ELEC3035 mindstorms

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Abstract

The exercises described in this document are inspired by Seymour Papert's book "Mindstorms: Children, Computers, And Powerful Ideas". Their purpose is to stimulate curiosity in systems and control related concepts via independent work on computer based projects. The projects cut across several disciplines—physics, mathematics, computer programming, as well as the subject of ELEC3035—systems and control theory.

Mindstorms (optional background information)

> What kind of curve is the trajectory of a stone thrown in the air?
> How should one throw a stone in order to reach as far as possible?

These are curiosity driven questions, which can be answered by first turning them into well defined mathematical problems (using high school physics) and then solving the mathematical problems (using analytical and numerical methods, studied in school as well as in the university). Solving mathematical problems is part of the ELEC3035 exercises, but it is not the main part and not the most important part either. More important than finding answers to math problems is the habit to ask questions (*i.e.*, aiming for open enquiring rather than following of authority) and the ability to turn vague questions into well defined problems (there is no recipe for this; it is more of an art than science). Only in combination with these skills, the ability to reason rigorously in the search for solutions becomes a powerful tool for solving real-life problems.

The philosophy of "mindstorms", described by Seymour Papert in his books [1, 2, 3], is that

> the only way to acquire any type of knowledge is to immerse in an environment that offers ample possibilities to practice this knowledge.

Using Papert's analogy, in the same way as you need France for learning french, you need "control land" for learning control theory. These exercises aim to serve the role of "control land" for ELEC30335. I like the exercise and hope you do too. There are however many possibilities for improvement. Your feedback (criticism, corrections of mistakes, and ideas for additional exercises) is a critical element in this process.

One needs motivation in order to pursue the search for answers to difficult questions. Such motivation comes only through personal interest and involvement with the questions. Unfortunately, the formal educational system reverses this causal relation—everyone is expected to acquire certain "pure" knowledge (the curriculum of one's education) in order to be prepared for solving "real-life" problems (or just for beginning an educated person). The "real-life" problems are usually not encountered until the education is over.

In ELEC3035, you are given a possibility to practice your systems and control knowledge by working on a series of exercises which are formulated as vague questions rather than as well defined problems. These questions are curiosity driven for me personally and I hope you too will be motivated to find answers. You have full freedom to choose the problem formulation and solution method that you deem relevant for the question at hand. Moreover, you are encouraged to find your own style in dealing with the assignments and ask questions that are not part of the assignments!

Given such freedom, you may come up with different answers to the original questions (as well as answers to questions that were never asked). As long as your answers are correct solutions to relevant problems, related to the original question, they are "valid answers". Clearly, you may come up with different valid answers. This is fine. In fact, this is what happens in real-life engineering practice.

Introduction and notation

In the exercises you will be dealing with questions related to free and controlled flight of an object in a gravitational field. For example, throwing a stone, we apply for a (short) period of time a force on the stone, which net effect is to give the stone an initial velocity. The magnitude and direction of the initial velocity is all that matters. How they are achieved is not important for the subsequent flight. From the moment of departing from the hand, the stone is falling freely; the forces that act on the stone are the gravitational force and a force from the impact with the air (friction and wind). To begin with, assume that the stone is flying in vacuum, so that the the only force acting on the stone is the gravitational force.

Note that the stone remains throughout its free falling flight in the plane determined by the initial velocity and the gravitational force. Therefore, although the stone is flying in a three dimensional space, it actually remains in a plane. Let $p(t)$ be the position of the stone at time t. We choose a reference moment of time $t = 0$ to be the moment when the stone starts its free fall and an orthogonal coordinate system in the plane of motion with vertical axis along the negative of the gravitational force and a perpendicular horizontal axis at the ground level, see Figure 1.

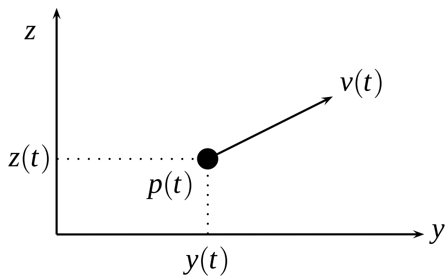

Figure 1: Problem setup.

The horizontal displacement of the stone at time t with respect to the origin is denoted by $y(t)$ and the vertical displacement by $z(t)$. More notation, used later on, is:

$p(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$	—	object's position and its coordinates at time t
$v(t)$	—	object's velocity at time t
$p_{\text{ini}}, v_{\text{ini}}$	—	initial (time $t = 0$) position and velocity
$x(t)$	—	state (position and velocity) at time t
m	—	object's mass
g	—	gravitational constant
$m\mathbf{g}$	—	gravitational force ($\mathbf{g}$ is a vector in the "negative vertical" direction with norm equal to g)

By the second law of Newton, for any $t > 0$, the position of the stone is described by the differential equation

$$m\ddot{p} = m\mathbf{g}, \qquad \text{where} \quad p(0) = p_{\text{ini}} \text{ and } \dot{p}(0) = v_{\text{ini}}. \tag{1}$$

Here p_{ini} is the initial position (the place from where the stone is thrown) and v_{ini} is the initial velocity (by which the stone is thrown). Note the following facts.

- (1) is a linear differential equation with constant coefficients. (This is good news because such equations can be solved explicitly.)
- (1) is of second order and is a vector equation (there is a scalar equation for y and another scalar equation for z).
- The equation does not depend on the mass m (since m cancels). This means that any stone, more generally any object, falling in a gravitational field without friction has the same trajectory.[1]

[1] The fact that the trajectory does not depend on the mass may be surprising and counter intuitive. Here is an instance when physics and mathematics reveal something that is not obvious.

Since it is no longer relevant that the thrown object is a stone, from now on, we will call it more generally "an object".

- The right hand side is constant. This further simplifies the solution of the equation.

Equation (1) "says" everything about the motion of the object. It is a concise and unambiguous description of any object falling in a gravitational field, starting from an arbitrary initial conditions—position and velocity. The equation, however, is not as clear as an explicit solution that exhibits the nature of the trajectory (p as a function of time). Therefore, your first task is to find the solution explicitly; both analytically (by hand) and numerically (using the computer). If you did not know the answer of the opening question "What kind of curve is the trajectory of a stone thrown in the air?", then you will discover it yourself, which is the best way to learn something.

Exercise 1: Trajectory of a freely falling object

In this exercise, we consider the trajectory of an object with initial condition $x(0) = x_{\text{ini}}$ (position and velocity) when no external force f is applied (*i.e.*, a freely falling object, thrown from some location $p(0) = p_{\text{ini}}$ with some initial velocity $v(0) = v_{\text{ini}}$). Using any method you deem appropriate, find an analytic expression for the resulting trajectory. Then, write a function (in your favourite programming language) that takes as an input the initial condition and returns as an output samples of the trajectory and a time vector of the corresponding moments of time. Test your function for some initial conditions and plot the resulting trajectories.

□

Exercise 2: Free fall with friction

In this exercise, we relax the assumption that the object is in vacuum. The net effect of the air on the object is a force f, acting on the object. The model equation (1) becomes

$$m\ddot{p} = m\mathbf{g} + f, \qquad \text{where} \quad p(0) = p_{\text{ini}} \text{ and } \dot{p}(0) = v_{\text{ini}}. \tag{2}$$

Without wind and turbulence, the force f is due to friction with the air and can be approximated by a linear function of the velocity, *i.e.*, we take

$$f = -\gamma v, \tag{3}$$

where γ is a constant depending on the physical properties of the environment as well as the size and shape of the object. Repeat exercise 1 for the case when a friction force (3) is present. Experiment with different values for the mass m and the friction constant γ.

□

Exercise 3: Bungee jumping

The object now is attached to one end of an elastic rope. The other end of the rope is fixed at a given location p_{r}, see Figure 2. The force exerted on the object from the rope is

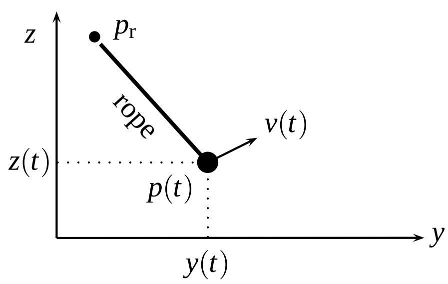

Figure 2: Setup for Exercise 3.

$$e(t) = \gamma\big(p_\mathrm{r} - p(t)\big), \tag{4}$$

where γ is a positive constant. Find the trajectory resulting from an initial condition x_ini. Compare the trajectories with and without friction in the air. □

In order to make the setup more realistic, modify the elastic force (4) taking into account the rope's length r, *i.e.*, the force is zero before the rope gets stretched. Can you still find the trajectory p in closed form? Revise the numerical simulation function for this case and observe how the trajectory differs from before.

Exercise 4: Hitting a stationary object

Knowing what the trajectory of a freely falling object is and being able to simulate it on a computer, our next objective is to choose the initial velocity v_ini, so that the object starting from initial position $p_\mathrm{ini} = 0$ reaches a given position p_des at a given moment of time $t = T$, *i.e.*, $p(T) = p_\mathrm{des}$. Solve the problem analytically in the case of no friction. Then write a function that takes as an input p_des and T and returns as an output the initial velocity v_ini that achieves a trajectory, such that $p(T) = p_\mathrm{des}$. Test your function for some p_des and T.

□

Modify your solution for the case when there is friction in the air.

□

Exercise 5: Throwing an object as far as possible

How far can you throw an object? In order to formulate the question mathematically, assume (which is a reasonable assumption) that you can give a maximal initial velocity to the object (by accelerating it for a period of time, after which period the object is freely falling). The question then is what direction should the initial velocity have in order for the object to reach as far as possible when it lands on the ground. The problem is considerably simpler when $z_\mathrm{ini} = 0$, *i.e.*, the object is thrown from the ground and there is no friction. Assume that this is the case. Once you come up with an answer, use your free falling simulation function to compute and plot the trajectory. Try some alternative admissible trajectories to make sure that your solution gives the best result.

□

Exercise 6: Deducing the initial state from observed trajectory

Suppose you observe a free falling object for a period of time. More precisely, you are given the positions at given moments of time

$$p(t_1), p(t_2), \dots, p(t_N), \qquad \text{where} \quad 0 < t_1 \leq t_2 \leq \cdots \leq t_N. \tag{5}$$

Can you find from this data the initial position $p(0)$ and the initial velocity $v(0)$? If so, write a function that accepts as an input data the positions (5) and returns as an output the initial state $x(0)$. Test your function for some simulated trajectories.

□

It is unrealistic to assume that the moments of time $t_1, \dots, t_N$ when the observations are taken are known relative to the moment of time when the target is thrown. In order to relax this assumption redo the problem when $t_1 = 0, \dots, t_N = t_\mathrm{obsv}$ are known relative to the moment of time when the first observation is taken. Can you still find the place and velocity from and by which the object is thrown as well as the moment of time, relative to the first observation, when this happened?

□

In practice, the observations are noisy and are not generated by a linear time-invariant system of low order. Test your estimation method with data (5) obtained as samples of a true trajectory $\bar{p}$ plus zero mean Gaussian additive noise. The estimated initial position and velocity fix an estimated trajectory $\widehat{p}$. Validate the accuracy of the method by evaluating the relative error function

$$e := \frac{\sqrt{\sum_{i=1}^{N} \|\bar{p}(t_i) - \widehat{p}(t_i)\|_2^2}}{\sqrt{\sum_{i=1}^{N} \|\bar{p}(t_i)\|_2^2}}.$$

□

Exercise 7: Hitting a moving object

Using the insights and functions that we obtained from the solution of exercises 2 and 4, we now turn to the seemingly challenging problem of hitting a moving object by an object thrown from initial position $p^1_{\text{ini}} = 0$. (The superscript index 1 shows that this is the "first" object; the one that is thrown. The target object's position, velocity, and state are denoted with superscript 2.) Suppose that we have observed the free fall of the target for a given period of time T_{ini}. The aim is to determine the initial velocity $v^1_{\text{ini}} = v^1(T_{\text{ini}})$, such that $p^1(T) = p^2(T)$ for some given $T > T_{\text{ini}}$, *i.e.*, the thrown object hits the target object. Write a function that implements the solution and test it on some simulated data.

□

Exercise 8: Effect of disturbances

In exercises 1–5, the trajectory of the object was described by equation (1). In this exercise, we simulate a free falling trajectory with a disturbance force

$$m\ddot{p} = m\mathbf{g} + d, \qquad \text{where} \quad p(0) = p_{\text{ini}} \text{ and } \dot{p}(0) = v_{\text{ini}}. \tag{6}$$

Think of the disturbance d as an effect of wind, exerting a force on the object (as a real wind does). The disturbance d is modelled by a zero mean white Gaussian random variable, acting in a horizontal direction.

- The zero mean property implies that for a long period of time the net effect of the disturbance is nil.
- The white noise property implies that the intensities of the wind in sequential moments of time are unrelated.
- The Gaussian property implies that the distribution of the wind intensity in any moment of time is the famous bell shaped curve.

Although these assumptions are somewhat arbitrary and certainly do not match all real wind scenarios, they are a convenient theoretical model to study and capture well some realistic types of disturbances.

Modify the free falling simulation function to correspond to the noisy setup (6). The noise d, described above, is determined by a single parameter—the noise variance v. The parameter v as well as the object's mass m be inputs to the simulation function. Experiment with different values of the parameters and comment on the results. Find the mean and variance of $p(T)$, corresponding to initial condition x_{ini}, *i.e.*, find the effect of the initial condition and disturbance on the landing location.

□

Exercise 9: Use of feedback control to reduce the effect of the disturbances

An effective and often used way to deal with disturbances (as well as model uncertainty) is feedback. Feedback requires measurement of the object's state or output (position) and possibility to apply corrective action via an input. In the setup of Exercise 2, the control input is an external force f. (Think of the object as a rocket equipped with a thruster.) In this exercise, the force acts in horizontal direction (thus counteracting the wind force).

Design a feedback controller that keeps the object "on target" despite of disturbances. "On target" means that the object is supposed to land as close as possible to the specified location y_{des}. Measure the effectiveness of the feedback control by computing the bias (mean value of $y_{\text{des}} - y(T)$) and the variance of $y(T)$. Compare different control design methods with the "no feedback control" solution of Exercise 2.

□

Exercise 10: Optimal open loop control

The object now is equipped with two thrusters: one acting (as in Exercise 7) in horizontal direction and the second one in vertical direction. The two thrusters can be controlled independently. Let $u_1(t)$ be the force that the first thruster applies on the object at time t and $u_2(t)$ the force that the second thruster applies on the object at time t. Modify the initial conditions simulation function for the present scenario. Then, find the minimum energy control, *i.e.*, the inputs u_1 and u_2 that minimize the function

$$\|u\| = \sqrt{\int_0^T u_1^2(t) + u_2^2(t)\mathrm{d}t}$$

and transfers the object from a given initial state $x(0) = x_{\text{ini}}$ to a given final state $x(T) = x_{\text{des}}$ for a given time T. Implement the solution in a function and experiment with different values for x_{ini}, x_{des}, and T. Can you explain the shape of the optimal control curve? Think of the function computing the optimal control as a mapping

$$(x_{\text{ini}}, x_{\text{des}}, T) \mapsto u_{\text{ln}}, \tag{7}$$

where u_{ln} is the minimum norm control achieving the transfer. Find input values $(x_{\text{ini}}, x_{\text{des}}, T)$ for which $u_{\text{ln}} = 0$ and describe *all* such values. What kind of map is (7)? Find the kernel of (7) and relate it to the verbal description of the input values $(x_{\text{ini}}, x_{\text{des}}, T)$, for which $u_{\text{ln}} = 0$.

□

References

[1] S. Papert. *Mindstorms: Children, Computers, And Powerful Ideas*. Basic Books, 1993.

[2] S. Papert. *The Children's Machine: Rethinking School In The Age Of The Computer*. Basic Books, 1994.

[3] S. Papert. *The Connected Family: Bridging the Digital Generation Gap*. Longstreet Press, 1996.