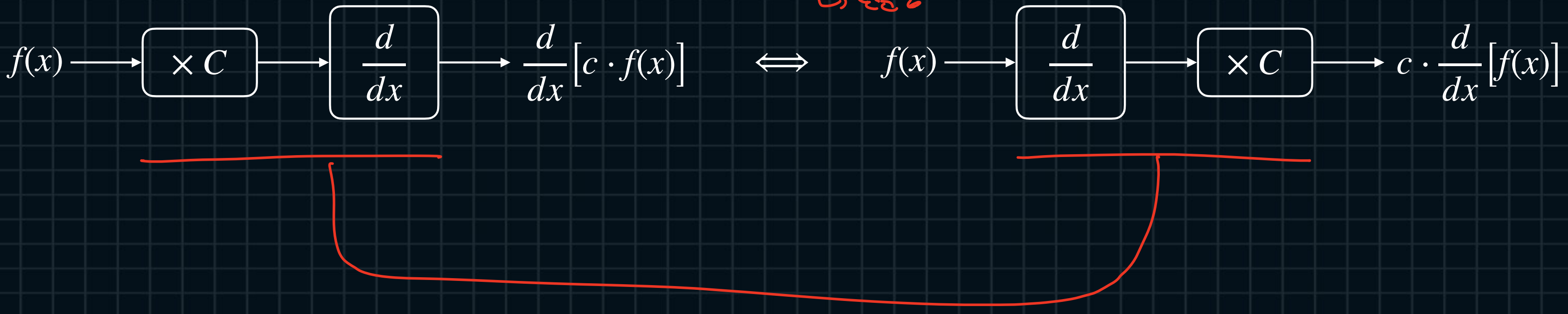
$$f(x) = x \implies 2 \cdot f(x) = 2x$$

$$f(x) = e^x \implies -3 \cdot f(x) = -3e^x$$

$$f(x) = sin(x) \implies e \cdot f(x) = e \cdot sin(x)$$

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)]$$

└ 탈출



$$f(x) \longrightarrow \boxed{\times C} \longrightarrow \boxed{\frac{d}{dx}} \longrightarrow \frac{d}{dx}[c \cdot f(x)] \iff f(x) \longrightarrow \boxed{\frac{d}{dx}} \longrightarrow \boxed{\times C} \longrightarrow c \cdot \frac{d}{dx}[f(x)]$$

순서 바꿔도 됨

Lecture.2
Basic Differentiation

- Differentiation Rules

Constant Multiple Rule

$$\frac{d}{dx}\left[c \cdot f(x)\right] = c \cdot \frac{d}{dx}\left[f(x)\right]$$

$f(x) = 2x$ $f'(x)=2$

$f(x) = -3e^x$ $f'(x)=-3e^x$

Lecture.2
Basic Differentiation

- Differentiation Rules

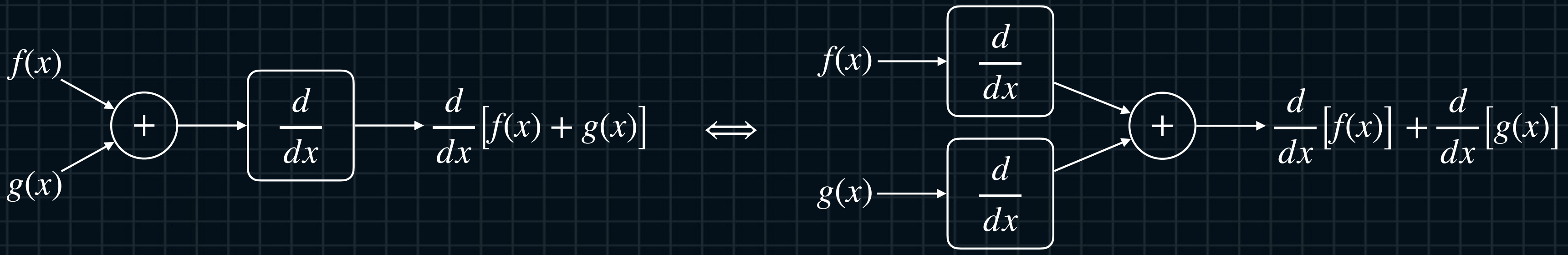
Sum Rule

다항함수의 경우

각 항으로 미분하여 더하기

$$f(x) = 3x^3 - 2x^2 + 10x - 20$$

$$\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{df}{dx} + \frac{dg}{dx}$$



$f(x)$
$g(x)$
$+$
$\frac{d}{dx}$
$\frac{d}{dx}\left[f(x) + g(x)\right]$
$\Longleftrightarrow$
$f(x)$
$g(x)$
$\frac{d}{dx}$
$\frac{d}{dx}$
$+$
$\frac{d}{dx}\left[f(x)\right] + \frac{d}{dx}\left[g(x)\right]$

Lecture.2
Basic Differentiation

- Differentiation Rules

Sum Rule

$$\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{df}{dx} + \frac{dg}{dx}$$

$$f(x) = 2x^2 - x + 7$$

$f'(x) = 4x - 1$

$$f(x) = sin(x) - \frac{1}{\sqrt{x}}$$

$f'(x) = cos(x) - x^{-\frac{1}{2}}$

$= cos(x) - (-\frac{1}{2}) \cdot x^{-\frac{3}{2}}$

$= cos(x) + \frac{1}{2} \cdot \frac{1}{x\sqrt{x}}$

$$f(x) = sinh(x) = \frac{e^x - e^{-x}}{2}$$

$= \frac{1}{2}e^x - \frac{1}{2}e^{-x}$

$f'(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x} = \frac{e^x + e^{-x}}{2}$

$y = f(g(x))$

$y' = f'(g(x)) \cdot g'(x)$

$$f(x) = cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\begin{tikzcd} sinh(x) \arrow[r, shift left] & cosh(x) \arrow[l, shift left, "\frac{d}{dx}"] \end{tikzcd}$$

상호 미분 관계

Lecture.2 Basic Differentiation

- Differentiation Rules

Linearity of Diff.

$$Sys\{\alpha \cdot f(t)\} = \alpha \cdot Sys\{f(t)\}$$ Homogeneity

$$Sys\{f(t) + g(t)\} = Sys\{f(t)\} + Sys\{g(t)\}$$ Additivity

$$Sys\{\alpha \cdot f(t) + \beta \cdot g(t)\} = \alpha \cdot Sys\{f(t)\} + \beta \cdot Sys\{g(t)\}$$

$$\frac{d}{dx}\left[c \cdot f(x)\right] = c \cdot \frac{d}{dx}\left[f(x)\right]$$ Constant Multiple Rule

$$\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{df}{dx} + \frac{dg}{dx}$$ Sum Rule

$$\frac{d}{dx}\left[\alpha \cdot f(x) + \beta \cdot g(x)\right] = \alpha \cdot \frac{d}{dx}\left[f(x)\right] + \beta \cdot \frac{d}{dx}\left[g(x)\right]$$

Lecture.2
Basic Differentiation

- Differentiation Rules

Time-invariance of Diff.

$$Sys\{f(t)\} = f'(t) \quad \Longrightarrow \quad Sys\{f(t-\tau)\} = f'(t-\tau)$$

$$\frac{df(t)}{dt} = f'(t) \quad \Longrightarrow \quad \frac{df(t-\tau)}{dt} = f'(t-\tau)$$

- Differentiation Rules

LTI Systems and Diff.

$$Sys\{\alpha \cdot f(t) + \beta \cdot g(t)\} = \alpha \cdot Sys\{f(t)\} + \beta \cdot Sys\{g(t)\}$$

$$Sys\{x(t)\} = y(t) \quad \Rightarrow \quad Sys\{x(t-\tau)\} = y(t-\tau)$$

$$\frac{d}{dx}\left[\alpha \cdot f(t-\tau) + \beta \cdot f(t-\tau)\right] = \alpha \cdot f'(t-\tau) + \beta \cdot g'(t-\tau)$$

Lecture.2
Basic Differentiation

- Differentiation Rules

Product Rule

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right] = \frac{d}{dx}\left[f(x)\right]\cdot g(x) + f(x)\cdot\frac{d}{dx}\left[g(x)\right]$$

$f(x) = x^3$

$f(x) = e^x ln(x)$

Lecture.2
Basic Differentiation

- Differentiation Rules

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)] \cdot g(x) - f(x) \cdot \frac{d}{dx}[g(x)]}{\left(g(x)\right)^2}$$

$$f(x) = \frac{1}{x}$$

$$f(x) = \frac{x^2}{e^x}$$

$$f(x) = tanh(x) = \frac{sinh(x)}{cosh(x)}$$

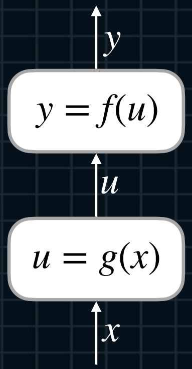
Lecture.2
Basic Differentiation
- Composite Functions and Chain Rule

Composite Functions

$$y = f(g(x))$$

$$u = g(x)$$

$$y = f(u)$$



Lecture.2
Basic Differentiation

- Composite Functions and Chain Rule

Chain Rule

$$y = f(g(x))$$

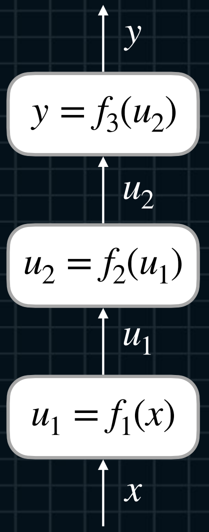
$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

Lecture.2
Basic Differentiation

- Composite Functions and Chain Rule

Chain Rule

$$y = f_1(f_2(f_3(x)))$$



y

$y = f_3(u_2)$

u_2

$u_2 = f_2(u_1)$

u_1

$u_1 = f_1(x)$

x

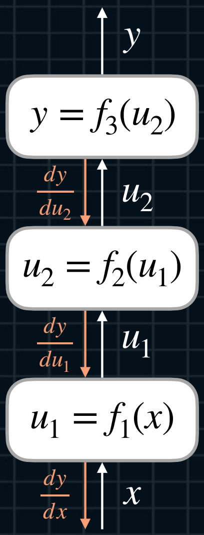
$$\frac{du_1}{dx} \qquad \frac{dy}{dx} = \frac{dy}{du_1}\frac{du_1}{dx}$$

$$\frac{du_2}{du_1} \qquad \frac{dy}{du_1} = \frac{dy}{du_2}\frac{du_2}{du_1}$$

$$\frac{dy}{du_2}$$

Chain Rule

$$y = f_1(f_2(f_3(x)))$$



y

$y = f_3(u_2)$

$\frac{dy}{du_2}$ u_2

$u_2 = f_2(u_1)$

$\frac{dy}{du_1}$ u_1

$u_1 = f_1(x)$

$\frac{dy}{dx}$ x

- Composite Functions and Chain Rule

Forward/Backward Calculations

$$y = f_4(f_3(f_2(f_1(x))))$$

Lecture.2
Basic Differentiation

- Composite Functions and Chain Rule

Exercises

$$f(x) = cos(x^3)$$

$$f(t) = sin(t - \tau)$$

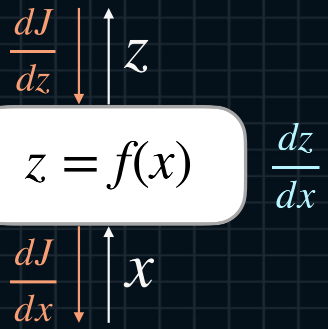
$$f(x) = (a - x)^2$$

$$f(x) = \frac{1}{1 + e^{-x}}$$

Lecture.2
Basic Differentiation

- Composite Functions and Chain Rule

Modules of Backpropagation

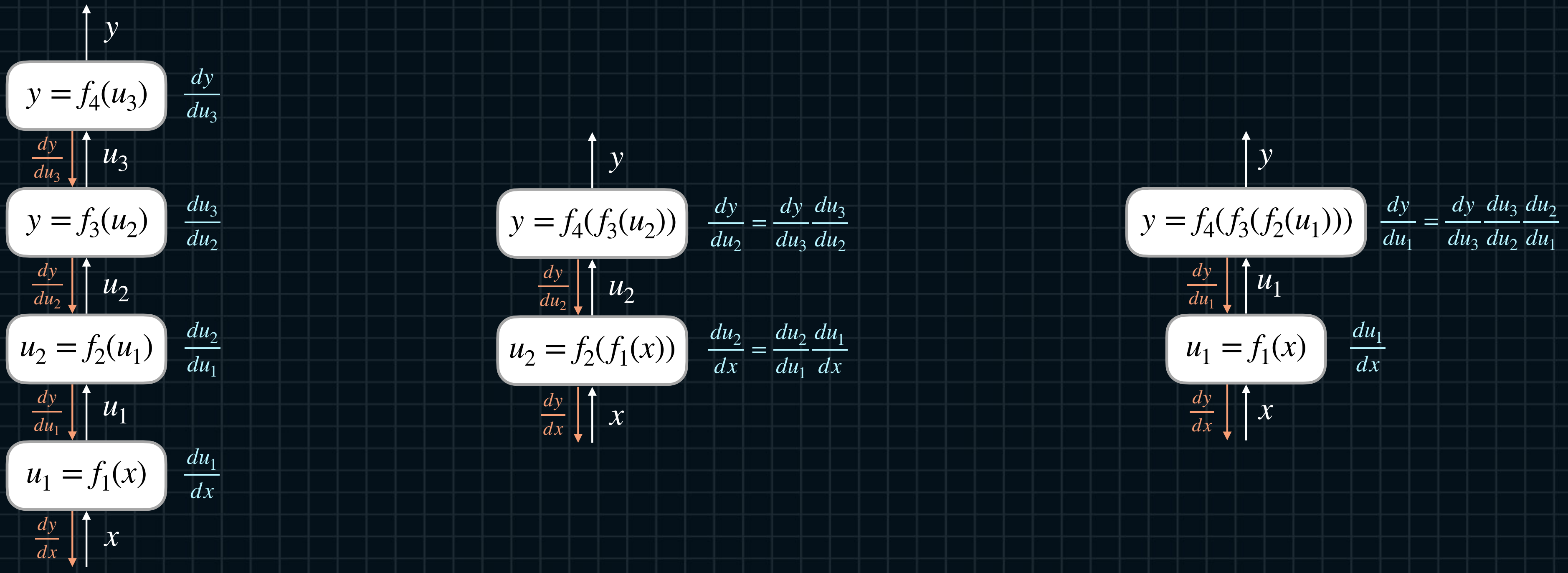

$\frac{dJ}{dz}$
z
$z = f(x)$
$\frac{dz}{dx}$
$\frac{dJ}{dx}$
x

$$\frac{dJ}{dz}\frac{dz}{dx} = \frac{dJ}{dx}$$

Lecture.2
Basic Differentiation

- Composite Functions and Chain Rule

Merging Modules



y

$y = f_4(u_3)$ $\quad \frac{dy}{du_3}$

$\frac{dy}{du_3}$ $\quad u_3$

$y = f_3(u_2)$ $\quad \frac{du_3}{du_2}$

$\frac{dy}{du_2}$ $\quad u_2$

$u_2 = f_2(u_1)$ $\quad \frac{du_2}{du_1}$

$\frac{dy}{du_1}$ $\quad u_1$

$u_1 = f_1(x)$ $\quad \frac{du_1}{dx}$

$\frac{dy}{dx}$ $\quad x$

y

$y = f_4(f_3(u_2))$ $\quad \frac{dy}{du_2} = \frac{dy}{du_3}\frac{du_3}{du_2}$

$\frac{dy}{du_2}$ $\quad u_2$

$u_2 = f_2(f_1(x))$ $\quad \frac{du_2}{dx} = \frac{du_2}{du_1}\frac{du_1}{dx}$

$\frac{dy}{dx}$ $\quad x$

y

$y = f_4(f_3(f_2(u_1)))$ $\quad \frac{dy}{du_1} = \frac{dy}{du_3}\frac{du_3}{du_2}\frac{du_2}{du_1}$

$\frac{dy}{du_1}$ $\quad u_1$

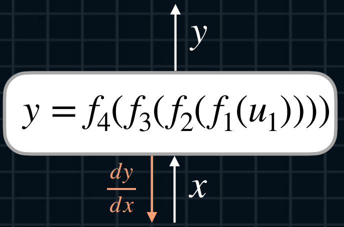
$u_1 = f_1(x)$ $\quad \frac{du_1}{dx}$

$\frac{dy}{dx}$ $\quad x$

Lecture.2
Basic Differentiation

- Composite Functions and Chain Rule

Merging Modules

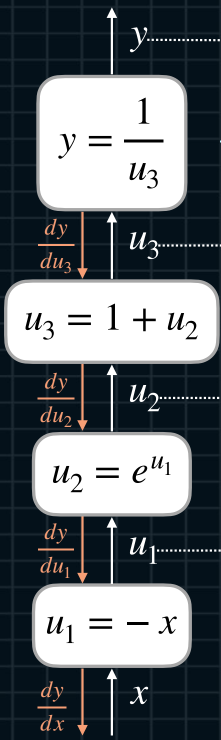

y
$y = f_4(f_3(f_2(f_1(u_1))))$
$\frac{dy}{dx}$
x

$$\frac{dy}{dx} = \frac{dy}{du_3}\frac{du_3}{du_2}\frac{du_2}{du_1}\frac{du_1}{dx}$$

Lecture.2
Basic Differentiation

- Composite Functions and Chain Rule

Merging Modules



y
$\frac{dy}{du_3}$
u_3
$\frac{dy}{du_2}$
u_2
$\frac{dy}{du_1}$
u_1
$\frac{dy}{dx}$
x

$$y = \frac{1}{u_3}$$

$$u_3 = 1 + u_2$$

$$u_2 = e^{u_1}$$

$$u_1 = -x$$

$$y = \frac{1}{1+e^{-x}}$$

$$\frac{dy}{du_3} = -\frac{1}{(u_3)^2}$$

$$u_3 = 1 + e^{-x}$$

$$\frac{du_3}{du_2} = 1$$

$$u_2 = e^{-x}$$

$$\frac{du_2}{du_1} = e^{u_1}$$

$$u_1 = -x$$

$$\frac{du_1}{dx} = -1$$

$$\frac{dy}{du_3} = -\frac{1}{(u_3)^2} = -\frac{1}{(1+e^{-x})^2}$$

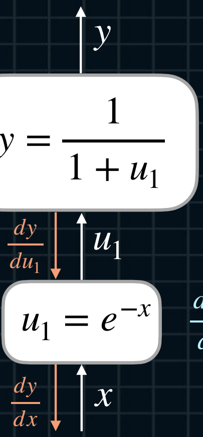
$$\frac{dy}{du_2} = \frac{dy}{du_3}\frac{du_3}{du_2} = -\frac{1}{(1+e^{-x})^2}$$

$$\frac{dy}{du_1} = \frac{dy}{du_2}\frac{du_2}{du_1} = -\frac{2}{(1+e^{-x})^2} \cdot e^{u_1}$$

$$= -\frac{e^{-x}}{(1+e^{-x})^2}$$

$$\frac{dy}{dx} = \frac{dy}{du_1}\frac{du_1}{dx} = \frac{e^{-x}}{(1+e^{-x})^2}$$

$$= y(1-y)$$



y
$\frac{dy}{du_1}$
u_1
$\frac{dy}{dx}$
x

$$y = \frac{1}{1+u_1}$$

$$u_1 = e^{-x}$$

$$\frac{dy}{du_1} = -\frac{1}{(1+u_1)^2}$$

$$\frac{du_1}{dx} = -e^{-x}$$

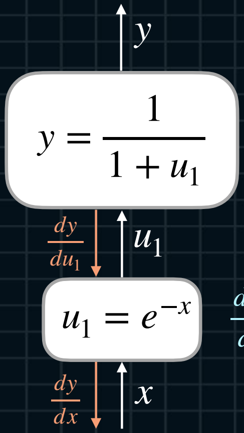
$$\frac{dy}{du_1} = -\frac{1}{(1+u_1)^2} = -\frac{1}{(1+e^{-x})^2}$$

$$\frac{dy}{du_1} = \frac{e^{-x}}{(1+e^{-x})^2} = y(1-y)$$

Lecture.2
Basic Differentiation

- Composite Functions and Chain Rule

Merging Modules

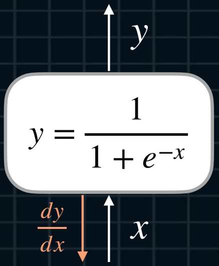

y
$y = \frac{1}{1+u_1}$
$\frac{dy}{du_1}$
u_1
$u_1 = e^{-x}$
$\frac{dy}{dx}$
x

$$\frac{dy}{du_1} = -\frac{1}{(1+u_1)^2}$$

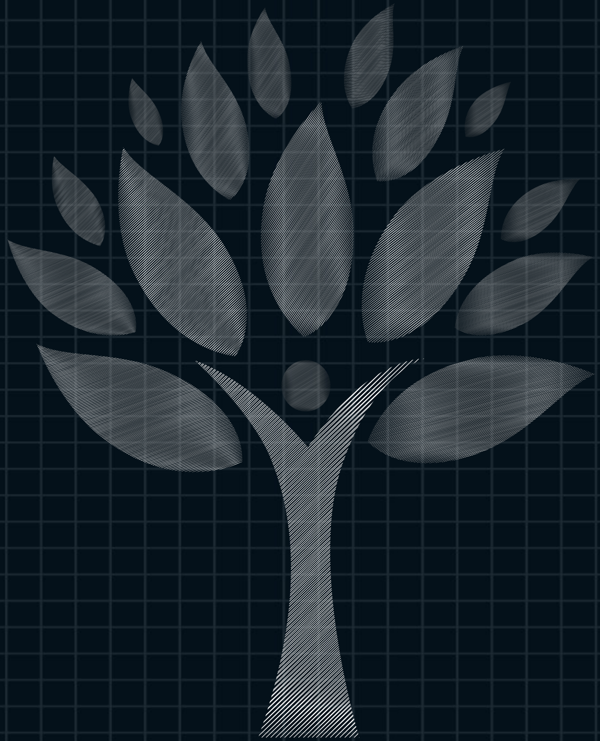
$$\frac{du_1}{dx} = -e^{-x}$$

$$\frac{dy}{du_1} = -\frac{1}{(1+u_1)^2} = -\frac{1}{(1+e^{-x})^2}$$

$$\frac{dy}{du_1} = \frac{e^{-x}}{(1+e^{-x})^2} = y(1-y)$$


y
$y = \frac{1}{1+e^{-x}}$
$\frac{dy}{dx}$
x

$$\frac{dy}{dx} = -y(1-y)$$



Backpropagation and Jacobian Matrices

Lecture.2

Basic Differentiation