Ringdown: analytic marginalization redux

The general model based $+|m|$ and $-|m|$ modes is:

$$h_n = C_{\ell mn} e^{-i\tilde{\omega}_{\ell mn} t} + C_{\ell -mn} e^{-i\tilde{\omega}^*_{\ell mn} t} \equiv C_{+n} e^{-i\tilde{\omega}_n t} + C_{-n} e^{-i\tilde{\omega}^*_n t}$$

$$= \left[|C_{+n}| e^{-i(\omega_n t - \phi_{+n})} + |C_{-n}| e^{i(\omega_n t + \phi_{-n})} \right] e^{-t/\tau_n}$$

$$= \Big[|C_{+n}| \cos(\omega_n t - \phi_{+n}) + |C_{-n}| \cos(\omega_n t + \phi_{-n}) +$$

$$- |C_{+n}| \sin(\omega_n t - \phi_{+n}) i + |C_{-n}| \sin(\omega_n t + \phi_{-n}) i \Big] e^{-t/\tau_n}$$

$$= \Big\{ |C_{+n}| (\cos\omega_n t \cos\phi_{+n} + \sin\omega_n t \sin\phi_{+n}) +$$

$$|C_{-n}| (\cos\omega_n t \cos\phi_{-n} - \sin\omega_n t \sin\phi_{-n}) +$$

$$-i |C_{+n}| (\sin\omega_n t \cos\phi_{+n} - \cos\omega_n t \sin\phi_{+n}) +$$

$$i |C_{-n}| (\sin\omega_n t \cos\phi_{-n} + \cos\omega_n t \sin\phi_{-n}) \Big\} e^{-t/\tau_n}$$

$$= \Big\{ \cos\omega_n t \left(|C_{+n}| \cos\phi_{+n} + |C_{-n}| \cos\phi_{-n} \right) + \sin\omega_n t \left(|C_{+n}| \sin\phi_{+n} - |C_{-n}| \sin\phi_{-n} \right) +$$

$$-i \Big[\cos\omega_n t \left(-|C_{+n}| \sin\phi_{+n} - |C_{-n}| \sin\phi_{-n} \right) + \sin\omega_n t \left(|C_{+n}| \cos\phi_{+n} - |C_{-n}| \cos\phi_{-n} \right) \Big\} e^{-t/\tau_n}$$

$$\equiv \left[\left(A^+_{c_n} \cos\omega_n t + A^+_{s_n} \sin\omega_n t \right) - i \left(A^\times_{c_n} \cos\omega_n t + A^\times_{s_n} \sin\omega_n t \right) \right] e^{-t/\tau_n}$$

$$h_+ = \sum_n \left(A^+_{cn} \cos\omega_n t + A^+_{sn} \sin\omega_n t \right) e^{-t/\tau_n} = \sum_n \sqrt{A^{+2}_{cn} + A^{+2}_{sn}} \cos\left(\omega_n t - \tan^{-1} \frac{A^+_{sn}}{A^+_{cn}} \right) e^{-t/\tau_n}$$

$$h_\times = \sum_n \left(A^\times_{cn} \cos\omega_n t + A^\times_{sn} \sin\omega_n t \right) e^{-t/\tau_n} = \sum_n \sqrt{A^{\times 2}_{cn} + A^{\times 2}_{sn}} \cos\left(\omega_n t - \tan^{-1} \frac{A^\times_{sn}}{A^+_{cn}} \right) e^{-t/\tau_n}$$

$$h_I = F_+ h_+ + F_x h_x$$

$$= \left[F_+^I \sum_n \left(A^+_{cn} \cos \omega_n t + A^+_{sn} \sin \omega_n t \right) + F_x^I \sum_n \left(A^x_{cn} \cos \omega_n t + A^x_{sn} \sin \omega_n t \right) \right] e^{-t/\tau_n}$$

$$= \sum_n \left[\cos \omega_n t \left(F_+^I A^+_{cn} + F_x^I A^x_{cn} \right) + \sin \omega_n t \left(F_+^I A^+_{sn} + F_x^I A^x_{sn} \right) \right] e^{-t/\tau_n}$$

$$= \sum_n^{(1,1)} \begin{pmatrix} F_+^I A^+_{cn} & F_+^I A^+_{sn} \\ F_x^I A^x_{cn} & F_x^I A^x_{sn} \end{pmatrix} \begin{pmatrix} \cos \omega_n t \\ \sin \omega_n t \end{pmatrix} e^{-t/\tau_n}$$

$$= \sum_n \left(F_+^I \; F_x^I \right) \begin{pmatrix} A^+_{cn} & A^+_{sn} \\ A^x_{cn} & A^x_{sn} \end{pmatrix} \begin{pmatrix} \cos \omega_n t \\ \sin \omega_n t \end{pmatrix} e^{-t/\tau_n}$$

check. $\left(F_+ \; F_x \right) \begin{pmatrix} A^+_{cn} \cos \omega_n t + A^+_{sn} \sin \omega_n t \\ A^x_{cn} \cos \omega_n t + A^x_{sn} \sin \omega_n t \end{pmatrix} = F_+ \left(A^+_{cn} \cos \omega_n t + A^+_{sn} \sin \omega_n t \right) + F_x \left(A^x_{cn} \cos \omega_n t + A^x_{sn} \sin \omega_n t \right)$ ✓

Let $h_{cn} \equiv \cos \omega_n t \, e^{-t/\tau_n}$ and $h_{sn} \equiv \sin \omega_n t \, e^{-t/\tau_n}$, then:

$$h_I = \left(F_+^I \; F_x^I \right) \begin{bmatrix} A^+_{c0} & A^+_{s0} \\ A^x_{c0} & A^+_{s0} \\ \vdots & \\ A^+_{cN} & A^+_{sN} \\ A^x_{cN} & A^x_{sN} \end{bmatrix} \begin{bmatrix} h_{c0} \\ h_{s0} \\ \vdots \\ h_{cN} \\ h_{sN} \end{bmatrix}$$

$$\equiv \vec{F}_I \cdot \overleftrightarrow{M} \cdot \vec{h} = F_{I_j} M_i^{\ j} h^i$$

w/ $\overleftrightarrow{M}$ an $(2N+2, 2)$ matrix, $\vec{h}$ an $(2N+2, 1)$ vector, and $\vec{F}_I$ a $(1,2)$ vector.