# A Brief Review of Quantum Liouville Theory

YUN-YEN, WU

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# Outline

- The Quantum Liouville Theory

- Degenerate Fields & Operator Algebra

- Logarithmic Conformal Field Theory

- Comparison with fermionic theory

- Conclusion & Future Work

# The Quantum Liouville Theory

# The Quantum Liouville Theory

▶ Setup: Partition function of 2D quantum gravity coupled to conformal matter

$$Z = \int D[g]e^{-S_{matter}[g]}$$

▶ Conformal gauge fixing using background metric

$$\hat{g}^{\mu\nu}(x) = e^{\hat{\sigma}(x)}g_{\mu\nu}^{(0)}(x)$$

▶ Problem and solution $\qquad\qquad\qquad \langle T_\mu^\mu \rangle \neq 0$

Sum over the scale factor $\sigma \implies$ Functional integral is not standard "linear" measure as well as trace anomaly!

Perform the Jacobian and found it can be exponentiated into a local action (by Polyakov (1981))

$$D[g] = J[\sigma]D[\sigma] = e^{S_L[\sigma]}D[\sigma]$$
anomaly

# The Quantum Liouville Theory

▶ The Classical Liouville Theory

$$S_{\text{cl}} = \frac{1}{4\pi} \int d^2 x \sqrt{g} \left( (\partial\varphi)^2 + \Lambda e^{\varphi} \right)$$

▶ Equation of Motion

$$\partial\bar{\partial}\varphi = \frac{\Lambda}{2} e^{\varphi} \Longrightarrow$$     surfaces of **constant negative curvature**

Solvable by Bäcklund transformation!

# The Quantum Liouville Theory

▶ The renormalized action in Liouville Theory

background charge

$$A_L[\hat{g}, \varphi] = \frac{1}{4\pi} \int \sqrt{\hat{g}} \left[ \hat{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + Q\hat{R}\varphi + 4\pi\mu e^{2b\varphi} \right] d^2x$$

kinetic                    Liouville Interaction

▶ Conformal gauge fixing using background metric

$$\hat{g}^{\mu\nu}(x) = e^{\hat{\sigma}(x)} g^{(0)}_{\mu\nu}(x) \quad \text{with geometric definition } \sigma(x) = 2b\varphi$$

▶ The functional integral (Matter + Ghosts + Liouville)

$$Z_{matter,ghosts}[\hat{g}] e^{-A_L[\hat{g},\varphi]} D_{\hat{g}}[\varphi]$$

# The Quantum Liouville Theory

▶ Requirement: Background independence of metric

▶ Constraint 1: Central charge balance --- **Conformal Invariant** with a total central charge of zero

$$c_{ghost}(-26) + c_{matter} + c_{Liouville} = 0 \quad \Longrightarrow \quad Q = \sqrt{\frac{25 - c_M}{6}}$$

▶ Constraint 2: Marginality --- Interaction term be conformal invariant and dimensionless ($\Delta = 1$)

$$\Longrightarrow \quad Q = \frac{1}{b} + b$$

# The Quantum Liouville Theory

▶ The holomorphic and anti-holomorphic energy-momentum tensor with respect to the background metric

$$T = -(\partial_z \varphi)^2 + Q\partial_z^2 \varphi$$
$$\bar{T} = -(\partial_{\bar{z}} \varphi)^2 + Q\partial_{\bar{z}}^2 \varphi$$

▶ Operator Product Expansion

$$T(z)T(w) = \frac{c_L/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \quad \text{with} \quad c_L = 1 + 6Q^2$$

▶ Duality

$$Q = b + \frac{1}{b} \quad \Longrightarrow \quad \text{Invariant for} \quad b \leftrightarrow \frac{1}{b}$$

# The Quantum Liouville Theory

▶ Primary field corresponding to momentum $\alpha$ in Liouville Theory

$$V_\alpha(\mathrm{x}) = e^{2\alpha\varphi(x)}$$

▶ Conformal dimension using OPE

Symmetry here!

$$\Delta_\alpha = \Delta_{Q-\alpha}$$
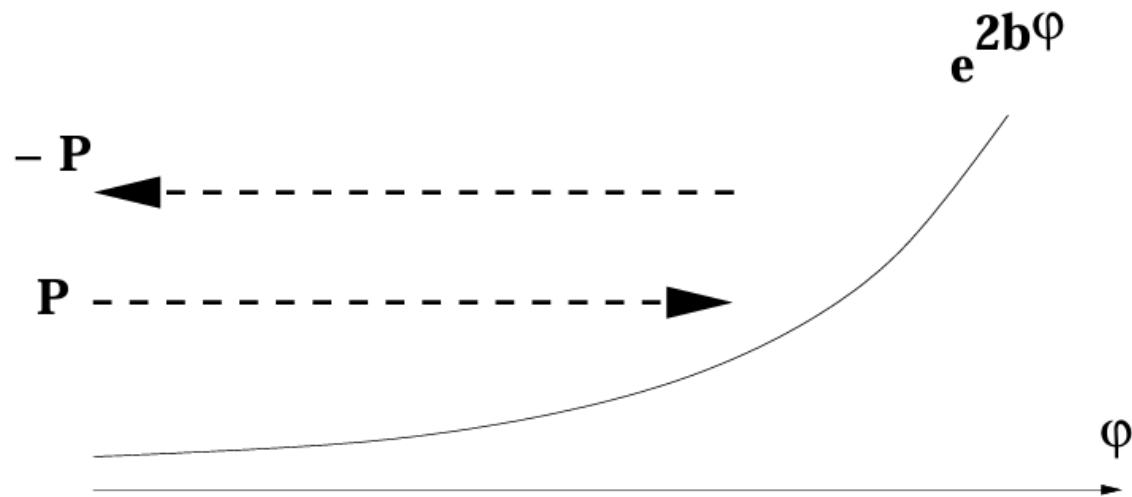
$$T(z)V_\alpha(w,\bar{w}) \sim \frac{\alpha(Q-\alpha)}{(z-w)^2}V_\alpha(w,\bar{w}) + \frac{1}{z-w}\partial_w V_\alpha(w,\bar{w}) \implies \Delta_\alpha = \alpha(Q-\alpha)$$

▶ Seiberg bound: physical spectrum is continuous with $\alpha = \frac{Q}{2} + iP$

● $\mathrm{Re}(\alpha) \leq Q/2$: Normalizable states

● $\mathrm{Re}(\alpha) > Q/2$: Non-normalizable (Local singularity)

# The Quantum Liouville Theory

▶ Reflection property: $\Delta_\alpha = \Delta_{Q-\alpha}$

$\quad V_\alpha = S(\alpha)V_{Q-\alpha}$  ( here $S(\alpha)$ has analytic form)

▶ Interpretation: The exponential potential acts like a "wall" at $\phi \to \infty$

# The Quantum Liouville Theory

▶ Three-point function in Liouville Theory: DOZZ Formula (1990)

$$\langle V_{\alpha_1}(x_1)V_{\alpha_2}(x_2)V_{\alpha_3}(x_3)\rangle = \frac{C(\alpha_1,\alpha_2,\alpha_3)}{(x_{12}\bar{x}_{12})^{\Delta_1+\Delta_2-\Delta_3}(x_{23}\bar{x}_{23})^{\Delta_3+\Delta_2-\Delta_1}(x_{31}\bar{x}_{31})^{\Delta_3+\Delta_1-\Delta_2}}$$

$$\text{with} \quad C(\alpha_i) \propto \frac{\Upsilon'(0)\prod\Upsilon(2\alpha_i)}{\Upsilon(\sum\alpha_i - Q)\prod(\alpha_i + \alpha_j - \alpha_k)}$$

have explicit form

# Degenerate Fields & Operator Algebra

# Degenerate Fields & Operator Algebra

▶ Classical Liouville Theory: Equation of motion

$$\partial\bar{\partial}\varphi = Me^{\varphi} \qquad \text{Non-linear!}$$

▶ Linear equation? Basic example of n=2 case:

$$\left(\partial^2 + T^{(c)}\right)e^{-\varphi/2} = 0$$

which satisfies second-order linear equation, manifesting of a "degenerate" field

▶ General construction of degenerate field

$$V_n^{(c)} = e^{(1-n)\varphi/2} \; with \; n = 1,2,3\dots \quad \Longrightarrow \quad D_n^{(c)}V_n^{(c)} = \overline{D}_n^{(c)}V_n^{(c)} = 0$$

null vector

with a unique differential operator

$$D_n^{(c)} = \partial^n + \cdots + (n-1)\partial^{n-2}T^{(c)}$$

# Degenerate Fields & Operator Algebra

▶ Next, consider the following field $\varphi V_n^{(c)}$

▶ Property of this new field

$$\overline{D}_n^{(c)} D_n^{(c)} \left(\varphi e^{(1-n)\varphi/2}\right) = B_n^{(c)} e^{(1+n)\varphi/2} \longrightarrow \quad \text{classical higher equation of motion}$$

with a particular coefficient

$$B_n^{(c)} = 2(-1)^{n+1} n! \, (n-1)! \left(\frac{M}{2}\right)^n$$

▶ The new field derives from the derivative with respect to n!

▶ The equation maps a "Logarithmic field" to a "dual" exponential field

# Degenerate Fields & Operator Algebra

▶ Next, consider the quantum Liouville field theory Lagrangian density

$$L = \frac{1}{4\pi}(\partial_a\phi)^2 + \mu e^{2b\phi} \implies \partial\bar{\partial}\phi = \pi b\mu e^{2b\phi}$$

▶ Parameter mapping

$$2b\phi \to \varphi, \qquad T \to b^{-2}T^{(c)}, \qquad 2\pi\mu b^2 \to M$$

▶ Discrete subset of primaries $V_{m,n} = V_{\alpha_{m,n}}$ with special value of positive m and n

$$\alpha_{m,n} = -(m-1)\frac{1}{2b} - (n-1)\frac{b}{2}$$

these fields called <span style="color:red">degenerate</span>, since the corresponding conformal dimension

$$\Delta_{m,n} = \frac{Q^2}{4} - \frac{(mb^{-1} + nb)^2}{4}$$

are the Kat dimensions of the degenerate representations of the Virasoro algebra

Duality!

$$b \leftrightarrow \frac{1}{b}$$

$$\Downarrow$$

$$\alpha_{m,n}(b) \leftrightarrow \alpha_{n,m}(1/b)$$

# Degenerate Fields & Operator Algebra

- Null vector property: $V_{m,n}$ has a null vector as level mn

$$D_{m,n}V_{m,n} = \overline{D}_{m,n}V_{m,n} = 0$$

with the operator constructed by

$$D_{m,n} = L_{-1}^{mn} + d_1^{m,n}(b)L_{-2}L_{-1}^{mn-2} + \cdots$$

- Example of Level 2 (1,2)
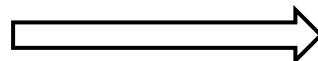
$$D_{1,2} = L_{-2} + b^2 L_{-1}^2 \qquad \Longleftrightarrow \qquad \partial^2 + T^{(c)} \quad \text{in classical}$$

# Logarithmic Conformal Field Theory

# Logarithmic Conformal Field Theory

▶ What is Logarithmic Conformal Field Theory?

Standard CFT:
Diagonalizable Hamiltonian ($L_0$)

$$L_0|\Delta\rangle = \Delta|\Delta\rangle$$

primary field

$\Longrightarrow$

Logarithmic CFT:
Non-diagonalizable Hamiltonian (Jordan Block)

$$L_0|\phi\rangle = \Delta|\phi\rangle \quad \text{primary field}$$
$$L_0|\psi\rangle = \Delta|\psi\rangle + |\phi\rangle$$

logarithmic partner

$$\begin{bmatrix} \Delta & 1 \\ 0 & \Delta \end{bmatrix}$$

# Logarithmic Conformal Field Theory

▶ Signature of Logarithmic CFT: correlation function (Scale invariance is broken by logarithmic terms)

$$\langle \phi(z)\phi(0)\rangle = 0 \quad \longrightarrow \quad \text{zero norm for primaries}$$
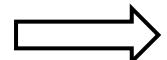
$$\langle \phi(z)\psi(0)\rangle = \frac{B}{z^{2\Delta}}$$

$$\langle \psi(z)\psi(0)\rangle = \frac{C - 2B \ln z}{z^{2\Delta}} \quad \longrightarrow \quad \text{logarithmic coupling (zero in standard CFT)}$$

# Liouville Theory as Logarithmic Conformal Field Theory

▶ What is the logarithmic partner of $V_\alpha$ in Liouville Theory?

$$V'_\alpha = \frac{1}{2}\frac{\partial}{\partial\alpha}V_\alpha = \phi e^{2\alpha\phi}$$

$\Longrightarrow$  logarithmic field (by Al.Zamolodchikov)

▶ OPE of this logarithmic field

$$T(z)V'_\alpha(w,\overline{w}) = \frac{\alpha(Q-\alpha)}{(z-w)^2}V'_\alpha(w,\overline{w}) + \frac{(Q/2-\alpha)}{(z-w)^2}V_\alpha(w,\overline{w}) + \frac{1}{z-w}\partial_w V'_\alpha(w,\overline{w})$$

Jordan block here!

# Liouville Theory as Logarithmic Conformal Field Theory

▶ Property of $V'_{m,n}$

$$D_{m,n}V_{m,n} = 0 \xrightarrow{\quad ? \quad} D_{m,n}V'_{m,n} = 0$$

▶ The answer is no! It is a primary field with different conformal dimension!

$$A_{m,n} = \lim_{\alpha \to \alpha_{m,n}} \frac{D_{m,n}\overline{D}_{m,n}V_{m,n}}{\alpha - \alpha_{m,n}}$$

is also a primary field

Proof: $D_{m,n}\overline{D}_{m,n}V_{m,n} = 0$ is a null vector

Consider the vicinity of $\alpha = \alpha_{m,n}$ $\longrightarrow$ $D_{m,n}\overline{D}_{m,n}V_\alpha = 0 + (\alpha - \alpha_{m,n})A_{m,n} + \cdots$

Apply the derivative $\longrightarrow$ $D_{m,n}\overline{D}_{m,n}V'_{m,n} = A_{m,n}$

Primary field with conformal dimension $\Delta_{m,n} + mn$ !

$$D_{m,n} = L_{-1}^{mn} + d_1^{m,n}(b)L_{-2}L_{-1}^{mn-2} + \cdots$$

# Liouville Theory as Logarithmic Conformal Field Theory

▶ What is the new primary field after mapping by $D_{m,n}\overline{D}_{m,n}$?

$$D_{m,n}\overline{D}_{m,n}V'_{m,n} = B_{m,n}\tilde{V}_{m,n} \longrightarrow \text{Denote} \quad \tilde{V}_{m,n} = V_{\tilde{\alpha}_{m,n}} \text{ with } \quad \tilde{\alpha}_{m,n} = -(m-1)\frac{1}{2b} + (n+1)\frac{b}{2}$$

Operator-valued relation

having conformal dimension $\Delta_{m,n} + mn$ as we expect!

↓

$$\tilde{V}_{m,n} = V_{m,-n}$$

Dual field by flipping the sigh of n

Analytic solution: $\quad B_{m,n} = \left(\pi\mu\gamma(b^2)\right)^n b^{1+2n-2m}\gamma(m-nb^2) \prod_{\substack{k=1-n \\ l=1-m \\ (k,l)\neq(0,0)}}^{\substack{m-1 \\ n-1}} (lb^{-1} + kb)$$

# Conclusion & Future Work

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▶ Conclusion

1. Logarithmic Liouville theory is the bosonization of symplectic fermions with non-Hermitian structure.

2. Liouville field theory has Jordan structure between primary fields and logarithmic fields

3. Operator-valued relation establish a dual primary field in Liouville field theory

# Reference

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