Practice problems for "Signal theory: Part 1"

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Linear algebra

Eigenvalues and eigenvectors of a companion matrix

The companion matrix related to the 3rd order polynomial

$$a(\lambda) := (\lambda-1)(\lambda-2)(\lambda-3) = -6 + 11\lambda - 6\lambda^2 + \lambda^3$$

is the 3×3 matrix

$$C_a := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}. \qquad (*)$$

1. Find 3 linearly independent eigenvectors of C_a.
2. Show that the eigenvalues of C_a are 1, 2, and 3.
3. Based on 1 and 2 conjecture and prove general properties of the eigenvalues and eigenvectors of an $n \times n$ companion matrix C_a related to an nth order polynomial a.

Orthogonality of left and right eigenvectors

A left eigenvector of a matrix A, related to an eigenvalue λ_i, is an $1 \times n$ vector w_i, such that

$$w_i A = \lambda_i w_i.$$

Find the left eigenvalues of $(*)$ and verify that they are orthogonal to the corresponding (right) eigenvectors v_i

$$A v_i = \lambda_i v_i.$$

Show that w_i, v_i can be chosen, so that $w_i v_i = 1$.

LTI systems

Harmonic oscillator

Find a state space representation of the harmonic oscillator $\frac{\mathrm{d}^2}{\mathrm{d}t^2} y = -\alpha y$, where $\alpha > 0$.

Multiple poles

Consider the autonomous system represented by a difference equation

$$y(t+2) - 2ay(t+1) + a^2 y(t) = 0.$$

(Its characteristic polynomial has both roots equal to $\lambda = a$.)

1. Show that both $y(t) = a^t$ and $y(t) = ta^t$ are solutions.
2. Find the trajectory generated from the initial conditions $y(0) = 1$ and $y(1) = 0$.

[Lue79, Chapter 2, Problem 2]

A bank offers 7% annual interest. What would be the overall annual rate if the 7% interest were compounded quarterly?

[Lue79, Chapter 2, Problem 5]

Find the second order linear homogeneous difference equation which generates the sequence 1, 2, 5, 12, 29, 70, 169. What is the limiting ratio of consecutive terms?

[Lue79, Chapter 2, Problem 10]

Consider the second order difference equation

$$y(t+2) - 2ay(t+1) + a^2 y(t) = 0.$$

Its characteristic polynomial has both roots equal to $\lambda = a$.

1. Show that both $y(t) = a^t$ and $y(t) = ta^t$ are solutions.
2. Find the solutions of this equation that satisfies the auxiliary conditions $y(0) = 1$ and $y(1) = 0$.

Embedded statics (Chapter 4, [Lue79, Problem 21])

Suppose a system is described by a set of equations of the form

$$\begin{bmatrix} T \\ 0 \end{bmatrix} x(t+1) = \begin{bmatrix} C \\ D \end{bmatrix} x(t) + \begin{bmatrix} u(t) \\ v(t) \end{bmatrix},$$

where $x(t)$ is an n-dimensional vector, T and C are $m \times n$ matrices, D is an $(n-m) \times n$ matrix, and $u(t)$ and $v(t)$ are m and $(n-m)$-dimensional vectors, respectively. Assume that the $n \times n$ matrix $\left[\begin{smallmatrix} T \\ D \end{smallmatrix}\right]$ is nonsingular. Following the steps below, it is possible to convert this system to state vector form.

1. Define $y = Tx$ and show that with this definition, and the lower part of the system equation, one may express $x(t)$ in the form
$$x(t) = Hy(t) - Gv(t)$$
Give an explicit definition of G and H.

2. Show that the top part of the original system can be written in the state vector form
$$y(t+1) = Ry(t) + Bv(t) + u(t)$$
and give expressions for R and B. Note that x can be recovered from y using part 1.

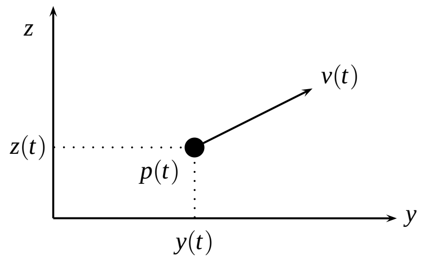
3. Apply this procedure to the following example.

   Samuelson proposed a model of the national economy based on the following assumptions. National income $Y(k)$ is equal to the sum of consumption $C(k)$, investment $I(k)$, and government expenditure $G(k)$. Consumption is proportional to the national income of the preceding year; and investment is proportional to the increase in consumer spending of that year over the preceding year. In equation form, the Samuelson model is

$$\begin{aligned} Y(k) &= C(k) + I(k) + G(k) \\ C(k+1) &= mY(k) \\ I(k+1) &= \mu\big(C(k+1) - C(k)\big) \end{aligned}$$

Trajectory of a freely falling object

Consider a free falling object in a 2-dimensional gravitational field. Let $p(t)$ be the position of the object at time t. We choose a reference moment of time $t = 0$ to be the moment when the object starts its free fall and an orthogonal coordinate system in the plane of motion with vertical axis along the negative of the gravitational force and a perpendicular horizontal axis at the ground level. The horizontal displacement of the object at time t with respect to the



origin is denoted by $y(t)$ and the vertical displacement by $z(t)$. More notation, used in the exercises is:

$p(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$	—	object's position and its coordinates at time t
$v(t)$	—	object's velocity at time t
$p_{\text{ini}}, v_{\text{ini}}$	—	initial (time $t = 0$) position and velocity
$x(t)$	—	state (position and velocity) at time t
m	—	object's mass
g	—	gravitational constant
$m\mathbf{g}$	—	gravitational force ($\mathbf{g}$ is a vector in the "negative vertical" direction with norm equal to g)

By the second law of Newton, for any $t > 0$, the position of the object is described by the differential equation

$$m\ddot{p} = m\mathbf{g}, \qquad \text{where} \quad p(0) = p_{\text{ini}} \text{ and } \dot{p}(0) = v_{\text{ini}}. \tag{1}$$

Find an analytic expression for the trajectories p.

Free fall with friction

In this exercise, we relax the assumption that the object is in vacuum. The net effect of the air on the object is a force f, acting on the object. The model equation (1) becomes

$$m\ddot{p} = m\mathbf{g} + f, \qquad \text{where} \quad p(0) = p_{\text{ini}} \text{ and } \dot{p}(0) = v_{\text{ini}}. \tag{2}$$

Without wind and turbulence, the force f is due to friction with the air and can be approximated by a linear function of the velocity, *i.e.*, we take

$$f = -\gamma v, \tag{3}$$

where γ is a constant depending on the physical properties of the environment as well as the size and shape of the object. Repeat the previous exercise for the case when a friction force (3) is present. Experiment with different values for the mass m and the friction constant γ.

Throwing an object as far as possible

How far can you throw an object? In order to formulate the question mathematically, assume that you can give a maximal initial velocity to the object (by accelerating it for a period of time, after which period the object is freely falling). The question then is what direction should the initial velocity have in order for the object to reach as far as possible when it lands on the ground. The problem is considerably simpler when $z_{\text{ini}} = 0$, *i.e.*, the object is thrown from the ground and there is no friction. Assume that this is the case. Once you come up with an answer, use your free falling simulation function to compute and plot the trajectory. Try some alternative admissible trajectories to make sure that your solution gives the best result.

Least-squares estimation

Existence and uniqueness of the least squares approximate solution

Consider the system of linear equations $Ax = b$, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ are given, and $x \in \mathbb{R}^n$ is unknown. A least squares approximate solution of the system is a solution to the optimization problem

$$\text{minimize} \quad \text{over } \widehat{x} \in \mathbb{R}^n \quad \|A\widehat{x} - b\|_2. \tag{LS}$$

When does a solution exists and when it is unique. Under the assumptions of existence and uniqueness, derive a closed form expression for the least squares approximate solution. Characterize all least squares approximate solutions in the general case. Solve the least-squares approximation problem

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 4 & 8 \end{bmatrix} x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Orthogonality principle for least-squares estimation

Show that

1. $\widehat{x}$ being a least squares approximate solution of the system $Ax = b$, and
2. $\widehat{x}$ being such that $b - A\widehat{x}$ is orthogonal to the span of the columns of A,

are equivalent. (This result is known as the orthogonality principle for least squares approximation.)

Weighted least-squares approximate solution

For a given positive definite matrix $W \in \mathbb{R}^{m\times m}$, define the weighted 2-norm

$$\|e\|_W = e^\top We.$$

The weighted least-squares approximation problem is

$$\text{minimize} \quad \text{over } \widehat{x} \in \mathbb{R}^n \quad \|A\widehat{x} - b\|_W. \tag{WLS}$$

When does a solution exist and when is it unique? Under the assumptions of existence and uniqueness, derive a closed form expression for the least squares approximate solution.

Orthogonality principle for weighted least-squares estimation

Formulate and prove a similar property to the orthogonality of $A\widehat{x} - b$ and $\operatorname{span}(A)$ for the weighted least-squares problem, *i.e.*, generalize the orthogonality principle to (WLS).

Stochastic signals

Finite Markov chain model ([Lue79, Chapter 7])

The weather in a certain city can be characterized as being either sunny, cloudy, or rainy. If it is sunny one day. then sun or clouds are equally likely the next day. If it is cloudy, then there is a fifty percent chance the next day will be sunny, a twenty-five percent chance of continued clouds, and a twenty-five percent chance of rain. If it is raining, it will not be sunny the next day, but continued rain or clouds are equally likely.

Denoting the three types of weather by S, C, and R, respectively, this model can be represented by an array of transition probabilities:

	S	C	R
S	1/2	1/2	0
C	1/2	1/4	1/4
R	0	1/2	1/2

This array is read by going down the left column to the current weather condition. The corresponding row of numbers gives the probabilities associated with the next weather condition. The process starts with some weather condition and moves, each day, to a new condition. There is no way, however, to predict exactly which transition will occur. Only probabilistic statements, presumably based on past experience, can be made.

We model the weather dynamics by a stochastic system, called a finite Markov chain. Let $x_1(t)$ be the probability that day-t is sunny, $x_1(t)$ be the probability that day-t is cloudy, and $x_3(t)$ be the probability that day-t is cloudy. Obviously $x(t)$ is nonnegative and its elements sum to one. (Such a vector is called a stochastic vector.)

1. Express the weather dynamics, described by the transition probabilities

$$P = \begin{bmatrix} .5 & 0.5 & 0 \\ 0.5 & 0.25 & 0.25 \\ 0 & 0.5 & 0.5 \end{bmatrix},$$

   as a deterministic linear time-invariant system for the vector of probabilities x.

2. What conditions needs to satisfy the matrix P so that $x(t)$ is a stochastic vector for all $t > 0$, provided $x(0)$ is a stochastic vector?

3. A limiting distribution of the Markov chain model is a distribution x_∞ such that $x(t) \to x_\infty$, as $t \to \infty$ (independent of the initial conditions). Is there a limiting distribution of the weather model? If so, find it.

Periodogram computation via FFT

The periodogram method for spectral estimation is

$$\widehat{\phi}(\omega) = \frac{1}{T} | \sum_{\tau=1}^{T} y(\tau) e^{-\mathbf{i}\omega\tau} |^2.$$

1. Explain how to compute $\widehat{\phi}(\omega_k)$, where

$$\omega_k = \frac{2\pi}{T} k, \qquad \text{for } k = 0, 1, \dots, T-1,$$

   using matrix-vector product. How many multiplication operations are required by this method?

2. Explain how to use the fast Fourier transform for fast computation of $\widehat{\phi}$ for $\omega = \omega_k$.

Covariance structure of an ARMA model

An auto-regressive moving-average (ARMA) system is a deterministic linear time-invariant system, "driven" by white noise input e.

$$\text{white noise} \longrightarrow \boxed{\text{deterministic system}} \longrightarrow y$$

In the scalar case, it can be represented by a difference equation

$$y(t) + \sum_{\tau=1}^{n} a_\tau y(t-\tau) = \sum_{\tau=0}^{m} b_\tau e(t-\tau),$$

with $b_0 = 1$. Prove that the auto-covariance r of y satisfies the equation

$$r(k) + \sum_{\tau=1}^{n} a_\tau r(k-i) = 0.$$

Yule-Walker equation

Yule-Walker equation is a method for estimation of the coefficients of an AR process

$$y(t) + \sum_{\tau=1}^{n} a_\tau y(t-\tau) = e(t).$$

Using the results from the previous problem, show that

$$\begin{bmatrix} r(0) & r(-1) & \cdots & r(-n) \\ r(1) & r(0) & \ddots & \vdots \\ \vdots & & \ddots & r(-1) \\ r(n) & \cdots & & r(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \sigma^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

(This equation is known as the Yule-Walker equation and forms the basis of many parametric estimation methods.) Explain how to estimation the coefficients a if the autocorrelation r was known.

Central limit theorem (Matlab)

Observe the smoothing effect of the convolution operation by convolving a rectangular pulse, centered at 0 with itself N times, where $N = 1, 2, \ldots, N$. Observe also that by proper normalization the resulting signal converges to a Gaussian distribution. Can you explanation these empirical observations?

References

[Lue79] D. G. Luenberger. *Introduction to Dynamical Systems: Theory, Models and Applications*. John Wiley, 1979.