ON FAITHFUL MODULAR REPRESENTATIONS

Let A be a complete discrete valuation ring with residue field $\mathbb{F}_q$ and uniformiser π. For $d, r \in \mathbb{Z}_{>0}$, we would like to ask:

Question 0.1. Does $G_r := \mathrm{GL}_d(A/\pi^r)$ admit an n-dimension faithful $\mathbb{F}_q$-representation (i.e. a group injection from G_r to $\mathrm{GL}_n(\mathbb{F}_q)$), where n depends on d, r, $\mathrm{char}(A)$, but not on q?

Indeed, if $\mathrm{char}(A) > 0$, then this question has a simple affirmative answer and the associated $\mathbb{F}_q$-representation can be used to link various aspects of the geometric representation theory of G_r. However, as illustrated in the following example, if $\mathrm{char}(A) = 0$, then this question is usually negative beyond the abelian case.

Example 0.2. Let us take $A = W(\mathbb{F}_q)$ and $r = 2$. Via the Teichmüller lift (see e.g. Fesenko–Vostokov) we know that the multiplicative group of $W_2(\mathbb{F}_q)$ is $C_{q-1} \times C_q$, which is the same with that of $\mathbb{F}_q[[\pi]]/\pi^2$, so there is a group embedding

$$\mathrm{GL}_1(W_2(\mathbb{F}_q)) \longrightarrow \mathrm{GL}_2(\mathbb{F}_q)$$

via

$$\mathrm{GL}_1(W_2(\mathbb{F}_q)) \cong C_{p-1} \times C_p \ni (x, y) \longmapsto \begin{bmatrix} x & xy \\ 0 & x \end{bmatrix}.$$

However, in general we do not have such an embedding for $d > 1$. Indeed, let $q = p$ be a prime and consider the group $\mathrm{GL}_2(W_2(\mathbb{F}_p))$: It has the subgroup

$$\begin{bmatrix} 1 & W_2(\mathbb{F}_p) = \mathbb{Z}/p^2 \\ 0 & 1 \end{bmatrix}$$

which admits p-elements of order p^2, however, $\mathrm{GL}_n(\mathbb{F}_p)$ cannot contain such a p-element for any $p > n$:

(Sketch of proof.) Suppose g is a matrix in $\mathrm{GL}_n(\mathbb{F}_p)$ of order p^2. Since we are doing computations over $\mathbb{F}_p$, we have $(x - I)^p = x^p - I$ for any matrix x. So, if $g^{p^2} = I$, then (by letting $x = g^p$) $(g - I)^{p^2} = 0$. By conjugating g into an upper triangular matrix $J(g)$ (for example, take $J(g)$ to be the Jordan normal form) we see this implies that the diagonal of $J(g)$ must be I, so, since $p > n$, we must have $(g - I)^p = 0$, which implies $g^p = I$, contradicting the requirement that g has order p^2.

Remark 0.3. We remark that, there is a complete classification of finite groups admitting *irreducible* faithful $\mathbb{F}_q$-representations, given by Tadasi Nakayama (see e.g. his paper "Note on faithful modular representations of a finite group").