```
import numpy as np
import pandas as pd
from scipy.linalg import logm, expm
from scipy.stats import norm
from scipy.optimize import fsolve
from matplotlib import pyplot as plt
```

Markov Chain

Let's consider a Markov Chain comprising three states: G, B, and D (representing "good," "bad," and "default" respectively). The following is the risk-neutral probability transition matrix M for a time period of 0.25 years.

$$M(T = 0.25) = \begin{pmatrix} 99\% & 0.75\% & 0.25\% \\ 1.75\% & 94.75\% & 3.5\% \\ 0 & 0 & 1 \end{pmatrix}$$

The exponential form e^{AT} provides a convenient way to express the transition matrix M, particularly when dealing with non-multiple time periods such as $M(T = 0.3)$.

$$e^{AT} = M(T)$$

```
# Parameters
T, M = 0.25, np.array([[0.99, 0.0075, 0.0025],
                       [0.0175, 0.9475, 0.035],
                       [0, 0, 1]])
A = logm(M)/T
print(f'Verify that the sum of each row equals one: {M.sum(axis=1)}')
```

```
Verify that the sum of each row equals one: [1. 1. 1.]
```

```
# Verify A
print(M @ M)
print(expm(A*0.5))
```

```
[[0.98023125 0.01453125 0.0052375 ]
 [0.03390625 0.8978875  0.06820625]
 [0.         0.         1.        ]]
[[0.98023125 0.01453125 0.0052375 ]
 [0.03390625 0.8978875  0.06820625]
 [0.         0.         1.        ]]
```

Let's consider a 6-year risky coupon bond that offers an annual coupon rate of 5% until it reaches maturity. The credit quality of the underlying firm can be characterized using the transition matrix M. In the event of a default by the underlying firm, the bond will become void, resulting in no recovery at the time of default.The risk-free (continuously compounded) interest rate is constant at 2%.

```
r = 0.02
Ts = [1, 2, 3, 4, 5, 6]
CFs = [5, 5, 5, 5, 5, 105]

price = np.array([0.0]*len(A))
for t, cf in zip(Ts, CFs):
  price += cf * np.exp(-r*t) * ( 1 - expm(A*t)[:, -1] )

print(f'when intial state is G, the bond price is {price[0]}')
print(f'when intial state is B, the bond price is {price[1]}')
print(f'when intial state is D, the bond price is {price[2]}')
```

```
when intial state is G, the bond price is 106.2244054724025
when intial state is B, the bond price is 63.2864736049127
when intial state is D, the bond price is 0.0
```

Encapsulate the bond pricing model within a class structure

```
class Markov_Chain:

    def __init__(self, M, T):
      self.A = logm(M) / T

    def transition_matrix(self, T):
      return expm(self.A * T)

    def bond_pricing(self, r, Ts, CFs):
      price = np.array([0.0]*len(self.A))
      for t, cf in zip(Ts, CFs):
        price += cf * np.exp(-r*t) * ( 1 - expm(A*t)[:, -1] )
      return price
```

```
# Verify the answer
markov_chain0 = Markov_Chain(M, T)
print(f'transition matrix M(T = 0.5) is: \n{markov_chain0.transition_matrix(0.5)}\n')
print(f'bond prices are: \n{markov_chain0.bond_pricing(r, Ts, CFs)}')
```

```
transition matrix M(T = 0.5) is:
[[0.98023125 0.01453125 0.0052375 ]
 [0.03390625 0.8978875  0.06820625]
 [0.         0.         1.        ]]

bond prices are:
[106.22440547  63.2864736    0.        ]
```

Let us consider $X(0,T)$ be the survival probability as a function of the horizon T. Define the term hazard rate $\lambda(T)$ as $X(0,T) = e^{-\lambda(T)T}$. Now, let's graph the term hazard rates for the initial states G and B.

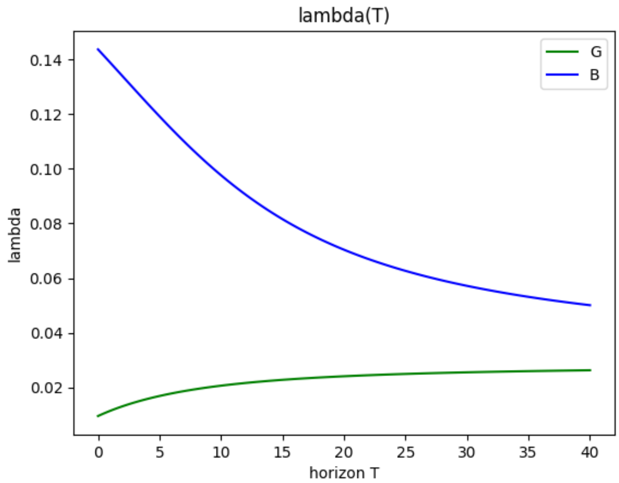
When the initial state is B, the term hazard rate $\lambda(T)$ exhibits a negative slope. Conversely, when the initial state is G, the term hazard rate $\lambda(T)$ displays a positive slope. This observation can be attributed to the reason that the impact of the initial state diminishes as the horizon T increases.

```
horizons = np.linspace(1e-5, 40, num=100)
X_G = []
X_B = []

for T in horizons:
    X_G.append(1-expm(A*T)[0, -1])
    X_B.append(1-expm(A*T)[1, -1])

lambda_G = -np.log(np.array(X_G))/horizons
lambda_B = -np.log(np.array(X_B))/horizons

plt.plot(horizons,lambda_G, color ='green', label='G')
plt.plot(horizons,lambda_B, color ='blue', label ='B')
plt.title('lambda(T)')
plt.xlabel('horizon T')
plt.ylabel('lambda')
plt.legend()
plt.show()
```



Structural Model

$$C(A, D_F, r, T, \sigma) = E = SN(d_1) - D_F e^{-rT} N(d_2)$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{A}{D_F}\right) + \left(r + \frac{\sigma^2}{2}\right)T\right]$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Let's consider a standard Merton model applied to a firm with an initial equity value of $E(t=0) = \$10$ and a face value of 5-year debt of $D_F = \$10$. The risk-free interest rate, compounded continuously, is denoted as $r = 2\%$. Additionally, we assume that the estimated volatility of the firm's assets is 30%.

```
# Parameter
E0 = 10
Df = 10
r = 0.02
T = 5
sigma = 0.3
```

```
def BS(A):
    d1 = (np.log(A/Df) +(r + 1/2 * sigma**2)*T) / (sigma * np.sqrt(T))
    d2 = (np.log(A/Df) +(r - 1/2 * sigma**2)*T) / (sigma * np.sqrt(T))
    return A * norm.cdf(d1) - Df * (np.exp(-r*T)) * norm.cdf(d2) - E0

solve = fsolve(BS, E0)
asset_initial = solve[0]
debt_initial = asset_initial - E0

d2 = (np.log(asset_initial/Df) +(r - 1/2 * sigma**2)*T) / (sigma * np.sqrt(T))
survival_rate = norm.cdf(d2)


print(f'The time-0 fair price of the firm’s assets is {asset_initial}')
print(f'The time-0 fair price of the firm’s debt is {debt_initial}')
print(f'The 5-year survival (risk-neutral )probability of the firm is {survival_rate}')
```

```
The time-0 fair price of the firm’s assets is 18.42882658967022
The time-0 fair price of the firm’s debt is 8.42882658967022
The 5-year survival (risk-neutral )probability of the firm is 0.7657676850600381
```

Now let's consider an extended Merton model with a constant continuous-time barrier $H_t = H = D_F$. In this model, we assume that the assets have drift $\mu = r$, and all other parameters remain the same as before. Let $\tau := \inf\{t > 0 \mid A(t) < H\}$ be the default time of the firm. It can be shown that (using reflection principal):

$$\Pr(\tau < t) = N(\frac{h - at}{\sigma\sqrt{t}}) + e^{\frac{2ah}{\sigma^2}} N(\frac{h + at}{\sigma\sqrt{t}})$$
$$\text{where } a = r - \frac{\sigma^2}{2} \text{ and } h = \log(\frac{H}{A(0)})$$

In the extended Merton model, any realization that reaches the boundary H and surpasses D_F is considered a default, while in the standard Merton model, it is not categorized as a default. These default realizations have a collective probability greater than zero. However, other realizations are consistently categorized in both models. Consequently, the 5-year survival rate in the extended Merton model is lower compared to the survival rate in the standard Merton model.

```
a = r - 0.5 * sigma**2
h = np.log(Df / asset_initial)

default_rate = norm.cdf((h - a*T)/(sigma*np.sqrt(T))) +\
               np.exp(2*h*a / sigma**2) * norm.cdf((h + a*T)/(sigma*np.sqrt(T)))
survival_rate = 1 - default_rate

print(f'The 5-year survival probability of the firm is {survival_rate}')
```

```
The 5-year survival probability of the firm is 0.5745171658288526
```

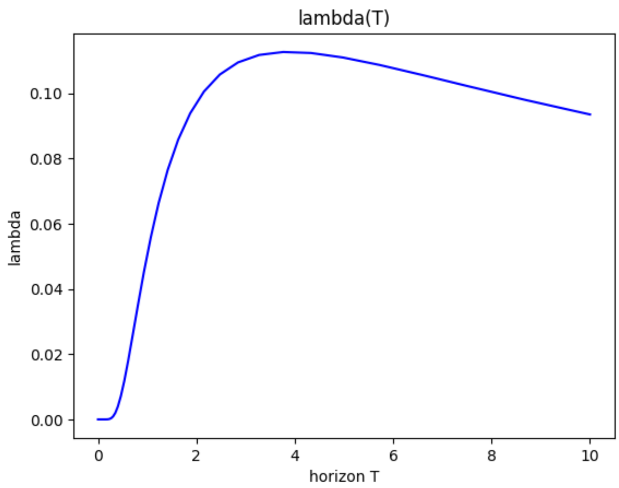
Contrary to real markets, the credit spread term structure $\lambda(T)$ generated by this model exhibits collapsing spreads for short maturities.

```
horizons = np.geomspace(1e-5, 10, num=100)
survival_rate = []
a = r - 0.5 * sigma**2
h = np.log(Df / asset_initial)

for T in horizons:
    X_T = 1 - (norm.cdf((h-a*T)/(sigma * np.sqrt(T))) +\
               np.exp(2*h*a / (sigma**2)) * norm.cdf((h+a*T)/(sigma*np.sqrt(T))))
    survival_rate.append(X_T)

lambda_A = (-np.log(survival_rate)/horizons)

plt.plot(horizons,lambda_A, color ='blue')
plt.title('lambda(T)')
plt.xlabel('horizon T')
plt.ylabel('lambda')
plt.show()
```



✓ 0 秒 完成時間：晚上11:35