ECON 3130 Notes

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Contents

1 Probabilities and Events

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Definition (Experiment)**.** A process that results in uncertain outcomes.

Definition (Outcome)**.** A possible result of the experiment.

Definition (Sample Space)**.** The set of all possible basic outcomes.

1.1 Events

An **event** is a set of basic outcomes, i.e., a subset of the sample space.

- Example 1: {Tails}
- Example 2: {Admitted}
- Example 3: {Dog, Cat}

1.2 Probabilities

Definition (Probability)**.** A numerical measure of the chance that an event will occur.

- $\Pr(E) = 1 \implies$ Event E is certain to occur.
- $\Pr(E) = 0 \implies$ Event E cannot occur.
- $\Pr(E) = 0.5 \implies$ Event E is equally likely to occur or not.

1.3 Interpretations of Probability

Relative Frequency Viewpoint

$$\Pr(E) = \lim_{n\to\infty} \frac{\text{Number of times } E \text{ occurs}}{n}$$

Probabilities represent long-run relative frequencies when an experiment is repeated independently.

Subjective Probability Probability can also represent an individual's degree of belief about an event.

- Example: "Will Lisa Cook be a Fed Governor on October 1?"
- Example: "Which city is further west: Reno or Las Vegas?"

1.4 Probability Distributions

Definition (Probability Distribution)**.** A table assigning probabilities to all basic outcomes of an experiment, where the probabilities sum to 1.

Example: Preferred Pet

Pet	Pr(Pet)
Dog	0.44
Cat	0.30
Other	0.07
None	0.19
Total	1.00

1.5 Probability Tables

When analyzing multiple attributes (e.g., gender and pet preference), we summarize data in a **probability table**.

Rules:

1. Two attributes: one on rows, one on columns.
2. Events are mutually exclusive and exhaustive.
3. Joint probabilities go in cells; marginal probabilities go in row and column totals.

Example Table: Preferred Pet vs. Gender

	Dog	Cat	Other	None	Pr(Gender)
Female	0.18	0.12	0.04	0.05	0.38
Male	0.32	0.15	0.05	0.09	0.62
Refuse/Other	0.00	0.00	0.00	0.00	0.00
Pr(Pet)	0.50	0.27	0.09	0.14	1.00

1.6 Set Theory and Venn Diagrams

Events can be represented as sets within a sample space.

- $A \cap B$ = Event that both A and B occur.
- $A \cup B$ = Event that A or B (or both) occur.
- A^c = Complement of A, i.e., A does not occur.

1.7 Three Useful Probability Rules

Rule 1: Intersection

$$\Pr(A \cap B) = \Pr(A \mid B)\Pr(B)$$

Rule 2: Union

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

Rule 3: Complement

$$\Pr(A^c) = 1 - \Pr(A)$$

1.8 Conditional Probability

Definition (Conditional Probability)**.** The probability of event A given event B:

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}, \quad \Pr(B) > 0$$

Multiplication Rule

$$\Pr(A \cap B) = \Pr(A \mid B) \Pr(B)$$

2 Discrete Random Variables

Date: Sep 3, 2025

2.1 Independence

Definition (Independence)**.** Two events A and B are **independent** if and only if

$$\Pr(A \cap B) = \Pr(A)\Pr(B).$$

Theorem (Independence and Joint Events)**.** *If A and B are independent, then*

$$Prob(A|B) = Prob(A)$$

The conditional probability = the marginal probability (of the unconditioned event).

2.2 Probability Trees

A way of deriving probabilities from conditional information. They are most useful when one has information that can be arranged sequentially.

Probability Tree Rules:

- The tree is drawn on its side.
- It starts from a circle called a chance node.
- The branches of the tree at each node correspond to possible outcomes. We label the branches with conditional probabilities.
- The probabilities at the end of the tree are joint probabilities obtained by multiplication.

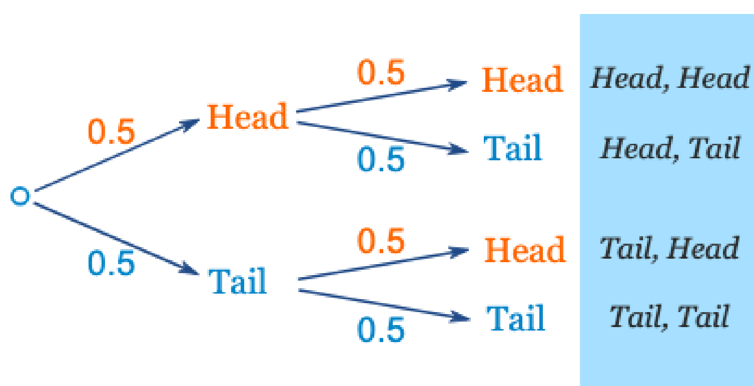


Figure 1: Example of a probability tree.

2.3 The Monty Hall Problem

A classic probability puzzle based on a game show:

- Three doors: behind one is a **car**, behind two are **goats**.

- You choose one door (say Door 1).
- The host, Monty Hall, who knows where the car is, opens another door showing a goat.
- You are given the choice to **stay** or **switch**.

Should you switch?

Solution Let C_i = event that the car is behind door i. Let H = event that Monty opens **Door 3**. Initially:

$$\Pr(C_1) = \Pr(C_2) = \Pr(C_3) = \tfrac{1}{3}.$$

Assume you pick Door 1 and Monty opens Door 3 revealing a goat. By Bayes' theorem:

$$\Pr(C_i \mid H) = \frac{\Pr(H \mid C_i)\Pr(C_i)}{\Pr(H)}.$$

Monty's behavior:

$$\Pr(H \mid C_1) = \tfrac{1}{2}, \qquad \Pr(H \mid C_2) = 1, \qquad \Pr(H \mid C_3) = 0.$$

Thus, the total probability of H:

$$\Pr(H) = \Pr(H \mid C_1)\Pr(C_1) + \Pr(H \mid C_2)\Pr(C_2) + \Pr(H \mid C_3)\Pr(C_3),$$

$$\Pr(H) = \tfrac{1}{2} \cdot \tfrac{1}{3} + 1 \cdot \tfrac{1}{3} + 0 = \tfrac{1}{2}.$$

Finally, the posterior probabilities:

$$\Pr(C_1 \mid H) = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2}} = \tfrac{1}{3}, \qquad \Pr(C_2 \mid H) = \frac{1 \cdot \frac{1}{3}}{\frac{1}{2}} = \tfrac{2}{3}.$$

The probability that your original choice is correct is $\frac{1}{3}$, while switching gives $\frac{2}{3}$. **Always switch.**

2.4 Random Variables

Definition (Random Variable)**.** An experiment with numerical outcomes (or outcomes that can be mapped to numbers).

Notation We denote a random variable with a capital letter, like X or Y.

Probability Mass Function (PMF)

$$f(x) = Pr(X = x)$$

Every discrete random variable has a PMF that describes the probability of each possible value.

2.5 Measures of Central Tendency

Expected Value

$$E[X] = \sum_{x \in S} x \cdot f(x)$$

Mode

$$Mode[X] = \text{value (or values) that maximize } f(x)$$

Median

$$Median[X] = \text{value } m \text{ such that } Pr(X \leq m) = Pr(X \geq m) = 0.5$$

2.6 Measures of Dispersion

Variance

$$Var[X] = E[(X - E[X])^2]$$
$$Var[X] = \sum_{x \in S} (x - E[X])^2 \cdot f(x)$$

A Variance Identity

$$\begin{aligned} Var[X] &= E[(X - E[X])^2] \\ &= E[X^2 - 2XE[X] + (E[X])^2] \\ &= E[X^2] - E[2XE[X]] + E[(E[X])^2] \\ &= E[X^2] - 2E[X]E[X] + (E[X])^2 \\ &= \boxed{E[X^2] - (E[X])^2} \end{aligned}$$

Variance of a Linear Function of X

$$\begin{aligned} Var[aX + b] &= E[(aX + b - E[aX + b])^2] \\ &= E[(aX + b - aE[X] - b)^2] \\ &= E[(aX - aE[X])^2] \\ &= E[a^2(X - E[X])^2] \\ &= a^2 E[(X - E[X])^2] \\ &= \boxed{a^2 Var[X]} \end{aligned}$$

Standard Deviation

$$\mathrm{SD}[X] = \sqrt{\mathrm{Var}[X]}$$

2.7 Conditional Expectations

Definition (Conditional Expectations)**.** The conditional expectation of a random variable X given another random variable Y is defined as:

$$E[X \mid Y] = \sum_{y \in S_Y} E[X \mid Y = y] \cdot P(Y = y)$$

3 More Discrete Random Variables

Date: Sep 8, 2025

3.1 Moments

Definition (Moment)**.** Numbers that represent qualities of a particular distribution. Expected values of a random variables raised to a different powers. rth moment centered around b of a random variable X is defined as

$$\mu_r = E[(X-b)^r]$$

The mean is the first moment centered around zero. The variance is the second moment centered around E[X].

3.2 Moment-generating function (MGF)

Definition (Moment-generating function)**.** The moment-generating function (MGF) of a random variable X is defined as

$$M_X(t) = E[e^{tX}] = \sum_x e^{tx} f(x) \quad \text{(pmf)}$$

for all t in an open interval containing 0 such that the expectation exists.

The mgf is not always defined for a random variable. If two random variables have the same mgf, then they also have the same pmf.

Finding moments using MGF The rth moment of a random variable X can be found by taking the rth derivative of the mgf and evaluating it at $t = 0$:

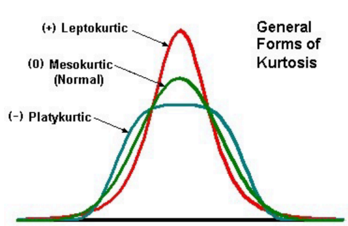
$$\mu_r = E[X^r] = M_X^{(r)}(0) = \sum_x x^r f(x)$$

3.3 Standardized moments

$$\hat{\mu}_k = \frac{E[(X-\mu)^k]}{E[(X-\mu)^2]^{k/2}}$$

kth centered moment divided by the standard deviation raised to the k. Standardized moments are unit-invariant.

Skew is standardized third moment. Negative skew = left tail. Positive skew = right tail. Kurtosis is standardized fourth moment.



Kurtosis measures how fat the tails are. Positive means more extreme values. Negative means fewer extreme values. Note that all three of these random variables have the same variance.

3.4 Bernoulli distribution

Definition (Beroulli distribution)**.** A Bernoulli random variable X takes the value 1 with probability p and the value 0 with probability $1 - p$.

$$P(X = 1) = p, \quad P(X = 0) = 1 - p$$

The mean of a Bernoulli random variable is $E[X] = p$. The variance of a Bernoulli random variable is $Var(X) = p(1 - p)$.

3.5 Binomial distribution

Definition (Binomial distribution)**.** A Binomial random variable X with parameters n and p counts the number of successes in n independent Bernoulli trials, each with success probability p.

The mean of a Binomial random variable is $E[X] = np$. The variance of a Binomial random variable is $Var(X) = np(1 - p)$ (by the linearity of expectation).

- There are a fixed number of independent trials.
- Each trial has two basic outcomes (e.g., "success"/"failure").
- The probability of a success, p, is constant across trials.

Binomial probability mass function (pmf) The probability mass function (pmf) for a Binomial random variable X is:

$$\Pr(X = k \mid n, p) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}$$

where:

- n = number of trials (maximum possible number of successes)
- k = number of successes
- $n - k$ = number of failures
- p = probability of a success on each trial
- $1 - p$ = probability of a failure on each trial

X can also be defined as the sum of n independent Bernoulli random variables, each with parameter p:

$$X = \sum_{i=1}^{n} X_i, \quad X_i \sim \text{Bernoulli}(p)$$

4 Continuous Random Variables

Date: Sep 10, 2025

Definition (Continuous Random Variable)**.** A random variable X is said to be continuous if there exists a function $f(x)$ such that for any set A,

$$P(X \in A) = \int_A f(x)\,dx$$

where $f(x)$ is called the probability density function (pdf) of X. Key properties of a legal pdf: $f(x) \geq 0\ \forall x \in S$ and $\int_S f(x)dx = 1$.

Cumulative Distribution Function (CDF)

$$F(a) = \Pr(X \leq a) = \int_{-\infty}^{a} f(x)dx$$

Expected Value The expected value (mean) of a continuous random variable X with pdf $f(x)$ is defined as:

$$E[X] = \int_{-\infty}^{+\infty} x f(x)\,dx$$

Variance The variance of a continuous random variable X with pdf $f(x)$ is defined as:

$$\mathrm{Var}[X] = \int_{-\infty}^{+\infty} (x - E[X])^2 f(x)\,dx$$

Moment-Generating Function (MGF) The moment-generating function (MGF) of a continuous random variable X with pdf $f(x)$ is defined as:

$$M(t) = \int_{-\infty}^{+\infty} e^{tx} f(x)\,dx, \qquad -h < t < h$$

where h is a constant such that $M(t)$ is finite for t in $(-h, h)$.

For discrete random variables, the MGF is:

$$M(t) = \sum_{x \in S} e^{tx} f(x)$$

The MGF characterizes the distribution if it is finite in some neighborhood around t.

The r-th moment of X can be obtained by differentiating $M(t)$ r times and evaluating at $t = 0$:

$$M^{(r)}(0) = E[X^r]$$

Quantiles The p'th quantile π_p is defined as the value such that

$$p = \int_{-\infty}^{\pi_p} f(x)\,dx$$

5 Normal Random Variables

Date: Sep 15, 2025

5.1 Definition and Properties

Definition (Normal Random Variable)**.** A normal random variable is a continuous random variable whose probability density function (pdf) is given by:

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

where μ is the mean and σ^2 is the variance (σ is the standard deviation).

- The normal distribution is symmetric about its mean μ.
- The total area under the pdf is 1.
- The standard normal distribution has $\mu = 0$ and $\sigma^2 = 1$.
- Probabilities are computed as areas under the curve:

$$\Pr(a < X < b) = \int_a^b f(x)dx$$

- $\mathbb{E}(X) = \mu$
- $\text{Var}(X) = \sigma^2$ (and thus $\text{SD}(X) = \sigma$)
- 68%, 95%, 99.7% of outcomes are within 1, 2, 3 standard deviations of the mean, respectively.

A normal random variable X is denoted by:

$$X \sim N(\mu, \sigma^2)$$

which we read as "X is normally distributed with mean μ and variance σ^2."

If $\mu = 0$ and $\sigma = 1$, then the random variable is a **standard normal**, denoted Z.

5.2 Moment Generating Function (mgf) of the Normal

In general, the moment generating function (mgf) of a random variable X is:

$$M(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{+\infty} e^{tx} f(x)dx$$

For the normal distribution:

$$M(t) = \int_{-\infty}^{+\infty} \frac{e^{tx}}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx$$

This simplifies to:

$$M(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

The mgf can be used to compute all moments of the normal distribution by differentiating $M(t)$ with respect to t and evaluating at $t = 0$.

5.3 Linear Combinations of Independent Normals

1. **Sums.** The sum of independent normally distributed random variables is also normally distributed.
   Example: If X and Y are independent and $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$, then $W = X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

2. **Constants.** If $Z \sim N(0, 1)$, then for constants a and b, $a + bZ \sim N(a, b^2)$. In general, multiplying a normal random variable by a constant and/or adding a constant yields another normal random variable.

3. **Standardizing.** If $X \sim N(\mu, \sigma^2)$, then the standardized variable $Z = \frac{X-\mu}{\sigma}$ follows a standard normal distribution: $Z \sim N(0, 1)$.

6 More Random Variables

Date: Sep 17, 2025