继而由引理*4.3.1*, 成立

$$\begin{aligned}&{S_n}'\Big(\tau_e\left({x_0}^{p^{-m}}\right),\cdots,\tau_e\left({x_n}^{p^{-m}}\right),\cdots;\tau_e\left({y_0}^{p^{-m}}\right),\cdots,\tau_e\left({y_n}^{p^{-m}}\right),\cdots\Big)^{p^{m-n}}\\ \equiv\ &\tau_e\Big(S_n\left({x_0}^{p^{-m}},\cdots,{x_n}^{p^{-m}},\cdots;{y_0}^{p^{-m}},\cdots,{y_n}^{p^{-m}},\cdots\right)\Big)^{p^{m-n}}\mod \boldsymbol{p}^{m-n+1}\Re.\end{aligned}$$

同余号两端再同时乘以 $\boldsymbol{p}^n$, $0\le n\le m$, 成立

$$\begin{aligned}&{S_n}'\Big(\tau_e\left({x_0}^{p^{-m}}\right),\cdots,\tau_e\left({x_n}^{p^{-m}}\right),\cdots;\tau_e\left({y_0}^{p^{-m}}\right),\cdots,\tau_e\left({y_n}^{p^{-m}}\right),\cdots\Big)^{p^{m-n}}\boldsymbol{p}^n\\ \equiv\ &\tau_e\Big(S_n\left({x_0}^{p^{-m}},\cdots,{x_n}^{p^{-m}},\cdots;{y_0}^{p^{-m}},\cdots,{y_n}^{p^{-m}},\cdots\right)\Big)^{p^{m-n}}\boldsymbol{p}^n\mod \boldsymbol{p}^{m+1}\Re.\end{aligned}$$

因此有

$$W_m^{(\Re)}\left(S'\begin{pmatrix}\tau_e\left({x_0}^{p^{-m}}\right),\cdots,\tau_e\left({x_n}^{p^{-m}}\right),\cdots;\\ \tau_e\left({y_0}^{p^{-m}}\right),\cdots,\tau_e\left({y_n}^{p^{-m}}\right),\cdots\end{pmatrix}\right)\tag{4.3.2.19a}$$

$$\equiv W_m^{(\Re)}\left(\tau_e\left(S\begin{pmatrix}{x_0}^{p^{-m}},\cdots,{x_n}^{p^{-m}},\cdots;\\ {y_0}^{p^{-m}},\cdots,{y_n}^{p^{-m}},\cdots\end{pmatrix}\right)\right)\mod \boldsymbol{p}^{m+1}\Re.\tag{4.3.2.19b}$$

3. 第三步, 针对式 *(4.3.2.19b)* 的出现, 我们考虑

$$\tau_e\left(S\left({x_0}^{p^{-m}},\cdots,{x_n}^{p^{-m}},\cdots;{y_0}^{p^{-m}},\cdots,{y_n}^{p^{-m}},\cdots\right)\right)$$

与

$$\tau_e\left(S(x_0,\cdots,x_n,\cdots;y_0,\cdots,y_n,\cdots)^{p^{-m}}\right)$$

的关系. 由于 $S\big({x_0}^{p^{-m}},\cdots,{x_n}^{p^{-m}},\cdots;{y_0}^{p^{-m}},\cdots,{y_n}^{p^{-m}},\cdots\big)^{p^m}\in\kappa$, 而 $\mathrm{char}\,(\kappa)=p$, 基于多项式幂的展开和引理*4.2.2.1*, 我们有

$$S\begin{pmatrix}{x_0}^{p^{-m}},\cdots,{x_n}^{p^{-m}},\cdots;\\ {y_0}^{p^{-m}},\cdots,{y_n}^{p^{-m}},\cdots\end{pmatrix}^{p^m}=S\left(x_0,\cdots,x_n,\cdots;y_0,\cdots,y_n,\cdots\right)\in\kappa.\tag{4.3.2.20}$$

又由于 κ 是一个完全环, κ 中任一个元素都存在唯一的一个 p^m-次根, $\forall m\ge 0$, 于是对式 *(4.3.2.20)* 左右两端取 p^m-次根, 可以得到

$$S\left({x_0}^{p^{-m}},\cdots,{x_n}^{p^{-m}},\cdots;{y_0}^{p^{-m}},\cdots,{y_n}^{p^{-m}},\cdots\right)=S(x_0,\cdots,x_n,\cdots;y_0,\cdots,y_n,\cdots)^{p^{-m}}.$$

也就有

$$\tau_e\left(S\begin{pmatrix}{x_0}^{p^{-m}},\cdots,{x_n}^{p^{-m}},\cdots;\\ {y_0}^{p^{-m}},\cdots,{y_n}^{p^{-m}},\cdots\end{pmatrix}\right)=\tau_e\left(S\begin{pmatrix}x_0,\cdots,x_n,\cdots;\\ y_0,\cdots,y_n,\cdots\end{pmatrix}^{p^{-m}}\right).\tag{4.3.2.21}$$

4. 第四步, 综合式 *(4.3.2.16)*, 式 *(4.3.2.19)*, 式 *(4.3.2.21)*, 我们总结得到

$$\begin{aligned}&W_m^{(\Re)}\left(\left\{\tau_e\left({x_n}^{p^{-m}}\right)\right\}\right)+W_m^{(\Re)}\left(\left\{\tau_e\left({y_n}^{p^{-m}}\right)\right\}\right)\\ \xlongequal{\text{式}(4.3.2.16)}\ &W_m^{(\Re)}\left(S'\left(\tau_e\left({x_0}^{p^{-m}}\right),\cdots,\tau_e\left({x_n}^{p^{-m}}\right),\cdots;\tau_e\left({y_0}^{p^{-m}}\right),\cdots,\tau_e\left({y_n}^{p^{-m}}\right),\cdots\right)\right)\\ \xlongequal{\text{式}(4.3.2.19)}\ &W_m^{(\Re)}\left(\tau_e\left(S\left({x_0}^{p^{-m}},\cdots,{x_n}^{p^{-m}},\cdots;{y_0}^{p^{-m}},\cdots,{y_n}^{p^{-m}},\cdots\right)\right)\right)\mod \boldsymbol{p}^{m+1}\Re.\\ \xlongequal{\text{式}(4.3.2.21)}\ &W_m^{(\Re)}\left(\tau_e\left(S(x_0,\cdots,x_n,\cdots;y_0,\cdots,y_n,\cdots)^{p^{-m}}\right)\right)\end{aligned}$$