ELEC 3035, Lecture 4: Controllability and state transfer

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- Definition of controllability
- State transfer and controllability matrix
- Least norm (minimum energy) control
- Controllability of input/output systems

Intuition behind controllability

Controllability — a property of the system ensuring that

> the system can be transferred from any given state x_{ini} to any desired state x_{des} over a period of time by proper choice of the input u.

Examples:

- autonomous systems (except for $\mathscr{B} = \{0\}$) are uncontrollable
- $\mathscr{B} = \{ (u,x) \in (\mathbb{R}^{m+n})^T \mid \sigma x = u \}$ is obviously controllable
- How about $\mathscr{B} = \left\{ (u,x) \mid \sigma x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \right\}$?

Definition of controllability

$\mathscr{B}$ is controllable if for any

- given trajectories $w_{\text{ini}}, w_{\text{des}} \in \mathscr{B}$,
- there exists a trajectory $w_{\text{ctr}} \in \mathscr{B}$ and a $\tau > 0$,

such that

- $w_{\text{ctr}}(t) = w_{\text{ini}}(t)$, for all $t < 0$ and
- $w_{\text{ctr}}(t) = w_{\text{des}}$, for all $t \geq \tau$.

Think of w_{ini} as a given past traj. and w_{des} as a desired future traj.

$\mathscr{B}$ controllable $\implies$ any given traj. can be steered to any desired trajectory

Controllability of $\mathscr{B}_{\text{ss}}(A,B) = \{ (u,x) \mid \sigma x = Ax + Bu \}$

In this case,

$$x_{\text{ini}} \leftrightarrow w_{\text{ini}} \qquad \text{and} \qquad x_{\text{des}} \leftrightarrow w_{\text{des}}$$

so the controllability question is

Can we transfer any given state $x_{\text{ini}} \in \mathbb{R}^n$ to any desired state $x_{\text{des}} \in \mathbb{R}^n$?

Furthermore,

- How do we find a control that transfers the state from x_{ini} to x_{des}?
- How do we find an efficient control ($\|u\|$ small)?
- What states are reachable by an arbitrary control input?
- What states are reachable by constrained control input?

State trajectories

The trajectories of the system

$$\mathscr{B}_{\mathrm{ss}}(A,B) = \{\, (u,x) \mid \sigma x = Ax + Bu \,\}$$

are in the DT case

$$x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{t-1-\tau} Bu(\tau) \tag{1}$$

and in the CT case

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) \mathrm{d}\tau \tag{2}$$

DT–CT analogy: $A^t \leftrightarrow e^{At}$ and $\sum_{\tau=0}^{t-1} (\cdot) \leftrightarrow \int_0^t (\cdot) \mathrm{d}\tau$

Reachable set

Define the set of states reachable from $x(0) = 0$ in t seconds

$$\text{DT:} \qquad \mathscr{R}_t := \left\{ \sum_{\tau=0}^{t-1} A^{t-1-\tau} B u(\tau) \mid u : \{0, \ldots, t-1\} \to \mathbb{R}^m \right\}$$

$$\text{CT:} \qquad \mathscr{R}_t := \left\{ \int_0^t e^{A(t-\tau)} B u(\tau) \mathrm{d}\tau \mid u : [0, t] \to \mathbb{R}^m \right\}$$

and the reachable set

$$\mathscr{R} := \mathscr{R}_\infty$$

i.e., with no time limit on the state transfer.

Facts:

- $\mathscr{R}_t$ is a subspace of $\mathbb{R}^n$
- $\mathscr{R}_{t_1} \subseteq \mathscr{R}_{t_2}$ for $t_1 \leq t_2$

Controllability matrix of the system $\mathscr{B}_{\mathrm{ss}}(A,B)$

In the DT case

$$x(t) = \sum_{\tau=0}^{t-1} A^{t-1-\tau} B u(\tau) = \underbrace{\begin{bmatrix} B & AB & \cdots & A^{t-1}B \end{bmatrix}}_{\mathscr{C}_t} \begin{bmatrix} u(t-1) \\ \vdots \\ u(0) \end{bmatrix} = \mathscr{C}_t U_t$$

so that

$$\mathscr{R}_t = \mathrm{image}(\mathscr{C}_t), \qquad \text{for } t \geq 0$$

By the Caley-Hamilton theorem A^t, for $t \geq n$, can be expressed as linear combination of $A^0, A^1, \ldots, A^{n-1}$. Therefore,

$$\mathscr{R}_t = \mathrm{image}(\mathscr{C}_n), \qquad \text{for } t \geq n$$

The matrix

$$\mathscr{C} := \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

is called controllability matrix of the system $\mathscr{B}_{\mathrm{ss}}(A,B)$.

The state transfer problem

For $t > 0$,

$$x(t) = A^t x(0) + \mathscr{C}_t U_t$$

so with $x(0) = x_{\text{ini}}$ and $x(t) = x_{\text{des}}$, we have

$$\text{state transfer } x_{\text{ini}} \mapsto x_{\text{des}} \text{ in } t \text{ seconds} \quad \Longleftrightarrow \quad x_{\text{des}} - A^t x_{\text{ini}} \in \mathscr{R}_t$$

Therefore state transfer reduces to reachability.

$$\boxed{\mathscr{B}(A, B) \text{ is controllable if and only if } \mathscr{R}_t = \mathbb{R}^n}$$

Regulation problem — special case of state transfer when $x_{\text{des}} = 0$.

Control inputs for state transfer

Consider a controllable system $\mathscr{B}_{\mathrm{ss}}(A,B)$ and $t \geq n$.

Q: What input sequences

$$U_t := \mathrm{col}\big(u(t-1),\dots,u(0)\big)$$

achieve the state transfer $x_{\mathrm{ini}} \mapsto x_{\mathrm{des}}$?

A: Any solution of the system

$$x_{\mathrm{des}} - A^t x_{\mathrm{ini}} = \mathscr{C}_t U_t$$

Controllability implies that $\mathscr{C}_t \in \mathbb{R}^{n\times tm}$ is full row rank,
so if $tm > n$, there are ∞ many solutions.

General solution: $\quad \mathscr{U} := \{\, U = U_{\mathrm{particular}} + z \mid z \in \ker(\mathscr{C}_t)\,\}$

Minimum energy state transfer

Among all solutions in $\mathscr{U}$, the least norm solution

$$U_{\ln,t} := \mathscr{C}_t^\top \left(\mathscr{C}_t \mathscr{C}_t^\top\right)^{-1} \left(x_{\text{des}} - A^t x_{\text{ini}}\right)$$
$$= \mathscr{C}_t^\top \left(\sum_{\tau=0}^{t-1} A^\tau B B^\top (A^\tau)^\top\right)^{-1} \left(x_{\text{des}} - A^t x_{\text{ini}}\right)$$

minimizes the 2-norm of the control signal.

$\|U_{\ln,t}\|_2^2$ is related to the control energy needed for state transfer.

The minimum "energy" needed for $x_{\text{ini}} \mapsto x_{\text{des}}$ in t seconds is

$$\mathscr{E}_{\min} := \|U_{\ln,t}\|_2^2 = \left(x_{\text{des}} - A^t x_{\text{ini}}\right)^\top \left(\sum_{\tau=0}^{t-1} A^\tau B B^\top (A^\tau)^\top\right)^{-1} \left(x_{\text{des}} - A^t x_{\text{ini}}\right)$$

Controllability Gramian

$\mathscr{E}_{\text{min}}$ shows how "hard" is to transfer the state and depends on t.

Assuming that the system is stable

$$G_{\text{c}} := \lim_{t \to \infty} \left(\sum_{\tau=0}^{t-1} A^{\tau} B B^{\top} (A^{\tau})^{\top} \right)$$

exists and gives the minimum energy

$$\mathscr{E}_{\text{min}} = \left(x_{\text{des}} - A^{t} x_{\text{ini}}\right)^{\top} G_{\text{c}}^{-1} \left(x_{\text{des}} - A^{t} x_{\text{ini}}\right)$$

for state transfer without time limit.

G_{c} is called the controllability Gramian of the system $\mathscr{B}_{\text{ss}}(A, B)$.
It satisfies the matrix equation

$$A G_{\text{c}} A^{\top} - G_{\text{c}} = -B B^{\top} \qquad \text{DT Lyapunov equation}$$

Continuous-time systems

The CT reachable set in t seconds is defined as

$$\mathscr{R}_t := \left\{ \int_0^t e^{A(t-\tau)} B u(\tau) \mathrm{d}\tau \mid u : [0,t] \to \mathbb{R}^m \right\}$$

It turns out that

$$\mathscr{R}_t = \text{image}(\mathscr{C}_t), \qquad \text{for } t > 0.$$

$\Longrightarrow$ the controllability condition in CT is the same as in DT

$$\mathscr{B}_{\text{ss}}(A,B) \text{ is controllable} \quad \Longleftrightarrow \quad \text{image}(\mathscr{C}_t) = \mathbb{R}^n$$

In CT any reachable x_{des} can be reached as fast as desired by large u.

Continuous-time minimum energy state transfer

$$u_{\text{ln}}(\tau) = B^\top (e^{A\tau})^\top \Bigg(\underbrace{\int_0^\tau e^{As} BB^\top (e^{As})^\top \mathrm{d}s}_{G_{\mathrm{c},t}} \Bigg)^{-1} \big(x_{\text{des}} - e^{A\tau} x_{\mathrm{ini}}\big), \text{ for } \tau \in [0, t]$$

Minimum energy for state transfer in t seconds

$$\int_0^t \|u_{\text{ln}}(\tau)\|_2^2 \mathrm{d}\tau = \big(x_{\text{des}} - e^{A\tau} x_{\mathrm{ini}}\big)^\top G_{\mathrm{c},t}^{-1} \big(x_{\text{des}} - e^{A\tau} x_{\mathrm{ini}}\big)$$

For a stable system

$$G_{\mathrm{c}} := \lim_{t\to\infty} \left(\int_0^s e^{A\tau} BB^\top (e^{A\tau})^\top \mathrm{d}\tau \right)$$

exists and satisfies the CT Lyapunov equation

$$AG_{\mathrm{c}} + G_{\mathrm{c}} A^\top = -BB^\top$$

Example

Consider the second order system

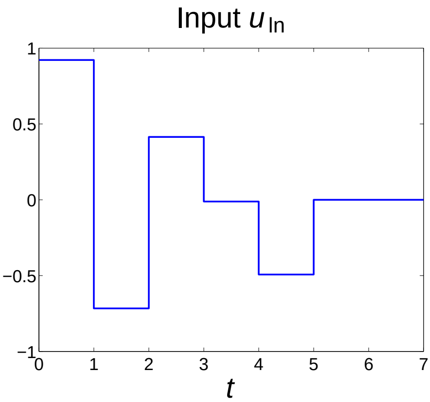
$$\mathscr{B}_{\mathrm{ss}}(A,b) = \left\{ (u,x) \mid \sigma x = \underbrace{\begin{bmatrix} -1.75 & -0.8 \\ 1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{b} u \right\}$$

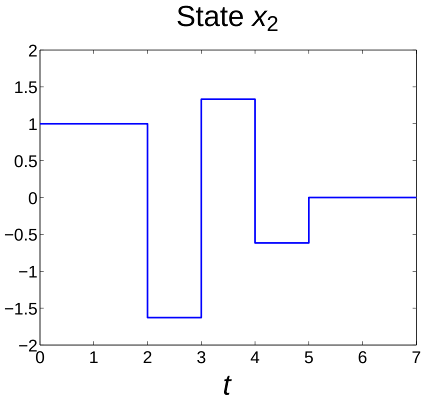
The controllability matrix is

$$\mathscr{C} = \begin{bmatrix} b & Ab \end{bmatrix} = \begin{bmatrix} 1 & -1.75 \\ 0 & 1 \end{bmatrix}$$

Is this system controllable?

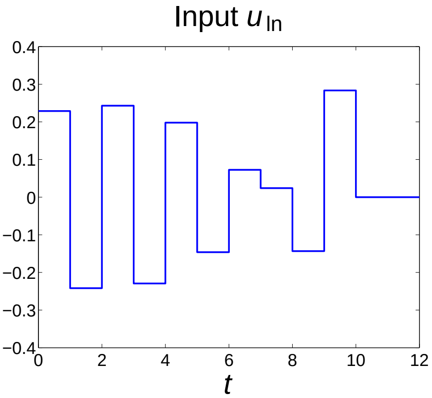
Minimum energy state transfer $\begin{bmatrix}1\\1\end{bmatrix} \mapsto \begin{bmatrix}0\\0\end{bmatrix}$ in $t = 5$ sec.

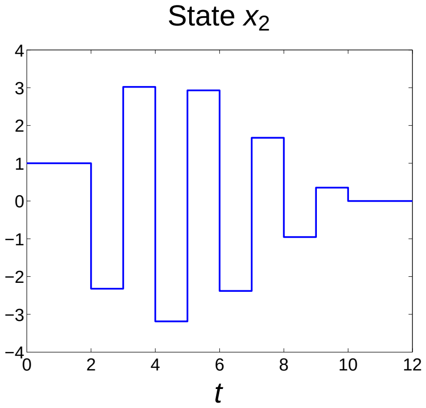




$$\mathscr{E}_{\text{min},5} = 1.7774$$

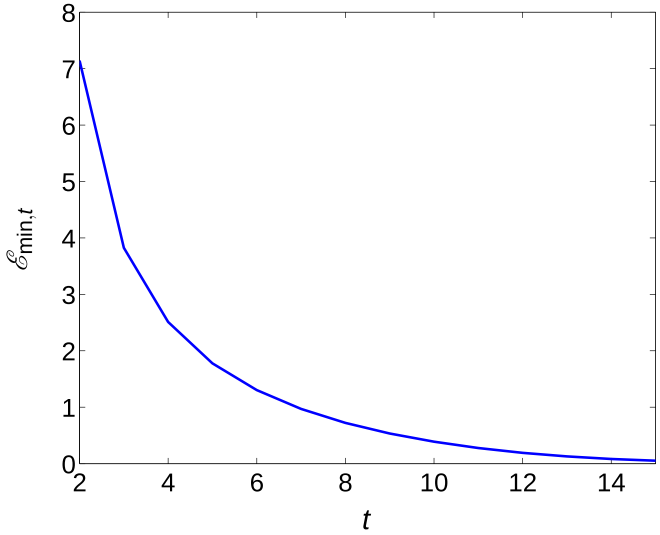
Minimum energy state transfer $\begin{bmatrix}1\\1\end{bmatrix} \mapsto \begin{bmatrix}0\\0\end{bmatrix}$ in $t = 10$ sec.



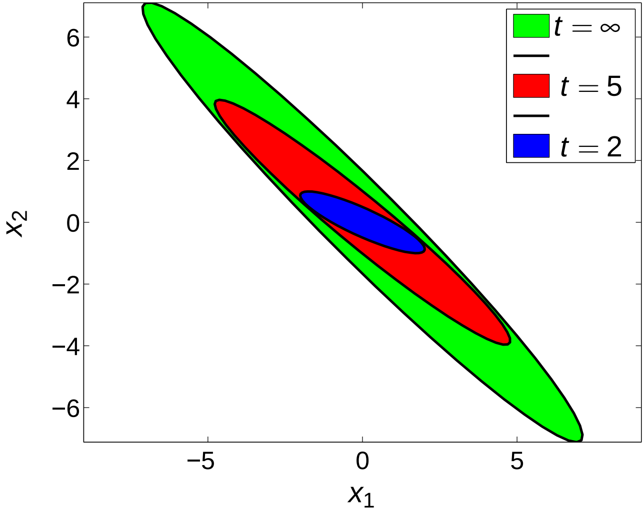


$$\mathscr{E}_{\mathrm{min},10} = 0.3898$$

Minimum control energy as a function of time



Reachable states with unit energy input



Controllability of
$\mathscr{B}_{\mathrm{i/o}}(P,Q) = \{\, (u,y) \mid P(\sigma)y = Q(\sigma)u \,\}$

Fact: $\mathscr{B}_{\mathrm{i/o}}(P,Q)$ is controllable iff P and Q have no common factor D, degree$(D) \geq 1$.

Consider the SISO case:

D is a common divisor of p and q iff there are $\overline{p}$ and $\overline{q}$, such that

$$p = d\overline{p} \qquad \text{and} \qquad q = d\overline{q}$$

Next we write these equations in a matrix form, which gives a

linear algebra condition for controllability of $\mathscr{B}_{\mathrm{i/o}}(p,q)$.

polynomial $\times$ polynomial $\iff$ Toeplitz matrix $\times$ vector

$$c(z) = a(z)b(z) \quad \iff \quad \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{\ell_c} \end{bmatrix} = \begin{bmatrix} a_0 & & & \\ a_1 & a_0 & & \\ \vdots & a_1 & \ddots & \\ a_{\ell_a} & \vdots & \ddots & a_0 \\ & a_{\ell_a} & & a_1 \\ & & \ddots & \vdots \\ & & & a_{\ell_a} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{\ell_b} \end{bmatrix}$$

$$\iff: \quad c = S_{\ell_b}(a)b \quad \iff \quad c = S_{\ell_a}(b)a$$

polynomial $c(z) \in \mathbb{R}[z]$, $\deg(c) = \ell_c$ $\longleftrightarrow$ vector $c \in \mathbb{R}^{\ell_c+1}$

polynomial operations $\longleftrightarrow$ structured matrix operations

$$p \in \mathbb{R}[z] \text{ and } q \in \mathbb{R}[z] \text{ have common divisor } d \in \mathbb{R}[z],\ \deg(d) = \ell_d \iff \begin{array}{c} \exists\ \overline{p} \in \mathbb{R}[z],\ \deg(\overline{p}) = \ell_p - \ell_d \\ \exists\ \overline{q} \in \mathbb{R}[z],\ \deg(\overline{q}) = \ell_q - \ell_d \\ \text{such that } p = d\overline{p} \text{ and } q = d\overline{q} \end{array}$$

$$\iff \quad q\overline{p} - p\overline{q} = 0$$

$$\iff \quad \begin{bmatrix} S_{\ell_{\overline{p}}}(q) & S_{\ell_{\overline{q}}}(p) \end{bmatrix} \begin{bmatrix} \overline{p} \\ -\overline{q} \end{bmatrix} = 0$$

$$\iff \quad \begin{bmatrix} S_{\ell_{\overline{p}}}(q) & S_{\ell_{\overline{q}}}(p) \end{bmatrix} \text{ is rank deficient}$$

($\begin{bmatrix} S_{\ell_{\overline{p}}}(q) & S_{\ell_{\overline{q}}}(p) \end{bmatrix}$ is $(\ell_p + \ell_q + 1 - \ell_d) \times (\ell_p + \ell_q + 2 - 2\ell_d)$)

Controllability test for $\mathscr{B}_{\mathrm{i/o}}(P,Q)$

GCD = greatest common divisor

Theorem The degree of the GCD d of p and q is equal to the rank deficiency of the Sylvester matrix $\begin{bmatrix} S_{\ell_p}(q) & S_{\ell_q}(p) \end{bmatrix}$, *i.e.*,

$$\deg(d) = \ell_p + \ell_q - \operatorname{rank}\left(\begin{bmatrix} S_{\ell_p}(q) & S_{\ell_q}(p) \end{bmatrix} \right).$$

Corollary $\mathscr{B}_{\mathrm{i/o}}(P,Q)$ is controllable iff $\begin{bmatrix} S_{\ell_p}(q) & S_{\ell_q}(p) \end{bmatrix}$ is full rank.