Akhmediev breathers, Kuznetsov–Ma solitons and rogue waves in a dispersion varying optical fiber

2016 Laser Phys. 26 035402

(http://iopscience.iop.org/1555-6611/26/3/035402)

IOP Publishing | Astro Ltd Laser Physics

Laser Phys. **26** (2016) 035402 (7pp) doi:10.1088/1054-660X/26/3/035402

Akhmediev breathers, Kuznetsov–Ma solitons and rogue waves in a dispersion varying optical fiber

Wen-Rong Sun, Bo Tian, Ya Sun, Jun Chai and Yan Jiang

State Key Laboratory of Information Photonics and Optical Communications, and School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, People's Republic of China

E-mail: tian_bupt@163.com

Received 27 November 2015
Accepted for publication 1 December 2015
Published 11 February 2016

Abstract

Dispersion varying fibres have applications in optical pulse compression techniques. We investigate Akhmediev breathers, Kuznetsov–Ma (KM) solitons and optical rogue waves in a dispersion varying optical fibre based on a variable-coefficient nonlinear Schrödinger equation. Analytical solutions in the forms of Akhmediev breathers, KM solitons and rogue waves up to the second order of that equation are obtained via the generalised Darboux transformation and integrable constraint. The properties of Akhmediev breathers, KM solitons and rogue waves in a dispersion varying optical fibre, e.g. dispersion decreasing fibre (DDF) or a periodically distributed system (PDS), are discussed: in a DDF we observe the compression behaviours of KM solitons and rogue waves on a monotonically increasing background. The amplitude of each peak of the KM soliton increases, while the width of each peak of the KM soliton gradually decreases along the propagation distance; in a PDS, the amplitude of each peak of the KM soliton varies periodically along the propagation distance on a periodic background. Different from the KM soliton, the Akhmediev breather and rogue waves repeat their behaviours along the propagation distance without the compression.

Keywords: dispersion varying optical fiber, breathers, optical rogue waves, generalised Darboux transformation

(Some figures may appear in colour only in the online journal)

1. Introduction

Rogue waves are giant waves occurring with low probability in the ocean [1]. Besides ocean waves, rogue waves have been reported in nonlinear optics, Bose–Einstein condensates and plasmas [2–8]. The term 'optical rogue waves' has been introduced based on the physical similarities between the extreme events of optical systems and oceanic rogue waves [2–8]. Laboratory experiments and theoretical approaches have been performed to study optical rogue waves [9–13]. For example, optical rogue waves in photonic crystal fibres, partially mode-locked fiber lasers, fibre Raman amplifiers and whispering-gallery-mode resonators have been observed [9–11]. Pulse propagation in the nonlinear fibre can be described by the nonlinear Schrödinger (NLS) equation [14, 15], which has certain types of breathers or solitons on finite backgrounds[1], i.e. Akhmediev breathers, Kuznetsov–Ma (KM) solitons and Peregrine solitons [5, 14–16].

Our aim here will be to study Akhmediev breathers, KM solitons and optical rogue waves in a dispersion varying optical fibre, which can be described by the following variable-coefficient NLS equation [5, 17, 18]:

$$\mathrm{i}\frac{\partial\psi}{\partial\xi}+\frac{D(\xi)}{2}\frac{\partial^2\psi}{\partial\tau^2}+2|\psi|^2\psi+\Omega(\xi)\tau^2\psi=0, \qquad (1)$$

where the envelope $\psi(\xi,\tau)$ is a function of the propagation distance ξ and temporal dimension τ, $D(\xi)$ denotes the varying

[1] The breathers or solitons are located on a constant background [5, 14–16].

dispersion and $\Omega(\xi)$ is the coefficient of parabolic potential. Dispersion varying fibres have their applications in optical pulse compression techniques [20]. Lax pair and soliton solutions of equation (1) have been obtained [17]. The propagation of the bound-state soliton has been discussed [17]. Dark soliton solutions of equation (1) have been obtained [18]. When $D(\xi)$ is a constant, equation (1) can be used to describe the pulse propagation in a fibre with uniform dispersion [5]. More on the NLS issue can be seen, e.g., in [19].

The possibility of controlling the solitons on finite background propagation with a dual-frequency input field has been reported [5, 20]. Studies have suggested that the longitudinal variation in a fibre's dispersive and nonlinear properties can modify the intensity fluctuations in the propagation of Akhmediev breathers [21]. Experimental study has also shown that the properties of Akhmediev breathers can be affected by the varying dispersion [22].

To our knowledge, the Akhmediev breather, KM soliton and rogue-wave solutions of equation (1) have not been constructed through the generalised Darboux transformation (DT) [23, 24]. Motivated by [20–22] and based on equation (1), in this paper we will show the properties of Akhmediev breathers, KM solitons and optical rogue waves in a dispersion varying optical fibre, i.e. the dispersion decreasing fiber (DDF) and periodically distributed system (PDS). Using the generalised DT, we will obtain the Akhmediev breather, KM soliton and rogue-wave solutions up to the second order with the limit process in section 2. The effects of the varying dispersion on the properties of Akhmediev breathers, KM solitons and rogue waves will be discussed in section 3. Section 4 will contain our conclusions.

2. DT-based iterative algorithm and solutions

In the frame of the 2×2 Ablowitz–Kaup–Newell–Segur inverse scattering formulation [25], the Lax pair associated with equation (1) under the constraint [17]

$$\Omega(\xi) = -\frac{1}{D(\xi)}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\ln D(\xi), \tag{2}$$

can be written as [17]

$$\Phi_\tau = \mathbf{U}\Phi, \quad \Phi_\xi = \mathbf{V}\Phi, \tag{3}$$

where $\Phi = (\Phi_1, \Phi_2)^T$ is the vector eigenfunction, Φ_1 and Φ_2 are the complex functions of ξ and τ, T denotes the transpose of a matrix, and $\mathbf{U}$ and $\mathbf{V}$ are expressible in the forms of [17]

$$\mathbf{U} = \begin{pmatrix} -\mathrm{i}\lambda(\xi) & \sqrt{2}\,\widetilde{\psi} \\ -\sqrt{2}\,\widetilde{\psi}^* & \mathrm{i}\lambda(\xi)\end{pmatrix},$$

$$\mathbf{V} = \mathrm{i}\begin{pmatrix} D(\xi)|\widetilde{\psi}|^2 & \sqrt{2}D(\xi)\left[\frac{1}{2}\widetilde{\psi}_\tau + \mathrm{i}\frac{D_\xi}{D^2(\xi)}\tau\widetilde{\psi}\right] \\ \sqrt{2}D(\xi)\left[\frac{1}{2}\widetilde{\psi}^*_\tau - \mathrm{i}\frac{D_\xi}{D^2(\xi)}\tau\widetilde{\psi}^*\right] & -D(\xi)|\widetilde{\psi}|^2 \end{pmatrix}$$
$$-\mathrm{i}\lambda(\xi)D(\xi)\begin{pmatrix} -\frac{D_\xi}{D^2(\xi)}\tau & \mathrm{i}\sqrt{2}\,\widetilde{\psi} \\ -\mathrm{i}\sqrt{2}\,\widetilde{\psi}^* & \frac{D_\xi}{D^2(\xi)}\tau \end{pmatrix} - \mathrm{i}\lambda^2(\xi)\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}, \tag{4}$$

where '*' denotes the complex conjugate, $\widetilde{\psi} = \frac{1}{\sqrt{D(\xi)}}\psi\, \mathrm{e}^{\frac{-\mathrm{i}\tau^2 D_\xi}{2D^2(\xi)}}$ and $\lambda(\xi) = \frac{\lambda(0)}{D(\xi)}$ is an eigenvalue of Lax Pair (3) with $\lambda(0)$ as a constant. One can check that the compatibility condition $\mathbf{U}_\xi - \mathbf{V}_\tau + \mathbf{UV} - \mathbf{VU} = 0$ is equivalent to equation (1).

Considering the following gauge transformation:

$$\Phi[1] = \mathbf{D}[1]\Phi, \tag{5}$$

where $[j]$ $(j = 0, 1, 2, \cdots N)$ represents the jth-iteration, $\mathbf{D}[1]$ is a 2×2 matrix and and $\Phi[1]$ is a 2×1 vector eigenfunction, through which we can cast Lax Pair (3) into

$$\Phi[1]_\tau = \mathbf{U}[1]\Phi[1], \quad \mathbf{U}[1] = (\mathbf{D}[1]_\tau + \mathbf{D}[1]\mathbf{U})\mathbf{D}[1]^{-1}, \tag{6a}$$

$$\Phi[1]_\xi = \mathbf{V}[1]\Phi[1], \quad \mathbf{V}[1] = (\mathbf{D}[1]_\xi + \mathbf{D}[1]\mathbf{V})\mathbf{D}[1]^{-1}, \tag{6b}$$

where $\mathbf{D}[1]^{-1}$ denotes the inverse matrix of $\mathbf{D}[1]$, $\mathbf{U}[1]$ and $\mathbf{V}[1]$ are the 2×2 matrices. The cross differentiation of Lax Pair (3) leads to

$$\mathbf{U}[1]_\xi - \mathbf{V}[1]_\tau + [\mathbf{U}[1], \mathbf{V}[1]] = \mathbf{D}[1](\mathbf{U}_\xi - \mathbf{V}_\tau + [\mathbf{U}, \mathbf{V}])\mathbf{D}[1]^{-1}, \tag{7}$$

which implies that in order to keep Lax Pair (3) invariant under Transformation (5), we need to obtain a matrix $\mathbf{D}[1]$ such that $\mathbf{U}[1]$ and $\mathbf{V}[1]$, respectively, possess the same forms as $\mathbf{U}$ and $\mathbf{V}$.

Assume that $(\Phi_{1,1}, \Phi_{2,1})^T$ is a complex eigenfunction of Lax Pair (3) with the eigenvalue $\lambda(\xi) = \lambda_1(\xi)$, and the DT can be given as

$$\psi[1] = \psi + \sqrt{2}\,\mathrm{i}\frac{[\lambda_1^*(\xi) - \lambda_1(\xi)]\Phi_{2,1}^*\Phi_{1,1}}{\Phi_{1,1}\Phi_{1,1}^* + \Phi_{2,1}\Phi_{2,1}^*}\sqrt{D(\xi)}\,\mathrm{e}^{\frac{\mathrm{i}\tau^2 D_\xi}{2D^2(\xi)}},$$

$$\Phi[1] = \mathbf{D}[1]\Phi,$$

$$\mathbf{D}[1] = \lambda(\xi)I - \begin{pmatrix}\Phi_{1,1} & -\Phi_{2,1}^* \\ \Phi_{2,1} & \Phi_{1,1}^*\end{pmatrix}\begin{pmatrix}\lambda_1(\xi) & 0 \\ 0 & \lambda_1^*(\xi)\end{pmatrix}\begin{pmatrix}\Phi_{1,1} & -\Phi_{2,1}^* \\ \Phi_{2,1} & \Phi_{1,1}^*\end{pmatrix}^{-1}, \tag{8}$$

where I is a 2×2 identity matrix, $\lambda_1(\xi)$ is an eigenvalue of Lax Pair (3), and $\Phi_{1,1}$ and $\Phi_{2,1}$ are the complex functions of ξ and τ. It can be verified that if $(\Phi_{1,1}, \Phi_{2,1})^T$ is an eigenfunction of Lax Pair (3) with the eigenvalue $\lambda(\xi) = \lambda_1(\xi)$, $(-\Phi_{2,1}^*, \Phi_{1,1}^*)^T$ is also an eigenfunction of Lax Pair (3) with the eigenvalue $\lambda(\xi) = \lambda_1^*(\xi)$.

Let $(\Phi_{1,1}, \Phi_{2,1})^T$, $(\Phi_{1,2}, \Phi_{2,2})^T$,$\cdots$, $(\Phi_{1,N}, \Phi_{2,N})^T$ be the N distinct solutions of Lax Pair (3) at $\lambda_1(\xi)$, $\cdots$, $\lambda_N(\xi)$, respectively, where $\Phi_{1,k}$'s and $\Phi_{2,k}$'s $(k = 1, 2, \cdots, N)$ are the functions of ξ and τ, and $\lambda_k(\xi)$'s are the eigenvalues of Lax Pair (3). Then the N-fold DT for equation (1) is

$$\psi[N] = \psi[0] + \sqrt{2}\,\mathrm{i}\sum_{k=1}^{N}\frac{[\lambda_k^*(\xi) - \lambda_k(\xi)]\Phi_{1,k}[k-1]\Phi_{2,k}[k-1]^*}{\Phi_{1,k}[k-1]\Phi_{1,k}[k-1]^* + \Phi_{2,k}[k-1]\Phi_{2,k}[k-1]^*}$$
$$\times\sqrt{D(\xi)}\,\mathrm{e}^{\frac{\mathrm{i}\tau^2 D_\xi}{2D^2(\xi)}}, \Phi[N] = \mathbf{D}[N]\mathbf{D}[N-1]\cdots\mathbf{D}[1]\Phi, \tag{9}$$

with

$$\mathbf{D}[k]=\lambda(\xi)I-\begin{pmatrix}\Phi_{1,k}[k-1] & -\Phi_{2,k}[k-1]^* \\ \Phi_{2,k}[k-1] & \Phi_{1,k}[k-1]^*\end{pmatrix}\begin{pmatrix}\lambda_k(\xi) & 0 \\ 0 & \lambda_k^*(\xi)\end{pmatrix}$$
$$\times\begin{pmatrix}\Phi_{1,k}[k-1] & -\Phi_{2,k}[k-1]^* \\ \Phi_{2,k}[k-1] & \Phi_{1,k}[k-1]^*\end{pmatrix}^{-1},\begin{pmatrix}\Phi_{1,k}[k-1] \\ \Phi_{2,k}[k-1]\end{pmatrix}$$
$$=(\mathbf{D}[k-1]\mathbf{D}[k-2]\cdots\mathbf{D}[1])|_{\lambda(\xi)=\lambda_k(\xi)}\begin{pmatrix}\Phi_{1,k} \\ \Phi_{2,k}\end{pmatrix},\psi[0]$$
$$=\psi,\begin{pmatrix}\Phi_{1,1}[0] \\ \Phi_{2,1}[0]\end{pmatrix}\begin{pmatrix}\Phi_{1,1} \\ \Phi_{2,1}\end{pmatrix}.$$

To derive the Akhmediev breather, KM soliton and rogue-wave solutions of equation (1), we derive the seed solutions as $\psi=\frac{1}{\sqrt{D(\xi)}}\mathrm{e}^{\mathrm{i}\left[\frac{\tau^2 D_\xi}{2D^2(\xi)}+2\int\frac{1}{D(\xi)}\mathrm{d}\xi\right]}$, and the corresponding solutions for Lax Pair (3) at $\lambda(\xi)=\sqrt{2}\frac{\mathrm{i}h}{D(\xi)}$ (For simplicity, we take $\lambda(0)=\sqrt{2}\,\mathrm{i}h$.) are

$$\varphi=\begin{pmatrix}\left[-\frac{(h-\sqrt{h^2-1})^{\frac12}}{\sqrt{h^2-1}}\mathrm{e}^{-\Delta}+\frac{(h+\sqrt{h^2-1})^{\frac12}}{\sqrt{h^2-1}}\mathrm{e}^{\Delta}\right]\mathrm{e}^{\mathrm{i}\left[\int\frac{1}{D(\xi)}\mathrm{d}\xi\right]} \\ \left[\frac{(h+\sqrt{h^2-1})^{\frac12}}{\sqrt{h^2-1}}\mathrm{e}^{-\Delta}-\frac{(h-\sqrt{h^2-1})^{\frac12}}{\sqrt{h^2-1}}\mathrm{e}^{\Delta}\right]\mathrm{e}^{-\mathrm{i}\left[\int\frac{1}{D(\xi)}\mathrm{d}\xi\right]}\end{pmatrix}, \quad (10)$$

where h is a real parameter and $\Delta=\sqrt{h^2-1}\frac{\sqrt{2}}{D(\xi)}\tau+2h\sqrt{h^2-1}\int\frac{1}{D(\xi)}\mathrm{d}\xi$.

Substituting Expression (10) into DT (8), we obtain the breather solutions of equation (1) under constraint (2),

$$\psi=\frac{1}{\sqrt{D(\xi)}}\mathrm{e}^{\mathrm{i}\left[\frac{\tau^2 D_\xi}{2D^2(\xi)}+2\int\frac{1}{D(\xi)}\mathrm{d}\xi\right]}+\sqrt{2}\,\mathrm{i}\frac{[\lambda_1^*(\xi)-\lambda_1(\xi)]\Phi_{2,1}^*\Phi_{1,1}}{\Phi_{1,1}\Phi_{1,1}^*+\Phi_{2,1}\Phi_{2,1}^*}\sqrt{D(\xi)}\,\mathrm{e}^{\frac{\mathrm{i}\tau^2 D_\xi}{2D^2(\xi)}}, \quad (11)$$

where

$$\Phi_{1,1}=\left[-\frac{(h-\sqrt{h^2-1})^{\frac12}}{\sqrt{h^2-1}}\mathrm{e}^{-\Delta}+\frac{(h+\sqrt{h^2-1})^{\frac12}}{\sqrt{h^2-1}}\mathrm{e}^{\Delta}\right]\mathrm{e}^{\mathrm{i}\left[\int\frac{1}{D(\xi)}\mathrm{d}\xi\right]},$$
$$\Phi_{2,1}=\left[\frac{(h+\sqrt{h^2-1})^{\frac12}}{\sqrt{h^2-1}}\mathrm{e}^{-\Delta}-\frac{(h-\sqrt{h^2-1})^{\frac12}}{\sqrt{h^2-1}}\mathrm{e}^{\Delta}\right]\mathrm{e}^{-\mathrm{i}\left[\int\frac{1}{D(\xi)}\mathrm{d}\xi\right]}.$$

Solution (11) includes the Akhmediev breather ($0<h<1$) and KM soliton ($h>1$) solutions, which will be discussed in section 3.

Then, we will obtain the rogue-wave solutions of equation (1). For Expression (10), assuming $h=1+f^2$, with f as the formal real parameter, and expanding φ at $f=0$, we obtain

$$\varphi=\varphi^{(0)}+\varphi^{(1)}f^2+\cdots, \quad (12)$$

where $\varphi^{(s)}=\frac{1}{(2s)!}\frac{\partial^{2s}}{\partial f^{2s}}\varphi(f)|_{f=0}$ $(s=0,1,\cdots)$. $\varphi^{(0)}$ can be expressed as

$$\varphi^{(0)}=\begin{pmatrix}\left[1+\frac{2\sqrt{2}\tau}{D(\xi)}+4\mathrm{i}\int\frac{1}{D(\xi)}\mathrm{d}\xi\right]\mathrm{e}^{\mathrm{i}\left[\int\frac{1}{D(\xi)}\mathrm{d}\xi\right]} \\ \left[1-\frac{2\sqrt{2}\tau}{D(\xi)}-4\mathrm{i}\int\frac{1}{D(\xi)}\mathrm{d}\xi\right]\mathrm{e}^{-\mathrm{i}\left[\int\frac{1}{D(\xi)}\mathrm{d}\xi\right]}\end{pmatrix}. \quad (13)$$

Substituting Expression (13) into DT (8), we obtain the first-order rogue-wave solutions of equation (1) under constraint (2),

$$\psi[1]=\frac{1}{\sqrt{D(\xi)}}\mathrm{e}^{\mathrm{i}\left[\frac{\tau^2 D_\xi}{2D^2(\xi)}+2\int\frac{1}{D(\xi)}\mathrm{d}\xi\right]}\frac{G_1}{F_1}, \quad (14)$$

where

$$G_1=8\tau^2+D^2(\xi)\left[-3-16\mathrm{i}\int\frac{1}{D(\xi)}\mathrm{d}\xi+16\left(\int\frac{1}{D(\xi)}\mathrm{d}\xi\right)^2\right],$$
$$F_1=8\tau^2+D^2(\xi)\left[1+16\left(\int\frac{1}{D(\xi)}\mathrm{d}\xi\right)^2\right].$$

In order to go to the next step of the DT, we introduce the following limit process:

$$\begin{pmatrix}\Phi_{1,2}[1] \\ \Phi_{2,2}[1]\end{pmatrix}=\lim_{f\to0}\frac{\left[\mathrm{i}\sqrt{2}\frac{1}{D(\xi)}f^2+\mathbf{D}[1]|_{\lambda(\xi)=\sqrt{2}\frac{\mathrm{i}}{D(\xi)}}\right]\varphi}{f^2}$$
$$=\sqrt{2}\frac{\mathrm{i}}{D(\xi)}\varphi(0)+\mathbf{D}[1]|_{\lambda(\xi)=\sqrt{2}\frac{\mathrm{i}}{D(\xi)}}\varphi^{(1)}, \quad (15)$$

where $\Phi_{1,2}[1]$ and $\Phi_{2,2}[1]$ are given by $\Phi_{1,k}[k-1]$ and $\Phi_{2,k}[k-1]$ when $k=2$.

Limit Process (15) allows the two-fold DT, namely,

$$\psi[2]=\psi[1]+\sqrt{2}\,\mathrm{i}\frac{[\lambda_1^*(\xi)-\lambda_1(\xi)]\Phi_{2,2}[1]^*\Phi_{1,2}[1]}{\Phi_{1,2}[1]\Phi_{1,2}[1]^*+\Phi_{2,2}[1]\Phi_{2,2}[1]^*}\sqrt{D(\xi)}\,\mathrm{e}^{\frac{\mathrm{i}\tau^2 D_\xi}{2D^2(\xi)}}. \quad (16)$$

By means of Expressions (14)–(16), we can obtain the second-order rogue-wave solutions of equation (1) under constraint (2),

$$\psi[2]=\frac{1}{\sqrt{D(\xi)}}\mathrm{e}^{\mathrm{i}\left[\frac{\tau^2 D_\xi}{2D^2(\xi)}+2\int\frac{1}{D(\xi)}\mathrm{d}\xi\right]}\frac{G_2}{F_2}, \quad (17)$$

where G_2 and F_2 can be expressed as

$$G_2=\left\{45-12288\mathrm{i}\left(\int\frac{1}{D(\xi)}\mathrm{d}\xi\right)^5+4096\left(\int\frac{1}{D(\xi)}\mathrm{d}\xi\right)^6\right.$$
$$-180\left(\frac{\sqrt{2}}{D(\xi)}\tau\right)^2-144\left(\frac{\sqrt{2}}{D(\xi)}\tau\right)^4+64\left(\frac{\sqrt{2}}{D(\xi)}\tau\right)^6$$
$$+768\left(\int\frac{1}{D(\xi)}\mathrm{d}\xi\right)^4\left[-11+4\left(\frac{\sqrt{2}}{D(\xi)}\tau\right)^2\right]$$
$$-1536\mathrm{i}\left[\int\frac{1}{D(\xi)}\mathrm{d}\xi\right)^3\left(1+4\left(\frac{\sqrt{2}}{D(\xi)}\tau\right)^2\right]$$
$$+48\left(\int\frac{1}{D(\xi)}\mathrm{d}\xi\right)^2\left[-39-120\left(\frac{\sqrt{2}}{D(\xi)}\tau\right)^2+16\left(\frac{\sqrt{2}}{D(\xi)}\tau\right)^4\right]$$
$$\left.-48\mathrm{i}\left(\int\frac{1}{D(\xi)}\mathrm{d}\xi\right)\left[-15-24\left(\frac{\sqrt{2}}{D(\xi)}\tau\right)^2+16\left(\frac{\sqrt{2}}{D(\xi)}\tau\right)^4\right]\right\},$$

$$F_2=\left\{9+4096\left(\int\frac{1}{D(\xi)}\mathrm{d}\xi\right)^6+108\left(\frac{\sqrt{2}}{D(\xi)}\tau\right)^2\right.$$
$$+48\left(\frac{\sqrt{2}}{D(\xi)}\tau\right)^4+64\left(\frac{\sqrt{2}}{D(\xi)}\tau\right)^6$$
$$+768\left(\int\frac{1}{D(\xi)}\mathrm{d}\xi\right)^4\left[9+4\left(\frac{\sqrt{2}}{D(\xi)}\tau\right)^2\right]$$
$$\left.+48\left(\int\frac{1}{D(\xi)}\mathrm{d}\xi\right)^2\left[33-24\left(\frac{\sqrt{2}}{D(\xi)}\tau\right)^2+16\left(\frac{\sqrt{2}}{D(\xi)}\tau\right)^4\right]\right\}.$$

3. Discussions

When $0<h<1$, the Akhmediev breather appears. We can observe the evolution from the plane wave (figure 1) to a train of localised pulses with periodicity in the temporal dimension τ (figures 2). With h $(0<h<1)$ increasing, the temporal separation between the adjacent peaks increases, as shown in figure 2.

When $h>1$, the KM soliton, which exhibits the localisation in the temporal dimension τ but periodicity along the propagation distance ξ, can be observed, as shown in figure 3. As h $(h>1)$ increases, the distance between the adjacent peaks decreases, as shown in figure 3.

The above discussion is based on a fibre with a uniform dispersion. We next discuss the effects of the varying dispersion $D(\xi)$ on the properties of KM solitons and Akhmediev breathers. In a DDF with $D(\xi)=\exp(-0.4\xi)$ and $\Omega(\xi)=-\frac{1}{D(\xi)}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\ln D(\xi)=0$, the KM soliton exists on a monotonically increasing background, as shown in figure 4. It can be seen that the KM soliton is gradually compressed along ξ. The amplitude of each peak increases, while the width of each peak gradually decreases along ξ, as shown in figure 4. Figure 4 also shows that the distance between the adjacent peaks varies along ξ, while this distance remains unchanged in a fibre with uniform dispersion. To investigate the properties of the KM soliton in a PDS, we consider $D(\xi)=1+0.6\cos(0.6\xi)$ and $\Omega(\xi)=-\frac{1}{[1+0.6\cos(0.6\xi)]}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\ln[1+0.6\cos(0.6\xi)]$. In a PDS,

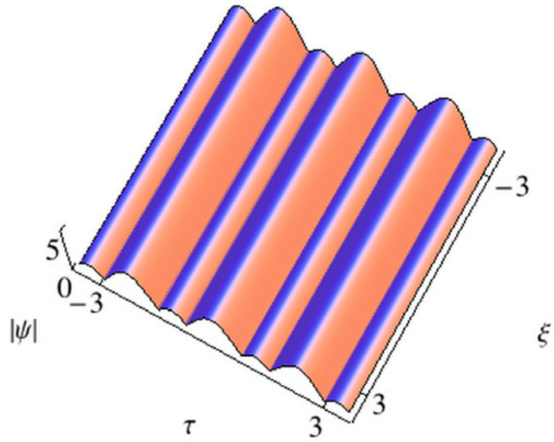

Figure 1. Plane wave via solutions (11) under constraint (2) with $D(\xi)=1,\Omega(\xi)=0$ and $(h=0)$.

the KM soliton exists on a periodic background, and the amplitude of each peak of the KM soliton varies periodically along ξ, as shown in figure 5(a). Moreover, in the PDS the Akhmediev breather repeats its behaviour along ξ. Different from the KM soliton, the Akhmediev breather does not suffer compression and the peak height remains unchanged along ξ, as seen in figure 5(b).

Increasing h leads to a stronger localisation in both dimensions until the limit $h\to 1$, which gives the rogue waves. The first and second-order rogue waves in a fibre with uniform dispersion are seen in figures 6(a) and 7(a). In a DDF it is found that the first and second-order rogue waves exist on a monotonically increasing background. The broadening and compression behaviours of the rogue waves are shown in figures 6(b) and 7(b). In the PDS, the first and second-order rogue waves exist on a periodic background, as shown in figures 6(c) and 7(c). We can observe that the rogue waves repeat their behaviours along ξ without broadening or compression.

4. Conclusion

Dispersion varying fibres, including the DDF and PDS, have their applications in optical pulse compression techniques. Akhmediev breathers, KM solitons and optical rogue waves in a dispersion varying optical fibre, which can be described by equation (1), i.e. a variable-coefficient NLS equation, have been reported in this paper. The results of this paper are as follows:

(A) N-Fold DT (9) of equation (1) has been obtained. Solving non-zero potential Lax Pair (3) and DT (9), we have obtained the solutions in the forms of Akhmediev breathers, KM solitons and rogue waves up to the second order, i.e. solutions (11), (14) and (17) under constraint (2). Limit process (15) allows us to go to 2-fold DT (16) and calculate the second-order rogue-wave solutions.

(B) The types of breathers or solitons on the finite backgrounds are shown:
Plane waves: $h=0$ (figure 1), Akhmediev breathers: $0<h<1$ (figure 2),

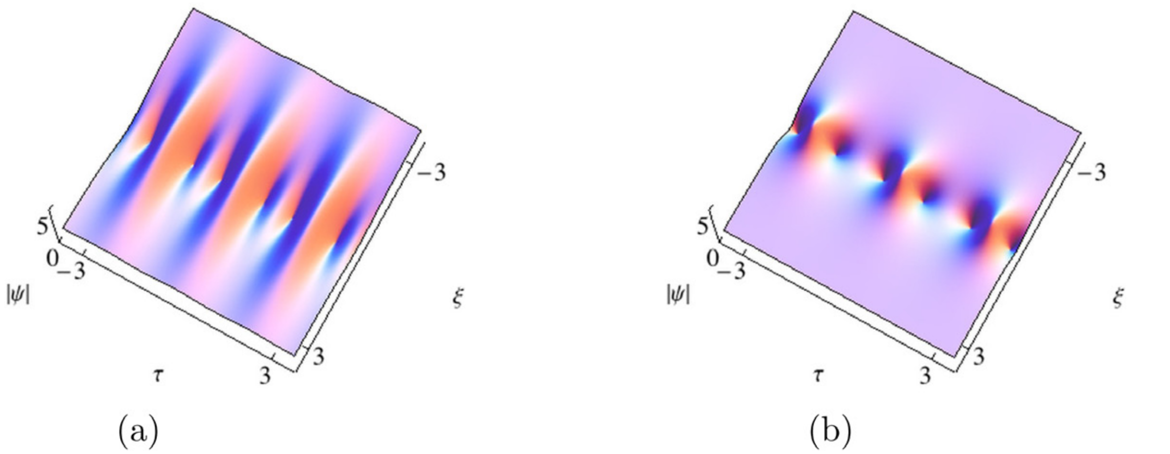

Figure 2. Akhmediev breathers via solutions (11) under constraint (2) with $D(\xi) = 1, \Omega(\xi) = 0$, (a) $h = 0.2$ and (b) $h = 0.6$.

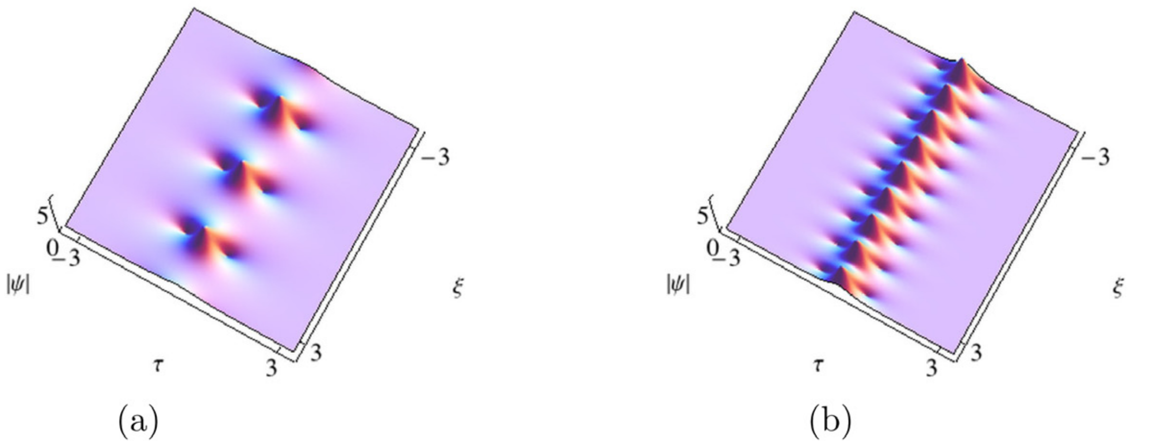

Figure 3. KM solitons via solutions (11) under constraint (2) with $D(\xi) = 1, \Omega(\xi) = 0$, (a) $h = 1.2$ and (b) $h = 1.6$.

KM solitons: $h > 1$ (figure 3), Rogue waves: $h \to 1$ (figures 6 and 7).

We have observed the evolution from the plane wave (figure 1) to a train of localised pulses with periodicity in the temporal dimension τ (figure 2). With h $(0 < h < 1)$ increasing, we have shown that the temporal separation between the adjacent peaks increases, as seen in figure 2. As h $(h > 1)$ increases, the distance between the adjacent peaks decreases, as seen in figure 3.

(C) Properties of the Akhmediev breathers, KM solitons and rogue waves in a dispersion varying optical fibre, i.e. the DDF or PDS, have been discussed:

(i) In a DDF, we have observed that the KM soliton exists on a monotonically increasing background and is gradually compressed along the propagation distance ξ, as shown in figure 4. Figure 4 also shows that the amplitude of each peak increases, while the width of each peak gradually decreases along ξ. In a PDS, figure 5(a) shows that the KM soliton exists on a periodic background. It can be seen that the amplitude of each peak of the KM soliton varies periodically along ξ.

(ii) We have observed that the Akhmediev breather repeats its behaviour without compression along ξ, as shown in figure 5(b).

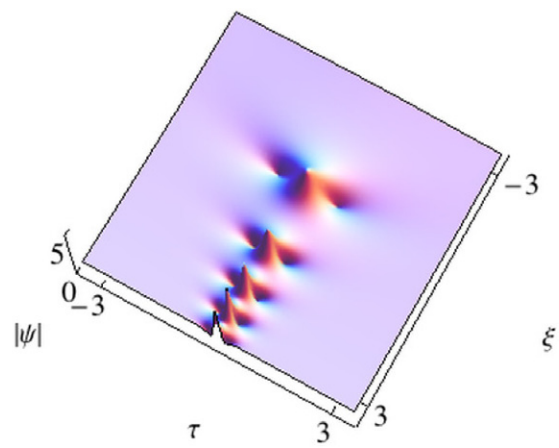

Figure 4. KM soliton in the DDF via solutions (11) under constraint (2) with $D(\xi) = \exp(-0.4\xi)$ and $h = 1.2$.

(iii) For the first and second-order rogue waves in a DDF, the broadening and compression of the rogue waves on a monotonically increasing background have been shown in figures 6(b) and 7(b), while in a PDS, the rogue waves repeat their behaviours on a periodic background without broadening or compression along ξ, as shown in figures 6(c) and 7(c).

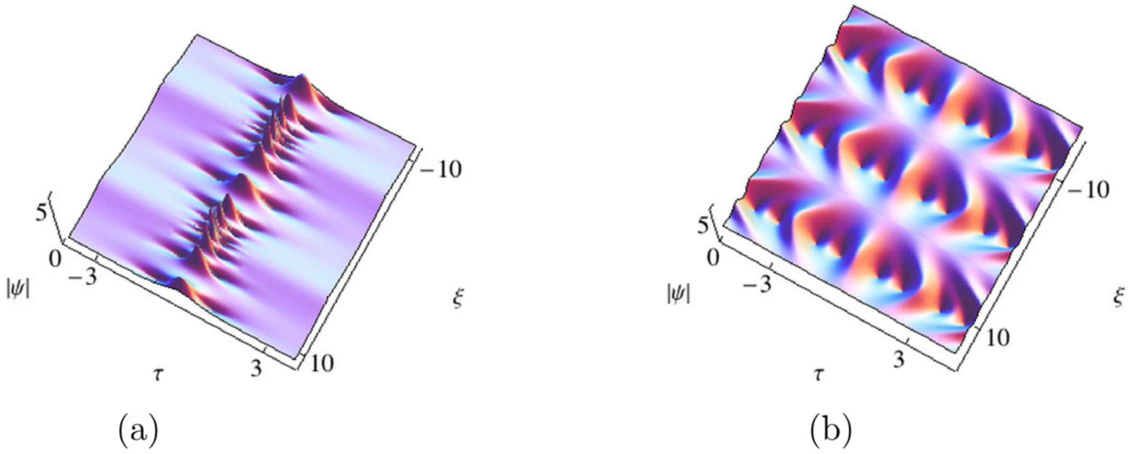

Figure 5. (a) KM soliton in the PDS via solutions (11) under constraint (2) with $D(\xi) = 1 + 0.6\cos(0.6\xi)$ and $h = 1.2$; (b) Akhmediev breather in the PDS via solutions (11) with $D(\xi) = 1 + 0.6\cos(0.6\xi)$ and $h = 0.2$.

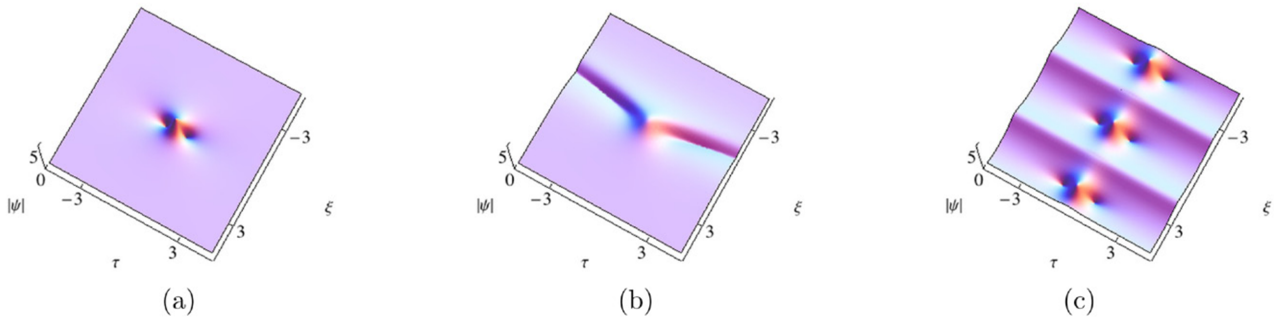

Figure 6. The first-order rogue waves via solutions (14) under constraint (2) with (a) $D(\xi) = 1$; (b) $D(\xi) = 1 + 0.8\exp(-1.9\xi)$; (c) $D(\xi) = 1 + 0.6\cos(0.6\xi)$.

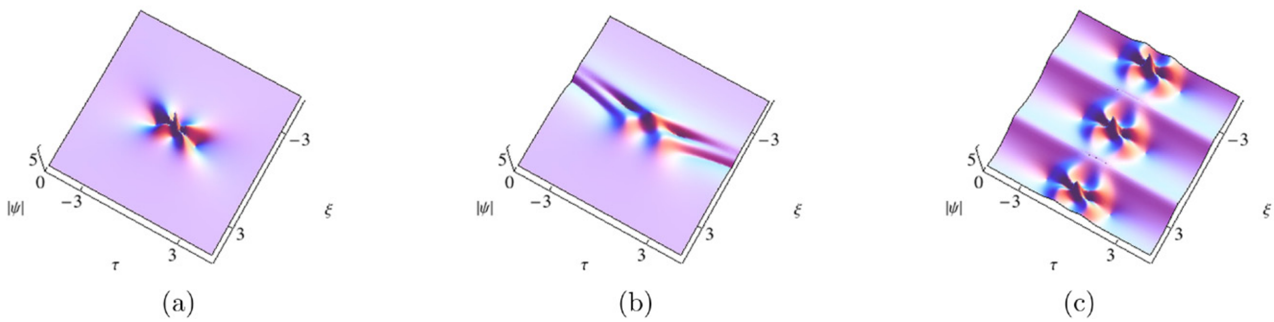

Figure 7. The second-order rogue waves via solutions (16) under constraint (2) with (a) $D(\xi) = 1$; (b) $D(\xi) = 1 + 0.8\exp(-1.9\xi)$; (c) $D(\xi) = 1 + 0.6\cos(0.6\xi)$

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant No. 11272023 and 11471050, by the Fund of State Key Laboratory of Information Photonics and the Optical Communications (Beijing University of Posts and Telecommunications), by the Fundamental Research Funds for the Central Universities of China under Grant Nos. 2011BUPTYB02 and 2015RC19 and by the BUPT Excellent PhD Students Foundation (No. CX2015303).

References

[1] Kharif C, Pelinovsky E and Slunyaev A 2009 *Rogue Waves in the Ocean* (New York: Springer)

[2] Yang Z P, Zhong W P and Belić M 2015 2D optical rogue waves in self-focusing Kerr-type media with spatially modulated coefficients *Laser Phys.* **25** 085402

Su C Q, Gao Y T, Xue L and Yu X 2015 Solitons and rogue waves for a higher-order nonlinear Schrodinger-Maxwell-Bloch system in an erbium-doped fiber *Z. Naturforsch.* A **70** 935

[3] Zhu H P and Pan Z H 2014 Combined Akhmediev breather and Kuznetsov-Ma solitons in a two-dimensional graded-index waveguide *Laser Phys.* **24** 045406

Wang Q M, Gao Y T, Su C Q, Shen Y J, Feng Y J and Xue L 2015 Higher-order rogue waves for a fifth-order dispersive nonlinear Schrodinger equation in an optical fibre *Z. Naturforsch.* A **70** 365

[4] Solli D R, Ropers C, Koonath P and Jalali B 2007 Optical rogue waves *Nature* **450** 1054

[5] Dudley J M, Dias F, Erkintalo M and Genty G 2014 Instabilities, breathers and rogue waves in optics *Nat. Photonics* **8** 755

[6] Dudley J M, Genty G and Eggleton B J 2008 Harnessing and control of optical rogue waves in supercontinuum generation *Opt. Exp.* **16** 3644

[7] Bludov Y V, Konotop V V and Akhmediev N 2010 Matter rogue waves *Phys. Rev.* A **80** 033610

[8] Bailung H, Sharma S K and Nakamura Y 2011 Observation of Peregrine solitons in a multicomponent plasma with negative ions *Phys. Rev. Lett.* **107** 255005

[9] Soto-Crespo J M, Grelu P H and Akhmediev N 2011 Dissipative rogue waves: extreme pulses generated by passively mode-locked lasers *Phys. Rev.* E **84** 016604

[10] Coillet A, Dudley J, Genty G, Larger L and Chembo Y K 2014 Optical rogue waves in whispering-gallery-mode resonators *Phys. Rev.* A **89** 013835

[11] Hammani K, Finot C, Dudley J M and Millot G 2008 Optical rogue-wave fluctuations in fiber Raman amplifiers *Opt. Exp.* **16** 16467

[12] Dudley J M, Genty G, Dias F, Kibler B and Akhmediev N 2009 Modulation instability, Akhmediev breathers and continuous wave supercontinuum generation *Opt. Exp.* **17** 21497

[13] Kibler B, Fatome J, Finot C, Millot G, Dias F, Genty G, Akhmediev N and Dudley J M 2010 The Peregrine soliton in nonlinear fibre optics *Nat. Phys.* **6** 790

[14] Akhmediev N and Ankiewicz A 1997 *Solitons: Non-linear Pulses and Beams* (London: Chapman and Hall)

[15] Peregrine D H 1983 Water-waves, nonlinear Schrödinger equations and their solutions *J. Aust. Math. Soc.* B **25** 16

[16] Kedziora D J, Ankiewicz A and Akhmediev N 2013 Classifying the hierarchy of nonlinear Schrödinger equation rogue-wave solutions *Phys. Rev.* E **88** 013207

[17] Serkin V N, Hasegawa A and Belyaeva T L 2007 Nonautonomous solitons in external potentials *Phys. Rev. Lett.* **98** 074102

[18] Serkin V N, Hasegawa A and Belyaeva T L 2010 Solitary waves in nonautonomous nonlinear and dispersive systems: nonautonomous solitons *J. Mod. Opt.* **57** 1456

[19] Yang J W, Gao Y T, Wang Q M, Su C Q, Feng Y J and Yu X 2016 Bilinear forms and soliton solutions for a fourth-order variable-coefficient nonlinear Schrodinger equation in an inhomogeneous Heisenberg ferromagnetic spin chain or an alpha helical protein *Physica* B **481** 148

Feng Y J, Gao Y T, Sun Z Y, Zuo D W, Shen Y J, Sun Y H, Xue L and Yu X 2015 Anti-dark solitons for a variable-coefficient higher-order nonlinear Schrodinger equation in an inhomogeneous optical fiber *Phys. Scr.* 90 045201

Wang Q M, Gao Y T, Su C Q, Mao B Q, Gao Z and Yang J W 2015 Dark solitonic interaction and conservation laws for a higher-order (2+1)-dimensional nonlinear Schrodinger-type equation in a Heisenberg ferromagnetic spin chain with bilinear and biquadratic interaction *Ann. Phys.* **363** 440

Su C Q, Gao Y T, Yu X, Xue L and Shen Y J 2016 Exterior differential expression of the (1+1)-dimensional nonlinear evolution equation with Lax integrability *J. Math. Anal. Appl.* **435** 735

[20] Agrawal G P 1995 *Nonlinear Fiber Optics* (New York: Academic)

[21] Tian Q, Yang Q, Dai C Q and Zhang J F 2011 Controllable optical rogue waves: recurrence, annihilation and sustainment *Opt. Commun.* **284** 2222

[22] Bendahmane A, Mussot A, Szriftgiser P, Zerkak O, Genty G, Dudley J M and Kudlinski A 2014 Experimental dynamics of Akhmediev breathers in a dispersion varying optical fiber *Opt. Lett.* **39** 4490

[23] Matveev V B and Salle M A 1991 *Darboux Transformation and Solitons* (Berlin: Springer)

[24] Guo B L, Ling L and Liu Q P 2012 Nonlinear Schrödinger equation: generalized darboux transformation and rogue wave solutions *Phys. Rev.* E **85** 026607

[25] Ablowitz M J and Clarkson P A 1992 *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge: Cambridge University Press)