ELEC 3035: Control systems design, Exam part I solutions

1. *Identification of an autonomous linear time-invariant system*

   (a) [15 marks] An nth order scalar autonomous linear time-invariant system can be represented by an nth order linear constant-coefficients difference equation

   $$p_0 y(t) + p_1 y(t+1) + \cdots + y(t+n) = 0, \qquad \text{for all } t \in \mathbb{Z}, \tag{1}$$

   where $p_0, p_1, \ldots, p_{n-1}$ are parameters of the model. Explain how to find the smallest n and corresponding parameters $p_0, p_1, \ldots, p_{n-1}$, such that the system defined by (1) is an exact model for a given sequence

   $$y_{\mathrm{d}} = \big(y_{\mathrm{d}}(1), \ldots, y_{\mathrm{d}}(T)\big),$$

   i.e., find the minimal n, for which there exist $p_0, p_1, \ldots, p_{n-1}$ satisfying the system of equations

   $$p_0 y_{\mathrm{d}}(t) + p_1 y_{\mathrm{d}}(t+1) + \cdots + y_{\mathrm{d}}(t+n) = 0, \qquad \text{for } t = 1, \ldots, T-n. \tag{2}$$

   (b) [5 marks] Apply the method of 1a on the sequence of the first nine Fibonacci numbers

   $$y_{\mathrm{d}} = (0, 1, 1, 2, 3, 5, 8, 13, 21).$$

   (c) [5 marks] Suppose that the system order n is fixed and there is no exact model for y_{d} of order n. How would you modify the method of step 1a for obtaining an approximate model? In what sense your approximate model approximates y_{d}?

   Solution:

   (a) For a given n, we aim to find parameters $p_0, p_1, \ldots, p_{n-1}$, so that (2) holds. Using matrix vector notation, the system of equations (2) can be written in a matrix form as

   $$\begin{bmatrix} p_0 & p_1 & \cdots & p_{n-1} & 1 \end{bmatrix} \begin{bmatrix} y_{\mathrm{d}}(1) & y_{\mathrm{d}}(2) & \cdots & y_{\mathrm{d}}(T-n) \\ y_{\mathrm{d}}(2) & y_{\mathrm{d}}(3) & \cdots & y_{\mathrm{d}}(T-n+1) \\ \vdots & \vdots & & \vdots \\ y_{\mathrm{d}}(n+1) & y_{\mathrm{d}}(n+2) & \cdots & y_{\mathrm{d}}(T) \end{bmatrix} = 0$$

   or in the standard form of a linear system of equations in the unknown parameters $p_0, p_1, \ldots, p_{n-1}$

   $$\underbrace{\begin{bmatrix} y_{\mathrm{d}}(1) & y_{\mathrm{d}}(2) & \cdots & y_{\mathrm{d}}(n) \\ y_{\mathrm{d}}(2) & y_{\mathrm{d}}(3) & \cdots & y_{\mathrm{d}}(n+1) \\ \vdots & \vdots & & \vdots \\ y_{\mathrm{d}}(T-n) & y_{\mathrm{d}}(T-n+1) & \cdots & y_{\mathrm{d}}(T-1) \end{bmatrix}}_{A_n} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{n-1} \end{bmatrix} = -\underbrace{\begin{bmatrix} y_{\mathrm{d}}(n+1) \\ y_{\mathrm{d}}(n+2) \\ \vdots \\ y_{\mathrm{d}}(T) \end{bmatrix}}_{b_n}. \tag{3}$$

   (The subscript n in A_n and b_n shows that the matrix A and the vector b depend on n.) Therefore, existence of an exact autonomous linear time-invariant model of order n for y_{d} is equivalent to existence of solution of the system (3).

   In order to find the smallest n, for which an exact model (1) exists, we need to iteratively check compatibility of the sequence of linear systems (3) obtained for increasing values of $n = 1, 2, \ldots$ and stop at the first instance when a compatible system is encountered. The resulting identification method is:

   - For $n = 1, 2, \ldots$
   - If $b_n \not\in \operatorname{Range}(A_n)$ (i.e., (3) has no solution) repeat.
   - Otherwise solve (3) and return the solution $p_0, \ldots, p_{n-1}$.

(b) For $n=1$,

$$A_1^\top = \begin{bmatrix} 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 \end{bmatrix} \qquad \text{and} \qquad b_1^\top = -\begin{bmatrix} 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 \end{bmatrix}.$$

Obviously $b_1 \notin \operatorname{Range}(A_1)$, so that there is no exact model of order 1.
For $n=2$,

$$A_2^\top = \begin{bmatrix} 0 & 1 & 1 & 2 & 3 & 5 & 8 \\ 1 & 1 & 2 & 3 & 5 & 8 & 13 \end{bmatrix} \qquad \text{and} \qquad b_2^\top = -\begin{bmatrix} 1 & 2 & 3 & 5 & 8 & 13 & 21 \end{bmatrix}.$$

We have $b_2 \in \operatorname{Range}(A_2)$, because

$$A_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix} = b_2,$$

so that

$$-y(t) - y(t+1) + y(t+2) = 0$$

defines an exact model for y_d.

(c) If a solution to (3) does not exist and the matrix A has full column rank, then we can approximately solve (3) in the least squares sense. The least squares approximate solution

$$x_\mathrm{ls} = (A_n^\top A_n)^{-1} A_n^\top b_n$$

defines an approximate model

$$x_{\mathrm{ls},1} y(t) + \cdots + x_{\mathrm{ls},n} y(t+n-1) + y(t+n) = 0$$

for y_d. The approximation is in the sense of minimising the residual

$$\|A_n x - b_n\|_2^2 = \sum_{t=1}^{T-n} \big(y_\mathrm{d}(t) + p_1 y_\mathrm{d}(t+1) + \cdots + p_n y_\mathrm{d}(t+n) \big)^2,$$

which is the same as the equation error in (1).

$\square$

2. *Reachability with bounded energy input*
Consider an nth order controllable discrete-time state space dynamical system defined by the difference equation

$$x(t+1) = Ax(t) + Bu(t).$$

We will call the squared 2-norm of the input sequence

$$U_T := \operatorname{col}\big(u(0), u(1), \dots, u(T-1)\big),$$

i.e., $\|U_T\|_2^2$ the "energy" of the input $\big(u(0), u(1), \dots, u(T-1)\big)$.

(a) [15 marks] Find the reachable set in $T \geq n$ seconds with bounded 2-norm input

$$\mathscr{R}_{T,\delta} := \{\, x(T) \mid x(t+1) = Ax(t) + Bu(t),\ x(0) = 0,\ \|U_T\|_2^2 \leq \delta \,\}.$$

You answer should be given in terms of the system parameters A, B, the time limit $T \geq n$, and the norm bound $\delta \geq 0$.

(b) [5 marks] Find the set of reachable states with bounded 2-norm input without time limit, i.e.,

$$\mathscr{R}_\delta := \lim_{T\to\infty} \mathscr{R}_{T,\delta}.$$

(c) [5 marks] What are the sets $\mathscr{R}_{T,\delta}$ and $\mathscr{R}_\delta$ in the cases of first and second order systems? Compute $\mathscr{R}_\delta$ for the system $x(t+1) = 0.5x(t) + u(t)$ and $\delta = 1$.

Solution:

(a) Without limit on the input signal's energy (equivalently, for $\delta = \infty$), the reachable set of a controllable system in $T \geq n$ seconds is $\mathbb{R}^n$, i.e.,

$$\mathscr{R}_{T,\infty} = \mathbb{R}^n, \qquad \text{for } T \geq n.$$

Therefore, for $T \geq n$ the only constraint on $\mathscr{R}_{T,\delta} \subseteq \mathbb{R}^n$ is imposed by the bound on the input energy

$$\mathscr{R}_{T,\delta} = \{\, x \in \mathbb{R}^n \mid \mathscr{E}_{\min,T}(x) \leq \delta \,\},$$

where $\mathscr{E}_{\min,T}(x)$ is the minimum energy required to reach x from the origin in at most T seconds. In class we derived the following expression for the minimum energy

$$\mathscr{E}_{\min,T}(x) = x^\top \Big(\sum_{t=0}^{T-1} A^t B B^\top (A^t)^\top \Big)^{-1} x,$$

so that

$$\mathscr{R}_{T,\delta} = \left\{ x \in \mathbb{R}^n \mid x^\top \Big(\sum_{t=0}^{T-1} A^t B B^\top (A^t)^\top \Big)^{-1} x \leq \delta \right\}.$$

(b) Without time limit and assuming that the system is stable

$$\mathscr{E}_{\min}(x) = x^\top G^{-1} x,$$

where the matrix G (the controllability Gramian) is a solution of the Lyapunov equation

$$AGA^\top - A = -BB^\top.$$

Therefore, assuming stability,

$$\mathscr{R}_{T,\delta} = \Big\{ x \in \mathbb{R}^n \mid x^\top G^{-1} x \leq \delta \Big\}.$$

(c) In the case of a first order system, $\mathscr{R}_{T,\delta}$ and $\mathscr{R}_\delta$ are intervals of $\mathbb{R}$ centered at the origin. The limit points of $\mathscr{R}_{T,\delta}$ are the solutions of the equation

$$\begin{aligned}
\mathscr{E}_{\min,T}(x) = \delta \quad & \Longrightarrow \quad x^2 \left(\sum_{t=0}^{T-1} a^{2t} b^2 \right)^{-1} = \delta \\
& \Longrightarrow \quad x^2 = \delta b^2 \sum_{t=0}^{T-1} a^{2t} \\
& \Longrightarrow \quad x^{1,2} = \pm b \sqrt{\delta \sum_{t=0}^{T-1} a^{2t}}.
\end{aligned}$$

Therefore,

$$\mathscr{R}_{T,\delta} = \left[-b \sqrt{\delta \sum_{t=0}^{T-1} a^{2t}},\ +b \sqrt{\delta \sum_{t=0}^{T-1} a^{2t}} \right].$$

For a stable system, the Lyapunov equation is

$$ga^2 - g = -b^2$$

and its solution is $g = b^2/(1-a^2)$. Therefore

$$\mathscr{R}_\delta = \begin{cases} \mathbb{R} & \text{if } |a| \geq 1 \\ [-b\sqrt{\delta/(1-a^2)},\ +b\sqrt{\delta/(1-a^2)}] & \text{otherwise.} \end{cases}$$

For the system $x(t+1) = 0.5x(t) + u(t)$ and $\delta = 1$,

$$\mathscr{R}_\delta = \left[-2/\sqrt{3},\ 2/\sqrt{3} \right].$$

In the case of a second order system $\mathscr{R}_{T,\delta}$ and $\mathscr{R}_{T,\delta}$ are ellipsoids in $\mathbb{R}^2$ centered at the origin defined by the controllability Gramian.

□

3. *Pole placement design*
   Design a dead-beat controller for the system defined by the difference equation

$$P(\sigma)y = Q(\sigma)u,$$

where $(\sigma y)(t) := y(t+1)$ is the shift operator,

$$P(z) = (z-1)^2 \qquad \text{and} \qquad Q(z) = z+1$$

using polynomial and state space methods.

   (a) *Design by polynomial methods* [10 marks]

      i. Define the desired closed-loop characteristic polynomial and the Diophantine equation corresponding to the controller input/output representation

$$R(\sigma)u = -S(\sigma)y.$$

      ii. Find the solution of the Diophantine equation of minimal degree.

      iii. Give the controller and the resulting closed-loop system.

   (b) *Design by state space methods* [10 marks]

      i. Derive a state space representation of the plant.

      ii. Calculate the state feedback gain.

      iii. Give the state-feedback controller and the resulting closed-loop system.

   (c) *Compare the results* [5 marks] Comment on the similarities and differences between the approaches used and results obtained in 3a and 3b. What is missing and would make the two approaches equivalent?

Solution:

   (a) *Design by polynomial methods*

      i. Since our goal is to derive a deadbeat controller, all poles of the closed-loop system should be at 0. Therefore, the desired characteristic polynomial of the closed-loop system is

$$P_{\text{cl}}(z) = z^{n_{\text{cl}}}$$

where n_{cl} is the order of the closed loop system.
Eliminating u from the plant and controller equations,

$$P(\sigma)y = Q(\sigma)u \qquad \text{and} \qquad R(\sigma)u = -S(\sigma)y.$$

we obtain a difference equation representation of the closed-loop system

$$\big(P(\sigma)R(\sigma) + Q(\sigma)S(\sigma)\big)y = 0.$$

The pole placement design aims to choose the controller polynomials R and S, so that the closed-loop characteristic polynomial is equal to the desired characteristic polynomial P_{cl}, i.e.,

$$P(z)R(z) + Q(z)S(z) = P_{\text{cl}}(z). \tag{4}$$

      ii. Controller of order zero, i.e., static feedback controller, can not achieve the desired pole location of a second order system. We attempt the simplest possible dynamic controller—the one of order one. In this case,

$$R(z) = z + R_0 \quad \text{and} \qquad S(z) = S_1 z + S_0.$$

Then equation (4) becomes

$$(\sigma^2 - 2\sigma + 1)(z + R_0) = (\sigma + 1)(S_1 z + S_0).$$

By equating the coefficients of the LHS polynomial to the coefficients of the RHS polynomial in the above equation, we obtain a system of three equations in the three unknowns R_0, S_1, and S_0

$$\begin{bmatrix} 1 & 1 & 0 \\ -2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} R_0 \\ S_0 \\ S_1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}.$$

It has a unique solution

$$R_0 = 3/4, \qquad S_0 = -3/4, \qquad S_1 = 5/4,$$

so that

$$R(z) = z + 3/4 \qquad \text{and} \qquad S(z) = -3/4z + 5/4.$$

is the solution of minimal degree to (4).

iii. The controller is defined by the difference equation

$$(\sigma + 3/4)u = -(3/4\sigma + 5/4)y$$

and the closed loop system is autonomous and is given by the difference equation

$$\sigma^3 y = 0.$$

(b) *Design by state space methods*

i. A state-space representation of the plant in the standard controllable form is

$$\sigma x = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 1 \end{bmatrix} x.$$

ii. The state-feedback controller is

$$u = -Kx,$$

where $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ is the state feedback gain matrix. Our goal is to select the parameters K_1 and K_2, so that the closed loop system has the desired pole locations.

A state space representation of the closed-loop system with the state-feedback controller is

$$\sigma x = (A - BK)x = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 - K_1 & 2 - K_2 \end{bmatrix}}_{A_c} x.$$

Since the matrix A_c is in a companion form, the closed-loop characteristic polynomial is

$$z^2 - (2 - K_2)z - (-1 - K_1).$$

In order to achieve the desired pole locations, we need to choose K_1 and K_2, so that

$$2 - K_2 = 0 \qquad \text{and} \qquad -1 - K_1 = 0,$$

Therefore

$$K = \begin{bmatrix} -1 & 2 \end{bmatrix}.$$

iii. The state-feedback controller is

$$u = \begin{bmatrix} 1 & -2 \end{bmatrix} x.$$

A state-space representation of the closed-loop system is

$$\sigma \hat{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \hat{x}$$

(c) *Compare the results*

The controller obtain in step 3a is output feedback dynamic of order one. The controller obtained in step 3b is state feedback static. In the state space approach we assumed that the state is measurable and did not derive an observer. With an observer, the state-feedback controller will also be output-feedback dynamic.

□