ELEC 3035, Lecture 6: Pole placement by state feedback

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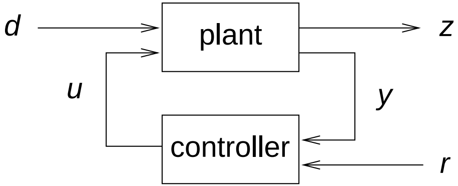
- Motivation for control by pole placement
- State feedback control
- Controller canonical form
- Solution of the state feedback pole placement problem

Feedback control



- r — reference signal
- u — control signal
- d — disturbance
- z — performance criterion
- y — measurement signal

Dynamical behaviour $\leftrightarrow$ pole location

LTI system dynamics is qualitatively determined by the pole locations

e.g., pole z_i with $|z_i| < 1$ (DT) or $z_i < 0$ (CT) $\quad\Longrightarrow\quad$ stable mode

Stability is a minimum requirement for control.

The poles of the closed-loop system should be in the

unit circle $\{z \mid \|z\| < 1\}$	(DT)	moreover	$\|z_i\|$ and $\angle z_i$	(DT)
half-plane $\{z \mid z < 0\}$	(CT)		$\Re(z_i)$ and $\Im(z_i)$	(CT)

determine how “fast” and “oscillatory” the mode is.

Motivation for pole placement

- The desired dynamics is specified by the pole locations

$$\{z_{\text{des},1}, \ldots, z_{\text{des},n_{\text{cl}}}\}$$

  of the closed-loop system

- or equivalently by the characteristic polynomial

$$p_{\text{des}}(z) = \prod_{i=1}^{n_{\text{cl}}}(z - z_{\text{des},i}) = p_{\text{des},0} + p_{\text{des},1}z + \cdots + p_{\text{des},n_{\text{cl}}}z^{n_{\text{cl}}}$$

  of the closed-loop system.

- Feedback affects the characteristic polynomial
- The aim of pole placement control is to choose the feedback so that the closed-loop system achieve the desired char. polynomial

Deadbeat control

In the DT case, if

$$z_{\mathrm{des},1} = \cdots = z_{\mathrm{des},n_{\mathrm{cl}}} = 0$$

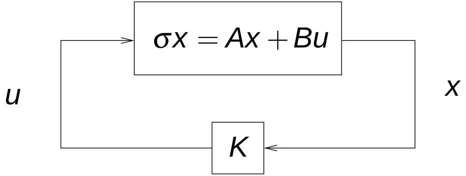
i.e.,

$$p_{\mathrm{des}}(z) = z^{n_{\mathrm{cl}}}$$

the closed-loop system state goes to zero in at most n_{cl} steps.

See the example on page 9.

State feedback



$$\sigma x = Ax + Bu, \quad u = Kx \qquad \Longrightarrow \qquad \sigma x = (A + KB)x$$

The closed-loop system is autonomous with state matrix

$$A_{\mathrm{c}} = A + KB.$$

Pole placement by state-feedback aims to choose K, so that

$$\det(zI - A_{\mathrm{c}}) = p_{\mathrm{des}}(z)$$

Example

Consider the system defined by the equation

$$\sigma x = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{b} u$$

and the desired characteristic polynomial $p_{\text{des}}(z) = z^2$.

$$\begin{aligned}
\det(zI - A - bk) &= \det\left(\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right) \\
&= \det\left(\begin{bmatrix} z - 1 - k_1 & -k_2 \\ -k_1 & z - 2 - k_2 \end{bmatrix} \right) \\
&= (z - 1 - k_1)(z - 2 - k_2) - k_2 k_1 \\
&= z^2 - (k_1 + k_2 + 3)z + 2 + 2k_1 + k_2
\end{aligned}$$

Equating the closed-loop char. polynomial to the desired char. polyn.

$$\det(zI - A - bk) = p_{\text{des}}(z)$$
$$\implies z^2 + (k_1 + k_2 + 3)z + 2 + 2k_1 + k_2 = z^2$$

gives two equations in the unknowns k_1 and k_2

$$k_1 + k_2 + 3 = 0$$
$$2 + 2k_1 + k_2 = 0$$

which unique solution is $k_1 = 1$, $k_2 = -4 \implies$ the control law is

$$u = \begin{bmatrix} 1 & -4 \end{bmatrix} x$$

and the closed-loop system is

$$\sigma x = (A + bk)x = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} x$$

Let's calculate two particular responses of the closed-loop system.

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto x(1) = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mapsto x(2) = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto x(1) = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix} \mapsto x(2) = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -4 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies \quad A_{\mathrm{c}}^2 = 0$$

Therefore, the state of the closed-loop system goes to zero in two steps from any initial condition (dead-beat control).

Controller canonical form (SISO case)

Fact: Any controllable system $\mathscr{B}_{\mathrm{i/o}}(p,q)$ can be represented in a state space form $\mathscr{B}_{\mathrm{i/s/o}}(A,b,c,d)$ with parameters

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -p_0 & -p_1 & \cdots & \cdots & -p_{n-1} \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$c = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \end{bmatrix}, \qquad d = q_n$$

where $c_0, c_1, \ldots, c_{n-1}$ are the coefficients of $q(z) - q_n p(z)$

The controllability matrix, associated with the pair A, b is

$$\mathscr{C}(A,b) := \begin{bmatrix} b & Ab & \cdots & A^{n-1}b \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & * \\ 0 & \iddots & \iddots & \vdots \\ 1 & * & \cdots & * \end{bmatrix}$$

$\implies \quad A, b$ is controllable

Similarity transformation for controller canonical form

A more general result:

Lemma:

- Let A, b and A', b' be two controllable pairs and
- assume that A and A' have the same char. polynomials.

Then there is a unique similarity transformation given by the matrix

$$T := \mathscr{C}(A, b)\big(\mathscr{C}(A', b')\big)^{-1}$$

such that

$$T^{-1}AT = A' \qquad \text{and} \qquad T^{-1}b = b'.$$

$\implies$ Any controllable representation of the system can be transformed to the controller canonical form.

Example

Consider again the system defined by the equation

$$\sigma x = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{b} u$$

The characteristic polynomial of A is

$$\det(Iz - A) = \det \begin{bmatrix} z-1 & 0 \\ 0 & z-2 \end{bmatrix} = (z-1)(z-2) = 2 - 3z + z^2$$

and the controllability matrix $\mathscr{C}(A,b) = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is full rank, so the system is controllable. Therefore,

$$A' = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, \qquad b' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is the controller canonical form of A, b.

The controllability matrix of the pair A', b' is

$$\mathscr{C}(A', b') = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$$

Therefore according to the lemma the similarity transformation that transforms A, b to A', b' is

$$T = \mathscr{C}(A, b)\big(\mathscr{C}(A', b')\big)^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}$$

Verify that $T^{-1}AT, T^{-1}b$ is indeed the controller canonical form, *i.e.*,

$$\begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$$

and

$$\begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

State-feedback pole placement in controller form

Let the plant be given by $\mathscr{B}_{\mathrm{i/s/o}}(A,b,c,d)$, with A,b in controller form.

Then

$$A_{\mathrm{cl}} := A + bk = \begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & & 0 \\ 0 & \cdots & \cdots & 0 & & 1 \\ k_1 - p_0 & k_2 - p_1 & \cdots & \cdots & & k_n - p_{n-1} \end{bmatrix}$$

and the closed-loop characteristic polynomial is

$$p_{\mathrm{cl}}(z) = (p_0 - k_1) + (p_1 - k_2)z + \cdots + (p_{n-1} - k_n)z^{n-1} + z^n$$

The equation $p_{\mathrm{cl}} = p_{\mathrm{des}}$ has the unique solution

$$k_1 = p_0 - p_{\mathrm{des},0}, \cdots, k_n = p_{n-1} - p_{\mathrm{des},n-1}$$

State-feedback pole placement in arbitrary basis

We know the solution in the controller form, so the approach is:

1. transform the given representation to controller canonical form,
2. solve the pole placement problem in the new basis,
3. convert the obtained state feedback gain to the original basis

Let the plant be given by $\mathscr{B}_{\mathrm{i/s/o}}(A, b, c, d)$. If A, b is controllable, there is similarity transf. T, s.t. $T^{-1}AT$, $T^{-1}b$ is in controller form

Let k' be the state-feedback in the new basis. Then

$$k = k'T^{-1}$$

is the state-feedback law that solves the pole placement problem in the original basis.

State-feedback pole placement in arbitrary basis

We showed that controllability is a sufficient condition for pole placement. It turns out that controllability is also necessary.

Theorem: The eigenvalues of $A + BK$ can be assigned choosing K to any locations in $\mathbb{C}$ if and only if A, B is controllable.

Notes:

- The theorem holds for general multivariable systems.
- The multi-input pole placement problem can be reduced to an equivalent single input pole placement problem.
- Good numerical methods for pole placement do not compute the controller canonical form.

Example

Consider the system example as before, but with A, b is controller form

$$\sigma x = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b} u$$

The state-feedback law that achieves the char. polynomial $p_{\text{des}}(z) = z^2$

$$k = \begin{bmatrix} 2 & -3 \end{bmatrix}.$$

In order to compare this state-feedback with the one obtained on page 8, we need to change the basis

$$kT^{-1} = \begin{bmatrix} 2 & -3 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -4 \end{bmatrix}$$