The most powerful unfalsified model

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Outline

- Exact identification problems
- Identifiability conditions
- Algorithms
  - from data to kernel representation
  - impulse response identification
  - N4SID-type algorithms
  - MOESP-type algorithms

Exact identification problems
($w_\mathrm{d} \mapsto \mathscr{B}$ such that $w_\mathrm{d} \in \mathscr{B}$)

An exact identification problem

Problem P1 (Exact identification)

Given two vector time series

$$u_\mathrm{d} = \big(u_\mathrm{d}(1), \dots, u_\mathrm{d}(T)\big) \in (\mathbb{R}^\mathtt{m})^T \qquad \text{“inputs”}$$
$$y_\mathrm{d} = \big(y_\mathrm{d}(1), \dots, y_\mathrm{d}(T)\big) \in (\mathbb{R}^\mathtt{p})^T \qquad \text{“outputs”}$$

find $\mathtt{n} \in \mathbb{N}$ and LTI system $\mathscr{B}$ of order $\mathtt{n}$, with $\mathtt{m}$ inputs and $\mathtt{p}$ outputs, s.t.

$$w_\mathrm{d} := (u_\mathrm{d}, y_\mathrm{d}) \in \mathscr{B},$$

i.e., w_d is a trajectory of $\mathscr{B}$.

How can we check that “$w_\mathrm{d} \in \mathscr{B}$”?

Checking that $w_d \in \mathscr{B} = \ker\big(R(\sigma)\big)$

$$w_d \in \mathscr{B} \iff R(\sigma)w_d = 0$$

$$\iff R_0 w_d(t) + R_1 w_d(t+1) + \cdots + R_\ell w_d(t+\ell) = 0 \quad \text{for } t = 1,\dots,T-\ell$$

$$\iff \begin{bmatrix} R_0 & R_1 & \cdots & R_\ell & & & \\ & R_0 & R_1 & \cdots & R_\ell & & \\ & & \ddots & \ddots & & \ddots & \\ & & & R_0 & R_1 & \cdots & R_\ell \end{bmatrix} \begin{bmatrix} w_d(1) \\ w_d(2) \\ \vdots \\ w_d(T) \end{bmatrix} = 0$$

$$\iff \begin{bmatrix} R_0 & R_1 & \cdots & R_\ell \end{bmatrix} \begin{bmatrix} w_d(1) & w_d(2) & \cdots & w_d(T-\ell) \\ w_d(2) & w_d(3) & \cdots & \\ \vdots & \vdots & & \vdots \\ w_d(\ell+1) & w_d(\ell+2) & \cdots & w_d(T) \end{bmatrix} = 0$$

Checking that $w_d \in \mathscr{B} = \operatorname{image}\big(M(\sigma)\big)$

$$w_d \in \mathscr{B} \iff \text{there is } v, \text{ such that } w_d = M(\sigma)v$$

$$\iff \text{there is } v, \text{ such that for } t = 1,\dots,T$$
$$w_d(t) = M_0 v(t) + M_1 v(t+1) + \cdots + M_\ell v(t+\ell)$$

$$\iff \text{there is solution } v \text{ of the system}$$

$$\begin{bmatrix} w_d(1) \\ w_d(2) \\ \vdots \\ w_d(T) \end{bmatrix} = \begin{bmatrix} M_0 & M_1 & \cdots & M_\ell & & & \\ & M_0 & M_1 & \cdots & M_\ell & & \\ & & \ddots & \ddots & & \ddots & \\ & & & M_0 & M_1 & \cdots & M_\ell \end{bmatrix} \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(T+\ell) \end{bmatrix}$$

Checking that $w_d \in \mathscr{B} = \mathscr{B}_{\mathrm{i/s/o}}(A,B,C,D)$

Let $\mathscr{B}$ be defined by a minimal input/state/output representation

$$\mathscr{B} := \mathscr{B}_{\mathrm{i/s/o}}(A,B,C,D) = \{\, (u,y) \mid \sigma x = Ax + Bu,\ y = Cx + Du \,\}$$

$(u_d, y_d) \in \mathscr{B}_{\mathrm{i/s/o}}(A,B,C,D) \iff$ there exists $x_{\mathrm{ini}} \in \mathbb{R}^{\mathtt{n}}$, such that

$$y_d = \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{T-1} \end{bmatrix}}_{\mathscr{O}_T(A,C)} x_{\mathrm{ini}} + \begin{bmatrix} D & & & & \\ CB & D & & & \\ CAB & CB & D & & \\ \vdots & \ddots & \ddots & \ddots & \\ CA^{T-1}B & \cdots & CAB & CB & D \end{bmatrix} u_d$$

(y_d is the response of $\mathscr{B}$ under input u_d and initial condition x_{ini})

Comments

- P1 is an exact fitting problem, a most basic SYSID problem
- easily generalizable to a set of N time series
  $u_{d,1},\dots,u_{d,N} \in (\mathbb{R}^{\mathtt{m}})^T$ and $y_{d,1},\dots,y_{d,N} \in (\mathbb{R}^{\mathtt{p}})^T$
- the realization problem

  $$\text{impulse response} \quad \mapsto \quad (A,B,C,D)$$

  is a special case of P1 for a set of $\mathtt{m}$ time series
- while $\mathtt{m}$ is given, finding $\mathtt{n}$ is part of the problem
  any observable system of order $\mathtt{n} \geq \mathtt{p}T$ is a (trivial) solution
- we are actually interested is a solution of a minimal order

Revised exact identification problem

Problem P1' (Exact identification)

Given two vector time series

$$u_\mathrm{d} = \big(u_\mathrm{d}(1),\dots,u_\mathrm{d}(T)\big) \in (\mathbb{R}^\mathtt{m})^T \qquad \text{“inputs”}$$
$$y_\mathrm{d} = \big(y_\mathrm{d}(1),\dots,y_\mathrm{d}(T)\big) \in (\mathbb{R}^\mathtt{p})^T \qquad \text{“outputs”}$$

find the smallest $\mathtt{n} \in \mathbb{N}$ and LTI system $\mathscr{B}$ of order $\mathtt{n}$, with $\mathtt{m}$ inputs and $\mathtt{p}$ outputs, such that

$$w_\mathrm{d} = (u_\mathrm{d}, y_\mathrm{d}) \in \mathscr{B}.$$

Set of LTI systems with a bounded complexity

Notation: $\mathscr{L}^{\mathtt{w},\mathtt{n}}_{\mathtt{m},\ell}$ is the set of all LTI systems with

- $\mathtt{w}$ (external) variables
- at most $\mathtt{m}$ inputs
- minimal state dimension at most $\mathtt{n}$ and
- lag (= observability index) at most ℓ

For $t \geq \mathtt{n}$, the set $\mathscr{B}|_t$ of all t samples long traj. of $\mathscr{B}$ has dimension

$$\dim(\mathscr{B}|_t) \;\leq\; t\mathtt{m} + \mathtt{n} \;\leq\; t\mathtt{m} + \mathtt{p}\ell$$

(where $\mathtt{p}(\ell-1) \leq \mathtt{n} \leq \mathtt{p}\ell$)

$\Longrightarrow$ $(\mathtt{m},\mathtt{n})$ and $(\mathtt{m},\ell)$ specify the complexity of the model class $\mathscr{L}^{\mathtt{w},\mathtt{n}}_{\mathtt{m},\ell}$

Another exact identification problem

Problem P2 (Exact identification)

Given a vector time series

$$w_\mathrm{d} = \big(w_\mathrm{d}(1),\dots,w_\mathrm{d}(T)\big) \in (\mathbb{R}^\mathtt{w})^T$$

find the smallest $\mathtt{m} \in \mathbb{N}$ and $\ell \in \mathbb{N}$ and LTI system $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}_{\mathtt{m},\ell}$, s.t. $w_\mathrm{d} \in \mathscr{B}$.

Comments:

- no separation between inputs and outputs
- the complexity is defined by $(\mathtt{m},\ell)$

Most powerful unfalsified model

The most powerful unfalsified model in the model class $\mathscr{L}^{\mathtt{w}}_{\mathtt{m},\ell}$ of a time series $w_\mathrm{d} \in (\mathbb{R}^\mathtt{w})^T$ is the system $\mathscr{B}_\mathrm{mpum}$ that is

1. in the model class, *i.e.*, $\mathscr{B}_\mathrm{mpum} \in \mathscr{L}^{\mathtt{w}}_{\mathtt{m},\ell}$,
2. unfalsified, *i.e.*, $w_\mathrm{d} \in \mathscr{B}_\mathrm{mpum}|_T$, and
3. most powerful among all LTI unfalsified systems, *i.e.*,

$$\mathscr{B}' \in \mathscr{L}^{\mathtt{w}}_{\mathtt{m},\ell} \text{ and } w_\mathrm{d} \in \mathscr{B}'|_T \quad\Longrightarrow\quad \mathscr{B}_\mathrm{mpum}|_T \subseteq \mathscr{B}'|_T.$$

MPUM may not exist, but if it does, then it is unique

Identifiability

Identifiability question

P2 is the problem of computing the MPUM of w_d in $\mathscr{L}^\mathtt{w}$

The following related question is of interest:

Suppose that

$$w_\mathrm{d} \in \overline{\mathscr{B}} \in \mathscr{L}^\mathtt{w}$$

and upper bounds n_max, ℓ_max of the order n and lag ℓ of $\overline{\mathscr{B}}$ are given.

> Under what conditions $\mathscr{B}_\mathrm{mpum}(w_\mathrm{d})$ is equal to the system $\mathscr{B}$?

the answer is given by the following lemma

Fundamental Lemma

> Let $\overline{\mathscr{B}} \in \mathscr{L}^{\mathtt{w},\mathrm{n}}_\mathtt{m}$ be controllable and let $w_\mathrm{d} := (u_\mathrm{d}, y_\mathrm{d}) \in \overline{\mathscr{B}}|_T$.
>
> Then, if u_d is persistently exciting of order $L + \mathrm{n}$,
>
> $$\text{image}\left(\begin{bmatrix} w_\mathrm{d}(1) & w_\mathrm{d}(2) & w_\mathrm{d}(3) & \cdots & w_\mathrm{d}(T-L+1) \\ w_\mathrm{d}(2) & w_\mathrm{d}(3) & w_\mathrm{d}(4) & \cdots & w_\mathrm{d}(T-L+2) \\ w_\mathrm{d}(3) & w_\mathrm{d}(4) & w_\mathrm{d}(5) & \cdots & w_\mathrm{d}(T-L+3) \\ \vdots & \vdots & \vdots & & \vdots \\ w_\mathrm{d}(L) & w_\mathrm{d}(L+1) & w_\mathrm{d}(L+2) & \cdots & w_\mathrm{d}(T) \end{bmatrix}\right) = \overline{\mathscr{B}}|_L$$

$\Longrightarrow$ under the conditions of the FL, any L samples long response y of $\mathscr{B}$ can be obtained as $y = \mathscr{H}_L(y_\mathrm{d})g$, for certain g $\rightsquigarrow$ algorithms

$\Longrightarrow$ with $L = \ell_\mathrm{max} + 1$, the FL gives conditions for identifiability

Persistency of excitation

$u_\mathrm{d} = \big(u_\mathrm{d}(1), \dots, u_\mathrm{d}(T)\big)$ is persistently exciting of order L if

$$\mathscr{H}_L(u_\mathrm{d}) := \begin{bmatrix} u_\mathrm{d}(1) & u_\mathrm{d}(2) & u_\mathrm{d}(3) & \cdots & u_\mathrm{d}(T-L+1) \\ u_\mathrm{d}(2) & u_\mathrm{d}(3) & u_\mathrm{d}(4) & \cdots & u_\mathrm{d}(T-L+2) \\ u_\mathrm{d}(3) & u_\mathrm{d}(4) & u_\mathrm{d}(5) & \cdots & u_\mathrm{d}(T-L+3) \\ \vdots & \vdots & \vdots & & \vdots \\ u_\mathrm{d}(L) & u_\mathrm{d}(L+1) & u_\mathrm{d}(L+2) & \cdots & u_\mathrm{d}(T) \end{bmatrix} \quad \text{is full row rank}$$

System theoretic interpretation:

u_d is persistently exciting of order L $\iff$ there is no LTI system with # of inputs $< \mathtt{m}$ and lag $< L$ for which u_d is a trajectory

Algorithms for exact identification ($w_{\mathrm{d}} \mapsto$ representation of the MPUM)

Overview of algorithms

1. $w_{\mathrm{d}} \mapsto R(\xi)$
2. $w_{\mathrm{d}} \mapsto$ impulse response H
3. $w_{\mathrm{d}} \mapsto (A,B,C,D)$ (possibly balanced)
   3.1 $w_{\mathrm{d}} \mapsto R(\xi) \mapsto (A,B,C,D)$ or $w_{\mathrm{d}} \mapsto H \mapsto (A,B,C,D)$
   3.2 $w_{\mathrm{d}} \mapsto \mathscr{O}_{\ell_{\max}+1}(A,C) \mapsto (A,B,C,D)$
   3.3 $w_{\mathrm{d}} \mapsto \big(x_{\mathrm{d}}(1),\dots,x_{\mathrm{d}}(\mathtt{n}_{\max}+\mathtt{m}+1)\big) \mapsto (A,B,C,D)$

$\boxed{w_{\mathrm{d}} \mapsto R(\xi)}$

under the assumptions of the FL, image $\big(\mathscr{H}_{\ell_{\max}+1}(w_{\mathrm{d}})\big) = \mathscr{B}|_{\ell_{\max}+1}$

$\implies$ a basis for left ker $\big(\mathscr{H}_{\ell_{\max}+1}(w_{\mathrm{d}})\big)$ defines a kernel repr. of $\mathscr{B}$

let

$$\begin{bmatrix} \widetilde{R}_0 & \widetilde{R}_1 & \cdots & \widetilde{R}_{\ell_{\max}} \end{bmatrix} \mathscr{H}_{\ell_{\max}+1}(w_{\mathrm{d}}) = 0, \quad \text{where } \widetilde{R}_i \in \mathbb{R}^{g\times \mathtt{w}}$$

and define $\widetilde{R}(\xi) = \sum_{i=0}^{\ell_{\max}} \xi^i \widetilde{R}_i$

then $\mathscr{B} = \ker\big(\widetilde{R}(\sigma)\big)$ is, in general, a nonminimal kernel representation

$\boxed{w_{\mathrm{d}} \mapsto R(\xi)}$

$\widetilde{R}$ can be made minimal by standard polynomial linear algebra alg.
find a unimodular matrix U, such that

$$U\widetilde{R} = \begin{bmatrix} R \\ 0 \end{bmatrix} \quad \text{and } R \text{ is full row rank}$$

then $\ker\big(R(\sigma)\big) = 0$ is minimal

Refinements:

- efficient recursive computation (exploiting the Hankel structure)
- as a byproduct find an input/output partition of the variables
- find a shortest lag kernel representation (*i.e.*, R row proper)

$w_{\mathrm{d}} \mapsto H$

Under the conditions of FL, there is G, such that $H = \mathscr{H}_t(y_{\mathrm{d}})G$

the problem reduces to the one of finding a particular G. Define

$$\begin{bmatrix}\mathscr{H}_{\ell_{\max}+t}(u_{\mathrm{d}})\\ \mathscr{H}_{\ell_{\max}+t}(y_{\mathrm{d}})\end{bmatrix} =: \begin{bmatrix}U_{\mathrm{p}}\\ U_{\mathrm{f}}\\ Y_{\mathrm{p}}\\ Y_{\mathrm{f}}\end{bmatrix} \qquad \begin{array}{lclcl} \operatorname{row dim}(U_{\mathrm{p}}) & = & \operatorname{row dim}(Y_{\mathrm{p}}) & = & \ell_{\max}\\ \operatorname{row dim}(U_{\mathrm{f}}) & = & \operatorname{row dim}(Y_{\mathrm{f}}) & = & t \end{array}$$

Let u_{d} be p.e. of order $t+\ell_{\max}+\mathrm{n}_{\max}$. Then there is G, such that

$$\begin{array}{rcll} \begin{bmatrix}U_{\mathrm{p}}\\ Y_{\mathrm{p}}\\ U_{\mathrm{f}}\end{bmatrix} G & = & \begin{bmatrix}0\\ 0\\ \begin{bmatrix}I_m\\ 0\end{bmatrix}\end{bmatrix} & \begin{array}{l}\left.\begin{array}{l}\\ \\ \end{array}\right\} \text{zero ini. conditions}\\ \leftarrow \quad \text{impulse input}\end{array} \\ {\color{red}Y_{\mathrm{f}} \quad G} & {\color{red}=} & {\color{red}H} & \end{array} \qquad (1)$$

$w_{\mathrm{d}} \mapsto H$

Block algorithm for computation of $\big(H(0),\dots,H(t-1)\big)$:

1. Input: u_{d}, y_{d}, $\ell_{\max}$, and t.
2. Solve the system of eqs (1). Let $\bar{G}$ be the computed solution.
3. Compute $H = Y_{\mathrm{f}}\bar{G}$.
4. Output: the first t samples of the impulse response H.

Refinements:

- solve (1) efficiently by exploiting the Hankel structure
- do the computations iteratively for pieces of H $\rightsquigarrow$ iterative alg.
- automatically choose t, for a sufficient decay of H

$w_{\mathrm{d}} \mapsto (A,B,C,D)$

- $w_{\mathrm{d}} \mapsto H(0:2\ell_{\max})$ or $R(\xi) \xrightarrow{\text{realization}} (A,B,C,D)$
- $w_{\mathrm{d}} \mapsto \mathscr{O}_{\ell_{\max}+1}(A,C) \xrightarrow{(2)} (A,B,C,D)$
- $w_{\mathrm{d}} \mapsto \big(x_{\mathrm{d}}(1),\dots,x_{\mathrm{d}}(\mathrm{n}_{\max}+\mathtt{m}+1)\big) \xrightarrow{(3)} (A,B,C,D)$

(2) and (3) are easy:

$$\mathscr{O}_{\ell_{\max}+1}(A,C) \mapsto (A,C) \quad \text{and} \quad (u_{\mathrm{d}},y_{\mathrm{d}},A,C) \mapsto (B,C,x_{\mathrm{ini}}) \qquad (2)$$

$$\begin{bmatrix}x_{\mathrm{d}}(2) & \cdots & x_{\mathrm{d}}(\mathrm{n}_{\max}+\mathtt{m}+1)\\ y_{\mathrm{d}}(1) & \cdots & y_{\mathrm{d}}(\mathrm{n}_{\max}+\mathtt{m})\end{bmatrix} = \begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}x_{\mathrm{d}}(1) & \cdots & x_{\mathrm{d}}(\mathrm{n}_{\max}+\mathtt{m})\\ u_{\mathrm{d}}(1) & \cdots & u_{\mathrm{d}}(\mathrm{n}_{\max}+\mathtt{m})\end{bmatrix} \qquad (3)$$

$\mathscr{O}_{\ell_{\max}+1}(A,C) \mapsto (A,B,C,D)$

C is the first block entry of $\mathscr{O}_{\ell_{\max}+1}(A,C)$ and A is given by

$$\big(\sigma^*\mathscr{O}_{\ell_{\max}+1}(A,C)\big)A = \big(\sigma\mathscr{O}_{\ell_{\max}+1}(A,C)\big) \qquad \text{shift equation}$$

(σ^* removes the last block entry and σ removes the first block entry)

Once C and A are known, the system of equations

$$y_{\mathrm{d}}(t) = CA^t x_{\mathrm{d}}(1) + \sum_{\tau=1}^{t-1} CA^{t-1-\tau}Bu_{\mathrm{d}}(\tau) + D\delta(t+1), \text{ for } t = 1,\dots,\ell_{\max}+1$$

is linear in D, B, $x_{\mathrm{d}}(1)$ (can be solved using Kronecker products)

$w_d \mapsto \mathscr{O}_{\ell_{max}+1}(A,C)$

The columns of $\mathscr{O}_{\ell_{max}+1}(A,C)$ are n linearly indep. free responses of $\mathscr{B}$

Under the conditions of FL, such resp. can be computed from data

$$\begin{bmatrix} \mathscr{H}_t(u_d) \\ \mathscr{H}_t(y_d) \end{bmatrix} G = \begin{bmatrix} 0 \\ Y_0 \end{bmatrix} \quad \begin{array}{l} \leftarrow \quad \text{zero inputs} \\ \leftarrow \quad \text{free responses} \end{array}$$

in order to obtain lin. indep. free responses, G should be maximal rank

Once we have a maximal rank matrix of free responses Y_0

$$Y_0 = \mathscr{O}_{\ell_{max}+1}(A,C) \underbrace{\begin{bmatrix} x_{\text{ini},1} & \cdots & x_{\text{ini},j} \end{bmatrix}}_{X_{\text{ini}}} \qquad \text{rank revealing factorization}$$

$\rightsquigarrow$ $\mathscr{O}_{\ell_{max}+1}(A,C)$ and X_{ini}, the factorization fixes the state space basis

$w_d \mapsto \big(x_d(1),\dots,x_d(n_{max}+m+1)\big)$

If the free responses are sequential, *i.e.*, if Y_0 is block-Hankel, then X_{ini} is a state sequence of $\mathscr{B}$

Computation of sequential free responses is achieved as follows

$$\underbrace{\begin{bmatrix} U_p \\ Y_p \\ U_f \end{bmatrix}}_{Y_f} G = \underbrace{\begin{bmatrix} U_p \\ Y_p \\ 0 \end{bmatrix}}_{Y_0} \quad \begin{array}{l} \left.\right\} \text{sequential ini. conditions} \\ \leftarrow \quad \text{zero inputs} \end{array} \tag{4}$$

Note: now we use the splitting of the data into “past” and “future”

$$Y_0 = \mathscr{O}_{\ell_{max}+1}(A,C) \begin{bmatrix} x_d(1) & \cdots & x_d(n_{max}+m+1) \end{bmatrix} \qquad \text{rank revealing factorization}$$

Refinements

- Solve (4) efficiently exploiting the Hankel structure
- Iteratively compute pieces of Y_0

  $\rightsquigarrow$ iterative algorithm

  - requires smaller persistency of excitation of u_d
  - could be more efficient

(Solve a few smaller systems of eqns instead of a single bigger one)

MOESP type algorithms

Orth. projection of the rows of $\mathscr{H}_{n_{max}}(y_d)$ on $\Big(\text{rowspan}\big(\mathscr{H}_{n_{max}}(u_d)\big)\Big)^{\perp}$

$$Y_0 := \mathscr{H}_{n_{max}}(y_d)\Pi^{\perp}_{u_d}$$

where

$$\Pi^{\perp}_{u_d} := \Big(I - \mathscr{H}^{\top}_{n_{max}}(u_d)\big(\mathscr{H}_{n_{max}}(u_d)\mathscr{H}^{\top}_{n_{max}}(u_d)\big)^{-1}\mathscr{H}_{n_{max}}(u_d)\Big)$$

Observe that $\Pi^{\perp}_{u_d}$ is maximal rank and

$$\begin{bmatrix} \mathscr{H}_{n_{max}}(u_d) \\ \mathscr{H}_{n_{max}}(y_d) \end{bmatrix} \Pi^{\perp}_{u_d} = \begin{bmatrix} 0 \\ Y_0 \end{bmatrix}$$

$\Longrightarrow$ the orthogonal projection computes free responses

Comments

- $T - \mathrm{n}_{\max} + 1$ free responses are computed via the orth. proj. while $\mathrm{n}_{\max}$ such responses suffice for the purpose of exact identification
- The orth. proj. is a geometric operation, whose system theoretic meaning is not revealed
- The condition for $\mathrm{rank}(Y_0) = \mathrm{n}$, given in the MOESP literature,

$$\mathrm{rank}\left(\begin{bmatrix} X_{\text{ini}} \\ \mathscr{H}_{\mathrm{n}_{\max}}(u_{\mathrm{d}}) \end{bmatrix}\right) = \mathrm{n} + \mathrm{n}_{\max}\mathtt{m}$$

is not verifiable from the data $(u_{\mathrm{d}}, y_{\mathrm{d}}) \implies$ can not be checked whether the computation gives $\mathscr{O}(A, C)$, *cf.*, p.e. condition of FL

N4SID-type algorithms

Consider the splitting of the data into "past" and "future"

$$\mathscr{H}_{2\mathrm{n}_{\max}}(u_{\mathrm{d}}) =: \begin{bmatrix} U_{\mathrm{p}} \\ U_{\mathrm{f}} \end{bmatrix}, \qquad \mathscr{H}_{2\mathrm{n}_{\max}}(y_{\mathrm{d}}) =: \begin{bmatrix} Y_{\mathrm{p}} \\ Y_{\mathrm{f}} \end{bmatrix}$$

with $\mathrm{row\,dim}(U_{\mathrm{p}}) = \mathrm{row\,dim}(U_{\mathrm{f}}) = \mathrm{row\,dim}(Y_{\mathrm{p}}) = \mathrm{row\,dim}(Y_{\mathrm{f}}) = \mathrm{n}_{\max}$ and let

$$W_{\mathrm{p}} := \begin{bmatrix} U_{\mathrm{p}} \\ Y_{\mathrm{p}} \end{bmatrix}$$

The key step of the N4SID algorithms is the oblique projection of the rows of Y_{f} along $\mathrm{row\,span}(U_{\mathrm{f}})$ onto $\mathrm{row\,span}(W_{\mathrm{p}})$

$$Y_0 := Y_{\mathrm{f}}/_{U_{\mathrm{f}}} W_{\mathrm{p}} := Y_{\mathrm{f}} \underbrace{\begin{bmatrix} W_{\mathrm{p}}^\top & U_{\mathrm{f}}^\top \end{bmatrix} \begin{bmatrix} W_{\mathrm{p}} W_{\mathrm{p}}^\top & W_{\mathrm{p}} U_{\mathrm{f}}^\top \\ U_{\mathrm{f}} W_{\mathrm{p}}^\top & U_{\mathrm{f}} U_{\mathrm{f}}^\top \end{bmatrix}^{+} \begin{bmatrix} W_{\mathrm{p}} \\ 0 \end{bmatrix}}_{\Pi_{\text{obl}}}$$

N4SID-type algorithms

Observe that

$$\begin{bmatrix} W_{\mathrm{p}} \\ U_{\mathrm{f}} \\ Y_{\mathrm{f}} \end{bmatrix} \Pi_{\text{obl}} = \begin{bmatrix} W_{\mathrm{p}} \\ 0 \\ Y_0 \end{bmatrix}$$

(in fact Π_{obl} is the least-norm, least-squares solution)

$\implies$ the oblique projection computes sequential free responses

Comments

- $T - 2\mathrm{n}_{\max} + 1$ sequential free responses are computed via the oblique projection while $\mathrm{n}_{\max} + \mathtt{m} + 1$ such responses suffice for exact ident.
- The oblique proj. is a geometric operation, whose system theoretic meaning is not revealed
- The conditions for $\mathrm{rank}(Y_0) = \mathrm{n}$, given in the N4SID literature,
  1. u_{d} persistently exciting of order $2\mathrm{n}_{\max}$ and
  2. $\mathrm{row\,span}(X_{\text{ini}}) \cap \mathrm{row\,span}(U_{\mathrm{f}}) = \{0\}$

  are not verifiable from the data $(u_{\mathrm{d}}, y_{\mathrm{d}})$

- System theoretic interpretation of the orth. and oblique proj.
- MOESP and N4SID alg. are computationally inefficient; more than what is necessary for exact ident. is computed $\rightsquigarrow$ cheaper algs
- The FL gives conditions for identifiability, verifiable from the data
- We clarified the role of the splitting: the "past" assigns the initial conditions and in the "future" a desired response is computed

$\Longrightarrow$ "past" should be chosen at least ℓ samples long; the length of "future" is free as long as the p.e. condition is satisfied

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Software

A MATLAB toolbox for exact SYSID is available from:

`ftp.esat.kuleuven.be/pub/SISTA/markovsky/abstracts/05-122.html`

In exercise 2 you will use the algorithms

- $w_\mathrm{d} \mapsto R(\xi)$ (`w2r`) and
- $w_\mathrm{d} \mapsto \big(x_\mathrm{d}(1),\dots,x_\mathrm{d}(\mathtt{n}_{\max}+\mathtt{m}+1)\big) \mapsto (A,B,C,D)$ (`uy2x2ss`)

in order to find the MPUM for given trajectory of an LTI system.