Abelian Bosonization in 1+1 dimensions

YUN-YEN, WU

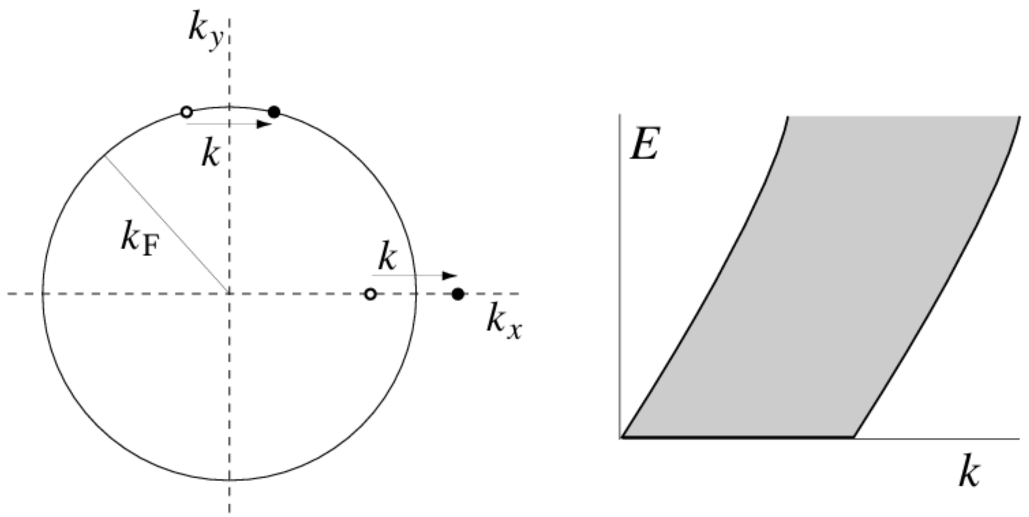
2025.12.16

Outline

- Motivation: The 1D Anomaly
- The Bosonization Dictionary
- Interactions & The Luttinger Liquid
- Summary & Conclusion
- Q & A

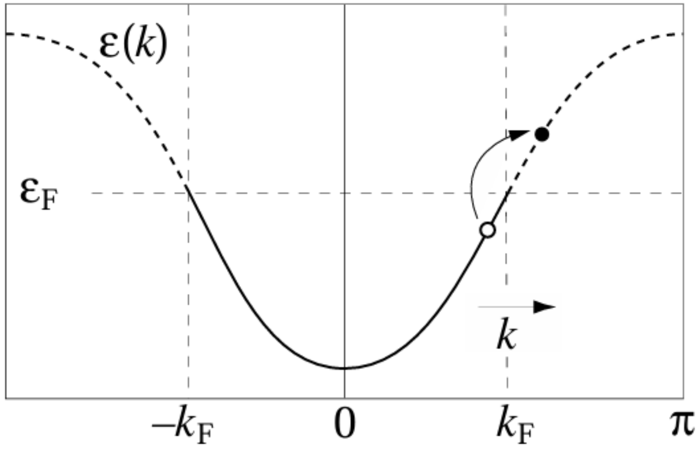
Motivation: The 2D Problem

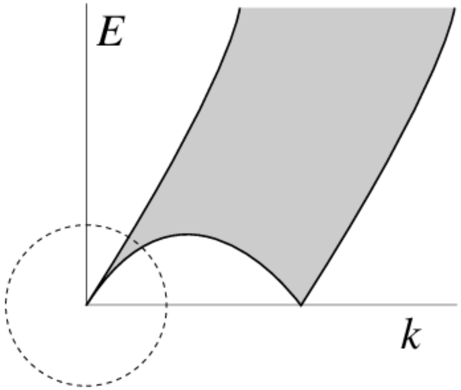
- **Geometry:**
  - Continuous Fermi surface (circle)
  - Particle & Hole group velocities are generally **not collinear**
- **Consequence (Landau Damping):**
  - Pairs dephase and drift apart
  - **Collective modes decay**
  - Single quasi-particles remain stable
- **Conclusion:**
  - Landau's Fermi Liquid Theory is valid



Why is 1D Special?

- **Kinematic Constraint:**
  - Discrete points of Fermi surface ($\pm k_F$)
  - Linear Dispersion: $\epsilon(k) \approx \pm v_F(k \mp k_F)$ near k_F
  - Velocity matching: $v_{particle} = v_{hole} = v_F$
- **Consequence:**
  - Particle-hole pair propagates **coherently**
  - Stable collective density modes
- **Conclusion:**
  - Fermi liquid theory fails → **Tomonaga-Luttinger Liquid**





Chiral Decomposition & Free Dirac Fermion in 1+1D

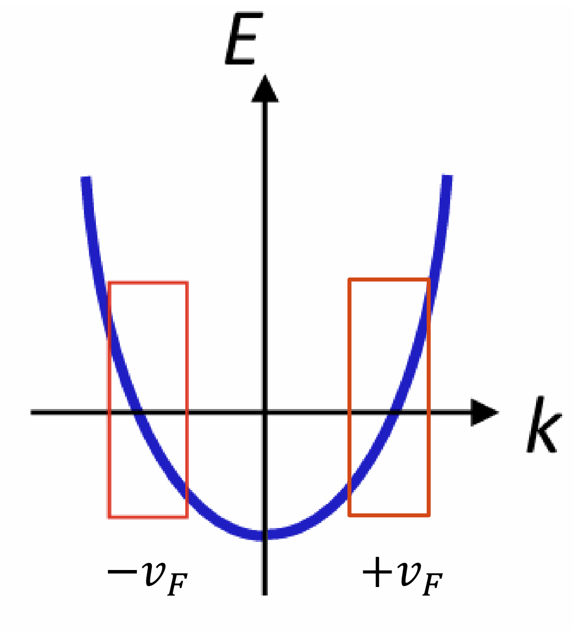
- **Linearization (slowly varying):**

$$\psi(x) \approx \underbrace{e^{ik_F x}\psi_R(x)}_{\text{right}} + \underbrace{e^{-ik_F x}\psi_L(x)}_{\text{left}}$$

- **Free Dirac Hamiltonian (linearized):**

$$H_0 = -iv_F \int dx \left(\psi_R^\dagger \partial_x \psi_R - \psi_L^\dagger \partial_x \psi_L\right)$$

- **Physical interpretation:**
  - Describe two **decoupled** massless fermions
  - Satisfy $U(1)_L \times U(1)_R$ symmetry



Current Algebra & Sugawara

- **Define Chiral Currents (Density):**

$$J_R(x) =: \psi_R^\dagger(x)\psi_R(x): \quad ; \quad J_L(x) =: \psi_L^\dagger(x)\psi_L(x):$$

- **Calculate current^2 (quantum mechanically):**

- $J_L(z)J_L(w) =: \psi_L^\dagger(z)\psi_L(z) :: \psi_L^\dagger(w)\psi_L(w):$

$$= \underbrace{\left(\frac{1}{2\pi(z-w)}\right)^2}_{\text{singular}} + \frac{i}{\pi} : \psi_L^\dagger(x)\partial_x\psi_L(x): + \ \ldots \ \xrightarrow[z \to w]{\text{normal ordering}} \ \begin{aligned} :(J_L)^2: &= +\frac{i}{\pi}\psi_L^\dagger(x)\partial_x\psi_L(x) \\ :(J_R)^2 &= -\frac{i}{\pi}\psi_R^\dagger(x)\partial_x\psi_R(x) \end{aligned}$$

- **Sugawara Construction:**

$$H_0 = -iv_F \int dx \left(\psi_R^\dagger\partial_x\psi_R - \psi_L^\dagger\partial_x\psi_L\right) = \pi v_F \int dx \{:(J_L)^2: + :(J_R)^2:\}$$

Sugawara form

"Hamiltonian becomes Bosonic !"

The Bosonic Signature

$$\boxed{\begin{aligned}\varphi &= \phi_L + \phi_R \\ \theta &= \phi_L - \phi_R\end{aligned}}$$

- **Free currents behave like "bosons"**

Fermion Current Algebra

$$[J_L(x), J_L(y)] = \frac{i}{2\pi}\partial_x \delta(x-y)$$

$$[J_R(x), J_R(y)] = -\frac{i}{2\pi}\partial_x \delta(x-y)$$

Schwinger term

$$J_{R,L} \leftrightarrow \frac{1}{2\pi}\frac{\partial}{\partial x}\phi_{R,L}$$

local correspondence

Free Boson Field derivative

$$[\partial_x \phi_L(x), \partial_x \phi_L(y)] = 2\pi i \partial_x \delta(x-y)$$

$$[\partial_x \phi_R(x), \partial_x \phi_R(y)] = -2\pi i \partial_x \delta(x-y)$$

- **Hamiltonian Mapping:**

$$H_0 = -iv_F \int dx \left\{\psi_R^\dagger \partial_x \psi_R - \psi_L^\dagger \partial_x \psi_L\right\} \quad \text{(Fermion)}$$

$$= \pi v_F \int dx \{:(J_L)^2: + :(J_R)^2:\} \quad \text{(Sugawara)}$$

Quadratic !!

$$= \frac{v_F}{4\pi}\int dx\{:(\partial_x \phi_R)^2: + :(\partial_x \phi_L)^2:\} = \frac{v_F}{8\pi}\int dx\{:(\partial_x \phi)^2: + :(\partial_x \theta)^2:\} \text{ (Bosonic)}$$

The Bosonization Dictionary

- **Field Operator Mapping:**

vertex operator

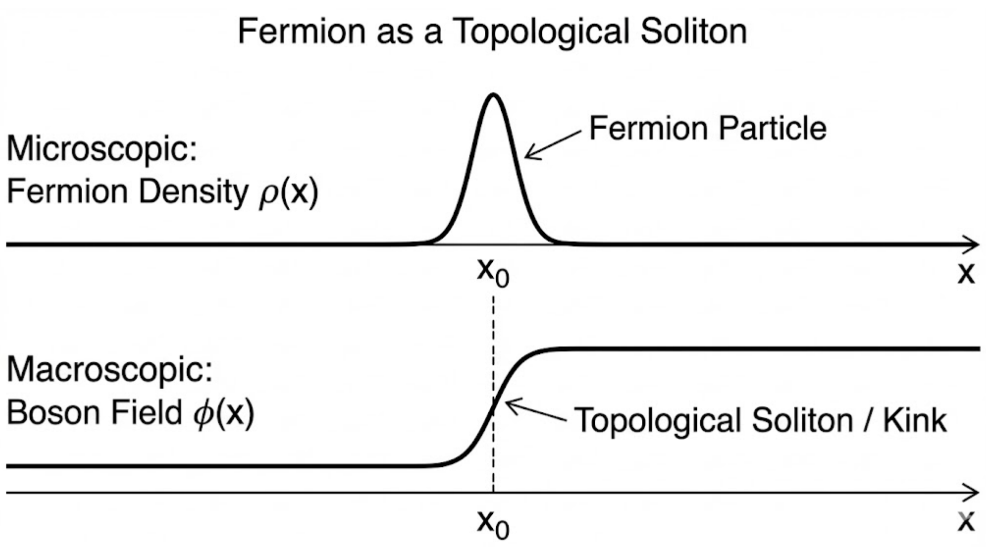
$$\psi_R(x) = \frac{\eta_R}{\sqrt{2\pi\alpha}} e^{-i\phi_R(x)} \quad ; \quad \psi_L(x) = \frac{\eta_L}{\sqrt{2\pi\alpha}} e^{+i\phi_L(x)}$$

- α : short-distance cutoff
- $\eta_{R,L}$: Klein factor

- **Density and Current Mapping:**

$$\rho = \psi^\dagger(x)\psi(x) = \rho_0 + \frac{1}{2\pi}\partial_x \varphi + \frac{1}{\pi\alpha}\cos\{2k_F x + \varphi(x)\}$$

$$J(x) = \psi_R^\dagger(x)\psi_R(x) - \psi_L^\dagger(x)\psi_L(x) = -\frac{1}{2\pi}\partial_x \theta(x)$$



From Topology to Z_2 Symmetry

- **Mismatch of Global boundary condition:**
- Bosonic field is compact: $\phi(\mathrm{x}+\mathrm{L}) = \phi + 2\pi\mathrm{m}$
- However, fermion operator can be periodic and anti-periodic **boundary conditions**

$$\psi(x+L) = \psi(x) \quad \text{or} \quad \psi(x+L) = -\psi(x)$$

- **Resolution: Z_2 symmetry (GSO projection):**

$$1 \equiv (-1)^F$$

- **Consequence: Summing Sectors:**

$$Z_{\text{boson}} = \frac{1}{2}\left(Z_{\text{periodic}} + Z_{\text{anti-periodic}}\right)$$

Ramond **Neveu-Schwarz**

Structure of the Hilbert Space

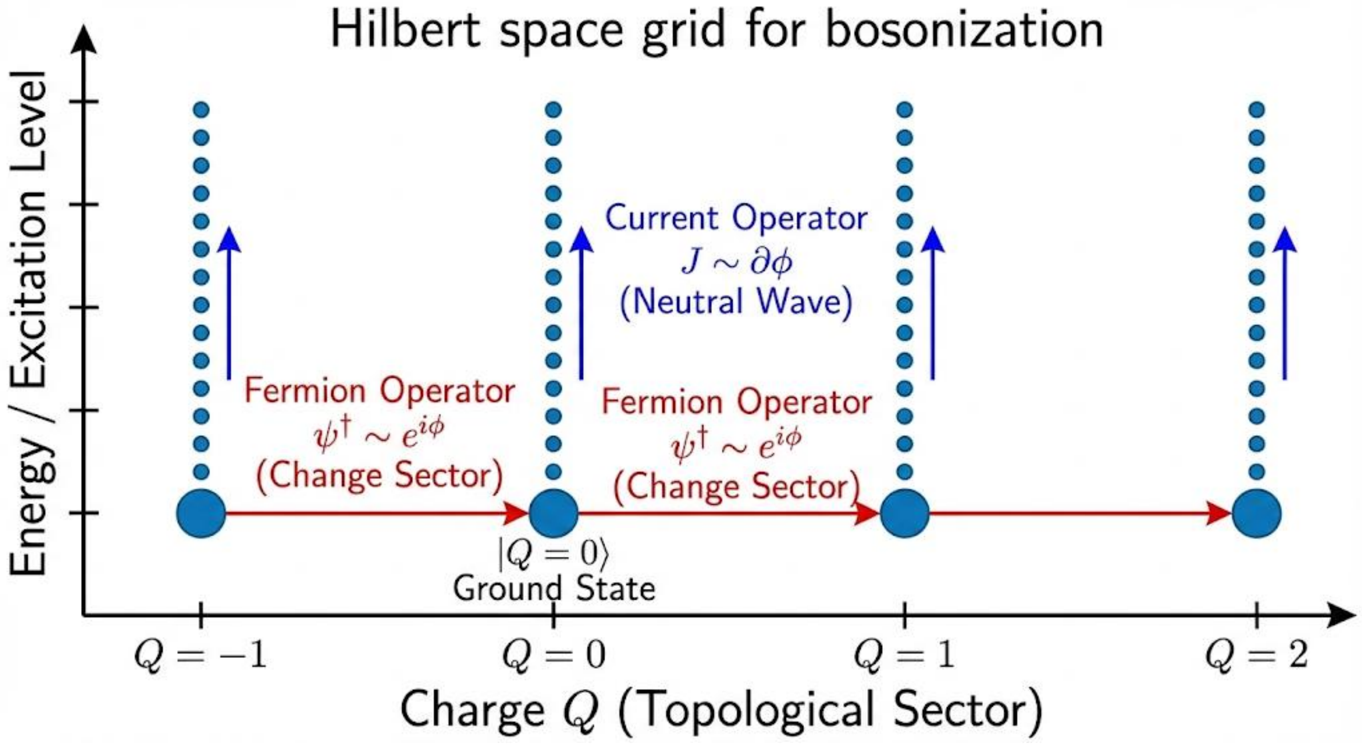
- **Two types of Excitations on bosonic side:**

- Zero mode excitation (topological):

1. Operator: $\psi^\dagger \sim e^{i\phi}$ (vertex op)
2. Change the total particle number N
3. Transitions between Topological Sectors

- Particle-hole excitation:

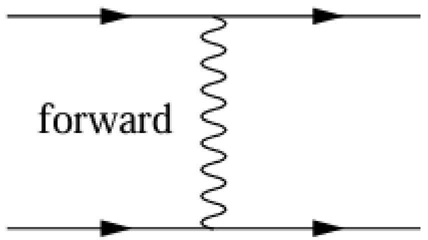
1. Operator: $J \sim \partial\phi$ (density/current op)
2. Change the energy level



Interactions & The Luttinger Liquid

- What about fermion-fermion interaction?
- Consider density-density coupling (**forward scattering**)

$$H_{\text{int}} = g_2 \int dx\, J_R(x) J_L(x) = g_2 \int dx \left(\psi_R^\dagger \psi_R\right)\left(\psi_L^\dagger \psi_L\right)$$



- **Bosonic language**

$$H_{\text{int}} \propto g_2 \int dx\, (\partial_x \phi_R)(\partial_x \phi_L) \sim (\partial_x \phi)^2 - (\partial_x \theta)^2$$

- **Full Hamiltonian**

$$H = \frac{v}{2\pi} \int dx \left\{ \frac{1}{K} (\partial_x \phi)^2 + K (\partial_x \theta)^2 \right\}$$

v: renormalized velocity
K: Luttinger parameter
- $K > 1$: Attractive fermion
- $K = 1$: Free fermion
- $K < 1$: Repulsive fermion

- **Tomonaga-Luttinger liquid**: translation-invariant, U(1)-symmetric fermion/boson systems

Beyond the Liquid: Sine-Gordon Model

- **Backscattering** & **Umklapp** interaction (mix right and left and transfer momentum):

$$\text{Fermion Term: } \psi_R^\dagger \psi_L + h.c.$$

- The Mapping after bosonization

$$H \sim \int dx \left[\frac{v}{2}(\partial_x \phi)^2 + g\cos(\beta\phi)\right]$$ **Sine-Gordon Hamiltonian**

- **RG analysis and Stability** (by conformal dimension Δ)
- **Irrelevant**: Potential scales to zero. Gapless Luttinger liquid remains stable.
- **Relevant**: Potential grows. The system undergoes a Berezinskii-Kosterlitz-Thouless (BKT) transition.

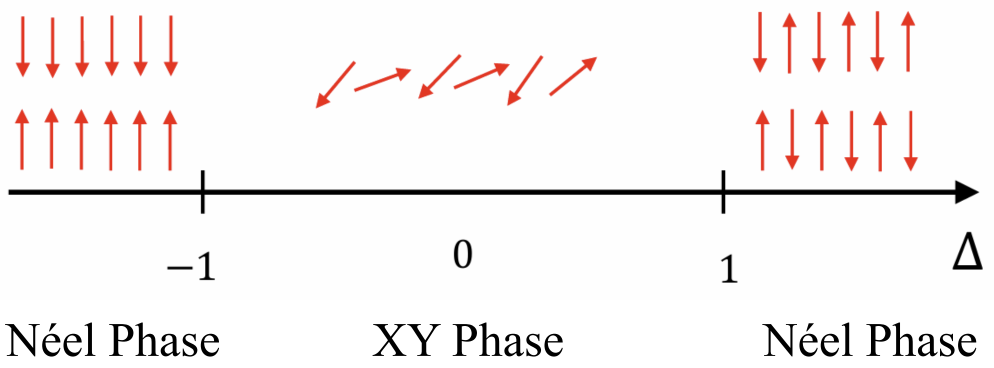
Application: The XXZ Spin Chain

- The Model of **XXZ Spin Chain**:

$$H = J\sum_j \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z\right)$$

- **The Mapping** (Fermion to Boson):
- Jordan-Wigner: Spins → Fermion
- Bosonization: XY term → Kinetic energy $(\partial\phi)^2$

  $S^z S^z$ term → $\cos(\beta\phi)$ potential (Umklapp)



- **Effective Hamiltonian**

$$H_{eff} \approx \int dx \left[\frac{v}{2}(\partial\phi)^2 + g\cos\left(\sqrt{16\pi}\phi\right)\right]$$

Sine-Gordon Model

Summary & Conclusion

- Collapse of Fermi surface to discrete points in 1+1 dimensions allows particle-hole pairs to propagate **coherently**.

- The Duality Dictionary:
- **Current** maps to **Boson Gradients** ($J \sim \partial\varphi$)
- **Fermions** map to **Topological Solitons** (Kinks) in the boson field ($\psi \sim e^{i\phi}$)

- Universal Physics and application:
- Forward scattering leads to the **Tomonaga-Luttinger Liquid**

Reference

- Shankar, R. (2017). *Quantum field theory and condensed matter: An introduction*. Cambridge University Press.
- Tong, D. (n.d.). *Quantum field theory on the line* [Lecture notes]. University of Cambridge.
- Sénéchal, D. (1999). *An introduction to bosonization*. arXiv preprint arXiv:9908262.
- von Delft, J., & Schoeller, H. (1998). *Bosonization for beginners—refermionization for experts*. arXiv preprint arXiv:9805275.
- Furusaki, A. (2024). *Bosonization of 1D interacting fermions and quantum spin chains* [Presentation slides]. RIKEN.

Q & A