```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from functools import partial
from scipy import integrate
from scipy.optimize import brentq, fsolve, minimize_scalar
from scipy.stats import norm, multivariate_normal
```

```
# Parameters (WTI Sep '23 (CLU23) on July 07 2023 10:44am CDT)
futures_price = 72.89
strike_price = 72.5
# risk_free_rate = 0.0535
risk_free_rate = 0.0
time_to_maturity = 41/365
vol_black = 0.33
vol_bachelier = vol_black*futures_price
```

Bachelier (1900)

$$dF_t = \sigma dW_t$$

```
# Bachelier (1900)
class Bachelier:
    def __init__(self, F, vol, r, tau):
      self.F = F
      self.vol = vol
      self.r = r
      self.tau = tau


    def option_pricer(self, vol = None, K = None, option_type = 'call'):
      '''
      Bachelier formula
      return call/put option price
      '''
      # default parameter
      if vol == None:
        vol = self.vol

      m = (self.F - K) / (vol * np.sqrt(self.tau))
      if option_type == 'call':
        return np.exp(-self.r * self.tau) * ((self.F - K)*norm.cdf(m) + vol * self.tau**0.5 * norm.pdf(m))
      elif option_type == 'put':
        return np.exp(-self.r * self.tau) * ((K - self.F)*(1-norm.cdf(m)) + vol * self.tau**0.5 * norm.pdf(m))


    def simulate_F(self, paths = 100000, steps = None):
      '''
      Monte Carlo simulation
      return underlying futures price and time vector
      '''
      # default parameter
      if steps == None:
        steps = 2

      # initialize vectors
      arr_F = np.zeros((paths, steps))
      arr_F[:,0] = self.F

      # generate Brownian motion
      arr_w = np.random.standard_normal(size = (paths, steps-1))

      # compute the corresponding price
      T_vec, dt = np.linspace(0, self.tau, steps, retstep=True)
      for t in range(0, steps-1):
        arr_F[:,t+1] = arr_F[:,t] + self.vol * dt**0.5 * arr_w[:,t]

      return arr_F, T_vec
```

```
bachelier = Bachelier(futures_price, vol_bachelier, risk_free_rate, time_to_maturity)
bachelier_call = bachelier.option_pricer(K = strike_price, option_type = 'call')
bachelier_put = bachelier.option_pricer(K = strike_price, option_type = 'put')

print(f'Bachelier formula: call is {bachelier_call:.4f}')
print(f'Bachelier formula: put is {bachelier_put:.4f}')
```

```
Bachelier formula: call is 3.4149
Bachelier formula: put is 3.0249
```

```
# verify through Monte Carlo simulation
paths = 1000000
arr_F, _ = bachelier.simulate_F(paths = paths)

arr_V = np.exp(-risk_free_rate*time_to_maturity) * np.maximum(arr_F[:,-1] - strike_price, 0)
mean_call, ste_call = arr_V.mean(), arr_V.std() / paths**0.5
arr_V = np.exp(-risk_free_rate*time_to_maturity) * np.maximum(strike_price - arr_F[:,-1], 0)
mean_put, ste_put = arr_V.mean(), arr_V.std() / paths**0.5

print(f'Monte Carlo simulation: call is {mean_call:.4f} with standard error {ste_call:.6f}')
print(f'Monte Carlo simulation: put is {mean_put:.4f} with standard error {ste_put:.6f}')
```

```
Monte Carlo simulation: call is 3.4112 with standard error 0.004843
Monte Carlo simulation: put is 3.0281 with standard error 0.004574
```

Black (1976)

$$dF_t = \sigma F_t dW_t$$

```
# Black (1976)
class Black:
    def __init__(self, F, vol, r, tau):
      self.F = F
      self.vol = vol
      self.r = r
      self.tau = tau


    def option_pricer(self, vol = None, K = None, option_type = 'call'):
      '''
      Black formula
      return call/put option price
      '''
      # default parameter
      if vol == None:
        vol = self.vol

      d1 = (np.log(self.F / K) +  0.5 * vol ** 2 * self.tau) / (vol * np.sqrt(self.tau))
      d2 = (np.log(self.F / K) -  0.5 * vol ** 2 * self.tau) / (vol * np.sqrt(self.tau))
      if option_type == 'call':
        return np.exp(-self.r * self.tau) * ( self.F * norm.cdf(d1) - K * norm.cdf(d2))
      elif option_type == 'put':
        return np.exp(-self.r * self.tau) * ( K * (1 - norm.cdf(d2)) - self.F * (1 - norm.cdf(d1)))


    def simulate_F(self, paths = 100000, steps = None):
      '''
      Monte Carlo simulation
      return underlying futures price and time vector
      '''
      # default parameter
      if steps == None:
        steps = 2

      # initialize vectors
      arr_F = np.zeros((paths, steps))
      arr_F[:,0] = self.F

      # generate Brownian motion
      arr_w = np.random.standard_normal(size = (paths, steps-1))

      # compute the corresponding price
      T_vec, dt = np.linspace(0, self.tau, steps, retstep=True)
      for t in range(0, steps-1):
        arr_F[:,t+1] = arr_F[:,t] * np.exp( -0.5 * self.vol**2 * dt + self.vol * dt**0.5 * arr_w[:,t] )

      return arr_F, T_vec
```

```
black = Black(futures_price, vol_black, risk_free_rate, time_to_maturity)
black_call = black.option_pricer(K = strike_price, option_type = 'call')
black_put = black.option_pricer(K = strike_price, option_type = 'put')

print(f'Black formula: call is {black_call:.4f}')
print(f'Black formula: put is {black_put:.4f}')
```

```
Black formula: call is 3.4047
Black formula: put is 3.0147
```

```
# verify through Monte Carlo simulation
paths = 1000000
arr_F, _ = black.simulate_F(paths = paths)

arr_V = np.exp(-risk_free_rate*time_to_maturity) * np.maximum(arr_F[:,-1] - strike_price, 0)
mean_call, ste_call = arr_V.mean(), arr_V.std() / paths**0.5
arr_V = np.exp(-risk_free_rate*time_to_maturity) * np.maximum(strike_price - arr_F[:,-1], 0)
mean_put, ste_put = arr_V.mean(), arr_V.std() / paths**0.5

print(f'Monte Carlo simulation: call is {mean_call:.4f} with standard error {ste_call:.6f}')
print(f'Monte Carlo simulation: put is {mean_put:.4f} with standard error {ste_put:.6f}')
```

```
Monte Carlo simulation: call is 3.4084 with standard error 0.005156
Monte Carlo simulation: put is 3.0115 with standard error 0.004277
```

Heston (1993)

$$\begin{cases} dF_t = \sqrt{v_t} F_t dW_t^{\,1} \\ dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^{\,2} \\ dW_t^{\,1} dW_t^{\,2} = \rho dt \end{cases}$$

```
# Heston (1993)

def cf(u, tau, mu, rho, v0, theta, kappa, sigma):
```

```
    """
    Heston characteristic function in Schoutens (2004)
    """

    d = ((kappa - rho*sigma*u*1j)**2 + sigma**2*(1j*u + u**2))**0.5
    g = (kappa - rho*sigma*u*1j - d)/(kappa - rho*sigma*u*1j + d)
    cf_ = np.exp( 1j*u*mu*tau +\
                 theta*kappa*sigma**-2*((kappa - rho*sigma*u*1j - d)*tau - 2*np.log((1 - g*np.exp(-d*tau))/( 1 - g))) +\
                 v0*sigma**-2*(kappa - rho*sigma*u*1j - d)*(1 - np.exp(-d*tau))/(1 - g*np.exp(-d*tau)) )
    return cf_

class Heston:
    def __init__(self, F, rho, v0, theta, kappa, sigma,  r, tau):
      self.F = F
      self.rho = rho
      self.v0 = v0 # spot variance
      self.theta = theta
      self.kappa = kappa
      self.sigma = sigma
      self.r = r
      self.tau = tau
      if 2*kappa * theta < sigma**2:
        print(f'The parameters entered fail the Feller condition')


    def option_pricer(self, K = None, option_type = 'call'):
      '''
      Fourier inversion
      return call/put option price
      '''

      k = np.log(K/self.F)
      cf_ = partial(cf, tau = self.tau, mu = 0, rho = self.rho, v0 = self.v0, theta = self.theta, kappa = self.kappa, sigma = self.sigma )

      integrand_Q1 = lambda u: np.real((np.exp(-u*k*1j) * cf_(u-1j) / (u*1j)) / cf_(-1j))
      integrand_Q2 = lambda u: np.real(np.exp(-u*k*1j) * cf_(u) / (u*1j))

      Q1 = 1/2 + 1/np.pi * integrate.quad(integrand_Q1, 1e-15, np.inf, limit=2000)[0]
      Q2 = 1/2 + 1/np.pi * integrate.quad(integrand_Q2, 1e-15, np.inf, limit=2000)[0]

      if option_type == 'call':
        return np.exp(-self.r * self.tau) * ( self.F * Q1 - K * Q2)
      elif option_type == 'put':
        return np.exp(-self.r * self.tau) * ( K * (1-Q2) - self.F * (1-Q1) )


    def simulate_F(self, paths = 10000, steps = None):
      '''
      Monte Carlo simulation
      return underlying futures price and time vector
      '''

      # default parameter
      if steps == None:
        steps = int(self.tau*252*4)

      # Initialize vectors
      arr_F = np.zeros((paths,steps)); arr_F[:,0] = self.F
      arr_v = np.zeros((paths,steps)); arr_v[:,0] = self.v0

      # Generate random 2D Brownian motions
      arr_w = multivariate_normal.rvs( mean = np.array([0, 0]),
                                       cov = np.array([[1, self.rho], [self.rho, 1]]),
                                       size = (paths, steps-1) )

      # Compute the corresponding paths
      T_vec, dt = np.linspace(0, self.tau, steps, retstep=True)
      for t in range(0,steps-1):
        arr_F[:,t+1] = arr_F[:,t] + arr_F[:,t] * ( arr_v[:,t] * dt )**0.5 * arr_w[:,t,0]
        arr_v[:,t+1] = arr_v[:,t] + self.kappa * ( self.theta - arr_v[:,t] ) * dt + self.sigma * ( arr_v[:,t] * dt )**0.5 * arr_w[:,t,1]
        arr_v[arr_v[:,t+1] < 0.0, t+1] = 1e-5 # prevent negative variance

      return arr_F, T_vec
```

```
# Heston parameters
rho = -0.30                                          # correlation coefficient
v0 = 0.30**2                                         # spot variance
theta = 0.40**2                                      # long-term variance
kappa = 5.0                                          # mean reversion coefficient
sigma = 0.7                                          # volatility of instantaneous variance (Vol of Vol)
```

```
heston = Heston(futures_price, rho, v0, theta, kappa, sigma, risk_free_rate, time_to_maturity)
heston_call = heston.option_pricer(K = strike_price, option_type = 'call')
heston_put = heston.option_pricer(K = strike_price, option_type = 'put')

print(f'Heston (foruier inversion): call is {heston_call:.4f}')
print(f'Heston (foruier inversion): put is {heston_put:.4f}')
```

```
Heston (foruier inversion): call is 3.3257
Heston (foruier inversion): put is 2.9357
```

```
# verify through Monte Carlo simulation
paths = 10000
arr_F, _ = heston.simulate_F(paths = paths)

arr_V = np.exp(-risk_free_rate*time_to_maturity) * np.maximum(arr_F[:,-1] - strike_price, 0)
mean_call, ste_call = arr_V.mean(), arr_V.std() / paths**0.5
```

```
arr_V = np.exp(-risk_free_rate*time_to_maturity) * np.maximum(strike_price - arr_F[:,-1], 0)
mean_put, ste_put = arr_V.mean(), arr_V.std() / paths**0.5

print(f'Monte Carlo simulation: call is {mean_call:.4f} with standard error {ste_call:.6f}')
print(f'Monte Carlo simulation: put is {mean_put:.4f} with standard error {ste_put:.6f}')
```

```
Monte Carlo simulation: call is 3.3095 with standard error 0.048315
Monte Carlo simulation: put is 2.9659 with standard error 0.044921
```

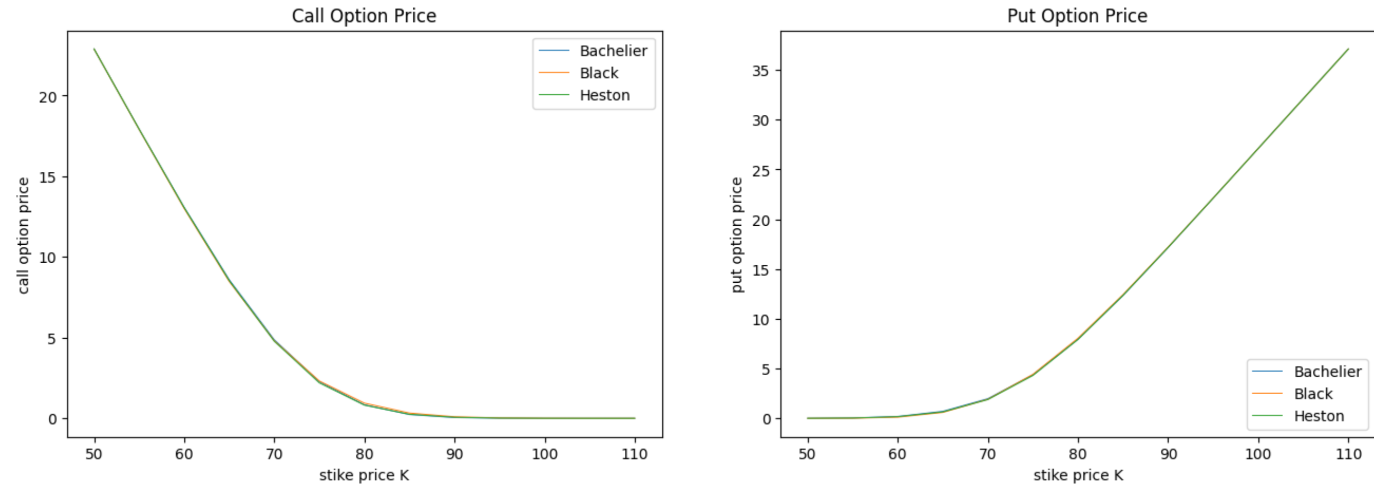
Quadratic Normal Model (2023)

$$\begin{cases} dF_t = \sigma(F_t)dW_t \\ \sigma(F_t) = cF^2 + bF + a = c(F - \frac{-b}{2c})^2 + a - \frac{b^2}{4c} \end{cases}$$

```
# Quadratic Normal Model (2023)
class QNM:
    def __init__(self, F, a, b, c, tau):
      self.F = F
      self.a = a
      self.b = b
      self.c = c
      self.tau = tau


    def option_pricer(self, K = None, option_type = 'call'):
      '''
      The method of linearization
      return call/put option price
      '''
      m = (self.F - K)/(self.a * self.tau**0.5)
      C_BC = (self.F - K)*norm.cdf(m) + self.a * np.sqrt(self.tau) * norm.pdf(m)
      U = self.tau**0.5 * norm.pdf((self.F-K)/self.tau**0.5) * (self.b*(self.F + K)/2 + self.c*(self.F**2 + self.F*K + K**2 + 0.5*self.a**2 * self.tau)/3)

      if option_type == 'call':
        return C_BC + U
      elif option_type == 'put':
        return C_BC + U + K - self.F


    def simulate_F(self, paths = 100000, steps = None):
      '''
      Monte Carlo simulation
      return underlying futures price and time vector
      '''
      # default parameter
      if steps == None:
        steps = 2

      # initialize vectors
      arr_F = np.zeros((paths, steps))
      arr_F[:,0] = self.F

      # generate Brownian motion
      arr_w = np.random.standard_normal(size = (paths, steps-1))

      # compute the corresponding price
      T_vec, dt = np.linspace(0, self.tau, steps, retstep=True)
      for t in range(0, steps-1):
        arr_F[:,t+1] = arr_F[:,t] + self.vol * dt**0.5 * arr_w[:,t]

      return arr_F, T_vec
```

```
arr_strike = np.arange(50,111,5)

# call option
arr_bachelier_call = bachelier.option_pricer(K = arr_strike, option_type = 'call')
arr_black_call = black.option_pricer(K = arr_strike, option_type = 'call')
list_heston_call = []
for strike_ in arr_strike:
  list_heston_call.append(heston.option_pricer(K = strike_, option_type = 'call'))

# put option
arr_bachelier_put = bachelier.option_pricer(K = arr_strike, option_type = 'put')
arr_black_put = black.option_pricer(K = arr_strike, option_type = 'put')
list_heston_put = []
for strike_ in arr_strike:
  list_heston_put.append(heston.option_pricer(K = strike_, option_type = 'put'))

# create subplots
fig = plt.figure(figsize=(16,5))

ax1 = fig.add_subplot(121)
ax1.plot(arr_strike, arr_bachelier_call, linewidth = 0.7, label = 'Bachelier')
ax1.plot(arr_strike, arr_black_call, linewidth = 0.7, label = 'Black')
ax1.plot(arr_strike, list_heston_call, linewidth = 0.7, label = 'Heston')
ax1.set_title('Call Option Price')
ax1.set_xlabel('stike price K')
ax1.set_ylabel('call option price')
ax1.legend(loc='upper right')

ax2 = fig.add_subplot(122)
ax2.plot(arr_strike, arr_bachelier_put, linewidth = 0.7, label = 'Bachelier')
```

```
ax2.plot(arr_strike, arr_black_put, linewidth = 0.7, label = 'Black')
ax2.plot(arr_strike, list_heston_put, linewidth = 0.7, label = 'Heston')
ax2.set_title('Put Option Price')
ax2.set_xlabel('stike price K')
ax2.set_ylabel('put option price')
ax2.legend(loc='lower right')

plt.show()
```

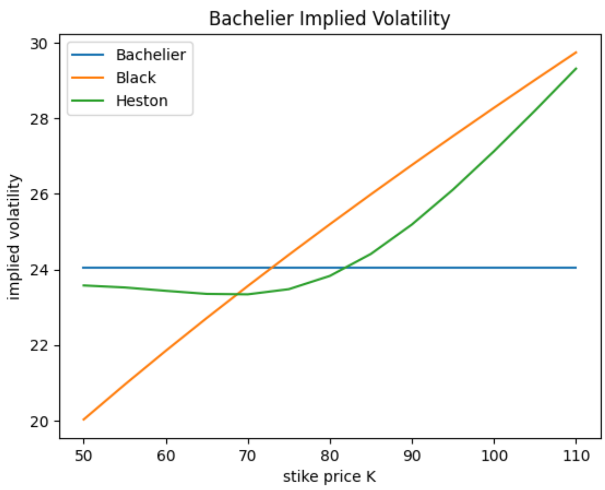


Implied Volatility

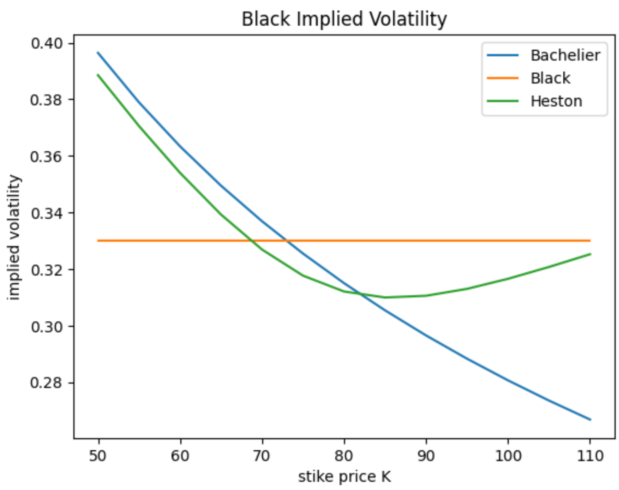
```
def implied_volatility(option_price, F, K, r, tau, option_type = 'call', model = 'black', method='fsolve', disp=True):
    """
       Return Implied volatility
       model: black (default), bachelier
       methods:  fsolve (default), brent, minimization
    """

    if model == 'bachelier':
      bachelier_ = Bachelier(F, F*0.1, r, tau)
      obj_fun = lambda vol : option_price - bachelier_.option_pricer(vol = vol, K = K, option_type = option_type)
      #obj_fun = lambda vol : option_price - bachelier(F, K, r, tau, vol, option_type)
    else: # model == 'black'
      black_ = Black(F, 0.1, r, tau)
      obj_fun = lambda vol : option_price - black_.option_pricer(vol = vol, K = K, option_type = option_type)
      #obj_fun = lambda vol : option_price - black(F, K, r, tau, vol, option_type)


    if method == 'minimization':
      obj_square = lambda vol : obj_fun(vol)**2
      res = minimize_scalar( obj_square, bounds=(1e-15, 8), method='bounded')
      if res.success == True:
        return res.x

    elif method == 'brent':
        x, r = brentq( obj_fun, a = 1e-15, b = 500, full_output = True)
        if r.converged == True:
            return x
    else:
        X0 = [0.1, 0.5, 1, 3]   # set of initial guess points
        for x0 in X0:
            x, _, solved, _ = fsolve(obj_fun, x0, full_output=True, xtol=1e-8)
            if solved == 1:
                return x[0]

    # display failed strikes
    if disp == True:
        print(method, K)
    return -1
```

```
# Bachelier implied volatility
list_iv_bachelier, list_iv_black, list_iv_heston = [], [], []
for bachelier_, black_, heston_, stirke_ in zip(arr_bachelier_call, arr_black_call, list_heston_call, arr_strike):
  list_iv_bachelier.append(implied_volatility(bachelier_, futures_price, stirke_, risk_free_rate, time_to_maturity,
                                                        option_type = 'call', model = 'bachelier', method='brent', disp=True))
  list_iv_black.append(implied_volatility(black_, futures_price, stirke_, risk_free_rate, time_to_maturity,
                                                        option_type = 'call', model = 'bachelier', method='brent', disp=True))
  list_iv_heston.append(implied_volatility(heston_, futures_price, stirke_, risk_free_rate, time_to_maturity,
                                                        option_type = 'call', model = 'bachelier', method='brent', disp=True))


plt.plot(arr_strike, list_iv_bachelier, label = 'Bachelier')
plt.plot(arr_strike, list_iv_black, label = 'Black')
plt.plot(arr_strike, list_iv_heston, label = 'Heston')
plt.title('Bachelier Implied Volatility')
plt.xlabel('stike price K')
plt.ylabel('implied volatility')
```

```
plt.legend()
plt.show()
```




```
# Black implied volatility
list_iv_bachelier, list_iv_black, list_iv_heston = [], [], []
for bachelier_, black_, heston_, stirke_ in zip(arr_bachelier_call, arr_black_call, list_heston_call, arr_strike):
  list_iv_bachelier.append(implied_volatility(bachelier_, futures_price, stirke_, risk_free_rate, time_to_maturity,
                                                        option_type = 'call', model = 'black', method='brent', disp=True))
  list_iv_black.append(implied_volatility(black_, futures_price, stirke_, risk_free_rate, time_to_maturity,
                                                        option_type = 'call', model = 'black', method='brent', disp=True))
  list_iv_heston.append(implied_volatility(heston_, futures_price, stirke_, risk_free_rate, time_to_maturity,
                                                        option_type = 'call', model = 'black', method='brent', disp=True))


plt.plot(arr_strike, list_iv_bachelier, label = 'Bachelier')
plt.plot(arr_strike, list_iv_black, label = 'Black')
plt.plot(arr_strike, list_iv_heston, label = 'Heston')
plt.title('Black Implied Volatility')
plt.xlabel('stike price K')
plt.ylabel('implied volatility')
plt.legend()
plt.show()
```




```
# Mount Google Drive
from google.colab import drive
drive.mount('/content/gdrive')

# Read CSV file from Google Drive
filename = '/content/gdrive/MyDrive/job preparation/data/' +\
              'clu23-options-american-options-exp-08_17_23-20-strikes-+_--side-by-side-%futuresoptionstime%-07-07-2023.csv'
```

```
Drive already mounted at /content/gdrive; to attempt to forcibly remount, call drive.mount("/content/gdrive", force_remount=True).
```

```
df_raw_data = pd.read_csv(filename, header=0, index_col = 5)
df_raw_data = df_raw_data.iloc[:-1]
df_raw_data.head()
```

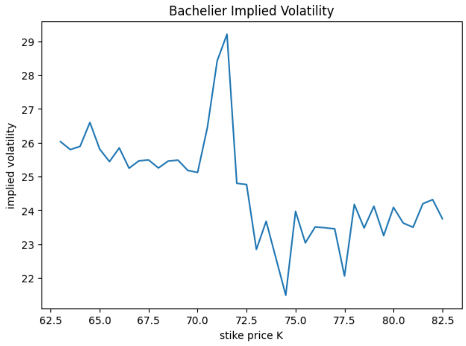
	Type	Last	Volume	Open Int	Premium	Type.1	Last.1	Volume.1	Open Int.1	Premium.1
Strike										

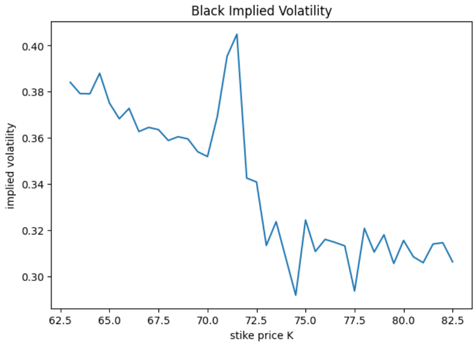
```
list_data = []
for index_ in df_raw_data.index:
  if index_ < futures_price:
    if not 's' in str(df_raw_data.loc[index_, 'Last.1']):
      list_data.append((index_, float(df_raw_data.loc[index_, 'Last.1']), 'put'))
  else:
    if not 's' in str(df_raw_data.loc[index_, 'Last']):
      list_data.append((index_, float(df_raw_data.loc[index_, 'Last']), 'call'))

arr_data = np.array(list_data)
df_data = pd.DataFrame({'price': arr_data.T[1].astype(float), 'option type': arr_data.T[2]}, index = arr_data.T[0].astype(float))
df_data
```

	price	option type
63.0	0.56	put
63.5	0.61	put
64.0	0.69	put
64.5	0.83	put
65.0	0.85	put
65.5	0.91	put
66.0	1.05	put
66.5	1.10	put
67.0	1.24	put
67.5	1.37	put
68.0	1.48	put
68.5	1.65	put
69.0	1.81	put
69.5	1.94	put
70.0	2.11	put
70.5	2.47	put
71.0	2.93	put
71.5	3.25	put
72.0	2.89	put
72.5	3.12	put
73.0	3.00	call
73.5	2.87	call
74.5	2.14	call
75.0	2.26	call
75.5	1.95	call
76.0	1.83	call
76.5	1.66	call
77.0	1.50	call
77.5	1.20	call
78.0	1.30	call
78.5	1.10	call
79.0	1.05	call
79.5	0.86	call
80.0	0.84	call
80.5	0.71	call
81.0	0.62	call
81.5	0.60	call
82.0	0.54	call
82.5	0.44	call

```
# Bachelier and Black implied volatility
list_mkt_iv_bachelier, list_mkt_iv_black = [], []
for strike_, price_, option_type_ in zip(df_data.index, df_data.loc[:,'price'], df_data.loc[:,'option type']):
  list_mkt_iv_bachelier.append(implied_volatility(price_, futures_price, strike_, risk_free_rate, time_to_maturity,
                                                  option_type = option_type_, model = 'bachelier', method='brent', disp=False))
  list_mkt_iv_black.append(implied_volatility(price_, futures_price, strike_, risk_free_rate, time_to_maturity,
                                              option_type = option_type_, model = 'black', method='brent', disp=False))

fig = plt.figure(figsize=(16,5))

ax1 = fig.add_subplot(121)
ax1.plot(df_data.index, list_mkt_iv_bachelier)
ax1.set_title('Bachelier Implied Volatility')
ax1.set_xlabel('stike price K')
ax1.set_ylabel('implied volatility')

ax2 = fig.add_subplot(122)
```

```
ax2.plot(df_data.index, list_mkt_iv_black)
ax2.set_title('Black Implied Volatility')
ax2.set_xlabel('stike price K')
ax2.set_ylabel('implied volatility')

plt.show()
```





✓ 2 秒　完成時間：下午4:10