

# Backpropagation and Jacobian Matrices

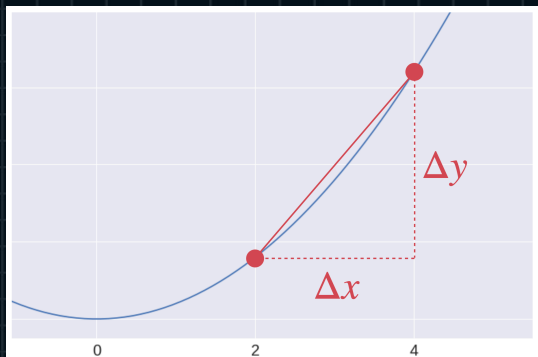
Lecture.2  
Basic Differentiation

## Lecture.2 Basic Differentiation

### - Rate of Changes

#### Average Rate of Change

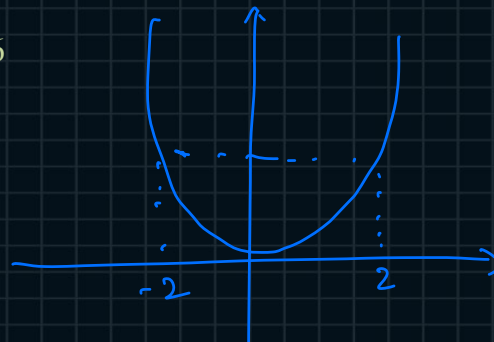
$$y = x^2$$



$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

함수의 변화량  
기저의 변화량 = 평균 변화율

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(4) - f(2)}{4 - 2} \\ &= \frac{16 - 4}{4 - 2} = 6\end{aligned}$$



$$\rightarrow \frac{\Delta y}{\Delta x} = 0$$

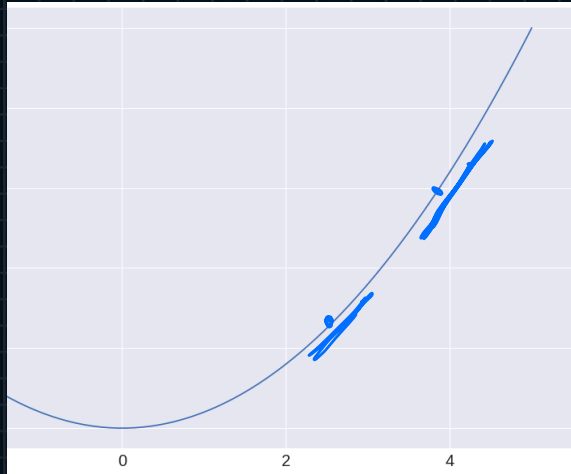
# Lecture.2 Basic Differentiation

## - Rate of Changes

Instantaneous Rate of Change

= 접선의 기울기

$\Delta x$  지대로 극한으로 접근



$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{(x + \Delta x) - x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

미분계수 = 접선의 기울기

$$= f'(a) = \left. \frac{dy}{dx} \right|_{x=a}$$

$$f'(2) = \lim_{h \rightarrow 0} \frac{(2 + h)^2 - 2^2}{h} = \lim_{h \rightarrow 0} \frac{\cancel{4} + 4h + h^2 - \cancel{4}}{h}$$

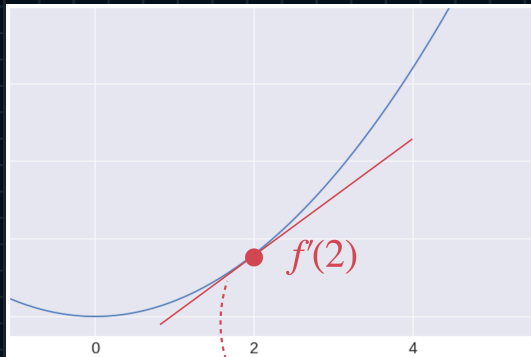
$$= \lim_{h \rightarrow 0} [4 + h] = 4$$

$$\leftarrow \lim_{h \rightarrow 0} \frac{f'(a+h)}{1}$$

## Lecture.2 Basic Differentiation

### - Rate of Changes

Instantaneous Rate of Change

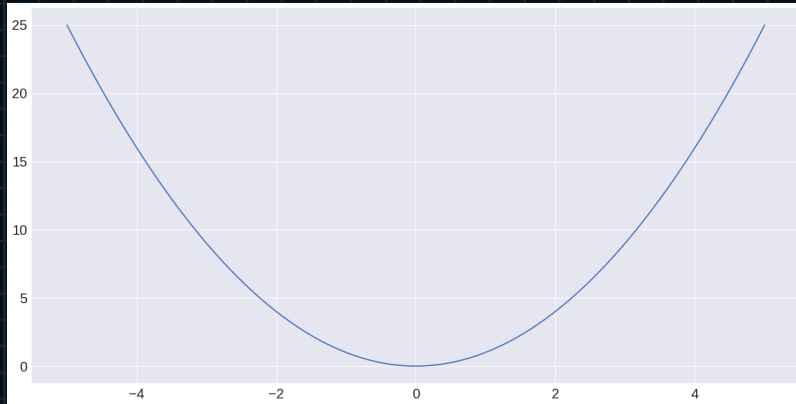


$$y - f(2) = f'(2)(x - 2)$$

## Lecture.2 Basic Differentiation

### - Derivatives and Differentiation

#### Derivatives



$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$y = x^2$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \rightarrow 0} [2x + h] = 2x$$

$$f'(-2) = 2 \cdot (-2) = -4$$

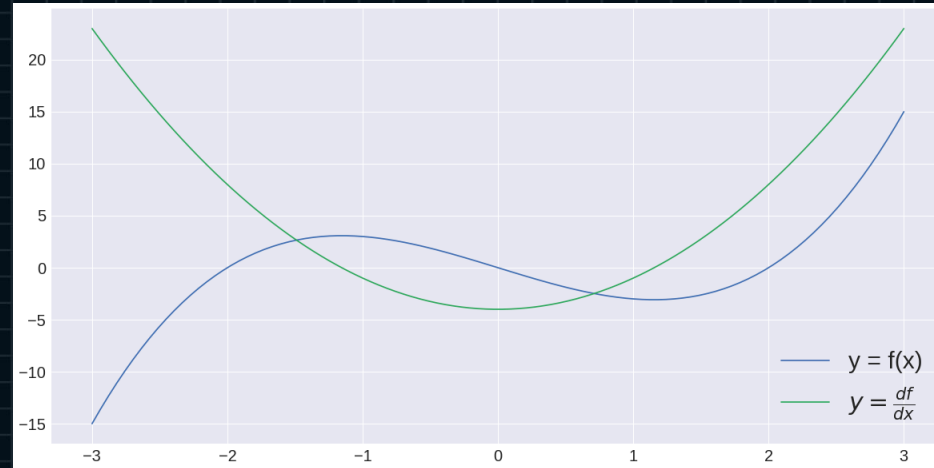
$$f'(0) = 2 \cdot 0 = 0$$

$$f'(2) = 2 \cdot 2 = 4$$

## Lecture.2 Basic Differentiation

### - Derivatives and Differentiation

#### Derivatives



$$f(x) = x(x + 2)(x - 2)$$

## Lecture.2 Basic Differentiation

### - Diff. of Basic Functions

#### Constant Functions

$$f(x) = c$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = 0$$

$$f(x) = c \implies f'(x) = 0$$

$$f(x) = 100$$

$$f(x) = e^2 - \ln(30)$$



## Lecture.2 Basic Differentiation

### - Diff. of Basic Functions

#### Power Functions

$$f(x) = x^c$$

$$f(x) = x^2$$

$$f(x) = x^{10}$$

$$f(x) = \frac{1}{x} \longrightarrow f(x) = x^{-1}$$

$$f(x) = \frac{1}{x^2} \longrightarrow f(x) = x^{-2}$$

$$f(x) = \sqrt{x} \longrightarrow f(x) = x^{\frac{1}{2}}$$

$$f(x) = \frac{1}{x \cdot \sqrt[3]{x}} \longrightarrow f(x) = x^{-\frac{4}{3}}$$



## Lecture.2 Basic Differentiation

### - Diff. of Basic Functions

#### Power Functions

$$f(x) = x^c \implies f'(x) = c \cdot x^{c-1}$$

$$f(x) = x$$

$$f(x) = x^2$$

$$f(x) = \frac{1}{x}$$

$$f(x) = \frac{-2}{x^3}$$

$$f(x) = \sqrt{x}$$

$$f(x) = \frac{1}{\sqrt[3]{x}}$$

## Lecture.2

### Basic Differentiation

### - Diff. of Basic Functions

#### Logarithmic Functions

$$f(x) = \log_a(x) \implies f'(x) = \frac{1}{x \cdot \ln(a)}$$

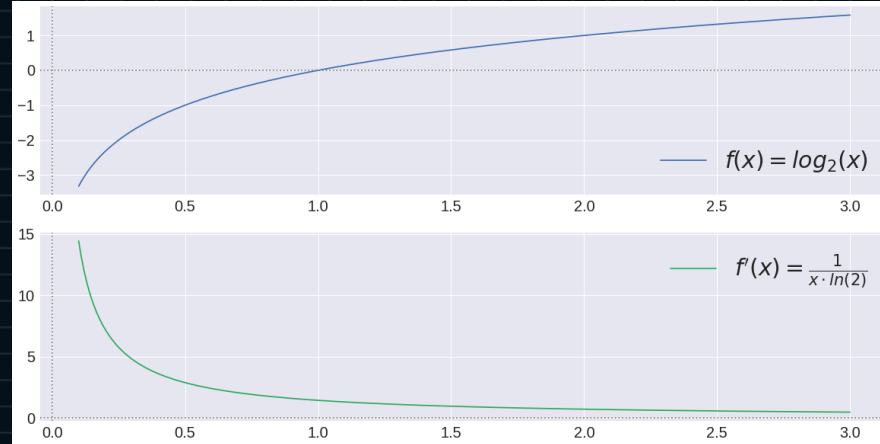
$$f(x) = \log_2(x)$$

$$f(x) = \log_e(x) = \ln(x)$$

## Lecture.2 Basic Differentiation

### - Diff. of Basic Functions

#### Logarithmic Functions



Monotonically Increasing Functions

$$x_1 < x_2 \implies f(x_1) < f(x_2)$$

## Lecture.2 Basic Differentiation

### - Diff. of Basic Functions

Inverse Function of Exponential

$$f(x) = a^x$$

$$\log(y) = \log(a^x) = x \cdot \log(a)$$

$$x = \frac{\log(y)}{\log(a)} = \log_a(y)$$

$$f(x) = \log_a(x)$$

## Lecture.2 Basic Differentiation

### - Diff. of Basic Functions

#### Exponential Functions

$$f(x) = a^x \implies f'(x) = \ln(a) \cdot a^x$$

$$f(x) = 2^x$$

$$f(x) = \ln^x(2)$$

$$f(x) = e^x$$

## Lecture.2

### Basic Differentiation

## - Diff. of Basic Functions

### Trigonometric Equalities

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1, \lim_{x \rightarrow 0} \frac{\tan(x)}{x} = 1$$

$$\sin(A) + \sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin(A) - \sin(B) = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos(A) + \cos(B) = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos(A) - \cos(B) = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

## Lecture.2 Basic Differentiation

### - Diff. of Basic Functions

#### Diff. of Sin Function

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2\cos\left(x + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cos\left(x + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} = \cos(x)$$



## Lecture.2 Basic Differentiation

### - Diff. of Basic Functions

#### Trigonometric Functions

$$f(x) = \sin(x) \implies f'(x) = \cos(x)$$

$$f(x) = \cos(x) \implies f'(x) = -\sin(x)$$

$$f(x) = \tan(x) \implies f'(x) = \sec^2(x) = \frac{1}{\cos^2(x)}$$

## Lecture.2 Basic Differentiation

### - Diff. of Basic Functions

#### Piecewise-defined Functions

$$f(x) = \begin{cases} f_1(x), & x \geq \alpha \\ f_2(x), & x < \alpha \end{cases} \implies f'(x) = \begin{cases} f'_1(x), & x \geq \alpha \\ f'_2(x), & x < \alpha \end{cases}$$

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$f(x) = \text{ReLU}(x) = \max(0, x)$$

## Lecture.2 Basic Differentiation

### - Differentiation Rules

#### Constant Multiple Rule

$$f(x) = x \implies 2 \cdot f(x) = 2x$$

$$f(x) = e^x \implies -3 \cdot f(x) = -3e^x$$

$$f(x) = \sin(x) \implies e \cdot f(x) = e \cdot \sin(x)$$

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)]$$



## Lecture.2

### Basic Differentiation

## - Differentiation Rules

Constant Multiple Rule

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)]$$

$$f(x) = 2x$$

$$f(x) = -3e^x$$

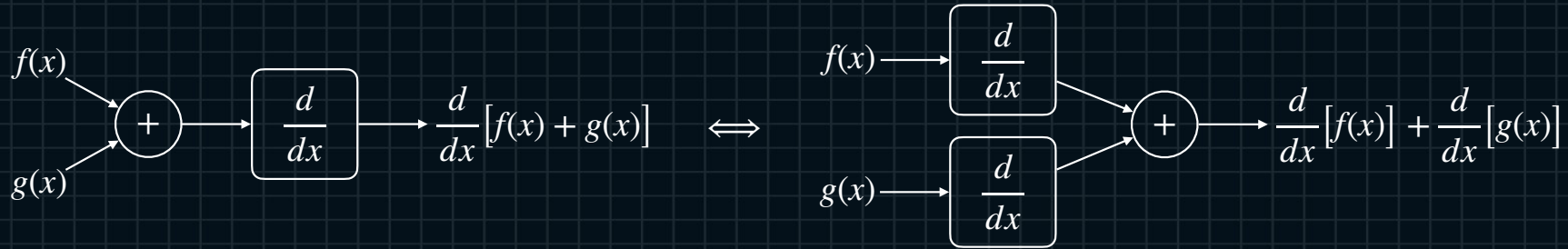
## Lecture.2 Basic Differentiation

### - Differentiation Rules

#### Sum Rule

$$f(x) = 3x^3 - 2x^2 + 10x - 20$$

$$\frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx} + \frac{dg}{dx}$$



## Lecture.2

### Basic Differentiation

## - Differentiation Rules

### Sum Rule

$$\frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx} + \frac{dg}{dx}$$

$$f(x) = 2x^2 - x + 7$$

$$f(x) = \sin(x) - \frac{1}{\sqrt{x}}$$

$$f(x) = \sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$f(x) = \cosh(x) = \frac{e^x + e^{-x}}{2}$$

$$\sinh(x) \xrightleftharpoons{\frac{d}{dx}} \cosh(x)$$

## Lecture.2 Basic Differentiation

### - Differentiation Rules

Linearity of Diff.

$$\text{Sys}\{\alpha \cdot f(t)\} = \alpha \cdot \text{Sys}\{f(t)\} \quad \text{Homogeneity}$$

$$\text{Sys}\{f(t) + g(t)\} = \text{Sys}\{f(t)\} + \text{Sys}\{g(t)\} \quad \text{Additivity}$$

$$\text{Sys}\{\alpha \cdot f(t) + \beta \cdot g(t)\} = \alpha \cdot \text{Sys}\{f(t)\} + \beta \cdot \text{Sys}\{g(t)\}$$

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}[f(x)] \quad \text{Constant Multiple Rule}$$

$$\frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx} + \frac{dg}{dx} \quad \text{Sum Rule}$$

$$\frac{d}{dx}[\alpha \cdot f(x) + \beta \cdot g(x)] = \alpha \cdot \frac{d}{dx}[f(x)] + \beta \cdot \frac{d}{dx}[g(x)]$$



## Lecture.2 Basic Differentiation

### - Differentiation Rules

Time-invariance of Diff.

$$\text{Sys}\{f(t)\} = f'(t) \implies \text{Sys}\{f(t - \tau)\} = f'(t - \tau)$$

$$\frac{df(t)}{dt} = f'(t) \implies \frac{df(t - \tau)}{dt} = f'(t - \tau)$$

## Lecture.2 Basic Differentiation

### - Differentiation Rules

LTI Systems and Diff.

$$\text{Sys}\{\alpha \cdot f(t) + \beta \cdot g(t)\} = \alpha \cdot \text{Sys}\{f(t)\} + \beta \cdot \text{Sys}\{g(t)\}$$

$$\text{Sys}\{x(t)\} = y(t) \quad \Rightarrow \quad \text{Sys}\{x(t - \tau)\} = y(t - \tau)$$

$$\frac{d}{dx} [\alpha \cdot f(t - \tau) + \beta \cdot g(t - \tau)] = \alpha \cdot f'(t - \tau) + \beta \cdot g'(t - \tau)$$

## Lecture.2 Basic Differentiation

### - Differentiation Rules

Product Rule

$$\frac{d}{dx} [f(x) \cdot g(x)] = \frac{d}{dx} [f(x)] \cdot g(x) + f(x) \cdot \frac{d}{dx} [g(x)]$$

$$f(x) = x^3$$

$$f(x) = e^x \ln(x)$$

## Lecture.2

### Basic Differentiation

### - Differentiation Rules

#### Quotient Rule

$$\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{\frac{d}{dx} [f(x)] \cdot g(x) - f(x) \cdot \frac{d}{dx} [g(x)]}{(g(x))^2}$$

$$f(x) = \frac{1}{x}$$

$$f(x) = \frac{x^2}{e^x}$$

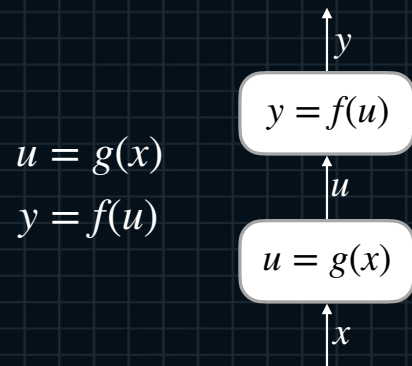
$$f(x) = \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

## Lecture.2 Basic Differentiation

### - Composite Functions and Chain Rule

#### Composite Functions

$$y = f(g(x))$$



## Lecture.2 Basic Differentiation

### - Composite Functions and Chain Rule

Chain Rule

$$y = f(g(x))$$

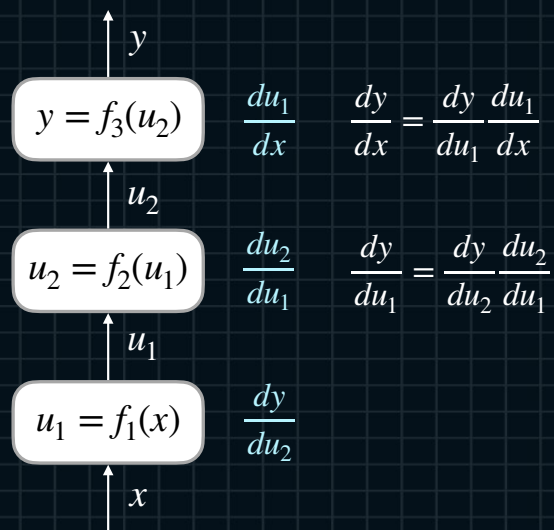
$$\frac{df}{dx} = \frac{df}{dg} \frac{dg}{dx}$$

## Lecture.2 Basic Differentiation

### - Composite Functions and Chain Rule

#### Chain Rule

$$y = f_1(f_2(f_3(x)))$$



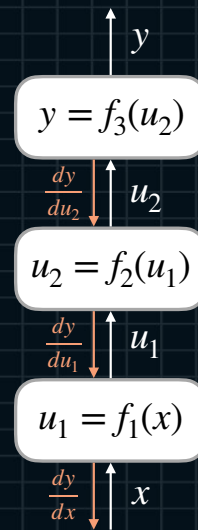


## Lecture.2 Basic Differentiation

### - Composite Functions and Chain Rule

#### Chain Rule

$$y = f_1(f_2(f_3(x)))$$



## Lecture.2 Basic Differentiation

### - Composite Functions and Chain Rule

Forward/Backward Calculations

$$y = f_4(f_3(f_2(f_1(x))))$$

## Lecture.2 Basic Differentiation

### - Composite Functions and Chain Rule

#### Exercises

$$f(x) = \cos(x^3)$$

$$f(t) = \sin(t - \tau)$$

$$f(x) = (a - x)^2$$

$$f(x) = \frac{1}{1 + e^{-x}}$$

## Lecture.2 Basic Differentiation

### - Composite Functions and Chain Rule

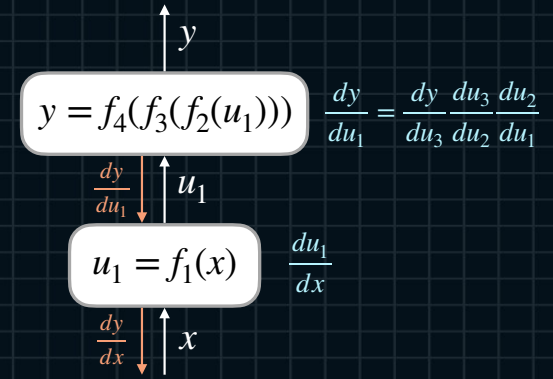
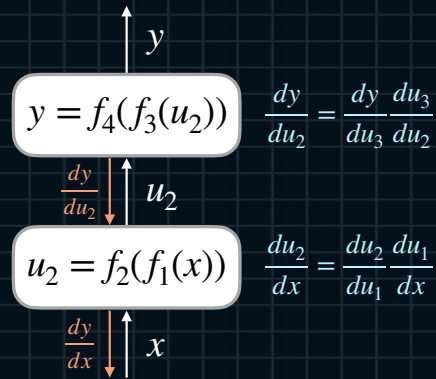
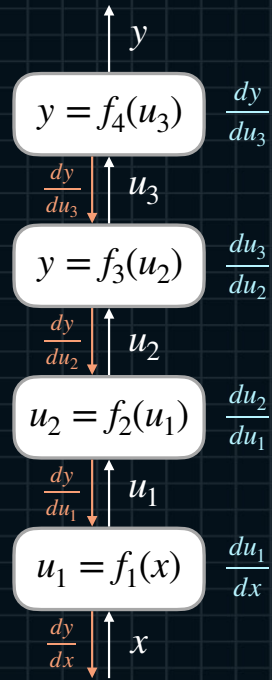
Modules of Backpropagation

$$\frac{dJ}{dz} \downarrow \uparrow z$$
$$z = f(x)$$
$$\frac{dJ}{dx} \downarrow \uparrow x$$
$$\frac{dz}{dx}$$
$$\frac{dJ}{dz} \frac{dz}{dx} = \frac{dJ}{dx}$$

## Lecture.2 Basic Differentiation

### - Composite Functions and Chain Rule

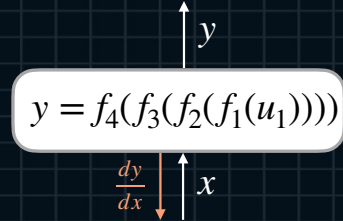
#### Merging Modules



## Lecture.2 Basic Differentiation

### - Composite Functions and Chain Rule

Merging Modules



A diagram showing a rounded rectangular box containing the function  $y = f_4(f_3(f_2(f_1(u_1))))$ . An upward-pointing arrow labeled  $y$  exits the top of the box. A downward-pointing arrow labeled  $\frac{dy}{dx}$  exits the bottom of the box. To the right of the box, the chain rule is written as  $\frac{dy}{dx} = \frac{dy}{du_3} \frac{du_3}{du_2} \frac{du_2}{du_1} \frac{du_1}{dx}$ .

$$y = f_4(f_3(f_2(f_1(u_1))))$$
$$\frac{dy}{dx} = \frac{dy}{du_3} \frac{du_3}{du_2} \frac{du_2}{du_1} \frac{du_1}{dx}$$

# Lecture.2 Basic Differentiation

## - Composite Functions and Chain Rule

### Merging Modules

$$\begin{array}{c}
 \begin{array}{c} \uparrow y \\ \boxed{y = \frac{1}{u_3}} \end{array} \quad \begin{array}{c} \cdots y = \frac{1}{1 + e^{-x}} \\ \frac{dy}{du_3} = -\frac{1}{(u_3)^2} \end{array} \\
 \begin{array}{c} \frac{dy}{du_3} = -\frac{1}{(u_3)^2} = -\frac{1}{(1 + e^{-x})^2} \\ \downarrow \frac{dy}{du_3} \end{array} \quad \begin{array}{c} \uparrow \\ \boxed{u_3 = 1 + u_2} \end{array} \quad \begin{array}{c} \cdots u_3 = 1 + e^{-x} \\ \frac{du_3}{du_2} = 1 \end{array} \\
 \begin{array}{c} \frac{dy}{du_2} = \frac{dy}{du_3} \frac{du_3}{du_2} = -\frac{1}{(1 + e^{-x})^2} \\ \downarrow \frac{dy}{du_2} \end{array} \quad \begin{array}{c} \uparrow \\ \boxed{u_2 = e^{u_1}} \end{array} \quad \begin{array}{c} \cdots u_2 = e^{-x} \\ \frac{du_2}{du_1} = e^{u_1} \end{array} \\
 \begin{array}{c} \frac{dy}{du_1} = \frac{dy}{du_2} \frac{du_2}{du_1} = -\frac{2}{(1 + e^{-x})^2} \cdot e^{u_1} \\ \downarrow \frac{dy}{du_1} \end{array} \quad \begin{array}{c} \uparrow \\ \boxed{u_1 = -x} \end{array} \quad \begin{array}{c} \cdots u_1 = -x \\ \frac{du_1}{dx} = -1 \end{array} \\
 \begin{array}{c} \frac{dy}{dx} = \frac{dy}{du_1} \frac{du_1}{dx} = \frac{e^{-x}}{(1 + e^{-x})^2} \\ = y(1 - y) \end{array} \quad \begin{array}{c} \downarrow \frac{dy}{dx} \\ \uparrow x \end{array}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{c} \uparrow y \\ \boxed{y = \frac{1}{1 + u_1}} \end{array} \quad \begin{array}{c} \frac{dy}{du_1} = -\frac{1}{(1 + u_1)^2} \\ \downarrow \frac{dy}{du_1} \end{array} \quad \begin{array}{c} \uparrow \\ \boxed{u_1 = e^{-x}} \end{array} \quad \begin{array}{c} \frac{du_1}{dx} = -e^{-x} \\ \downarrow \frac{du_1}{dx} \end{array} \quad \begin{array}{c} \uparrow \\ x \end{array} \\
 \frac{dy}{dx} = \frac{dy}{du_1} \frac{du_1}{dx} = -\frac{1}{(1 + e^{-x})^2} \cdot (-e^{-x}) = \frac{e^{-x}}{(1 + e^{-x})^2} = y(1 - y)
 \end{array}$$



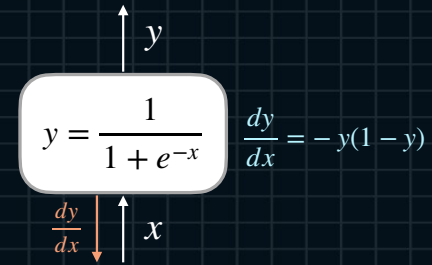
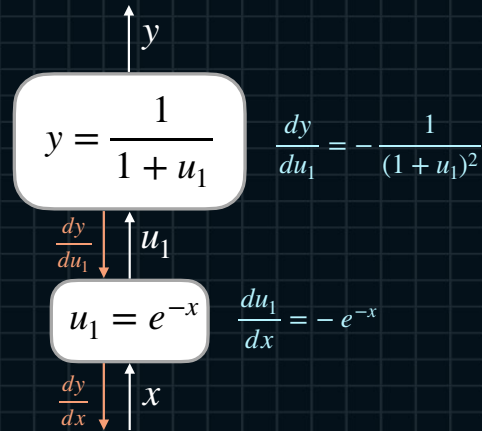
## Lecture.2 Basic Differentiation

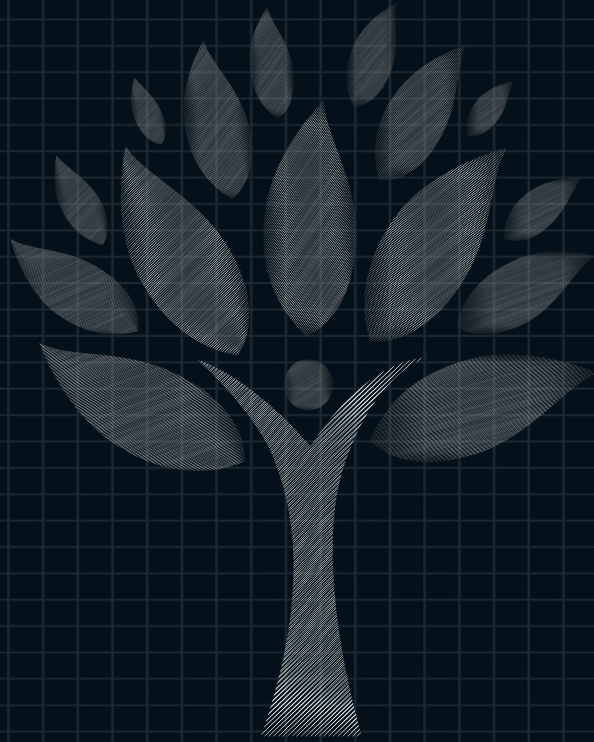
### - Composite Functions and Chain Rule

#### Merging Modules

$$\frac{dy}{du_1} = -\frac{1}{(1+u_1)^2} = -\frac{1}{(1+e^{-x})^2}$$

$$\frac{dy}{du_1} = \frac{e^{-x}}{(1+e^{-x})^2} = y(1-y)$$





# Backpropagation and Jacobian Matrices

Lecture.2  
Basic Differentiation