What I saw at NIPS

Ivan Markovsky

University of Southampton

Classical sampling theory

- Shannon-Nyquist theorem: $\omega_{\text{sampling}} \geq 2\omega_{\text{Nyquist}}$
- A principle underlying most (all?) signal acquisition protocols and analog-to-digital converters.
- If the signal is not band-limited, it is low-pass filter (anti-aliasing) prior to being sampled.
- The NIPS tutorial of Emmanuel Candés (Caltech) presented the theory of compressive sampling.
- Sparse signals can be recovered from far fewer samples than required by the classical theory.

Sparse signal recovery

- Discrete-time signal $x \in \mathbb{R}^n$ is called sparse if it has (far) fewer degrees of freedom than samples n.
- Special case: x has only $k \ll n$ nonzero elements.
- Let $f = Ax$, with $\dim(f) = m \ll n$, be observations from x.
- Problem: recover x from f and A.
- Solution: for uniform random A, if $m \geq ck \log n$, then

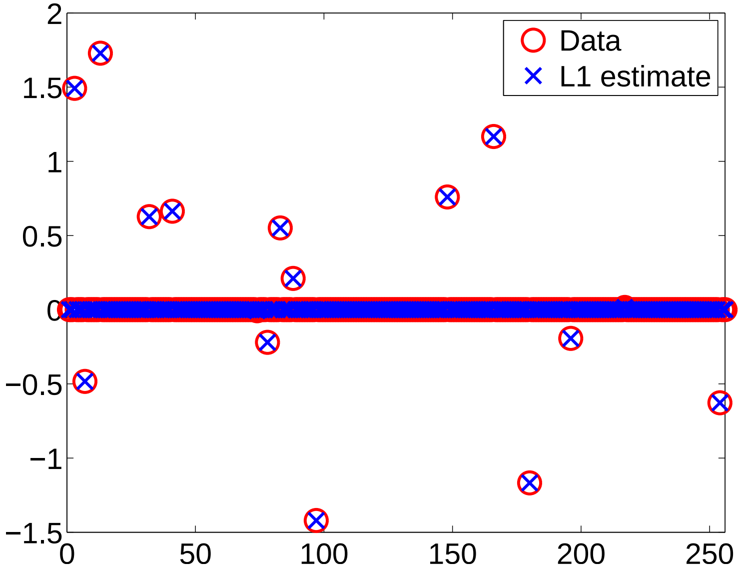
$$\widehat{x}^* := \arg\min \|\widehat{x}\|_1 \quad \text{subject to} \quad f = A\widehat{x}$$

  is equal to x with high probability.
- Sufficiently sparse x can be recovered exactly from a few samples f by solving a convex programme.

Simulation example

```
n = 256; m = 80; k = 16;

x = zeros(n,1); rp = randperm(n);
x(rp(1:k)) = randn(k,1);

A = randn(m,n);
f = A * x;

cvx_begin
    variable xh(n);
    minimize(norm(xh,1))
    subject to
    A*xh == f
cvx_end

norm(x - xh)
```

$\|x-\hat{x}\| = 10^{-8}$ — it works!



Low-rank matrix completion

- Problem: Recover rank k matrix $X \in \mathbb{R}^{p\times q}$ from $m < pq$ of its entries.
- E.g., infer the answers in a partially filled in survey (Netfix).
- More generally, $f = A\,\mathrm{vec}(X)$, with $m = \dim(f)$ is observed.
- In many cases X is low-rank, $k \ll n = \max(p,q)$. (Corresponds to the sparsity assumption in the 1D case.)
- Solution: For uniform random A, if $m \geq ckn^{5/4}\log n$,

$$\widehat{X}^* := \arg\min \|\widehat{X}\| \quad \text{subject to} \quad \widehat{f} = A\,\mathrm{vec}(\widehat{X})$$

  is equal to X with high probability.

Sensor network localization

- Consider n sensors placed at $x_1,\ldots,x_n$ in $\mathbb{R}^2$.
- The distance between ith and jth sensor is

$$d_{ij} = \|x_i - x_j\|$$

- Localization problem: given some d_{ij}'s find the rest.
- A matrix completion problem for $D = [d_{ij}] \in \mathbb{R}^{n\times n}$.
- Key observation: $\mathrm{rank}(D) \leq 4$

$$D = \mathbf{1}_n z^\top + z\mathbf{1}_n z^\top + 2XX^\top \quad \text{where} \quad \begin{matrix} z := \mathrm{col}(x_1^\top x_1,\ldots,x_n^\top x_n) \\ X := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \end{matrix}$$

Structure from motion

- Setup: Static object is observed through a moving camera and n points are tracked over m frames.
- Problem: reconstruct the 3D location of the points from their 2D image coordinates $w_{mn} = (u_{mn}, v_{mn})$.
- Theorem (Thomasi & Kanade, 1993): $\mathrm{rank}([w_{mn}]) \leq 4$
- In a rank revealing factorization $[w_{mn}] = MS + c\mathbf{1}^\top$
  - M contains the camera coordinates in 3D (motion)
  - S contains the point coordinates in 3D (the shape)
  - $c = \mathrm{col}(\frac{1}{n}\sum_{i=1}^n w_{1n},\ldots,\frac{1}{n}\sum_{i=1}^n w_{mn})$