ELEC 3035: Quiz on controllability **(15 min)**

Name (optional):

Consider the system $\mathscr{B} = \Big\{ (u,y) \in (\mathbb{R}^2)^{\mathbb{N}} \mid \text{there is } x \in (\mathbb{R}^2)^{\mathbb{N}} \text{ such that } \sigma x = \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b} u,\ y = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{c} x \Big\}.$

1. *Transfer function, order, stability* Find transfer function and difference equation representations of $\mathscr{B}$. Suggest a name for $\mathscr{B}$.

   Transfer function:

$$H(z) = c(Iz - A)^{-1}b + d = \begin{bmatrix} 1 & 1 \end{bmatrix} \left(\begin{bmatrix} z & \\ & z \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1/(z-2) & \\ & 1/z \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1/z$$

   Difference equation: $y(t) = u(t-1)$, i.e., the system delays the input with one time step. Name for $\mathscr{B}$: unit delay.

   What is the order of $\mathscr{B}$? Is $\mathscr{B}$ stable?

   $\mathscr{B}$ is a second order unstable system. E.g., the output $y(t) = 2^t$ generated by $x(0) = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $u = 0$ is unbounded.

2. *Simulation* Find the impulse response of $\mathscr{B}$ (i.e., the response to the unit pulse under zero initial conditions).

   The impulse response h of $\mathscr{B}$ is the one step delayed unit step, i.e., $h(t) = \begin{cases} 1 & \text{if } t = 1 \\ 0 & \text{otherwise} \end{cases}$

3. *Controllability* Is the system $\mathscr{B}_x = \{ (u,x) \in (\mathbb{R}^3)^{\mathbb{N}} \mid \sigma x = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 0 \end{smallmatrix}\right] x + \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] u \}$ controllable? Is it possible to transfer the state from $x(0) = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ to $x(2) = \left[\begin{smallmatrix} 0 \\ 5 \end{smallmatrix}\right]$?

   The controllability matrix is $\mathscr{C} = \begin{bmatrix} b & Ab \end{bmatrix} = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$, which is not full row rank, so that $\mathscr{B}_x$ is uncontrollable. The initial state $x(0) = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ can be transferred to $x(2) = \left[\begin{smallmatrix} 0 \\ 5 \end{smallmatrix}\right]$ because

$$\left(x(2) - \begin{bmatrix} c \\ cA \end{bmatrix} x(0) \right) = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \in \operatorname{image}(\mathscr{C}) = \left\{ \begin{bmatrix} 0 \\ \alpha \end{bmatrix} | \alpha \in \mathbb{R} \right\}.$$

   If so, give a control input that achieves the transfer.

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix} \quad \Longrightarrow \quad \text{any } u(0) \in \mathbb{R} \text{ and } u(1) = 5 \text{ achieves the transfer.}$$

4. *Singular system of equations* Solve the system $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} u = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} u = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \Longleftrightarrow \quad \begin{bmatrix} 1 & 3 \end{bmatrix} u = 2 \quad \Longleftrightarrow \quad u_1 + 3u_2 = 2 \quad \Longleftrightarrow \quad u_1 = 2 - 3u_2$$

   Therefore, the general solution is $u = \operatorname{col}(2 - 3\alpha, \alpha)$ for any $\alpha \in \mathbb{R}$.

   Note that a solution can not be obtained by elimination because the matrix $\left[\begin{smallmatrix} 1 & 3 \\ 2 & 6 \end{smallmatrix}\right]$ is singular. Equivalently, the inverse matrix does not exist. The reason for this is that a solution is not unique.