Linear dynamic filtering with noisy input and output

Ivan Markovsky and Bart De Moor

ESAT-SCD (SISTA), K.U.Leuven, Belgium

Outline

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State estimation with noisy input and output

in this talk, we pose and answer the question:

how should we modify the Kalman filter when both the input and the output of the system are noisy?

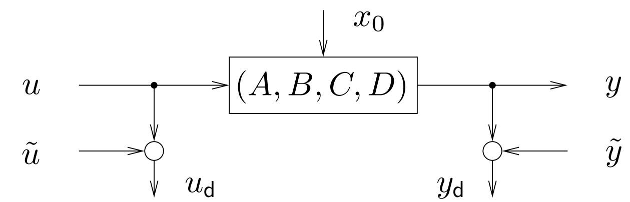
we consider the deterministic discrete-time LTI state-space system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \qquad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t), \qquad t = 0, 1, \ldots \end{aligned} \tag{1}$$

together with the measurement errors model

$$u_{\mathrm{d}}(t) = u(t) + \tilde{u}(t), \qquad y_{\mathrm{d}}(t) = y(t) + \tilde{y}(t) \tag{2}$$

State estimation with noisy input and output (cont.)



note: (1,2) is an errors-in-variables model when the problem is to estimate A, B, C, D but since our purpose is to estimate x, knowing A, B, C, D, we call it a noisy I/O model

related work is done by Guidorzi *et. al.*, the SISO noisy I/O estimation problem is solved and new algorithms are derived

our result is: the noisy I/O state estimation problem is a Kalman filtering problem, so that the optimal estimate is given by a Kalman filter

Problem formulation

define: $$u := [u^\top(0) \ \cdots \ u^\top(t_f-1)]^\top, \quad x := [x^\top(0) \ \cdots \ x^\top(t_f)]^\top$$
$$V := \operatorname{blk\,diag}\big(V(0),\dots,V(t_f-1)\big)$$

assume that the measurement errors $\tilde{u}$ and $\tilde{y}$ are random, centered, normal, uncorrelated, and white with known covariance matrices

$$\operatorname{cov}\big(\tilde{u}(t)\big) =: V_{\tilde{u}}(t), \qquad \operatorname{cov}\big(\tilde{y}(t)\big) =: V_{\tilde{y}}(t), \tag{3}$$

and that the initial condition x_0 is unknown

optimal noisy I/O smoothing problem:

$$\min_{\hat{u},\hat{y},\hat{x}} \left\| \begin{bmatrix} V_{\tilde{u}} & \\ & V_{\tilde{y}} \end{bmatrix}^{-\frac{1}{2}} \begin{bmatrix} \hat{u}-u_\mathrm{d} \\ \hat{y}-y_\mathrm{d} \end{bmatrix} \right\|^2 \quad \text{s.t.} \quad \begin{array}{r} \hat{x}(t+1) = A\hat{x}(t) + B\hat{u}(t) \\ \hat{y}(t) = C\hat{x}(t) + D\hat{u}(t) \\ \text{for } t = 0,1,\dots,t_f-1 \end{array} \tag{4}$$

optimal smoothed state estimate $\hat{x}(\cdot,t_f)$ is the solution of (4).

Problem formulation (cont.)

optimal noisy I/O filtering problem: find a dynamical system,

$$\begin{aligned} z(t+1) &= A_\mathrm{f}(t)z(t) + B_\mathrm{f}(t)\begin{bmatrix} u_\mathrm{d}(t) \\ y_\mathrm{d}(t) \end{bmatrix} \\ \hat{x}(t) &= C_\mathrm{f}(t)z(t) + D_\mathrm{f}(t)\begin{bmatrix} u_\mathrm{d}(t) \\ y_\mathrm{d}(t) \end{bmatrix} \end{aligned} \tag{5}$$

such that $\hat{x}(t) = \hat{x}(t,t+1)$, where $\hat{x}(\cdot)$ is the solution of (5), *i.e.*, the optimal filtered state estimate, and $\hat{x}(\cdot,t+1)$ is the optimal smoothed state estimate with a time horizon $t+1$

optimal noisy I/O filter is the solution (5) of the optimal noisy I/O filtering problem

Solution of the smoothing problem

the I/O dynamics of (1), for $0,\dots,t_f-1$, is given by $y = \Gamma x_0 + Tu$,

$$\Gamma := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{t_f-2} \end{bmatrix}, \quad T := \begin{bmatrix} H(0) & 0 & \cdots & 0 \\ H(1) & H(0) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ H(t_f-1) & H(t_f-2) & \cdots & H(0) \end{bmatrix}, \quad \begin{array}{c} H(0) = D, \\ H(t) = CA^{t-1}B, \\ \text{for } t = 1,2,\dots \end{array}$$

we see that the optimal noisy I/O smoothing problem (4) is

$$\min_{\hat{x}_0,\hat{u}} \left\| \begin{bmatrix} V_{\tilde{u}} & \\ & V_{\tilde{y}} \end{bmatrix}^{-\frac{1}{2}} \left(\begin{bmatrix} u_\mathrm{d} \\ y_\mathrm{d} \end{bmatrix} - \begin{bmatrix} 0 & I \\ \Gamma & T \end{bmatrix} \begin{bmatrix} \hat{x}_0 \\ \hat{u} \end{bmatrix} \right) \right\|^2, \tag{6}$$

i.e., a weighted least-squares problem

The modified Kalman filter

we convert the noisy I/O model (1,2) in the form

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + w(t) \\ y(t) &= Cx(t) + Du(t) + v(t) \end{aligned} \tag{7}$$

by substituting $u_\mathrm{d}(t) - \tilde{u}$ for $u(t)$ and $y_\mathrm{d}(t) - \tilde{y}$ for $y(t)$ in (1)

$$\begin{aligned} x(t+1) &= Ax(t) + Bu_\mathrm{d}(t) - B\tilde{u}(t) \\ y_\mathrm{d}(t) &= Cx(t) + Du_\mathrm{d}(t) - D\tilde{u}(t) + \tilde{y}(t) \end{aligned}$$

and defining (fake) process noise w and measurement noise v by

$$w := -B\tilde{u} \quad \text{and} \quad v := -D\tilde{u} + \tilde{y}$$

The modified Kalman filter (cont.)

the resulting system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu_{\mathsf{d}}(t) + w(t) \\ y_{\mathsf{d}}(t) &= Cx(t) + Du_{\mathsf{d}}(t) + v(t) \end{aligned} \tag{8}$$

is in the form (7), where

$$\mathbf{E} \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \begin{bmatrix} w^\top(t+\tau) & v^\top(t+\tau) \end{bmatrix} = \begin{bmatrix} -B & 0 \\ -D & I \end{bmatrix} \begin{bmatrix} V_u(t) & \\ & V_y(t) \end{bmatrix} \begin{bmatrix} -B & 0 \\ -D & I \end{bmatrix}^\top$$

we call the Kalman filter of the modified system (8) the modified Kalman filter

The modified Kalman filter ≡ optimal noisy I/O filter

in the paper, we prove by a linear algebra argument that the modified Kalman filter is indeed the optimal noisy I/O filter

Conclusion

- a new problem was reduced to a classical problem with known solution
- thus, one can take advantage of many numerically stable and reliable algorithms for Kalman filtering and use available software

note: these conclusions are in contrast to the results of Guidorzi *et. al.*, where the solution is not recognized as a (modified) Kalman filter and new algorithms for optimal noisy I/O filter are derived

References

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