Computation of LTI system responses directly from input/output data

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Outline

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Motivation: find models from data

in this talk, the data $\tilde{w} = (\tilde{u}, \tilde{y})$ is an exact I/O traj. of an LTI system $\mathcal{S}$

(think of deterministic subspace identification: from $\tilde{w}$ to (A, B, C, D))

the impulse response H of $\mathcal{S}$ is a particular representation of $\mathcal{S}$

$\Rightarrow$ an algorithm that computes H from w is an identification algorithm

to go from H to (A, B, C, D) is realization (*e.g.*, Kung's algorithm)

in subspace identification, sequential free responses of $\mathcal{S}$ are computed
$\rightsquigarrow$ oblique projection

the oblique proj. is equivalent of the block algorithm, presented here

The fundamental lemma

the proposed algorithm is based on the following fundamental fact:

(under reasonable assumptions, stated latter) the l samples long windows of $\tilde{w}$ span the set of all l samples long trajectories of $\mathcal{S}$

notation:

$\mathcal{H}_l(\bullet)$ is a Hankel matrix with l block-rows, *e.g.*, with $\tilde{u}(1), \dots, \tilde{u}(T)$,

$$\mathcal{H}_l(\tilde{u}) = \begin{bmatrix} \tilde{u}(1) & \tilde{u}(2) & \cdots & \tilde{u}(T-l+1) \\ \tilde{u}(2) & \tilde{u}(3) & \cdots & \tilde{u}(T-l+2) \\ \vdots & \vdots & & \vdots \\ \tilde{u}(l) & \tilde{u}(l+1) & \cdots & \tilde{u}(T) \end{bmatrix}$$

$\mathtt{n}_{\max}$ is an upper bound on the order $\mathtt{n}$ of $\mathcal{S}$

The fundamental lemma

persistency of excitation plays an important role in system identification

$\tilde{u}$ is persistently exciting of order l $:\iff$ $\mathcal{H}_l(\tilde{u})$ is of full row rank

the assumptions needed in the fundamental lemma are:
(i) $\tilde{u}$ is persistently exciting of order $l + \mathtt{n}$, and (ii) $\mathcal{S}$ is controllable

define: $U_\mathsf{p} \in \mathbb{R}^{\mathtt{n}_{\max}\mathtt{m}\times\bullet}$ $U_\mathsf{f} \in \mathbb{R}^{l\mathtt{m}\times\bullet}$, $Y_\mathsf{p} \in \mathbb{R}^{\mathtt{n}_{\max}\mathtt{p}\times\bullet}$, $Y_\mathsf{f} \in \mathbb{R}^{l\mathtt{p}\times\bullet}$ by

$$\mathcal{H}_{\mathtt{n}_{\max}+l}(\tilde{u}) =: \begin{bmatrix} U_\mathsf{p} \\ U_\mathsf{f} \end{bmatrix} \quad \text{and} \quad \mathcal{H}_{\mathtt{n}_{\max}+l}(\tilde{y}) =: \begin{bmatrix} Y_\mathsf{p} \\ Y_\mathsf{f} \end{bmatrix}$$

The fundamental lemma

Theorem 1. Let $\tilde{w} = (\tilde{u}, \tilde{y})$ be a trajectory of a controllable LTI system $\mathcal{S}$ of order $\mathtt{n} \leq \mathtt{n}_{\max}$ and let $\tilde{u}$ be persistently exciting of order $l + 2\mathtt{n}_{\max}$. Then the system of equations

$$\begin{bmatrix} U_\mathsf{p} \\ U_\mathsf{f} \\ Y_\mathsf{p} \end{bmatrix} g = \begin{bmatrix} 0 \\ u_\mathsf{f} \\ 0 \end{bmatrix},$$

is solvable for any u_f and any particular solution $\bar{g}$ allows the computation of the response y_f of $\mathcal{S}$ due to the input u_f and zero initial conditions as $y_\mathsf{f} = Y_\mathsf{f}\bar{g}$.

Block algorithm

Theorem 1 gives a block algorithm for the computation of the response y_f

if $u_\mathsf{f} = \delta(t)e_i$, where e_i is the ith unit vector, $y_\mathsf{f} = H(:, i)$

let $\tilde{w}$ be finite, *e.g.*, $\tilde{w} = \big(\tilde{w}(1), \dots, \tilde{w}(T)\big)$

the persistency of excitation assumption implies that

$$l \leq \frac{T+1}{\mathtt{m}+1} - 2\mathtt{n}_{\max}$$

$\Rightarrow$ using the block algorithm, we are limited in the length of the response y_f that can be computed from $\tilde{w}$

it is possible, however, to find an arbitrary many samples of the response

Iterative algorithm

input: $\tilde{w} = (\tilde{u}, \tilde{y})$, u_f, $\mathtt{n}_{\max}$

1. initialization: choose l and set $k := 0$, $f_\mathsf{u}^{(0)} := \left[\begin{smallmatrix} 0 \\ u_\mathsf{f}(1:l) \end{smallmatrix}\right]$, $f_{\mathsf{y,p}}^{(0)} := 0$

2. repeat

2.1. solve $\left[\begin{smallmatrix} U_\mathsf{p} \\ U_\mathsf{f} \\ Y_\mathsf{p} \end{smallmatrix}\right] g^{(k)} = \left[\begin{smallmatrix} f_\mathsf{u}^{(k)} \\ f_{\mathsf{y,p}}^{(k)} \end{smallmatrix}\right]$ and let $y_\mathsf{f}^{(k)} := Y_\mathsf{f} g^{(k)}$

2.2. shift $f_\mathsf{u}^{(k+1)} := \left[\begin{smallmatrix} \sigma^l f_\mathsf{u}^{(k)} \\ u(kl+1:(k+1)l) \end{smallmatrix}\right]$, $f_{\mathsf{y,p}}^{(k+1)} := \sigma^l \left[\begin{smallmatrix} f_{\mathsf{y,p}}^{(k)} \\ y_\mathsf{f}^{(k)} \end{smallmatrix}\right]$

2.3. $k := k+1$

until $t < kl$ $\quad$ ($t :=$ # of samples in u_f)

output $y_\mathsf{f} := \operatorname{col}\big(y_\mathsf{f}^{(0)}, \dots, y_\mathsf{f}^{(k-1)}\big)$

Iterative algorithm

the iterative algorithm works by matching the initial conditions of the responses $y_{\mathrm{f}}^{(0)}, y_{\mathrm{f}}^{(1)}, \ldots$, so that they become sequential pieces of y_{f}

the parameter l can be chosen between 1 and $(T+1)/(\mathtt{m}+1) - 2\mathtt{n}_{\max}$

different values affect the numerical stability, efficiency, and sensitivity to noise in the data (illustrated by simulation example)

the freedom to choose the parameter l allows to relax the persistency of excitation condition of Theorem 1

Sequential free responses

sequential free resp.—a sequence of free responses with a corresponding sequence of initial conditions being a valid state sequence of $\mathcal{S}$

free responses can be computed from $\tilde{w}$ by solving the system

$$\begin{bmatrix} U_{\mathrm{p}} \\ Y_{\mathrm{p}} \\ U_{\mathrm{f}} \end{bmatrix} G = \begin{bmatrix} U_{\mathrm{p}} \\ Y_{\mathrm{p}} \\ 0 \end{bmatrix}, \qquad \text{and setting } Y_0 = Y_{\mathrm{f}} G \tag{1}$$

moreover, the Hankel structure of U_{p} and Y_{p} implies sequentiality of Y_0

this gives a block algorithm for the computation of Y_0

the general iterative algorithm can be used for the computation of Y_0 (with a small modification for the initial conditions)

Link with the oblique projection

consider the least squares least norm solution of (1)

$$\bar{G} = \begin{bmatrix} W_{\mathrm{p}}^{\top} & U_{\mathrm{f}}^{\top} \end{bmatrix} \begin{bmatrix} W_{\mathrm{p}} W_{\mathrm{p}}^{\top} & W_{\mathrm{p}} U_{\mathrm{f}}^{\top} \\ U_{\mathrm{f}} W_{\mathrm{p}}^{\top} & U_{\mathrm{f}} U_{\mathrm{f}}^{\top} \end{bmatrix}^{+} \begin{bmatrix} W_{\mathrm{p}} \\ 0 \end{bmatrix}, \quad W_{\mathrm{p}} := \begin{bmatrix} U_{\mathrm{p}} \\ Y_{\mathrm{p}} \end{bmatrix}$$

$\Rightarrow$ $Y_{\mathrm{f}} \bar{G} = Y_{\mathrm{f}} /_{U_{\mathrm{f}}} W_{\mathrm{p}}$, the oblique projection of Y_{f} along U_{f} on W_{p}

$\Rightarrow$ the oblique projection is an implementation of the block algorithm for the computation of the matrix Y_0 of sequential free responses

Simulation example

$\mathcal{S}$ is of order 3, $\quad\tilde{u}$ is a unit variance white noise, $\quad T = 100$

$\tilde{w} = (\tilde{u}, \tilde{y})$ is a trajectory of $\mathcal{S}$ + white noise with std. deviation σ

$e := \|Y_0 - \hat{Y}_0\|_F$, $\quad Y_0$ — exact, $\quad \hat{Y}_0$ — estimated from data

f — amount of operations in mega flops

	$\sigma = 0.0$		$\sigma = 0.2$		$\sigma = 0.4$	
Method	e	f	e	f	e	f
Iterative alg. with QR	10^{-14}	130	2.5257	132	4.7498	132
Obl. proj. from def.	10^{-10}	182	3.2063	187	6.0915	189
Obl. proj. with QR	10^{-14}	251	3.2063	251	6.0915	252

Conclusions

we presented:

- an algorithm for computation of an arbitrary response of an LTI system from an exact finite I/O trajectory of the system
- the algorithm is based on a fundamental fact $+$ a refinement that allows to construct a long response from several short ones
- a special case of the block version of the algorithm corresponds to the oblique projection
- the parameter l of the iterative alg. allows to relax the persistency of excitation condition and to gain numerical and statistical efficiency