On the linear quadratic data-driven control

Ivan Markovsky

University of Southampton

Joint work with Paolo Rapisarda

"Data-driven" control methods

data-driven = model-free = model-less = $\cdots$

1995	Skelton, Furuta	LQG design using Markov par.
1996	Hjalmarsson, Gevers	iterative feedback tuning
1997	Safonov	unfalsified control
1999	Favoreel, De Moor	subspace based LQG control
2001	Woodley	subspace based $\mathscr{H}_\infty$ control
2002	Campi, Lecchini	virtual reference feedback tuning
2004	Ikeda, Fujisaki	subspace based LQ tracking

Our work is in the spirit of the "subspace based" methods.

Notation

$\mathtt{w}$ — # of external variables, $\mathtt{m}$ — # of inputs, $\mathtt{p}$ — # of outputs

$\mathbb{N} := \{1, 2, \dots\}$ — time axis

$\mathscr{B}(A,B,C,D)$ — the system defined by $\begin{aligned} \sigma x &= Ax + Bu \\ y &= Cx + Du \end{aligned}$

Restriction of the system behavior $\mathscr{B}$ to the interval $1, 2, \dots, t$

$$\mathscr{B}_t := \{\, w_\mathrm{p} \in (\mathbb{R}^\mathtt{w})^t \mid \text{there is } \; w_\mathrm{f} \text{ such that } (w_\mathrm{p}, w_\mathrm{f}) \in \mathscr{B} \,\}$$

$\mathrm{lag}(\mathscr{B})$	—	lag of $\mathscr{B}$	(observability index of I/S/O repr.)
$\mathrm{order}(\mathscr{B})$	—	order of $\mathscr{B}$	

We assume that an input/output partition of the variables is given.

The simulation problem

Classical simulation problem: Given

- system $\mathscr{B} := \mathscr{B}(A,B,C,D)$,
- input $u \in (\mathbb{R}^\mathtt{m})^t$, and
- initial conditions $x_\mathrm{ini} \in \mathbb{R}^\mathtt{n}$,

find the response y of $\mathscr{B}$ to u and ini. cond. x_ini.

Data-driven simulation problem: Given

- trajectory $w_\mathrm{d} = (u_\mathrm{d}, y_\mathrm{d}) \in (\mathbb{R}^\mathtt{w})^T$ of $\mathscr{B}$,
- input $u \in (\mathbb{R}^\mathtt{m})^t$, and
- initial trajectory $w_\mathrm{ini} \in (\mathbb{R}^\mathtt{w})^{T_\mathrm{ini}}$, $w_\mathrm{ini} \in \mathscr{B}_{T_\mathrm{ini}}$,

find the response y of $\mathscr{B}$ to u, such that $\big(w_\mathrm{ini}, (u, y)\big) \in \mathscr{B}_{T_\mathrm{ini}+t}$.

Notes:

- $\mathscr{B}$ is specified implicitly by $w_{\rm d}$,
- the initial condition $x_{\rm ini}$ is specified implicitly by $w_{\rm ini}$.

Algorithm 1: data-driven simulation, using I/S/O repr.

1. identification $\quad w_{\rm d} \mapsto (A,B,C,D)$
2. observer $\quad \big(w_{\rm ini},(A,B,C,D)\big) \mapsto x_{\rm ini}$
3. classical simulation $\quad \big(u,x_{\rm ini},(A,B,C,D)\big) \mapsto y$

Can we find y without deriving an explicit representation of $\mathscr{B}$?

Notation for Hankel matrices

Given a signal $w = \big(w(1),\dots,w(T)\big)$ and $t \leq T$, define

$$\mathscr{H}_t(w) := \begin{bmatrix} w(1) & w(2) & w(3) & \cdots & w(T-t+1) \\ w(2) & w(3) & w(4) & \cdots & w(T-t+2) \\ w(3) & w(4) & w(5) & \cdots & w(T-t+3) \\ \vdots & \vdots & \vdots & & \vdots \\ w(t) & w(t+1) & w(t+2) & \cdots & w(T) \end{bmatrix}$$

block-Hankel matrix with t block-rows, composed of w

Basic idea

Assuming that $w_{\rm d}$ is a trajectory of $\mathscr{B}$ (exact data),

lin. comb. of the columns of $\mathscr{H}_t(w_{\rm d})$ are trajectories of $\mathscr{B}$, *i.e.*,

for all g, $\quad \mathscr{H}_t(w_{\rm d})g \in \mathscr{B}_t$

Under additional conditions—persistency of excitation of $u_{\rm d}$ and controllability of $\mathscr{B}$—every trajectory can be generated that way.

In what follows, we assume that these conditions are satisfied.

$\Longrightarrow$ computing the response of $\mathscr{B}$ to given input and initial conditions from data $w_{\rm d}$, requires choosing a suitable g

Construction of responses from data

Problem: Find y, such that $\big(w_{\rm ini},(u,y)\big) \in \mathscr{B}$, where $w_{\rm ini}, u$ are given, and $\mathscr{B}$ is implicitly defined by $w_{\rm d}$.

There is g, such that

$$\mathscr{H}_{T_{\rm ini}+t}(w_{\rm d})g = \big(w_{\rm ini},(u,y)\big).$$

The eqns with RHS y, define y, for given g. The others restrict g.

Generic data-driven simulation algorithm:

1. compute any solution g of the equations with RHS $w_{\rm ini}$, u
2. substitute g in the equations for y

Define

$$U := \mathscr{H}_{T_{\text{ini}}+t}(u_{\text{d}}), \qquad Y := \mathscr{H}_{T_{\text{ini}}+t}(y_{\text{d}})$$

and the partitionings

$$U =: \begin{bmatrix} U_{\text{p}} \\ U_{\text{f}} \end{bmatrix}, \qquad Y =: \begin{bmatrix} Y_{\text{p}} \\ Y_{\text{f}} \end{bmatrix}.$$

Algorithm 2: data-driven simulation

1. compute the least norm solution of

$$\begin{bmatrix} U_{\text{p}} \\ Y_{\text{p}} \\ U_{\text{f}} \end{bmatrix} g = \begin{bmatrix} u_{\text{ini}} \\ y_{\text{ini}} \\ u \end{bmatrix}.$$

2. compute $y := Y_{\text{f}} g$.

Special case $u = 0$: free response

Allows to compute an observability matrix $\mathscr{O}$ of $\mathscr{B}$ from data, by finding n $\geq$ order$(\mathscr{B})$ linearly indep. free responses.

Let $\mathtt{l}_{\text{max}}$ be an upper bound for the lag of $\mathscr{B}$ and take $T_{\text{ini}} = \mathtt{l}_{\text{max}}$.

Algorithm 3: compute an observability matrix $\mathscr{O}$

1. compute the least norm solution of

$$\begin{bmatrix} U_{\text{p}} \\ Y_{\text{p}} \\ U_{\text{f}} \end{bmatrix} G = \begin{bmatrix} U_{\text{p}} \\ Y_{\text{p}} \\ 0 \end{bmatrix}$$

2. compute $Y := Y_{\text{f}} G$
3. compute a rank revealing factorization $Y = \mathscr{O} X_{\text{ini}}$

Special case $w_{\text{ini}} = 0$: zero initial cond. response

Let h be the impulse response of $\mathscr{B}$, and define

$$\mathscr{T}_t(h) := \begin{bmatrix} h(0) & & & & \\ h(1) & h(0) & & & \\ h(2) & h(1) & h(0) & & \\ \vdots & \ddots & \ddots & \ddots & \\ h(t-1) & \cdots & \cdots & h(1) & h(0) \end{bmatrix}$$

For any $w = \text{col}(u, y) \in \mathscr{B}_t$,

$$y = \mathscr{O} x_{\text{ini}} + \mathscr{T}_t(h) u$$

We can compute a basis for $\mathscr{B}_{0,t} := \text{image}\left(\mathscr{T}_t(h)\right)$ from data, by finding $t\mathtt{m}$ lin. indep. zero initial cond. responses.

Algorithm 4: compute a basis of $\mathscr{B}_{0,t}$

1. compute the least norm solution of

$$\begin{bmatrix} U_{\text{p}} \\ Y_{\text{p}} \\ U_{\text{f}} \end{bmatrix} G = \begin{bmatrix} 0 \\ 0 \\ \mathscr{H}_{t,t\mathtt{m}}(u_{\text{d}}) \end{bmatrix}$$

2. compute $Y_0 := Y_{\text{f}} G$

Then $\text{image}(Y_0) = \text{image}\left(\mathscr{T}_t(h)\right) = \mathscr{B}_{0,t}$.

Special case $w_{\text{ini}} = 0$, $u = I\delta$: impulse response

With the same construction we can find the first t Markov parameters of $\mathscr{B}$, which is a system identification method.

Algorithm 5: compute the impulse response

1. compute the least norm solution of

$$\begin{bmatrix} U_{\text{p}} \\ Y_{\text{p}} \\ U_{\text{f}} \end{bmatrix} G = \begin{bmatrix} 0 \\ 0 \\ \text{col}(I_{\mathtt{m}}, 0) \end{bmatrix}$$

2. compute $h := Y_{\text{f}} G$

Linear quadratic tracking problem

Given

- trajectory $w_{\text{d}} = (u_{\text{d}}, y_{\text{d}}) \in (\mathbb{R}^{\mathtt{w}})^T$ of $\mathscr{B}$,
- initial trajectory $w_{\text{ini}} \in (\mathbb{R}^{\mathtt{w}})^{T_{\text{ini}}}$, $w_{\text{ini}} \in \mathscr{B}_{T_{\text{ini}}}$,
- reference trajectory $w_{\text{r}} \in (\mathbb{R}^{\mathtt{w}})^{T_{\text{r}}}$, and
- positive definite matrix $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$,

find a trajectory of $\mathscr{B}$ that is optimal with respect to the criterion

$$J(w_{\text{r}}, w) := (w_{\text{r}} - w)^\top \Phi (w_{\text{r}} - w)$$

and has as a prefix the initial trajectory w_{ini}, *i.e.*, find

$$w_{\text{f}}^* := \arg\min_{w_{\text{f}}} J(w_{\text{r}}, w_{\text{f}}) \quad \text{subject to} \quad (w_{\text{ini}}, w_{\text{f}}) \in \mathscr{B}_{T_{\text{ini}} + T_{\text{f}}}.$$

Solution using an I/S/O representation

The classical but indirect solution is:

Algorithm 7: data-driven LQ tracking, using I/S/O repr.

1. $w_{\text{d}} \xrightarrow{\text{Identification}} (A, B, C, D)$
2. $\big(w_{\text{ini}}, (A, B, C, D)\big) \xrightarrow{\text{Observer (1,2)}} x_{\text{ini}}$
3. $\big(\Phi, w_{\text{r}}, x_{\text{ini}}, (A, B, C, D)\big) \xrightarrow{\text{Synthesis (3,4,5)}} w_{\text{f}}^*$

We aim to find algorithms that do not derive a repr. of $\mathscr{B}$.

Observer

Let h be the impulse response of $\mathscr{B}$. We have,

$$y_{\text{ini}} = \mathscr{O}(A, C) x(1) + \mathscr{T}_{T_{\text{ini}}}(h) u_{\text{ini}}, \tag{1}$$

where

$$\mathscr{O}(A, C) := \text{col}(C, CA, \dots, CA^{T_{\text{ini}}-1})$$

defines a system of equations for the initial state $x(1)$.

$w_{\text{ini}} \in \mathscr{B}_{T_{\text{ini}}} \implies$ existence of solution
(A, B, C, D) minimal $\implies$ uniqueness

$$x_{\text{ini}} := x(T_{\text{ini}} + 1) = A^{T_{\text{ini}}} x(1) + \begin{bmatrix} A^{T_{\text{ini}}-1}B & A^{T_{\text{ini}}-2}B & \cdots & B \end{bmatrix} u_{\text{ini}}. \tag{2}$$

LQ Regulator

LQ tracking problem:

$$\min_{x,u,y} \; (w_\mathrm{r} - \mathrm{col}(u,y))^\top \Phi (w_\mathrm{r} - \mathrm{col}(u,y))$$

$$\text{subject to} \quad \begin{array}{ll} x(t+1) = Ax(t) + Bu(t), & x(1) = x_\mathrm{ini} \\ y(t) = Cx(t) + Du(t), & \text{for } t = 1, \ldots, T_\mathrm{r}. \end{array}$$

The solution for the $w_\mathrm{r} = 0$ case (regulation problem) is

$$\begin{aligned} x^*(t+1) &= (A - BL_t)x^*(t), \quad x(1) = x_\mathrm{ini} \\ w_\mathrm{f}^*(t) &= \begin{bmatrix} -L_t \\ C - DL_t \end{bmatrix} x^*(t) \end{aligned} \tag{3}$$

a state feedback.

Define

$$\Phi =: \begin{bmatrix} \Phi_u & \Phi_{uy} \\ \Phi_{yu} & \Phi_y \end{bmatrix}.$$

The optimal input is a state feedback with time-varying gain

$$\begin{aligned} L_t := {} & (B^\top S_{t+1} B + \Phi_u + \Phi_{uy} D + D^\top \Phi_{uy}^\top + D^\top \Phi_y D)^{-1} \\ & \times (B^\top S_{t+1} A + \Phi_{uy} C + D^\top \Phi_y C) \end{aligned} \tag{4}$$

where S is given by the Riccati difference equation

$$\begin{aligned} S_t = {} & A^\top S_{t+1} A + C^\top \Phi_y C - (B^\top S_{t+1} A + \Phi_{uy} C + D^\top \Phi_y C)^\top \\ & \times (B^\top S_{t+1} B + \Phi_u + \Phi_{uy} D + D^\top \Phi_{uy}^\top + D^\top \Phi_y D)^{-1} \\ & \times (B^\top S_{t+1} A + \Phi_{uy} C + D^\top \Phi_y C), \qquad S_{T_\mathrm{r}} = 0. \end{aligned} \tag{5}$$

Solution using the impulse response representation

LQ tracking problem:

$$\min_{w_\mathrm{f}} (w_\mathrm{r} - w_\mathrm{f})^\top \Phi (w_\mathrm{r} - w_\mathrm{f}) \quad \text{subject to} \quad (w_\mathrm{ini}, w_\mathrm{f}) \in \mathscr{B}_{T_\mathrm{ini}+T_\mathrm{f}}$$

Let $y_{\mathrm{f},0}$ be the free response of $\mathscr{B}$ initiated by w_ini.

$$y_\mathrm{f} = y_{\mathrm{f},0} + \mathscr{T}_{T_\mathrm{r}}(h) u_\mathrm{f}$$

so that the tracking problem becomes

$$\min_{u_\mathrm{f}} (w_\mathrm{r} - w_\mathrm{f})^\top \Phi (w_\mathrm{r} - w_\mathrm{f}) \quad \text{subject to} \quad y_\mathrm{f} = \mathscr{T}_{T_\mathrm{r}}(h) u_\mathrm{f} + y_{\mathrm{f},0}$$

a weighted least squares problem.

With $\widetilde{h} := \mathrm{col}(I\delta, h)$,

$$w_\mathrm{f} := \Big(\mathrm{col}\,(u_\mathrm{f}(1), y_\mathrm{f}(1)), \ldots, \mathrm{col}\,(u_\mathrm{f}(T_\mathrm{r}), y_\mathrm{f}(T_\mathrm{r}))\Big) = \mathscr{T}_{T_\mathrm{r}}(\widetilde{h}) u_\mathrm{f}$$

Then

$$w_\mathrm{f} = \mathscr{T}_{T_\mathrm{r}}(\widetilde{h}) u_\mathrm{f} + w_{\mathrm{f},0}, \quad \text{where} \quad w_{\mathrm{f},0} := \mathrm{col}(0, y_{\mathrm{f},0})$$

The tracking problem becomes

$$\min_{u_\mathrm{f}} \left(w_\mathrm{r} - w_{\mathrm{f},0} - \mathscr{T}_{T_\mathrm{r}}(\widetilde{h}) u_\mathrm{f}\right)^\top \Phi \left(w_\mathrm{r} - w_{\mathrm{f},0} - \mathscr{T}_{T_\mathrm{r}}(\widetilde{h}) u_\mathrm{f}\right)$$

and the solution is

$$\begin{aligned} u_\mathrm{f}^* &= \left(\mathscr{T}_{T_\mathrm{r}}^\top(\widetilde{h}) \Phi \mathscr{T}_{T_\mathrm{r}}(\widetilde{h})\right)^{-1} \mathscr{T}_{T_\mathrm{r}}^\top(\widetilde{h}) \Phi (w_\mathrm{r} - w_{\mathrm{f},0}) \\ y_\mathrm{f}^* &= \mathscr{T}_{T_\mathrm{r}}(h) u_\mathrm{f}^* + y_{\mathrm{f},0} \end{aligned} \tag{6}$$

Ingredients of the solution:

- the free response $y_{\mathrm{f},0}$ and
- the impulse response h.

We can compute them directly from w_{d}.

Algorithm 8: data-driven LQ tracking, using impulse resp. repr.

1. $(w_{\text{ini}}, w_{\mathrm{d}}, T_{\mathrm{r}}) \xrightarrow{\text{Algorithm 2}} y_{\mathrm{f},0}$
2. $(w_{\mathrm{d}}, T_{\mathrm{r}}) \xrightarrow{\text{Algorithm 6}} h$
3. $(\Phi, w_{\mathrm{r}}, w_{\mathrm{f},0}, h) \xrightarrow{(6)} w_{\mathrm{f}}^*$

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Data-driven solution

Define the zero initial conditions subbehavior of $\mathscr{B}$

$$\mathscr{B}_{0,T_{\mathrm{r}}} := \left\{ w \in (\mathbb{R}^{\mathtt{w}})^{T_{\mathrm{r}}} \mid (\underbrace{0,\dots,0}_{\text{lag}(\mathscr{B})}, w) \in \mathscr{B}_{\text{lag}(\mathscr{B})+T_{\mathrm{r}}} \right\}$$

Theorem: Let $W_0 \in \mathbb{R}^{T_{\mathrm{r}}\mathtt{w} \times \bullet}$ be a matrix, such that

$$\text{image}\,(W_0) = \mathscr{B}_{0,T_{\mathrm{r}}}$$

Then the LQ optimal trajectory is

$$w_{\mathrm{f}}^* = W_0 \left(W_0^\top \Phi W_0\right)^+ W_0^\top \Phi (w_{\mathrm{r}} - w_{\mathrm{f},0}) + w_{\mathrm{f},0} \qquad (7)$$

where $w_{\mathrm{f},0}$ is the free response of $\mathscr{B}$, caused by w_{ini}.

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Proof

Any zero initial conditions trajectory $w = \text{col}(u, y) \in (\mathbb{R}^{\mathtt{w}})^{T_{\mathrm{r}}}$ is of the form $w = \mathscr{T}_{T_{\mathrm{r}}}(\widetilde{h})u$. Therefore,

$$\mathscr{B}_{0,T_{\mathrm{r}}} = \text{image}\left(\mathscr{T}_{T_{\mathrm{r}}}(\widetilde{h})\right) = \text{image}\,(W_0)$$

Consider the space $\mathscr{W} = (\mathbb{R}^{\mathtt{w}})^{T_{\mathrm{r}}}$ with inner product defined by $\langle w_1, w_2 \rangle = w_1^\top \Phi w_2$. The projector on $\mathscr{B}_{0,T_{\mathrm{r}}}$ in $\mathscr{W}$ is

$$\mathscr{T}_{T_{\mathrm{r}}}(\widetilde{h})\left(\mathscr{T}_{T_{\mathrm{r}}}^\top(\widetilde{h})\Phi\mathscr{T}_{T_{\mathrm{r}}}(\widetilde{h})\right)^{-1}\mathscr{T}_{T_{\mathrm{r}}}^\top(\widetilde{h})\Phi = W_0\left(W_0^\top \Phi W_0\right)^+ W_0^\top \Phi$$

Then the data-driven solution (7) follows from the solution (6), using the impulse response representation.

Algorithm for data-driven LQ tracking

Algorithm 9: data-driven LQ tracking

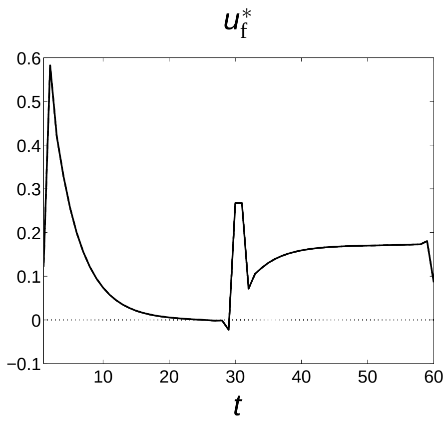
1. $(w_{\text{ini}}, w_{\mathrm{d}}, T_{\mathrm{r}}) \xrightarrow{\text{Algorithm 2}} y_{\mathrm{f},0}$
2. $(w_{\mathrm{d}}, T_{\mathrm{r}}) \xrightarrow{\text{Algorithm 4}} W_0$
3. $(\Phi, w_{\mathrm{r}}, y_{\mathrm{f},0}, W_0) \xrightarrow{(7)} w_{\mathrm{f}}^*$

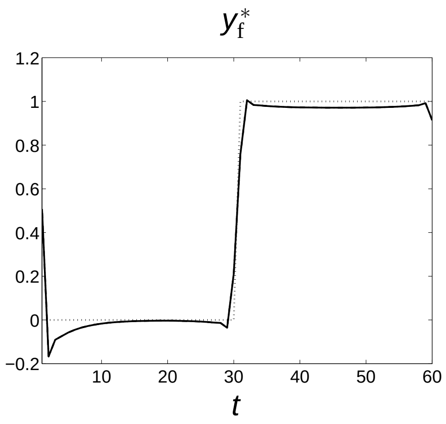
Simulation showing the equivalence of the three methods

$\mathscr{B}$	—	2nd order, $\mathtt{m} = 1$ input, $\mathtt{p} = 1$ output
w_{d}	—	random trajectory of $\mathscr{B}$ with $T = 200$ samples
Φ	—	identity (assign equal weights to the variables)

step tracking: $u_{\mathrm{r}} = 0, \quad y_{\mathrm{r}} = \text{step}, \quad w_{\text{ini}} = (1, 1)$

Simulation example





w_r — dotted line, w_f^* — solid line, $J(w_r, w_f^*) = 2.1034$

Conclusions

- Given $w_d \in \mathscr{B}_T$, we can compute feedforward LQ tracking control without deriving a repr. of $\mathscr{B}$. (data-driven control)
- For doing this we need
  - $\mathscr{B}$ to be controllable,
  - u_d to be persistently exciting of sufficient order.
- The construction of the optimal control is based on
  - free response $y_{f,0}$ of $\mathscr{B}$ under w_{ini}, and
  - zero ini. cond. trajectories W_0 (a basis for $\mathscr{B}_{0,T_r}$).

- Compared with previous work on subspace control, we
  - give conditions under which the problem is solvable,
  - relate the problem to data-driven simulation, and
  - derive new computational algorithms.

Thank you