Bosonization in 1+1 dimension and Sine-Gordon Model

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Outline

Introduction

Both free massless bosonic and fermionic theories are CFT with $c = 1$.
Bosonization is the procedure that maps the fermion case to the boson case.

Free Massless Fermion $\longrightarrow$ Free Massless Boson

A boson corresponds to a particle-hole excitation in fermionic theory.

Quantization of Fermion

Massless Dirac Fermion

The action for a free massless Dirac fermion in 1+1d is

$$S = \int d^2x \, i\bar{\psi}(\gamma_0 \partial_t - \gamma_1 \partial_x)\psi, \quad \psi = (\psi_-, \psi_+)^T$$

obeying the anti-commutation relation

$$\{\psi_\pm^\dagger(x), \psi_\pm(y)\} = \delta(x-y)$$

In terms of the Fourier transforms

$$\psi_\pm(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \psi_\pm(p) e^{ipx} e^{-\frac{1}{2}\epsilon|p|}$$

obeying

$$\{\psi_\pm^\dagger(p), \psi_\pm(q)\} = 2\pi\delta(p-q)$$

Massless Dirac Fermion

Compute some two point functions in position space

$$\left\langle \psi_+(x)\psi_+^\dagger(y) \right\rangle = \frac{i}{2\pi} \frac{1}{(x-y)+i\epsilon}$$
$$\left\langle \psi_-(x)\psi_-^\dagger(y) \right\rangle = \frac{-i}{2\pi} \frac{1}{(x-y)-i\epsilon}$$

We can also define our vector and axial current

$$j_V^\mu = \bar{\psi}\gamma^\mu\psi$$
$$j_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$$

Quantization of Boson

Massless Compact Boson

The action of the massless compact boson in 1+1d is

$$S = \int d^2x \, \frac{\beta^2}{2} (\partial_\mu \phi)^2, \quad \phi \in [0, 2\pi)$$

with radius $R = \sqrt{2\pi}\beta l_s$
The action is invariant under $\phi \to \phi + constant$, the associated current is

$$j^\mu_{shift} = \beta^2 \partial^\mu \phi$$

There is also another symmetry in 1+1 dimension corresponding to winding, with associated current

$$j^\mu_{wind} = \frac{1}{2\pi} \epsilon^{\mu\nu} \partial_\nu \phi$$

which the charge Q_{wind} is the number of times go around the spatial circle.

Massless Compact Boson

There is another $\tilde{\phi}$ satisfy the periodic condition of the original theory,

$$S = \int d^2x \, \frac{\tilde{\beta}^2}{2} (\partial_\mu \tilde{\phi})^2 \quad \tilde{\phi} \in [0, 2\pi)$$

with $\tilde{\beta}$ and $\tilde{R}$ is

$$\tilde{\beta}^2 = \frac{1}{4\pi\beta^2}, \quad \tilde{R} = \frac{l_s^2}{R}$$

The relation of ϕ and $\tilde{\phi}$ is

$$\partial_\mu \phi = \frac{1}{2\pi\beta^2} \epsilon_{\mu\nu} \partial^\nu \tilde{\phi}$$

which is referred to as T-duality.

Massless Compact Boson

Do the canonical quantization of ϕ

$$\phi(x,t) = \frac{1}{\beta}\int \frac{dp}{2\pi}\frac{1}{\sqrt{2\,|p|}}(a_p e^{ipx} + a_p^\dagger e^{-ipx})e^{-\frac{1}{2}\epsilon|p|}$$

Introduce the chiral boson

$$\begin{aligned}
\phi_\pm(x,t) &= \frac{1}{2}\left[\phi(x,t) \pm \frac{1}{2\pi\beta^2}\tilde{\phi}(x,t)\right] \\
&= \frac{1}{2}\left[\phi(x,t) \mp \frac{1}{\beta^2}\int_{-\infty}^{x}\Pi(x')dx'\right] \\
&= \pm\frac{1}{\beta}\int_0^{\pm\infty}\frac{dp}{2\pi\sqrt{2\,|p|}}\left[a_p e^{ipx} + h.c.\right]e^{-\frac{1}{2}\epsilon|p|}
\end{aligned}$$

which satisfy $\partial_+\phi_- = 0$ and $\partial_-\phi_+ = 0$

Massless Compact Boson

The chiral boson satisfy the commutation relation:

$$[\phi_-(x), \phi_-(y)] = -\frac{i}{4\beta^2} sgn(x-y)$$

$$[\phi_+(x), \phi_+(y)] = \frac{i}{4\beta^2} sgn(x-y)$$

$$[\phi_+(x), \phi_-(y)] = \frac{i}{4\beta^2}$$

Now, compute some Green's function defined below

$$G_-(x,y) = \left\langle \phi_-(x)\phi_-(y) - \phi_-^2(0) \right\rangle = \frac{1}{4\pi\beta^2} \ln \frac{\epsilon}{\epsilon + i(x-y)}$$

$$G_+(x,y) = \left\langle \phi_+(x)\phi_+(y) - \phi_+^2(0) \right\rangle = \frac{1}{4\pi\beta^2} \ln \frac{\epsilon}{\epsilon - i(x-y)}$$

Massless Compact Boson

To compute the correlation function, we use vertex operator with the form

$$e^{i\phi} =: e^{i\phi} :$$

which satisfy the periodic condition.
Finally, we compute the correlator

$$\left\langle e^{i\sqrt{4\pi}\beta\phi_-(x)} e^{-i\sqrt{4\pi}\beta\phi_-(y)} \right\rangle = e^{4\pi\beta^2 G_-(x,y)} = \frac{\epsilon}{\epsilon + i(x-y)}$$

$$\left\langle e^{i\sqrt{4\pi}\beta\phi_+(x)} e^{-i\sqrt{4\pi}\beta\phi_+(y)} \right\rangle = e^{4\pi\beta^2 G_+(x,y)} = \frac{\epsilon}{\epsilon - i(x-y)}$$

Bosonization

Bosonization

Compare the right moving of fermions and bosons correlation function

$$\left\langle \psi_+(x)\psi_+^\dagger(y) \right\rangle = \frac{i}{2\pi} \frac{1}{(x-y)+i\epsilon}$$

$$\left\langle e^{i\sqrt{4\pi}\beta\phi_+(x)} e^{-i\sqrt{4\pi}\beta\phi_+(y)} \right\rangle = \frac{\epsilon}{\epsilon - i(x-y)}$$

We can find the identity relationship

$$\psi_+(x) \longleftrightarrow \sqrt{\frac{1}{2\pi\epsilon}} e^{i\sqrt{4\pi}\beta\phi_+(x)}$$

$$\psi_-(x) \longleftrightarrow \sqrt{\frac{1}{2\pi\epsilon}} e^{-i\sqrt{4\pi}\beta\phi_-(x)}$$

Bosonization

Now, let's compare the commutation relation of both case.
In fermion case,

$$
\begin{aligned}
\{\psi_+^\dagger(x), \psi_+(y)\} &= \frac{1}{2\pi\epsilon}\{e^{-i\sqrt{4\pi}\beta\phi_+(x)}, e^{i\sqrt{4\pi}\beta\phi_+(y)}\} \\
&= \frac{1}{2\pi\epsilon}\left(e^{-i\sqrt{4\pi}\beta\phi_+(x)}e^{i\sqrt{4\pi}\beta\phi_+(y)} + e^{i\sqrt{4\pi}\beta\phi_+(y)}e^{-i\sqrt{4\pi}\beta\phi_+(x)}\right) \\
&= \frac{1}{2\pi\epsilon}\left(\frac{\epsilon}{\epsilon - i(x-y)} + \frac{\epsilon}{\epsilon + i(x-y)}\right) \\
&= \delta(x-y)
\end{aligned}
$$

Also,

$$\{\psi_-^\dagger(x), \psi_-(y)\} = \delta(x-y)$$

Both satisfy the anti-commutation relation of the fermion case.

Bosonization Dictionary

1. Another constraint?

$$
\begin{aligned}
e^{in\phi+iw\tilde{\phi}} &= e^{i(n+2\pi\beta^2 w)\phi_-}e^{i(n-2\pi\beta^2 w)\phi_+} \quad with \quad n, w \in \mathbf{Z} \\
&= e^{i(n+w/2)\phi_-}e^{i(n-w/2)\phi_+}
\end{aligned}
$$

If we want to satisfy the periodicity condition, we need the fermion to coupled the $\mathbf{Z}_2$ symmetry, summing over both periodic and anti-periodic condition (Ramond and Neveu-Schwarz boundary conditions).

2. Other dimensions?
The particle-hole spectrum is a continuum throughout and interactions have a harder time forming coherently propagating particle-hole pairs, so the spectrum cannot be described by such a simple form.

Bosonization Dictionary

	Bosonic theory	Fermionic theory
Action	$\int d^2x \frac{\beta^2}{2}(\partial_\mu \phi)^2$	$\int d^2x i\bar{\psi}\, \partial\!\!\!/ \psi$
Vertex Operator	$\sqrt{\frac{1}{2\pi\epsilon}} e^{-i\sqrt{4\pi}\beta\phi_-}$	ψ_-
Vertex Operator	$\sqrt{\frac{1}{2\pi\epsilon}} e^{i\sqrt{4\pi}\beta\phi_+}$	ψ_+
Mass	$-\frac{1}{\pi\epsilon}\cos\sqrt{4\pi}\beta\phi$	$\bar{\psi}\psi$
Mass	$\frac{1}{\pi\epsilon}\sin\sqrt{4\pi}\beta\phi$	$\bar{\psi} i\gamma^5\psi$
Vector current	$j^\mu_{wind} = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi$	$j^\mu_V = \bar{\psi}\gamma^\mu\psi$
Axial current	$j^\mu_{shift} = \beta^2\partial^\mu\phi$	$j^\mu_A = \bar{\psi}\gamma^\mu\gamma^5\psi$

Klein Factor

If more than one fermion species, we need to introduce the "Klein Factor".

$$\psi_-^\mu(x) \longleftrightarrow \sqrt{\frac{1}{2\pi\epsilon}}\eta_-^\mu e^{-i\sqrt{4\pi}\beta\phi_-^\mu(x)}$$

$$\psi_+^\mu(x) \longleftrightarrow \sqrt{\frac{1}{2\pi\epsilon}}\eta_+^\mu e^{i\sqrt{4\pi}\beta\phi_+^\mu(x)}$$

which obey the Clifford algebra

$$\{\eta_-^\mu, \eta_-^\nu\} = \{\eta_+^\mu, \eta_+^\nu\} = 2\delta^{\mu\nu}, \quad \{\eta_-^\mu, \eta_+^\nu\} = 0$$

Massless Thirring Model

Consider the massless, interacting Thirring model, with action

$$S = \int d^2x \, \bar{\psi} i \not{\partial} \psi - g(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi)$$

Using Bosonization rule, the result is

$$S = \int d^2x \frac{\beta^2}{2} (\partial_\mu \phi)^2$$

with

$$\beta^2 = \frac{1}{4\pi} + \frac{g}{2\pi^2}$$

Sine-Gordon Model

Sine-Gordon Model

Now, we consider the action of the massive fermionic model

$$S = \int d^2x \, \bar{\psi}(i \not{\partial} - m)\psi$$

Using bosonization rule, the result is

$$S = \int d^2x \, \frac{\beta^2}{2} (\partial_\mu \phi)^2 + \frac{m}{\pi \epsilon} \cos \sqrt{4\pi} \beta \phi$$

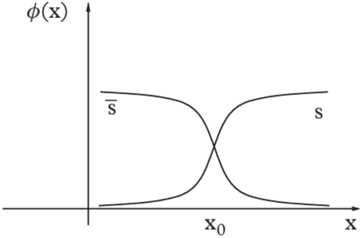
which is called the sine-Gordon model.

Sine-Gordon Model

The solution of Sine-Gordon model is

$$\phi(x) = 4 \arctan \left[e^{m(x - x_0 - vt)} \right]$$

It can be understood as soliton and anti-soliton with topological charge $T = \pm 1$



The fermion is identified with a kink in the Sine-Gordon model.

Sine-Gordon Model with Non-Hermition Mass

Consider the fermion case with non-Hermition mass

$$S = \int d^2x\, \bar{\psi}(i\,\partial\!\!\!/)\psi - \Delta\psi_-^\dagger\psi_+ - \Gamma\psi_+^\dagger\psi_-$$

Using the bosonization rule, the result is

$$\begin{aligned} S &= \int d^2x\, \frac{\beta^2}{2}(\partial_\mu\phi)^2 + \frac{\Delta}{2\pi\epsilon}e^{-i\sqrt{4\pi}\beta\phi} + \frac{\Gamma}{2\pi\epsilon}e^{i\sqrt{4\pi}\beta\phi} \\ &= \int d^2x\, \frac{\beta^2}{2}(\partial_\mu\phi)^2 + \frac{\Delta+\Gamma}{2\pi\epsilon}\cos\sqrt{4\pi}\beta\phi + i\frac{\Gamma-\Delta}{2\pi\epsilon}\sin\sqrt{4\pi}\beta\phi \end{aligned}$$

which may not has the exact solution. (vs. the exact solution in fermionic model)

Conclusion and Outlook

Conclusion and Outlook

Conclusion

- In 1+1 dimension, the fermionic theory coupled with $\mathbf{Z}_2$ symmetry can match with the bosonic theory through the bosonization rule
- In Sine-Gordon model, a fermion can be identified as a kink

Outlook

- Solve and Calculate the correlation function of Sine-Gordon model, especially do it at exception point

Reference

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Thank you