A new method for the computation of the STLS estimator

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Outline

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Motivation

existence of a linear model, explaining the data, is equivalent to the rank deficiency of a certain matrix containing the data

static linear model: data $C =: [A\ B] \in \mathbb{R}^{m\times(n+d)}$, model $AX = B$

existence of a static linear model $AX = B$ implies that $\text{rank}(C) = n$

for noisy data $C = \bar{C} + \tilde{C}$, $\text{rank}(C) = n + d$, so that, there is no linear model $AX = B$ explaining the data.

assuming $\text{vec}(\tilde{C}) \sim \text{N}(0, \alpha W)$, the maximum likelihood estimator is

$$\min_{\Delta C, \hat{X}} \ \text{vec}^\top(\Delta C)\, W^{-1} \text{vec}(\Delta C) \quad \text{s.t.} \quad (C - \Delta C) \begin{bmatrix} \hat{X} \\ -I \end{bmatrix} = 0$$

note: unstructured problem

Motivation (cont.)

linear dynamic model: data $w_0, \ldots, w_{t_f}$, auto regressive model of order n

$$w_t + H_1 w_{t-1} + \cdots + H_n w_{t-n} = 0$$

existence of an auto regressive model is equivalent to $\text{rank}\big(\mathcal{H}(w)\big) = n$

$$\mathcal{H}(w) = \begin{bmatrix} w_0^\top & w_1^\top & \cdots & w_n^\top \\ w_1^\top & w_2^\top & \cdots & w_{n+1}^\top \\ \vdots & \vdots & & \vdots \\ w_{t_f-n}^\top & w_{t_f-n+1}^\top & \cdots & w_{t_f}^\top \end{bmatrix}$$

assuming $w_t = \bar{w}_t + \tilde{w}_t$, where $\bar{w}_t$ is generated by an AR model of order n, and $\tilde{w}_t \sim \text{N}(0, \alpha W_k)$, the maximum likelihood estimation problem is

$$\min_{\Delta w, \hat{X}} \ \Delta w^\top \text{blk diag}^{-1}(W_0, \ldots, W_{t_f}) \Delta w \quad \text{s.t.} \quad \mathcal{H}(w - \Delta w) \begin{bmatrix} \hat{X} \\ -I \end{bmatrix} = 0$$

note: structured problem

Problem formulation

linear structured multivariate errors-in-variables (EIV) model

$$AX \approx B, \qquad A = \bar{A} + \tilde{A}, \qquad B = \bar{B} + \tilde{B}, \qquad \bar{A}\bar{X} = \bar{B}$$

$\begin{array}{l} A \in \mathbb{R}^{m\times n} \\ B \in \mathbb{R}^{m\times d} \end{array}$, $nd \ll m$, are observations, and $X \in \mathbb{R}^{n\times d}$ is a parameter

$$\begin{aligned} C &:= \begin{bmatrix} A & B \end{bmatrix} = \mathcal{S}(p) \\ \bar{C} &:= \begin{bmatrix} \bar{A} & \bar{B} \end{bmatrix} = \mathcal{S}(\bar{p}) \\ \tilde{C} &:= \begin{bmatrix} \tilde{A} & \tilde{B} \end{bmatrix} = \textstyle\sum_{l=1}^{n_p} S_l \tilde{p}_l \end{aligned} \qquad \text{where} \qquad \begin{aligned} &\mathcal{S}(p) = S_0 + \textstyle\sum_{l=1}^{n_p} S_l p_l \\ &\mathbf{E}\,\tilde{p} = 0, \quad \mathbf{E}\,\tilde{p}\tilde{p}^\top = \alpha W \end{aligned}$$

STLS problem: $\min_{X,\Delta p} \; \Delta p^\top W^{-1} \Delta p \quad \text{s.t.} \quad \mathcal{S}(p - \Delta p) \begin{bmatrix} X \\ -I_d \end{bmatrix} = 0$

Existing methods

general notes:

- non-convex optimization problem
- the existing methods are local optimization algorithms

approaches:

- constrained total least squares (CTLS) approach
- structured total least norm (STLN) approach
- Riemannian singular value decomposition (RiSVD)
- alternating projections

Existing methods (cont.)

new approach: derive an equivalent optimization problem $\min_X f_0(X)$ and apply standard optimization methods to solve it

our aim: efficient cost function $f(X)$ and first derivative $f'(X)$ evaluation possible in $O(m)$ operations for the case:

$$C = \begin{bmatrix} C^{(1)} & \cdots & C^{(q)} \end{bmatrix}, \quad \text{where } C^{(l)},\ l = 1, \ldots, q, \text{ is} \qquad \text{(A.1)}$$

block-Toeplitz, block-Hankel, unstructured, or noise free,

the repeated blocks in all block-Toeplitz/Hankel blocks $C^{(l)}$ are $n_y \times n_u$

$$W = I \quad \textit{i.e.} \quad \mathbf{E}\,\tilde{p}\tilde{p}^\top = \sigma^2 I \qquad \text{(A.2)}$$

the noise variance σ^2 need not be known

Equivalent optimization problem

elimination of the correction Δp by analytically minimizing over it

$$f_0(X) := \arg\min_{\Delta p} \Delta p^\top W^{-1} \Delta p \quad \text{s.t.} \quad \mathcal{S}(p - \Delta p) \left[\begin{smallmatrix} X \\ -I \end{smallmatrix}\right] = 0 \qquad (1)$$

define $R(X) := AX - B = C \left[\begin{smallmatrix} X \\ -I \end{smallmatrix}\right]$,

$$r(X) := \operatorname{vec}\big(R^\top(X)\big) = \operatorname{vec}\big(\begin{bmatrix} r_1(X) & \cdots & r_m(X) \end{bmatrix} \big) = \begin{bmatrix} r_1(X) \\ \vdots \\ r_m(X) \end{bmatrix}$$

and random part $\tilde{R} := R - \mathbf{E}\,R = \tilde{A}X - \tilde{B} = \tilde{C} X_{\text{ext}}$ of the residual

$\mathcal{S}$ is affine, so that the constraint of (1) is linear in Δp

$$\mathcal{S}(p - \Delta p) X_{\text{ext}} = 0 \iff r(X) = G(X)\Delta p,$$

Equivalent optimization problem (cont.)

where $G(X) := \big[\text{vec}\big((S_1 X_{\text{ext}})^\top\big) \;\; \cdots \;\; \text{vec}\big((S_{n_p} X_{\text{ext}})^\top\big)\big]$

then (1) is a least norm problem and its solution is

$$\Delta p_{\text{min}}(X) = W G^\top(X)\big(G(X) W G^\top(X)\big)^{-1} r(X)$$

$$\begin{aligned} f_0(X) &= \Delta p_{\text{min}}^\top(X) W^{-1} \Delta p_{\text{min}}(X) \\ &= r^\top(X)\big(G(X)WG^\top(X)\big)^{-1} r(X) =: r^\top(X)\Gamma^{-1}(X) r(X) \end{aligned}$$

we have $\sigma^2 G(X) W G^\top(X) = \mathbf{E}\,\big(G(X)\tilde{p}\big)\big(G(X)\tilde{p}\big)^\top$,
but $\tilde{r}(X) = \text{vec}\big(\tilde{R}^\top(X)\big) = G(X)\tilde{p}$, so that

$$\Gamma(X) = G(X) W G^\top(X) = \frac{1}{\sigma^2}\mathbf{E}\,\tilde{r}(X)\tilde{r}^\top(X) =: \frac{1}{\sigma^2} V_{\tilde{r}}(X)$$

Notation

with $\mathbf{m} := m/n_y$, define

$$\tilde{C}^\top =: \big[\tilde{C}_1 \;\; \cdots \;\; \tilde{C}_{\mathbf{m}}\big] =: \begin{bmatrix} \tilde{C}_1^{(1)} & \cdots & \tilde{C}_{\mathbf{m}}^{(1)} \\ \vdots & & \vdots \\ \tilde{C}_1^{(q)} & \cdots & \tilde{C}_{\mathbf{m}}^{(q)} \end{bmatrix}, \quad \text{where} \quad \begin{array}{l} \tilde{C}_i \in \mathbb{R}^{(n+d)\times n_y} \\ \tilde{C}_i^{(l)} \in \mathbb{R}^{n_l \times n_y} \end{array}$$

let $V_{\tilde{\mathbf{c}},ij} := \mathbf{E}\,\tilde{\mathbf{c}}_i \tilde{\mathbf{c}}_j^\top$, where $\tilde{\mathbf{c}}_i := \text{vec}(\tilde{C}_i^\top)$

$V_{\tilde{\mathbf{c}},ij}$ is the (i,j)-th block of $V_{\tilde{c}} := \mathbf{E}\,\tilde{c}\tilde{c}^\top$, where $\tilde{c} := \text{vec}(\tilde{C}^\top)$

$$V_{\tilde{\mathbf{c}},k} := V_{\tilde{\mathbf{c}},k1} \qquad \text{and} \qquad W_{\tilde{\mathbf{c}},k} := \frac{1}{\sigma^2} V_{\tilde{\mathbf{c}},k}, \quad \text{for } k = 1, \ldots, s$$

occasionally we drop the explicit dependence of r and Γ on X

Properties of the weight matrix Γ

Theorem Assume that (A.1) and (A.2) hold, then

$$\Gamma(X) = \begin{bmatrix} \Gamma_0 & \Gamma_{-1} & \cdots & \Gamma_{-s} & & 0 \\ \Gamma_1 & \ddots & \ddots & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \Gamma_{-s} \\ \Gamma_s & \ddots & \ddots & \ddots & \ddots & \vdots \\ & \ddots & \ddots & \ddots & \ddots & \Gamma_{-1} \\ 0 & & \Gamma_s & \cdots & \Gamma_1 & \Gamma_0 \end{bmatrix}, \tag{2}$$

where $\Gamma_k(X) = \Gamma_{-k}^\top(X) = (I_{n_y} \otimes X_{\text{ext}}^\top)\, W_{\tilde{\mathbf{c}},k}\, (I_{n_y} \otimes X_{\text{ext}}^\top)^\top$ and

$$s := \max_{l \in \{1,\ldots,q\}} \left\{ \frac{\mathcal{D}_l(2)}{n_u} \;:\; \mathcal{D}_l(1) = \mathtt{T} \text{ or } \mathcal{D}_l(1) = \mathtt{H} \right\} - 1, \tag{3}$$

for structured problems and $s := 0$ for unstructured problems.

Proof By definitions $\Gamma := \mathbf{E}\,\tilde{r}\tilde{r}^\top/\sigma^2$, and $\tilde{r} := \text{vec}\big(X_{\text{ext}}^\top \tilde{C}^\top\big)$

$$\begin{aligned} \Gamma_{ij}(X) &= \frac{1}{\sigma^2}\mathbf{E}\; \text{vec}(X_{\text{ext}}^\top \tilde{C}_i)\, \text{vec}^\top(X_{\text{ext}}^\top \tilde{C}_j) \\ &= \frac{1}{\sigma^2}(I_{n_y} \otimes X_{\text{ext}}^\top)\, V_{\tilde{\mathbf{c}},ij}\, (I_{n_y} \otimes X_{\text{ext}}^\top)^\top. \end{aligned}$$

Next we analyze the structure of $V_{\tilde{c}} := \mathbf{E}\,\text{vec}(\tilde{C}^\top)\text{vec}^\top(\tilde{C}^\top)$.

$\tilde{C} = \mathcal{S}(\tilde{p})$ is a function of $\tilde{p}$. By assumption (A.1), any element $\tilde{C}_{ij}$ is equal to an element of $\tilde{p}$ (or 0 if C_{ij} is noise free). We write explicitly the common elements of $\tilde{p}$, between two block rows $\tilde{C}_i$ and $\tilde{C}_j$, $i \geq j$.

For certain matrix Z_{ij}, to be specified, we have

$$\tilde{C}_i = Z_{ij}\tilde{C}_j + \tilde{O}_{ij},$$

where $\tilde{O}_{ij}$ contains the elements of $\tilde{p}$ that are present in $\tilde{C}_i$ but not in $\tilde{C}_j$. By assumption (A.2), $\mathbf{E}\,\tilde{p}\tilde{p}^\top = \sigma^2 I$, we have

$$V_{\tilde{\mathbf{c}},ij} = \mathbf{E}\,\tilde{\mathbf{c}}_i\tilde{\mathbf{c}}_j^\top = \mathbf{E}\left((I_{n_y}\otimes Z_{ij})\tilde{\mathbf{c}}_j + \text{vec}(\tilde{O}_{ij})\right)\tilde{\mathbf{c}}_j^\top = \sigma^2(I_{n_y}\otimes Z_{ij}). \quad (4)$$

$$\left(\tilde{\mathbf{c}}_i = \text{vec}(\tilde{C}_i) = \text{vec}(Z_{ij}\tilde{C}_j + \tilde{O}_{ij})\right)$$

We specify Z_{ij}. If $\tilde{p}_i$ appears in the block $C^{(l)}$, then it does not appear in any other block $C^{(k)}$, $k \neq l$. With $\mathbf{E}\,\tilde{p}\tilde{p}^\top = \sigma^2 I$, this implies that

$$Z_{ij} = \text{blk diag}(Z_{ij}^{(1)}, \ldots, Z_{ij}^{(q)}),$$

where $Z_{ij}^{(l)}$ depends only on the type of structure of the block $C^{(l)}$. Thus we analyze $Z_{ij}^{(l)}$, for the four basic structures of assumption (A.1).

Define the $n_l \times n_l$ shift matrix $J_{n_l} := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$. Then

$$Z_{ij}^{(l)} = \begin{cases} J_{n_l}^{n_u(i-j)\top} & , \text{ if } C^{(l)} \text{ is Toeplitz} \\ J_{n_l}^{n_u(i-j)} & , \text{ if } C^{(l)} \text{ is Hankel} \\ \delta(i-j)I_{n_l} & , \text{ if } C^{(l)} \text{ is unstructured} \\ 0_{n_l} & , \text{ if } C^{(l)} \text{ is noise free,} \end{cases} \quad \text{for } i \geq j. \quad (5)$$

Substituting back in (4), we see that $V_{\tilde{c},ij}$ depends only on the difference $k = i - j$. Thus $V_{\tilde{c}}$ is block-Toeplitz.

In addition, from the definition (3) of s, we have

$$s \geq \frac{n_l}{n_u} - 1, \quad \text{for all } l \implies n_u(s+1) \geq n_l, \quad \text{for all } l$$

$$\implies J_{n_l}^{n_u k} = 0, \quad \text{for all } l \text{ and for all } k > s.$$

Thus $Z_{ij} = 0$, for $i - j > s$ and $V_{\tilde{c}}$ is block-banded. □

check $\tilde{C}_i = Z_{ij}\tilde{C}_i + O_{ij}$ on a Hankel block with $n_l = 4$, $n_u = 1$

$$\tilde{C} = \begin{bmatrix} \tilde{p}_1 & \tilde{p}_2 & \tilde{p}_3 & \tilde{p}_4 \\ \tilde{p}_2 & \tilde{p}_3 & \tilde{p}_4 & \tilde{p}_5 \\ \tilde{p}_3 & \tilde{p}_4 & \tilde{p}_5 & \tilde{p}_6 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$i = 2,\ j = 1: \quad \begin{bmatrix} \tilde{p}_2 \\ \tilde{p}_3 \\ \tilde{p}_4 \\ \tilde{p}_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \\ \tilde{p}_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \tilde{p}_5 \end{bmatrix} \implies Z_{21} = J_4$$

$$i = 3,\ j = 1: \quad \begin{bmatrix} \tilde{p}_3 \\ \tilde{p}_4 \\ \tilde{p}_5 \\ \tilde{p}_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \\ \tilde{p}_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \tilde{p}_5 \\ \tilde{p}_6 \end{bmatrix} \implies Z_{31} = J_4^2$$

another example with a block Hankel matrix

check $\tilde{C}_i = Z_{ij}\tilde{C}_i + O_{ij}$ on a Hankel block with $n_l = 4$, $n_u = 2$

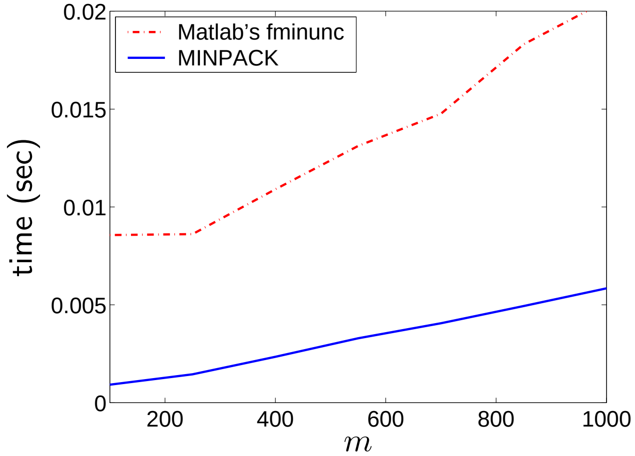
$$\tilde{C} = \begin{bmatrix} \tilde{p}_1 & \tilde{p}_2 & \tilde{p}_3 & \tilde{p}_4 \\ \tilde{p}_3 & \tilde{p}_4 & \tilde{p}_5 & \tilde{p}_6 \\ \tilde{p}_5 & \tilde{p}_6 & \tilde{p}_7 & \tilde{p}_8 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$i = 2,\ j = 1: \quad \begin{bmatrix} \tilde{p}_3 \\ \tilde{p}_4 \\ \tilde{p}_5 \\ \tilde{p}_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \\ \tilde{p}_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \tilde{p}_5 \\ \tilde{p}_6 \end{bmatrix} \implies Z_{21} = J_4^2$$

Simulation example

C—Hankel, $m = 100, \ldots, 1000$, $n = 2$, $\sigma = 0.01$, $N = 10$ repetitions

average execution time as a function of m



Conclusion

- efficient (*i.e.*, $O(m)$) local optimization methods for STLS estimation
- deal with flexible structure specification $C = [C^{(1)} \cdots C^{(q)}]$, where the blocks $C^{(i)}$ are (block) Toeplitz, Hankel, unstructured, or noise free
- applications: system identification, model reduction, . . .

extensions:

- generalize the algorithms for W diagonal
- add regularization
- efficiently compute Δp