Realization and identification of autonomous linear periodically time-varying systems

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Abstract

Subsampling of a linear periodically time-varying system results in a collection of linear time-invariant systems with common poles. This key fact, known as "lifting", is used in a two step realization method. The first step is the realization of the time-invariant dynamics (the lifted system). Computationally, this step is a rank-revealing factorization of a block-Hankel matrix. The second step derives a state space representation of the periodic time-varying system. It is shown that no extra computations are required in the second step. The computational complexity of the overall method is therefore equal to the complexity for the realization of the lifted system. A modification of the realization method is proposed, which makes the complexity independent of the parameter variation period. Replacing the rank-revealing factorization in the realization algorithm by structured low-rank approximation yields a maximum likelihood identification method. Existing methods for structured low-rank approximation are used to identify efficiently linear periodically time-varying system.

Keywords: linear periodically time-varying systems, lifting, realization, Kung's algorithm, Hankel low-rank approximation, maximum likelihood estimation.

1 Introduction

1.1 Overview of the literature

Without additional restrictions or prior knowledge, identification of a linear time-varying system is an ill-posed problem. Indeed, the number of model parameters is a multiple of the number of data points and is therefore equal to or exceeds the number of constraints imposed by the data. For example, in a static, single-input single-output, linear time-varying model $\widehat{y}(t) = h(t)u(t)$, for $t = 1,\dots,T$, the number of model parameters $h(1),\dots,h(T)$ is equal to the number of linear equations. Assuming that $u(t) \neq 0$ for all t, any data $\big(u(t),y(t)\big)$ is fitted exactly, *i.e.*, $\widehat{y}(t) = y(t)$, by a linear time-varying model with parameter $h(t) := y(t)/u(t)$.

A well-posed linear time-varying identification problem *must* involve additional prior knowledge about the model, which restricts the effective number of parameters. A general way to formulate such prior knowledge and related reduction of the model parameters is expansion of the model parameters as a linear combination of a small (compared with the number of samples) number of given basis functions. Two common types of prior knowledge, related to specific choices of the basis functions, are:

- *"slowly" varying model parameters:* From an algorithmic point of view, the "slow variation" assumption results in regularization of the ill-posed computational problem. This type of prior knowledge is quantified by a hyper-parameter, related to the rate of change of the parameters. In the statistical literature, the slow parameter variation assumption is presented in a Bayesian setting as a prior.

- *periodic parameter variation:* Another way of regularizing the ill-posed linear time-varying identification problem is to assume that the time-varying coefficients are periodic, where the period is "small" in comparison with the number of data points, or has a small number of harmonics. In this paper we consider the first case, *i.e.*, the parameter variation within a period is modeled nonparametrically. Then, the period of the parameter time-variation serves as a hyper-parameter. Systems with periodic coefficients appear naturally in mechanical systems with rotating parts [BC08]; biomedical applications, where periodicity comes from the hearth beat and/or breathing [IKK10, SLJ$^+$13]; and econometrics, where the periodicity comes from seasonality [HW01]. Linear periodically time-varying systems also appear when a nonlinear system is linearized about a periodic trajectory [SA11].

Input-output identification methods for linear periodically time-varying systems are proposed in [Hen95, VY95, Liu97, MC02, YM10, XSY12]. Less attention is devoted to the output-only case. A method for exact output-only data, based on polynomial algebra, is proposed in [Kui99] and a frequency domain method for output-only identification of linear periodically time-varying systems is developed in [All09, AG06]. Both the method of [Kui99, KW97] and the method of [All09] are based on a lifting approach, *i.e.*, the time-varying system is represented equivalently as a multivariable time-invariant system. The number of outputs p' of the lifted system is equal to the number of outputs p of the original periodic system times the number of samples P in a period of the parameter variation.

1.2 Aim and contribution of the paper

In the paper, we consider the periodic parameter variation approach for autonomous dynamical systems. First, we study the realization problem: given an exact trajectory of an unknown linear periodically time-varying system, find a state space representation of that system. A formal statement of the problem is given in Section 2 (see, Problem 1). Second, we study the problem of maximum likelihood estimation in the output error setting (see, Problem 2).

Most methods proposed in the literature consist of the following main steps (see also Figure 1):

1. *preprocessing* — lifting of the data,
2. *main computation* — derivation of a linear time-invariant model for the lifted data,
3. *postprocessing* — derivation of an equivalent linear periodically time-varying model.

$$\begin{matrix}\text{data}\\ y\end{matrix} \xrightarrow{\text{lifting}} \begin{matrix}\text{lifted}\\ \text{data } y'\end{matrix} \xrightarrow[]{\substack{\text{LTI system}\\ \text{realization/identification}}} \begin{matrix}\text{LTI model}\\ \text{for } y'\end{matrix} \xrightarrow[]{\substack{\text{LPTV parameter}\\ \text{computation}}} \begin{matrix}\text{LPTV model}\\ \text{for } y\end{matrix}$$

Figure 1: Main steps of the linear periodic time-varying system realization/identification methods. (LTI — linear time-invariant, LPTV — linear periodically time-varying)

The key in solving the linear periodic time-varying realization and identification problem is the lifting operation, which converts the time-varying dynamics into time-invariant dynamics of a system with $\mathrm{p}' = \mathrm{p}P$ outputs. From a computational point of view, the realization of the lifted dynamics is a rank-revealing factorization of a block-Hankel matrix. A numerically stable way of doing this operation is the singular value decomposition of a $\mathrm{p}'L \times (T-L)$ matrix, where L is an upper bound on the order, p is the number of outputs, and T is the number of time samples. Its computational complexity is $O(L^2\mathrm{p}^2PT)$ operations.

Once the linear time-invariant dynamics of the lifted model is obtained, it is transformed back to a linear periodically time-varying model in a postprocessing step. In the subspace identification literature, see, *e.g.*, [Hen95], this operation is done indirectly by computing shifted versions of the state sequence of the model and solving linear systems of equations for the model parameters. This method, refered to as the "indirect method" is Algorithm 1 in the paper, has computational complexity $O(L^2\mathrm{p}^2P^2T)$.

The main shortcoming of the indirect method is that it requires extra computations for the derivation of the shifted state sequences and the solution of the systems of equations for the model parameters. This increases the computational complexity by a factor of P compared with the complexity of the realization of the lifted system. We show in Section 4 that the linear periodically time-varying model's parameters can be obtained directly from the lifted model's parameters without extra computations. The resulting method, refered to as the "direct method" is Algorithm 2 in the paper. Its computational complexity is $O(L^2\mathrm{p}^2PT)$. A further improvement of the indirect method (Algorithm 3) operates on a $L \times \mathrm{p}(T-L)$ Hankel matrix and requires only $O(L^2\mathrm{p}T)$ operations.

The maximum-likelihood estimation problem is considered in Section 5. Using the results relating the realization problem to rank revealing factorization of a Hankel matrix constructed from the data, we show that the maximum-likelihood identification problem is equivalent to structured low-rank approximation. Subsequently, we use existing efficient local optimization algorithms [UM13] for structured low-rank approximation.

The main contributions of the paper are:

1. reduction of the computational cost of linear periodically time-varying system realization from $O(L^2\mathtt{p}^2P^2T)$ to $O(L^2\mathtt{p}T)$, and

2. maximum-likelihood method for linear periodically time-varying system identification.

2 Preliminaries, problem formulation, and notation

An autonomous discrete-time linear time-varying system $\mathscr{B}$ can be represented by a state space model

$$\mathscr{B} = \mathscr{B}(A,C) := \{\, y \mid x(t+1) = A(t)x(t),\ y(t) = C(t)x(t),\ \text{for all } t, \text{ with } x(1) = x_{\text{ini}} \in \mathbb{R}^{\mathtt{n}} \,\}, \tag{1}$$

where $A(t) \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and $C(t) \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$ are the model coefficient functions — A is the state transition matrix and C is the output matrix. A state space representation $\mathscr{B}(A,C)$ of the model $\mathscr{B}$ is not unique due to a change of basis, *i.e.*,

$$\mathscr{B} = \mathscr{B}(A,C) = \mathscr{B}(\widehat{A},\widehat{C}),$$

where, for all t

$$\widehat{A}(t) = V(t+1)A(t)V^{-1}(t) \quad \text{and} \quad \widehat{C}(t) = C(t)V^{-1}(t), \tag{2}$$

with a nonsingular matrix $V(t) \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$.

In this paper, we consider the subclass of autonomous linear time-varying systems, for which the coefficient functions A and C are periodic with period P

$$A(t) = A(t+kP) \quad \text{and} \quad C(t) = C(t+kP), \qquad \text{for all } t \text{ and } k.$$

Such systems are called linear periodically time-varying and are parameterized in state space by two matrix sequences

$$\big(A_1,\dots,A_P\big) \qquad \text{and} \qquad \big(C_1,\dots,C_P\big),$$

such that

$$A(t) = A_{(t-1) \bmod P+1} \qquad \text{and} \qquad C(t) = C_{(t-1) \bmod P+1}.$$

The nonuniqueness of the coefficients functions (A,C) of a periodic time-varying system's state space representation is given by (2). In order to preserve the periodicity of the coefficient functions, however, we restrict to state transformations V to periodic, *i.e.*, $V(t) = V_{(t-1) \bmod P+1}$, for some

$$\big(V_1,\dots,V_P\big), \qquad \text{where } V_i \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}} \text{ and } \det(V_i) \neq 0.$$

The class of autonomous linear periodically time-varying systems with order at most $\mathtt{n}$ and period P is denoted by $\mathscr{L}_{0,\mathtt{n},P}$. (The zero subscript index stands for zero inputs.)

Problem 1 (Realization of an autonomous linear periodicaly time-varying system)**.** *Given a trajectory*

$$y = \big(y(1),\dots,y(T)\big),$$

of an autonomous linear periodically time-varying system $\mathscr{B}$, the period P of $\mathscr{B}$, and the state dimension $\mathtt{n}$ of $\mathscr{B}$, find a state space representation $\mathscr{B}(\widehat{A},\widehat{C})$ of the system $\mathscr{B}$, i.e.*,*

$$\textit{find} \qquad \widehat{\mathscr{B}} \in \mathscr{L}_{0,\mathtt{n},P} \quad \textit{such that} \quad y \in \widehat{\mathscr{B}}.$$

The assumption that the order $\mathtt{n}$ of $\mathscr{B}$ is given can be relaxed, see Note 2.

Notation

- $\mathscr{B}(A,C)$, defined in (1), is a linear autonomous periodically time-varying system with state space parameters (A,C). When A and C are constant matrices (rather than matrix sequences) the system is linear time-invariant.
- For a vector time series $y = \big(y(1),\dots,y(T)\big)$, $y(t) \in \mathbb{R}^{\mathtt{p}}$ we define the block-Hankel matrix with L, $1 \le L \le T$ block-rows

$$\mathscr{H}_L(y) := \begin{bmatrix} y(1) & y(2) & y(3) & \cdots & y(T-L+1) \\ y(2) & y(3) & \iddots & & y(T-L+1) \\ y(3) & \iddots & & & \vdots \\ \vdots & & & & \\ y(L) & y(L+1) & \cdots & & y(T) \end{bmatrix} \in \mathbb{R}^{\mathtt{p}L\times(T-L+1)}.$$

- The extended observability matrix of a linear periodically time-varying system with a state space representation $\mathscr{B}(A,C)$ is

$$\mathscr{O}_L(A,C) := \begin{bmatrix} C(1) \\ C(2)A(1) \\ C(3)A(2)A(1) \\ \vdots \\ C(L)A(L-1)A(L-2)\cdots A(1) \end{bmatrix}.$$

- The “lifting” operator

$$\operatorname{lift}_P(y) = \big(y'(1),\dots,y'(T')\big) = \left(\begin{bmatrix} y(1) \\ \vdots \\ y(P) \end{bmatrix}, \begin{bmatrix} y(P+1) \\ \vdots \\ y(2P) \end{bmatrix}, \dots, \begin{bmatrix} y((T'-1)P) \\ \vdots \\ y(T'P) \end{bmatrix} \right), \qquad T' := \left\lfloor \frac{T}{P} \right\rfloor, \tag{3}$$

  ($\lfloor a \rfloor$ is the largest integer smaller than a) sub-samples the $\mathtt{p}$-dimensional vector sequence y at a period P starting from the 1st, 2nd, ..., Pth sample and stacks the resulting P sequences in an augmented $\mathtt{p}' := \mathtt{p}P$-dimensional vector sequence y'—the lifted sequence. Applied on a system $\mathscr{B}$, the operator lift_P acts on all trajectories of the system.

3 Realization of the lifted system

As shown in [BC08, Section 6.2.3], the lifted system $\operatorname{lift}_P\big(\mathscr{B}(A,C)\big)$ admits an $\mathtt{n}$th order linear time-invariant representation

$$\mathscr{B}(\widehat{\Phi},\widehat{\Psi}) = \operatorname{lift}_P\big(\mathscr{B}(A,C)\big), \qquad \text{with } \widehat{\Phi} \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}} \text{ and } \widehat{\Psi}^{\mathtt{p}'\times\mathtt{n}}.$$

(An alternative derivation of this result is given in the appendix.) The problem of obtaining the parameters $\widehat{\Phi}$ and $\widehat{\Psi}$ from the lifted trajectory y' of the periodically time-varying system is a classical linear time-invariant realization problem. We use Kung's method [Kun78], which is based on the Hankel matrix $\mathscr{H}_L(y')$. The number of block-rows L must be such that both the number of rows and the number of columns of $\mathscr{H}_L(y')$ are greater than $\mathtt{n}$.

Note 1 (On the choice of the parameter L). From the point of view of minimizing the computational cost, L is chosen as small as possible, *i.e.*,

$$L = L_{\min} := \left\lceil \frac{\mathtt{n}+1}{\mathtt{p}'} \right\rceil, \qquad (\lceil a \rceil \text{ is the smallest integer larger than } a).$$

In the presence of noise, however, the accuracy of the algorithm is improved by increasing L. In [KT82a], it is shown that best approximation is achieved by choosing L so that the Hankel matrix $\mathscr{H}_L(y')$ is close to square, *i.e.*,

$$L = L_{\mathrm{sq}} := \left\lfloor \frac{T'+1}{\mathtt{p}'+1} \right\rfloor.$$

Let

$$\mathscr{H}_L(y') = \mathscr{O}\mathscr{C}, \qquad \text{where} \quad \mathscr{O} \in \mathbb{R}^{L\mathtt{p}' \times \mathtt{n}} \text{ and } \mathscr{C} \in \mathbb{R}^{\mathtt{n} \times (T'-L)}, \tag{4}$$

be a rank revealing factorization of the block-Hankel matrix $\mathscr{H}_L(y')$. Such a factorization can be obtained, for example, from the singular value decomposition

$$\mathscr{H}_L(y') = USV'^{\top} = \underbrace{U\sqrt{S}}_{\mathscr{O}}\,\underbrace{\sqrt{S}V'^{\top}}_{\mathscr{C}}, \qquad \text{where} \quad \sqrt{S} := \operatorname{diag}(\sqrt{s_1}, \dots, \sqrt{s_n}). \tag{5}$$

The parameter $\widehat{\Psi}$ is set equal to the first $\mathtt{p}'$ rows of the matrix $\mathscr{O}$ and $\widehat{\Phi}$ is computed from the shift equation

$$\underline{\mathscr{O}}\widehat{\Phi} = \overline{\mathscr{O}}, \tag{6}$$

where $\overline{\mathscr{O}}$ is $\mathscr{O}$ with the first $\mathtt{p}'$ rows removed and $\underline{\mathscr{O}}$ is $\mathscr{O}$ with the last $\mathtt{p}'$ rows removed.

Note 2 (Unknown order $\mathtt{n}$). If the order of the linear periodically time-varying system is not given a priori, it can be determined from the rank of the block-Hankel matrix $\mathscr{H}_L\big(\mathrm{lift}_P(y)\big)$.

Note 3 (Inexact data and model reduction). Truncation of the singular value decomposition (5) is a method to do (unstructured) low-rank approximation, which has the system theoretic interpretation of identifying reduced order model. In the case of perturbation of exact data by noise, truncation of the singular value decomposition to the order of the true system has the effect of signal de-noising. $L = L_{\mathrm{sq}}$ has empirical justification of making the algorithm less sensitive to noise [KT82b].

The $\mathscr{C}$ factor of the factorization (4) has the form

$$\mathscr{C} = \begin{bmatrix} \widehat{x}_{\mathrm{ini}} & \widehat{\Phi}\widehat{x}_{\mathrm{ini}} & \cdots & \widehat{\Phi}^{T'-L+1}\widehat{x}_{\mathrm{ini}} \end{bmatrix} = \begin{bmatrix} \widehat{x}'(1) & \widehat{x}'(2) & \cdots & \widehat{x}'(T'-L) \end{bmatrix} =: \begin{bmatrix} \widehat{X}_1' & \widehat{x}'(T'-L) \end{bmatrix}, \tag{7}$$

where $\widehat{x}_{\mathrm{ini}}$ is the initial condition and $\widehat{x}'(1), \widehat{x}'(2), \dots$ is the state sequence of the linear time-invariant model $\mathscr{B}(\widehat{\Phi}, \widehat{\Psi})$. The initial condition $\widehat{x}_{\mathrm{ini}}$ can be obtained directly from $\mathscr{C}$ or it can be re-estimated back from the data by solving the overdetermined system of linear equations

$$y' = \mathscr{O}_{T'}(\widehat{\Phi}, \widehat{\Psi})\widehat{x}_{\mathrm{ini}}, \tag{8}$$

and defining

$$\widehat{x}'(t') := \Phi^{t'-1}\widehat{x}_{\mathrm{ini}}, \quad \text{for } t' = 1, 2, \dots$$

Note 4 (Inexact data and model reduction). In the case of noisy data or true system that is not in the model class (see Notes 3), (8) generically has no exact solution. Then, the least-squares approximate solution can be used as a means of estimating the initial condition from (8).

4 Computation of the linear time-varying system's parameters

4.1 Indirect method

Define the matrices

$$\widehat{X}_i' := V_i \begin{bmatrix} x(i) & x(i+P) & x(i+2P) & \cdots & x\big(i+(T'-L-1)P\big) \end{bmatrix}, \qquad \text{for } i = 1, \dots, P,$$

constructed from the state sequence $\big(x(1), x(2), \dots\big)$ in a state-space basis, defined by V_i. The derivation of $\widehat{X}_1'$ is a by-product of the realization of the lifted system $\mathscr{B}(\widehat{\Phi}, \widehat{\Psi})$, see (7). The shifted state sequences $\widehat{X}_2', \dots, \widehat{X}_P'$ can also be computed from (4) by using the i-steps shifting data $\big(y(i), \dots, y(T)\big)$ instead of $\big(y(1), \dots, y(T)\big)$. Note that the computation of $\widehat{X}_i'$ through (4) results in general in a basis V_i that is different from V_j, for $i \neq j$.

The model parameters $(\widehat{A}, \widehat{C})$ are computed from the equations

$$\begin{bmatrix} \widehat{X}_{i+1}' \\ Y_i \end{bmatrix} = \begin{bmatrix} \widehat{A}_i \\ \widehat{C}_i \end{bmatrix} \widehat{X}_i', \qquad \text{for } i = 1, \dots, P, \tag{9}$$

where

$$Y_i := \begin{bmatrix} y(i) & y(i+P) & y(i+2P) & \cdots & y\big(i+(T'-L-1)P\big) \end{bmatrix}.$$

The matrix $\widehat{X}'_{P+1}$ is obtained from $\widehat{X}'_1$ by pre-multiplication with Φ (*i.e.*, shift with P steps forward)

$$\widehat{X}'_{P+1} := \Phi \widehat{X}'_1.$$

This guarantees that $\widehat{X}'_{P+1}$ is in the same basis as $\widehat{X}'_1$, which implies that $V_1 = V_{P+1}$.

Algorithm 1 summarizes the method for realization of linear periodically time-varying systems, described above.

Algorithm 1 Indirect algorithm for linear periodically time-varying system realization.	# of operations
Input: Sequence $y \in (\mathbb{R}^{\mathtt{p}})^T$ and natural numbers P and $\mathtt{n}$.	
1: *lifting:* (3)	0
2: *realization of the lifted system:* $(y', \mathtt{n}) \mapsto \mathscr{B}\big(\widehat{\Phi}, \widehat{\Psi}\big)$ ((4) and (6))	$O\big((L\mathtt{p}')^2 T'\big)$
3: *state estimation:* compute the state sequence matrices $\widehat{X}'_1, \ldots, \widehat{X}'_P$ and define $\widehat{X}'_{P+1} := \widehat{\Phi}\widehat{X}'_1$	$O\big(P(L\mathtt{p}')^2 T'\big)$
4: *parameter estimation:* solve the systems (9)	$O\big(P(\mathtt{n}+\mathtt{p}')^2\big)$
Output: Parameters $\widehat{A}$ and $\widehat{C}$ of the linear periodically time-varying system.	
	overall cost: $O(L^2\mathtt{p}^2P^2T)$

4.2 Direct method

The most expensive step of Algorithm 1 is the computation of the shifted state sequences $\widehat{X}'_i$, which requires P factorizations of block-Hankel matrices. As proven in the following proposition, the parameters $(\widehat{A}, \widehat{C})$ of the linear periodically time-varying system can be obtained directly from the parameters $(\widehat{\Phi}, \widehat{\Psi})$ of the linear time-invariant system, without extra computation.

Proposition 1. *The linear periodically time-varying system $\mathscr{B}(\widehat{A}, \widehat{C})$, with parameters*

$$\widehat{A}_1 := I_n, \; \ldots, \; \widehat{A}_{P-1} := I_n, \; \widehat{A}_P := \widehat{\Phi} \qquad \text{and} \qquad \widehat{\Psi} =: \operatorname{col}(\widehat{C}_1, \ldots, \widehat{C}_P), \quad \text{where } \widehat{C}_i \in \mathbb{R}^{\mathtt{p} \times \mathtt{n}}, \tag{10}$$

is equivalent to the linear time-invariant system $\mathscr{B}(\widehat{\Phi}, \widehat{\Psi})$, i.e., *$\mathscr{B}(\widehat{\Phi}, \widehat{\Psi}) = \text{lift}_P\big(\mathscr{B}(\widehat{A}, \widehat{C})\big)$.*

Proof. We have to show that a response y of the system $\text{lift}_P\big(\mathscr{B}(\widehat{A}, \widehat{C})\big)$ is also a response of the linear time-invariant system $\mathscr{B}(\widehat{\Phi}, \widehat{\Psi})$. Let x_{ini} be the initial condition of the linear periodically time-varying system $\mathscr{B}(\widehat{A}, \widehat{C})$ that generates y. We have

$$\begin{gathered} y(1) = \widehat{C}_1 x_{\text{ini}}, \; \ldots, y(P) = \widehat{C}_P x_{\text{ini}} \\ y(P+1) = \widehat{C}_1 \widehat{\Phi} x_{\text{ini}}, \; \ldots, y(2P) = \widehat{C}_P \widehat{\Phi} x_{\text{ini}} \\ \vdots \\ y(t'P+1) = \widehat{C}_1 \widehat{\Phi}^{t'} x_{\text{ini}}, \; \ldots, y(2P) = \widehat{C}_P \widehat{\Phi}^{t'} x_{\text{ini}}. \end{gathered}$$

On the other hand, the response of the linear time-invariant system $\mathscr{B}(\widehat{\Phi}, \widehat{\Psi})$ to the initial condition x_{ini} is

$$y'(1) = \widehat{\Psi} x_{\text{ini}} = \begin{bmatrix} \widehat{C}_1 \\ \vdots \\ \widehat{C}_P \end{bmatrix} x_{\text{ini}}, \quad y'(2) = \widehat{\Psi}\widehat{\Phi} x_{\text{ini}} = \begin{bmatrix} \widehat{C}_1 \\ \vdots \\ \widehat{C}_P \end{bmatrix} \widehat{\Phi} x_{\text{ini}}, \quad \ldots, \quad y'(t') = \widehat{\Psi}\widehat{\Phi}^{t'} x_{\text{ini}} = \begin{bmatrix} \widehat{C}_1 \\ \vdots \\ \widehat{C}_P \end{bmatrix} \widehat{\Phi}^{t'} x_{\text{ini}}.$$

It follows that $\text{lift}_P(y) = y'$. □

Algorithm 2 summarizes the direct method for realization of linear periodically time-varying systems.

Algorithm 2 Direct algorithm for linear periodically time-varying system realization. # of operations

Input: Sequence $y \in (\mathbb{R}^{\mathtt{p}})^T$ and natural numbers P and $\mathtt{n}$.

1: *lifting:* (3) — 0

2: *modeling:* $(y', \mathtt{n}) \mapsto \mathscr{B}(\widehat{\Phi}, \widehat{\Psi})$ (*e.g.*, Kung's algorithm: (4) and (6)) — $O\big((L\mathtt{p}')^2 T'\big)$

3: define $\widehat{A}$ and $\widehat{C}$ via (10) — 0

Output: Parameters $\widehat{A}$ and $\widehat{C}$ of the linear periodically time-varying system.

overall cost: $O(L^2\mathtt{p}^2 PT)$

4.3 Modification of the direct method

Consider the "transposed" lifted sequence

$$y'^{\top} := \big(y'^{\top}(1), \dots, y'^{\top}(T')\big), \qquad y'^{\top}(t) \in \mathbb{R}^{1\times \mathtt{p}'}$$

and the associated block-Hankel matrix

$$\mathscr{H}_L(y'^{\top}) := \begin{bmatrix} y'^{\top}(1) & y'^{\top}(2) & y'^{\top}(3) & \cdots & y'^{\top}(T'-L) \\ y'^{\top}(2) & y'^{\top}(3) & \iddots & & y'^{\top}(T'-L+1) \\ y'^{\top}(3) & \iddots & & & \vdots \\ \vdots & & & & \\ y'^{\top}(L) & y'^{\top}(L+1) & \cdots & & y'^{\top}(T') \end{bmatrix} \in \mathbb{R}^{L\times \mathtt{p}'(T'-L+1)}. \tag{11}$$

The parameter L satisfies the constraints $L > \mathtt{n}$ and $\mathtt{p}(T'-L-1) > \mathtt{n}$. As before, for best approximation, L is selected to make $\mathscr{H}_L(y'^{\top})$ as square as possible. For minimal computational cost, L is chosen as small as possible, which in the case of (11) is $\mathtt{n}+1$.

Since y is a trajectory of the linear periodically time-varying system $\mathscr{B}(A,C)$, we have that

$$\mathscr{H}_L(y'^{\top}) = \mathscr{O}_L(\widehat{\Phi}^{\top}, x_{\text{ini}}^{\top}) \cdot \mathscr{O}_{T'-L+1}^{\top}(\widehat{\Phi}, \widehat{\Psi}).$$

Therefore, the parameters $(\widehat{\Psi}, \widehat{\Phi})$ of the lifted system can be identified from the rank revealing factorization

$$\mathscr{H}_L(y'^{\top}) = USV^{\top} = \underbrace{U\sqrt{S}}_{\mathscr{O}}\,\underbrace{\sqrt{S}V^{\top}}_{\mathscr{C}}. \tag{12}$$

The initial condition x_{ini} is the transposed first row of the $\mathscr{O}$ factor, Ψ is the transposed first $\mathtt{n}\times\mathtt{p}$ block element of $\mathscr{C}$, and $\widehat{\Phi}^{\top}$ is a solution of the shift equation

$$\underline{\mathscr{O}}\widehat{\Phi}^{\top} = \overline{\mathscr{O}}. \tag{13}$$

The resulting identification method is summarized in Algorithm 3.

Algorithm 3 Modified direct algorithm for linear periodically time-varying realization. # of operations

Input: Sequence $y \in (\mathbb{R}^{\mathtt{p}})^T$ and natural numbers P and $\mathtt{n}$.

1: *lifting:* (3) — 0

2: *modeling:* $(y', \mathtt{n}) \mapsto \mathscr{B}(\widehat{\Phi}, \widehat{\Psi})$ ((12) and (13)) — $O\big(L^2 p' T'\big)$

3: define $\widehat{A}$ and $\widehat{C}$ via (10) — 0

Output: Parameters $\widehat{A}$ and $\widehat{C}$ of the linear periodically time-varying system.

overall cost: $O(L^2\mathtt{p}T)$

Note that the computational cost of Algorithm 3 is independent of the period P and is linear in the number of outputs $\mathtt{p}$. This is a significant improvement over Algorithm 2. In addition, as discussed in the next section, using the matrix $\mathscr{H}_L(y'^{\top})$ instead of $\mathscr{H}_L(y)$ has an important advantage in the case of optimal approximate identification.

5 Maximum likelihood identification

Assume that the data is generated in the output error setup:

$$y = \bar{y} + \widetilde{y}, \qquad \text{where} \quad \bar{y} \in \bar{\mathscr{B}} \in \mathscr{L}_{0,\mathtt{n},P} \\ \text{and } \widetilde{y} \text{ is zero mean white Gaussian process with covariance matrix } s^2 I_{\mathtt{p}}. \tag{14}$$

The "true value" $\bar{y}$ of the data y is generated by a linear periodically time-varying system $\mathscr{B}(\bar{A}, \bar{C})$, refered to as the "true system". Our aim is to estimate the true linear periodically time-varying system $\mathscr{B}(\bar{A}, \bar{C})$ from the data y and the prior knowledge that the true system belongs to the model class $\mathscr{L}_{0,\mathtt{n},P}$. The maximum likelihood identification principle applied to the data generating model (14) leads to the following optimization problem.

Problem 2 (Maximum liklihood identification of an autonomous linear periodicaly time-varying system)**.** *Given a trajectory*

$$y = \big(y(1), \ldots, y(T)\big),$$

and a model class $\mathscr{L}_{0,\mathtt{n},P}$, specified by natural numbers $\mathtt{n}$ and P,

$$\text{minimize} \quad \textit{over } \widehat{y} \textit{ and } \widehat{\mathscr{B}} \quad \|y - \widehat{y}\|_2 \quad \textit{subject to} \quad \widehat{y} \in \widehat{\mathscr{B}} \in \mathscr{L}_{0,\mathtt{n},P}. \tag{15}$$

As shown in Section 4.3,

$$\widehat{y} \in \widehat{\mathscr{B}} \in \mathscr{L}_{0,\mathtt{n},P} \qquad \iff \qquad \operatorname{rank}\big(\mathscr{H}_{\mathtt{n}+1}\big(\operatorname{lift}_P(\widehat{y}^\top)\big)\big) \leq \mathtt{n},$$

so that the optimization problem (15) is equivalent to a Hankel structured low-rank approximation problem [Mar08].

Proposition 2 (Optimal identification of linear parameter varying system via structured low-rank approximation)**.** *Problem 2 is equivalent to the structured low-rank approximation problems*

$$\text{minimize} \quad \textit{over } \widehat{y} \quad \|y - \widehat{y}\|_2 \quad \textit{subject to} \quad \operatorname{rank}\big(\mathscr{H}_{\mathtt{n}+1}\big(\textit{lift}_P(\widehat{y}^\top)\big)\big) \leq \mathtt{n}. \tag{SLRA}$$

For the solution of the structured low-rank approximation problems (SLRA) we use the method of [UM13]. It is based on the kernel representations of the rank constraint

$$\operatorname{rank}\big(\mathscr{H}_{\mathtt{n}+1}\big(\operatorname{lift}_P(\widehat{y}^\top)\big)\big) \leq \mathtt{n} \qquad \iff \qquad \text{there is an } R^{1\times(\mathtt{n}+1)}, \text{ such that } R\mathscr{H}_{\mathtt{n}+1}\big(\operatorname{lift}_P(\widehat{y}^\top)\big) = 0 \text{ and } RR^\top = 1.$$

In the method of [UM13], the optimization variable $\widehat{y}$ is eliminated by analytically minimizing over it. The resulting nonlinear least squares problem for R is solved by local optimization methods.

The solution $\widehat{y}$ of the structured low-rank approximation is by construction an exact trajectory of a system in the model class $\mathscr{L}_{0,\mathtt{n},P}$. Therefore, the remaining problem of finding the model for $\widehat{y}$ (which is the optimal approximate model for y) is an exact identification problem and can be solved by Algorithm 3. The structured low-rank approximation methods of [UM13], however, computes as a byproduct the kernel matrix R. Therefore, a rank revealing factorization (12) can be computed without using the computationally more expensive singular value decomposition.

The orthogonal complement $R^\perp$ of R is equal to the left factor $\mathscr{O}$ in (5). Knowledge of $\mathscr{O}$ is sufficient to determine the parameters $(\widehat{\Phi}, \widehat{\Psi})$ of the lifted system. The resulting optimal identification method is summarized in Algorithm 3.

Algorithm 4 Algorithm for optimal linear periodically time-varying system identification.	# of operations
Input: Sequence $y \in (\mathbb{R}^{\mathtt{p}})^T$ and natural numbers P and $\mathtt{n}$.	
1: *lifting:* (3)	0
2: *modeling:* $(y', \mathtt{n}) \mapsto (R, \widehat{y}')$ (SLRA)	$O((\mathtt{n}+1)^3 \mathtt{p}' T')$ per iteration
3: Compute $\mathscr{O} = R^\perp$ and define $\widehat{A}$ and $\widehat{C}$ via (10).	$O((\mathtt{n}+1)^3)$
Output: Parameters $\widehat{A}$ and $\widehat{C}$ of the linear periodically time-varying system.	
	overall cost (for K iterations): $O((\mathtt{n}+1)^3 \mathtt{p} T K)$

The methods in the paper are implemented in Matlab and are available in the `ident` directory of the structured low-rank approximation package [MU12]:

`http://homepages.vub.ac.be/~imarkovs/slra/software.html`

The simulation results presented in the following section can be reproduced with the m-file `pltv_all_examples`.

6 Numerical examples

In all simulation examples, the data is generated according to the output error model (14). A range of noise variances are chosen and for each value of the noise variance 100 Monte Carlo repetitions of the identification with different noise realizations are done. For each noisy data set, the Algorithms 1–4 are applied on the first 3/4 of the data (identification data). The identified models are evaluated in terms of their fitting accuracy on the remaining 1/4 of the data (validation data)

$$e = \frac{\|y_{\text{val}} - \widehat{y}_{\text{val}}\|_2}{\|y_{\text{val}} - \text{mean}(y_{\text{val}})\|_2}.$$

The reported results are the average fitting accuracy over the 100 repetitions.

6.1 Example from [Hen95]

The first example is Example 1 from [Hen95]: a second order linear time-varying system with parameters[1]

$$\begin{aligned} \bar{A}_1 &= \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, & \bar{C}_1 &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \\ \bar{A}_2 &= \tfrac{1}{5}\begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}, & \bar{C}_2 &= \begin{bmatrix} 2 & 0 \end{bmatrix}, \\ \bar{A}_3 &= \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}, & \bar{C}_3 &= \begin{bmatrix} 1 & 1 \end{bmatrix}. \end{aligned}$$

The number of periods is `Tp` and the noise standard deviations are specified by the vector `s_vec`. In an example with simulation parameters

⟨*example Hench*⟩≡

```
clear all; ex = 'hench'; Tp = 20; N = 7; s_vec = linspace(0, 0.5, N); test_pltv
```

the obtained relative approximation errors e by the methods are:

'noise std'	[0]	[0.0833]	[0.1667]	[0.2500]	[0.3333]	[0.4167]	[0.5000]
'Alg. 1'	[4.2809e-15]	[0.3253]	[0.5699]	[0.7769]	[0.8451]	[0.9813]	[1.0273]
'Alg. 2'	[5.2142e-15]	[0.3246]	[0.5737]	[0.7555]	[0.8261]	[0.9341]	[0.9747]
'Alg. 3'	[2.6446e-15]	[0.3103]	[0.5564]	[0.7345]	[0.8159]	[0.9023]	[0.9410]
'Alg. 4'	[1.0029e-14]	[0.3079]	[0.5539]	[0.7277]	[0.8162]	[0.8993]	[0.9395]

The approximation error at the level of the numerical precision for zero noise variance is an empirical confirmation of the theoretical results in the paper. A plot of the data and the approximating trajectories for the last value of the noise standard deviation is shown in Figure 2. The bias $|\bar{y} - \text{mean}(\widehat{y})|$ and standard deviation $\text{std}(\widehat{y})$ of the approximations (see Figure 3) shows that the direct method is more robust to noise in the data.

6.2 Mathieu oscillator

The second example is Mathieu oscillator—a spring-mass-damper system with time-periodic spring stiffness. A state-space representation of Mathieu oscillator is

$$\bar{A}_\tau = \begin{bmatrix} 0 & 1 \\ \bar{a}_1 & \bar{a}_{2,\tau} \end{bmatrix}, \quad \bar{C}_\tau = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad \text{for } \tau = 1, \dots, P.$$

In the simulation example, we have the following parameter values

$$\bar{a}_1 = -0.9, \qquad \bar{a}_{2,\tau} = -\big(0.1 + 0.4\cos(2\pi\tau/P)\big),$$

where the period P is 3.

⟨*example Mathieu oscillator*⟩≡

```
clear all; ex = 'mathieu'; P = 3; Tp = 20; N = 7; s_vec = linspace(0, 0.30, N); test_pltv
```

[1] Example 1 in [Hen95] is an input-output system. We use it here to generate a free response, so that only the A and C parameters are relevant.

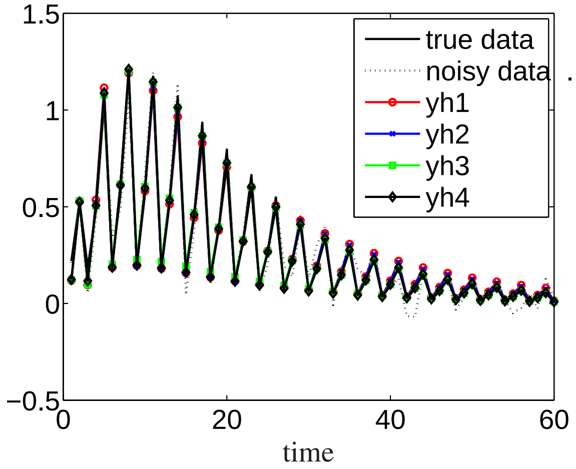

Figure 2: Results for the simulation example of [Hen95]: true model output $\bar{y}$, noisy data y, and outputs $\widehat{y}$ of the identified model by Algorithms 1 and 2.

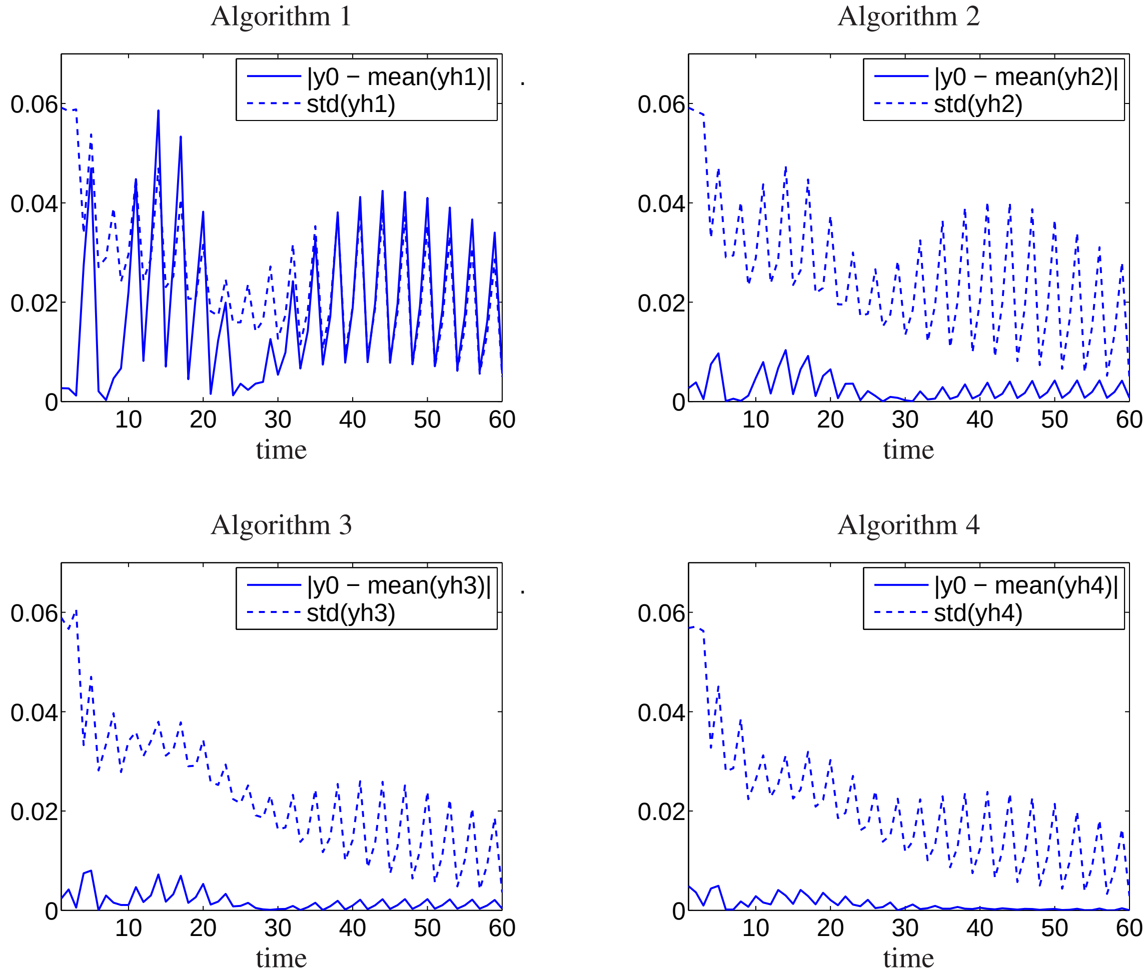

Figure 3: Results for the simulation example of [Hen95]: bias $|\bar{y} - \text{mean}(\widehat{y})|$ and standard deviation terms $\text{std}(\widehat{y})$ of the modeled response.

The obtained relative approximation errors e are

'noise std'	[0]	[0.0500]	[0.1000]	[0.1500]	[0.2000]	[0.2500]	[0.3000]
'Alg. 1'	[8.8811e-15]	[0.4794]	[0.7700]	[0.8808]	[0.9624]	[0.9920]	[1.0000]
'Alg. 2'	[3.0398e-15]	[0.4797]	[0.7707]	[0.8824]	[0.9657]	[0.9966]	[1.0056]
'Alg. 3'	[5.5009e-15]	[0.4793]	[0.7636]	[0.8814]	[0.9582]	[0.9918]	[1.0020]
'Alg. 4'	[2.3834e-15]	[0.4789]	[0.7632]	[0.8806]	[0.9558]	[0.9879]	[0.9992]

Figure 4 shows the fit of the data by the models for the last value of the noise variance, and Figure 5 shows the bias and variance components of the error.

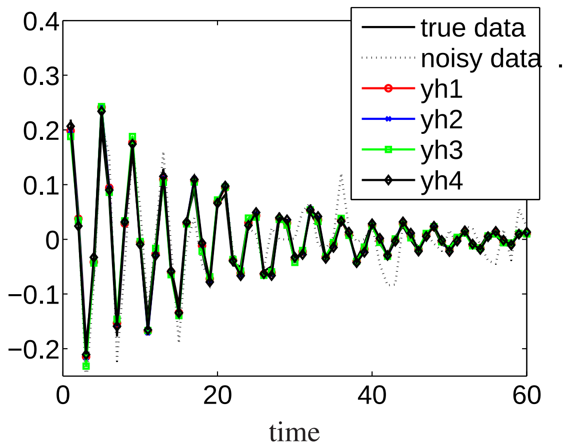

Figure 4: Results for the simulation example of Mathieu oscillator: true model output $\bar{y}$, noisy data y, and outputs $\widehat{y}$ of the identified model by Algorithms 1 and 2.

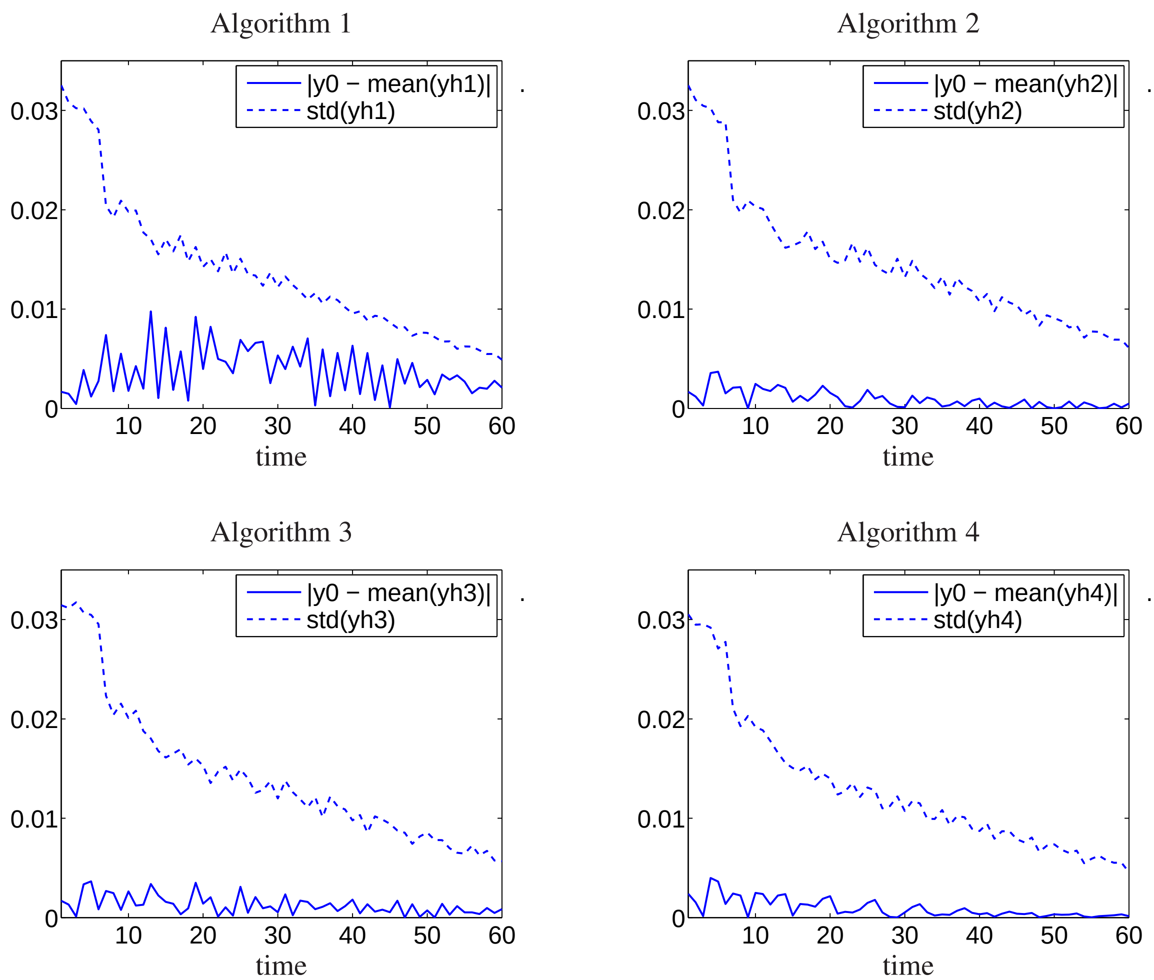

Figure 5: Results for Mathieu oscillator: bias $|\bar{y} - \text{mean}(\widehat{y})|$ and standard deviation $\text{std}(\widehat{y})$ of the modeled response.

Converting the identified system $\mathscr{B}(\widehat{A},\widehat{C})$ into companion canonical form [BC08, Section 3.4], we extract estimates $\widehat{a}_1$ and $\widehat{a}_2$ of the true parameter values $\bar{a}_1$ and $\bar{a}_2$. The parameter relative approximation errors

$$e_\theta = \frac{\|\bar{\theta}-\widehat{\theta}\|_2}{\|\bar{\theta}\|_2}, \qquad \text{where } \widehat{\theta} = \begin{bmatrix}\widehat{a}_1\\ \widehat{a}_2\end{bmatrix} \text{ and } \bar{\theta} = \begin{bmatrix}\bar{a}_1\\ \bar{a}_2\end{bmatrix}$$

are as follows:

```
'noise std'    [         0]    [0.0500]    [0.1000]    [0.1500]    [0.2000]    [0.2500]    [0.3000]
'Alg. 1'       [5.1046e-16]    [0.0401]    [0.0799]    [0.1301]    [0.1681]    [0.2134]    [0.2686]
'Alg. 2'       [4.6643e-16]    [0.0401]    [0.0800]    [0.1301]    [0.1683]    [0.2121]    [0.2684]
'Alg. 3'       [5.5389e-16]    [0.0435]    [0.0878]    [0.1350]    [0.1739]    [0.2271]    [0.2894]
'Alg. 4'       [8.8132e-16]    [0.0396]    [0.0796]    [0.1275]    [0.1637]    [0.2042]    [0.2637]
```

7 Conclusions

Realization and maximum likelihood identification algorithms for autonomous linear periodically time-varying systems are developed. The algorithms are based on 1) lifting of the original time series, 2) modeling of the lifted time-series by a linear time-invariant system, 3) transition from the time-invariant system's parameters to the ones of the periodic time-varying system. It is shown that the derivation of the periodic time-varying system's state space parameters in step 3 can be done without extra computations and, in step 2, the realization problem can be solved by rank revealing factorization of a block Hankel matrix with $\mathtt{n}+1$ rows, where $\mathtt{n}$ is the order to the system. These facts lead to a new realization algorithm, which cost is independent of the parameter variation's period P and to a maximum likelihood identification algorithm, based on Hankel structured low-rank approximation. Consequently, readily available robust and efficient optimization methods and software can be used for identifying periodic linear time-varying systems.

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A The lifting operator

The advantage of the lifting transformation

$$\big(y(1),\dots,y(T)\big) = y \mapsto y' = \big(y'(1),\dots,y'(T')\big) := \operatorname{lift}_P(y)$$

is that if y is a trajectory of a linear periodically time-varying system with period P, then the lifted sequence y' is a trajectory of a linear time-invariant system. Therefore, problems about the original linear periodically time-varying system is reduced to equivalent problems for a linear time-invariant system. The following proposition shows a state-space representation of a lifted linear periodically time-varying system.

Proposition 3. *The lifted autonomous linear periodically time-varying system* $\text{lift}_P\big(\mathscr{B}(A,C)\big)$ *is a linear time-invariant system* $\mathscr{B}(A',C')$*, with*

$$A' := \text{diag}\big(\Phi_1,\ldots,\Phi_P\big) \qquad \textit{and} \qquad C' := \text{diag}\big(C_1,\ldots,C_P\big), \tag{16}$$

where

$$\begin{aligned} \Phi_1 &:= A_P A_{P-1} A_{P-2} \cdots A_3 A_2 A_1, \\ \Phi_2 &:= A_1 A_P A_{P-1} \cdots A_4 A_3 A_2, \\ &\ \ \vdots \\ \Phi_P &:= A_{P-1} A_{P-2} \cdots A_2 A_1 A_P. \end{aligned} \tag{17}$$

and initial condition

$$x'_{\text{ini}} = \mathscr{O}_P(A,I)x_{\text{ini}}, \tag{18}$$

where $x_{\text{ini}} \in \mathbb{R}^{\mathtt{n}}$ *is the initial condition of the linear periodically time-varying system* $\mathscr{B}(A,C)$.

Proof. The sequences

$$y^{(i)} = \big(y(i), y(i+P), \ldots, y(i+(T'-1)P)\big), \qquad \text{for} \quad i = 1,\ldots,P,$$

are generated by the linear time-invariant systems $\mathscr{B}\big(\Phi_i, C_i\big)$, under initial conditions

$$x^{(i)}(1) = A_{i-1} \cdots A_2 A_1 x_{\text{ini}}.$$

Note that the sampling time of the linear time-invariant systems is $P\Delta t$, where Δt is the sampling time of the linear periodically time-varying system.

Collecting the systems $\mathscr{B}\big(\Phi_i, C_i\big)$, $i = 1,\ldots,P$ and stacking their outputs together, we obtain the linear time-invariant system $\mathscr{B}(A',C')$, which output y' is by construction $\text{lift}_P(y)$, provided the initial condition is specified as in (18). □

Note 5. The transformation $(A,C) \mapsto (A',C')$ from a linear periodically time-varying system to an equivalent linear time-invariant system is explicitly given by (17) and is computationally easy to perform. The reverse transformation $(A',C') \mapsto (A,C)$, however, is a difficult computational problem. In the proposed identification method, we do not solve equations (17) directly for the linear periodically time-varying system parameters. The method for transforming the linear time-invariant system parameter to the linear periodically time-varying system parameters is the main contribution of this paper.

The following proposition shows that the circulant products $\Phi_1,\ldots,\Phi_P$ of the matrices $A_1,\ldots,A_P$ have the same eigenvalues.

Proposition 4. *Assuming that* A_t *is nonsingular for all* t*, the matrices* $\Phi_1,\ldots,\Phi_P$*, defined in (17), have the same eigenvalues.*

Proof. Let λ be an eigenvalue of Φ_1. Then there is a vector v, such that

$$\Phi_1 v = \lambda v. \tag{19}$$

Pre-multiplying (19) with $A^{-1}(P)$, we have

$$\underbrace{A_{P-1} A_{P-2} \cdots A_2 A_1 A_P}_{\Phi_P} A^{-1}(P)v = \lambda A^{-1}(P)v,$$

so that λ is also an eigenvalue of Φ_P (with a corresponding eigenvector $A^{-1}(P)v$). The same argument can be used to show the converse: an eigenvalue of Φ_P is an eigenvalue of Φ_1. This proves that Φ_1 and Φ_P have the same eigenvalues.

Recursive application of the above procedure establishes the equivalence of the eigenvalues of $\Phi_1,\ldots,\Phi_P$. □

As a consequence of the fact that $\Phi_1,\ldots,\Phi_P$ have the same eigenvalues, we see that the linear time-invariant representation $\mathscr{B}(A',C')$, with A' and C' given in (16), of the lifted system $\text{lift}_P\big(\mathscr{B}(A,C)\big)$ is nonminimal. The state dimension of $\mathscr{B}(A',C')$ is $\mathtt{n}P$, while the system has only $\mathtt{n}$ modes.

Corollary 1. *Let $\mathscr{B}(A,C)$ be an autonomous linear periodically time-varying system of order* $\mathtt{n}$*, with* $\mathtt{p}$ *outputs and period P. Then, the lifted system $\mathrm{lift}_P\big(\mathscr{B}(A,C)\big)$ is linear time-invariant of order* $\mathtt{n}$ *with* $\mathtt{p}P$ *outputs. It admits a representation $\mathscr{B}(\Phi,\Psi)$, with*

$$\Phi = \Phi_1 \qquad \textit{and} \qquad \Psi = \mathscr{O}_P(A,C).$$

Proof. Let x be the state vector of the linear periodically time-varying system $\mathscr{B}(A,C)$ and x' be the state vector of the linear time-invariant system $\mathscr{B}(\Phi,\Psi)$. Consider a trajectory of y of $\mathscr{B}(A,C)$, generated under initial condition $x(0) = x_{\mathrm{ini}} \in \mathbb{R}^{\mathtt{n}}$. The vector of the stacked outputs $y(1),\dots,y(P)$ of $\mathscr{B}(A,C)$ is $\mathscr{O}_P(A,C)x_{\mathrm{ini}}$. By the definition, the output $y'(1)$ of $\mathscr{B}(\Phi,\Psi)$ under initial conditions $x'(0) = x_{\mathrm{ini}}$ is $y'(1) = \mathscr{O}_P(A,C)x_{\mathrm{ini}}$. For $x(0) = x'(0) = x_{\mathrm{ini}}$, we have that

$$x(Pt') = \Phi_1^{t'} x_{\mathrm{ini}} = x'(t').$$

This proves that $\mathscr{B}(\Phi,\Psi) = \mathrm{lift}_P\big(\mathscr{B}(A,C)\big)$. $\square$