ELEC 3035: Review of Part I

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- Representations
- Autonomous systems and stability
- Controllability and observability
- Pole placement

Input/output representation $\mathscr{B}_{\mathrm{i/o}}(P,Q)$

The difference equation in DT

$$P_0 y(t) + P_1 y(t+1) + \cdots + P_n y(t+n) = Q_0 u(t) + Q_1 u(t+1) + \cdots + Q_m u(t+m)$$

or differential equation in CT

$$P_0 y(t) + P_1 \frac{\mathrm{d}}{\mathrm{d}t} y(t) + \cdots + P_n \frac{\mathrm{d}^n}{\mathrm{d}t^n} y(t+n) = Q_0 u(t) + \frac{\mathrm{d}}{\mathrm{d}t} Q_1 u(t+1) + \cdots + Q_m \frac{\mathrm{d}^m}{\mathrm{d}t^m} u(t+m)$$

where $m \leq n$ defines linear time-invariant (LTI) system

The class of system that admit such a repr. is called finite dimensional.

Transfer function

Consider a system $\mathscr{B}_{\mathrm{i/o}}(P,Q)$ and let $\mathscr{L}$ be the Laplace transform.

$$P(\sigma)y = Q(\sigma)u \quad \Longrightarrow \quad P(s)Y(s) = Q(s)U(s)$$

where $Y := \mathscr{L}(y)$ and $U := \mathscr{L}(u)$.

The rational function

$$Y(s)U^{-1}(s) = P^{-1}(s)Q(s) =: H(s)$$

is called transfer function.

In the SISO case

$$\frac{Y(s)}{U(s)} = \frac{Q(s)}{P(s)} =: h(s).$$

Input/state/output (I/S/O) representation

A finite dimensional LTI system admits a representation via

$$\sigma x = Ax + Bu, \qquad y = Cx + Du$$

- x — an auxiliary variable called state
- $\mathtt{n} := \dim(x)$ — state dimension, $\mathbb{R}^{\mathtt{n}}$ — state space
- $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$, $C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$, $D \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$ — parameters of $\mathscr{B}$
- $\mathtt{m} := \dim(u)$ — input dimension, $\mathtt{p} := \dim(y)$ — output dimension

single input single output (SISO) systems — $\dim(u) = \dim(y) = 1$
multi input multi output (MIMO) systems — $\dim(u) \geq 1$, $\dim(y) \geq 1$

Nonuniqueness of an I/S/O representation

There are two sources of nonuniqueness of an I/S/O representation:

1. redundant states — $\mathtt{n} := \dim(x)$ bigger than "necessary"
2. nonuniqueness of A, B, C, D — choice of state space basis

minimal I/S/O representations — $\dim(x)$ is as small as possible

For any nonsingular matrix $T \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ and

$$\widetilde{A} = T^{-1}AT, \quad \widetilde{B} = T^{-1}B, \quad \widetilde{C} = CT, \quad \widetilde{D} = D$$

we have that

$$\mathscr{B}_{\text{i/s/o}}(A,B,C,D) = \mathscr{B}_{\text{i/s/o}}(\widetilde{A},\widetilde{B},\widetilde{C},\widetilde{D}).$$

Nonuniqueness of an I/O representation

There are two sources of nonuniqueness of an I/O representation:

1. redundant equations — $g := \operatorname{row}\dim(P)$ bigger than "necessary"
2. nonuniquencess of P, Q — equivalence of equations

minimal I/O representations — $\operatorname{row}\dim(P)$ is as small as possible

In the single output case, P, Q are unique up do a scaling factor, *i.e.*,

$$\widetilde{P} = \alpha P, \qquad \widetilde{Q} = \alpha Q, \qquad \text{for } \alpha \in \mathbb{R}$$

we have that

$$\mathscr{B}_{\text{i/o}}(P,Q) = \mathscr{B}_{\text{i/o}}(\widetilde{P},\widetilde{Q}).$$

For multi output systems the nonuniqueness of P, Q is more essential.

I/S/O $\mapsto$ transfer function

The transfer function corresponding to a system $\mathscr{B}_{\text{i/s/o}}(A,B,C,D)$ is

$$H(s) = C(sI - A)^{-1}B + D.$$

For the opposite direction "transfer function $\mapsto$ I/S/O", see page 23.

State transition matrix

The dynamics of the sate vector x of the system

$$\sigma x = Ax, \qquad y = Cx$$

is given by the equation

$$x(t_2) = \Phi(t_2 - t_1)x(t_1)$$

where $\Phi(t) = A^t$ in DT and $\Phi(t) = e^{At}$ in CT.

The matrix $\Phi(t)$ is called state transition matrix.

$\Phi(t)$ shows how the initial state $x(t_1)$ is propagated in $t_1 + t$ time steps

Note: if $t < 0$, $\Phi(t)$ propagates backwards in time.

State construction

Consider a scalar autonomous system defined by the equation

$$P_0 y + P_1 \sigma y + \cdots + P_{n-1}\sigma^{n-1} y + I\sigma^n y = 0.$$

How can we represent this system in a state space form

$$\sigma x = Ax, y = Cx$$

Choose, for example, $x(t) = \operatorname{col}\big(y(t-1), \ldots, y(t-n)\big)$. Then

$$A = \begin{bmatrix} -P_{n-1} & -P_{n-2} & \cdots & -P_1 & -P_0 \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & I & 0 \end{bmatrix} \qquad \text{companion matrix of } P$$

$$C = \begin{bmatrix} -P_{n-1} & -P_{n-2} & \cdots & -P_1 & -P_0 \end{bmatrix}$$

Characteristic polynomial of a matrix

The polynomial equation

$$\det(\lambda I_n - A) = c_0\lambda^0 + c_1\lambda^1 + \cdots + c_n\lambda^n = 0$$

is called the characteristic equation of the matrix $A \in \mathbb{R}^{n\times n}$.

The roots of the characteristic polynomial

$$c(z) = c_0 z^0 + c_1 z^1 + \cdots + c_n z^n$$

are equal to the eigenvalues of A.

Cayley-Hamilton thm: Every matrix satisfies its own char. polynomial

$$c_0 A^0 + c_1 A^1 + \cdots + c_n A^n = 0.$$

Stability

An autonomous system

$$\mathscr{B} = \{\, x \mid \sigma x = f(x) \,\}$$

is stable if $x \in \mathscr{B}$ implies $x(t) \to 0$ as $t \to \infty$.

For a linear time-invariant system,

$$\mathscr{B} = \{\, x \mid \sigma x = Ax \,\}$$

the eigenvalues of A determine the stability property of the system.

CT LTI system is stable iff all eigenvalues have negative real parts.

DT LTI system is stable iff all eigenvalues have absolute value < 1.

State trajectories

The trajectories of the system

$$\mathscr{B}_{\mathrm{ss}}(A,B) = \{\, (u,x) \mid \sigma x = Ax + Bu \,\}$$

are in the DT case

$$x(t) = A^t x(0) + \sum_{\tau=0}^{t-1} A^{t-1-\tau} B u(\tau) \tag{1}$$

and in the CT case

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau)\,\mathrm{d}\tau \tag{2}$$

DT–CT analogy: $\quad A^t \leftrightarrow e^{At} \quad$ and $\quad \sum_{\tau=0}^{t-1} (\cdot) \leftrightarrow \int_0^t (\cdot)\,\mathrm{d}\tau$

Controllability and state transfer

Controllability — a property of the system ensuring that

> the system can be transferred from any given state x_{ini} to any desired state x_{des} over a period of time by proper choice of the input u.

$\mathscr{B}(A,B)$ is controllable if and only if $\mathscr{R}_t = \mathbb{R}^n$

Minimum energy state transfer:

$$U_{\text{ln},t} = \mathscr{C}_t^\top \left(\mathscr{C}_t \mathscr{C}_t^\top\right)^{-1} \left(x_{\text{des}} - A^t x_{\text{ini}}\right)$$

The minimum "energy" needed for $x_{\text{ini}} \mapsto x_{\text{des}}$ in t seconds is

$$\mathscr{E}_{\text{min}} := \|U_{\text{ln},t}\|_2^2 = \left(x_{\text{des}} - A^t x_{\text{ini}}\right)^\top \left(\sum_{\tau=0}^{t-1} A^\tau B B^\top (A^\tau)^\top\right)^{-1} \left(x_{\text{des}} - A^t x_{\text{ini}}\right)$$

Controllability Gramian

$\mathscr{E}_{\text{min}}$ shows how "hard" is to transfer the state and depends on t.

Assuming that the system is stable

$$G_{\text{c}} := \lim_{t\to\infty} \left(\sum_{\tau=0}^{t-1} A^\tau B B^\top (A^\tau)^\top\right)$$

exists and gives the minimum energy

$$\mathscr{E}_{\text{min}} = \left(x_{\text{des}} - A^t x_{\text{ini}}\right)^\top G_{\text{c}}^{-1} \left(x_{\text{des}} - A^t x_{\text{ini}}\right)$$

for state transfer without time limit.

G_{c} is called the controllability Gramian of the system $\mathscr{B}_{\text{ss}}(A,B)$.
It satisfies the matrix equation

$$A G_{\text{c}} A^\top - G_{\text{c}} = -BB^\top \qquad \text{DT Lyapunov equation}$$

Controllability test for $\mathscr{B}_{\text{i/o}}(P,Q)$

GCD = greatest common divisor

Theorem The degree of the GCD d of p and q is equal to the rank deficiency of the Sylvester matrix $\begin{bmatrix} S_{\ell_p}(q) & S_{\ell_q}(p) \end{bmatrix}$, *i.e.*,

$$\deg(d) = \ell_p + \ell_q - \operatorname{rank}\left(\begin{bmatrix} S_{\ell_p}(q) & S_{\ell_q}(p) \end{bmatrix}\right).$$

Corollary $\mathscr{B}_{\text{i/o}}(P,Q)$ is controllable iff $\begin{bmatrix} S_{\ell_p}(q) & S_{\ell_q}(p) \end{bmatrix}$ is full rank.

Output trajectories

The trajectories of the system

$$\mathscr{B}_{\text{i/s/o}}(A,B,C,D) = \{\, (u,x) \mid \sigma x = Ax + Bu,\ y = Cx + Du \,\}$$

are in the DT case

$$y(t) = CA^t x(0) + C \sum_{\tau=0}^{t-1} A^{t-1-\tau} B u(\tau) + Du(t)$$

and in the CT case

$$y(t) = Ce^{At} x(0) + C \int_0^t e^{A(t-\tau)} B u(\tau) \mathrm{d}\tau + Du(t)$$

DT–CT analogy: $\quad A^t \leftrightarrow e^{At} \quad$ and $\quad \sum_{\tau=0}^{t-1} (\cdot) \leftrightarrow \int_0^t (\cdot) \mathrm{d}\tau$

Observability of DT systems

Suppose we have observed u and y over the period $[0, t-1]$.

The system of equations

$$y(\tau) = CA^{\tau}x(0) + C\sum_{s=0}^{\tau-1} A^{\tau-1-s}Bu(s) + Du(\tau), \qquad \text{for } \tau = 0, 1, \ldots, t-1$$

written in a matrix form is

$$\underbrace{\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(t-1) \end{bmatrix}}_{Y_t} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{t-1} \end{bmatrix}}_{\mathscr{O}_t} x(0) + \underbrace{\begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ CA^{t-1}B & \cdots & CB & D \end{bmatrix}}_{\mathscr{T}_t} \underbrace{\begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t-1) \end{bmatrix}}_{U_t}$$

$$Y_t = \mathscr{O}_t x(0) + \mathscr{T}_t U_t$$

- $\mathscr{O}_t$ maps the initial state to the output over $[0, t-1]$
- $\mathscr{T}_t$ maps the input to the output over $[0, t-1]$

Estimating the initial state requires to solve for x_0

$$\mathscr{O}_t x(0) = Y_t - \mathscr{T}_t U_t$$

Therefore, x_0 can be reconstructed uniquely if and only if $\ker(\mathscr{O}_t) = \{0\}$

$\mathscr{B}_{\mathrm{i/s/o}}(A,B,C,D)$ is observable if and only if $\ker(\mathscr{O}) = \{0\}$, where

$$\mathscr{O} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Least squares observer

Assume that the output is observed with measurement noise v, *i.e.*,

$$y = Cx + Du + v$$

Then the system of equations for x_0

$$\mathscr{O}_t x(0) = Y_t - \mathscr{T}_t U_t$$

(generically) has no exact solution. The least-squares observer is

$$\widehat{x}_{\mathrm{ls}}(0) = \underbrace{(\mathscr{O}_t^{\top}\mathscr{O}_t)^{-1}\mathscr{O}_t}_{F_{\mathrm{ls}}}(Y_t - \mathscr{T}_t U_t)$$

It minimizes the output estimation error $\|Y - \widehat{Y}\|_2$, where

$$\widehat{Y} := \mathscr{O}_t \widehat{x}_{\mathrm{ls}}(0) + \mathscr{T}_t U_t$$

Duality between observability and controllability

The system $\mathscr{B}(A^{\top}, C^{\top}, B^{\top}, D^{\top})$ is called the dual of $\mathscr{B}_{\mathrm{i/s/o}}(A,B,C,D)$.

The observability matrix of $\mathscr{B}_{\mathrm{i/s/o}}(A,B,C,D)$ is

$$\mathscr{O}(A,C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \mathscr{C}^{\top}(A^{\top}, C^{\top})$$

equal to the transposed of the controllability matrix of

$$\mathscr{B}(A^{\top}, C^{\top}, B^{\top}, D^{\top})$$

Motivation for pole placement

- The desired dynamics is specified by the pole locations

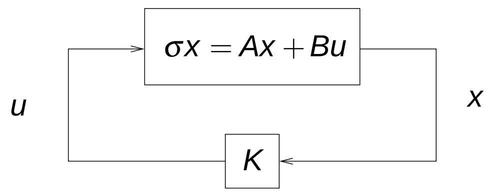
$$\{z_{\text{des},1},\dots,z_{\text{des},n_{\text{cl}}}\}$$

  of the closed-loop system
- or equivalently by the characteristic polynomial

$$p_{\text{des}}(z)=\prod_{i=1}^{n_{\text{cl}}}(z-z_{\text{des},i})=p_{\text{des},0}+p_{\text{des},1}z+\cdots+p_{\text{des},n_{\text{cl}}}z^{n_{\text{cl}}}$$

  of the closed-loop system.
- Example: in deadbeat control $z_{\text{des},i}=0$, for all i, *i.e.*, $p_{\text{des}}(z)=z^{n_{\text{cl}}}$
- The aim of pole placement control is to choose the feedback so that the closed-loop system achieve the desired char. polynomial

State feedback



$$\sigma x = Ax+Bu,\quad u=Kx \qquad \Longrightarrow \qquad \sigma x=(A+KB)u$$

The closed-loop system is autonomous with state matrix

$$A_{\text{c}}=A+KB.$$

Pole placement by state-feedback aims to choose K, so that

$$\det(zI-A_{\text{c}})=p_{\text{des}}(z)$$

Controller canonical form (SISO case)

Fact: Any controllable system $\mathscr{B}_{\text{i/o}}(p,q)$ can be represented in a state space form $\mathscr{B}_{\text{i/s/o}}(A,b,c,d)$ with parameters

$$A=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \ddots & \vdots\\ \vdots & & \ddots & \ddots & 0\\ 0 & \cdots & \cdots & 0 & 1\\ -p_0 & -p_1 & \cdots & \cdots & -p_{n-1}\end{bmatrix},\qquad b=\begin{bmatrix}0\\0\\ \vdots\\1\end{bmatrix}$$

$$c=\begin{bmatrix}c_0 & c_1 & \cdots & c_{n-1}\end{bmatrix},\qquad d=q_n$$

where $c_0,c_1,\dots,c_{n-1}$ are the coefficients of $q(z)-q_np(z)$

Similarity transformation for controller canonical form

A more general result:

Lemma:

- Let A,b and A',b' be two controllable pairs and
- assume that A and A' have the same char. polynomials.

Then there is a unique similarity transformation given by the matrix

$$T:=\mathscr{C}(A,b)\big(\mathscr{C}(A',b')\big)^{-1}$$

such that

$$T^{-1}AT=A' \qquad \text{and} \qquad T^{-1}b=b'.$$

$\Longrightarrow$ Any controllable representation of the system can be transformed to the controller canonical form.

State-feedback pole placement in controller form

Let the plant be given by $\mathscr{B}_{\mathrm{i/s/o}}(A, b, c, d)$, with A, b in controller form.

Then

$$A_{\mathrm{cl}} := A + bk = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ k_1 - p_0 & k_2 - p_1 & \cdots & \cdots & k_n - p_{n-1} \end{bmatrix}$$

and the closed-loop characteristic polynomial is

$$p_{\mathrm{cl}}(z) = (p_0 - k_1) + (p_1 - k_2)z + \cdots + (p_{n-1} - k_n)z^{n-1} + z^n$$

The equation $p_{\mathrm{cl}} = p_{\mathrm{des}}$ has the unique solution

$$k_1 = p_0 - p_{\mathrm{des},0}, \cdots, k_n = p_{n-1} - p_{\mathrm{des},n-1}$$

Observer design

The observer design is based on the following principles:

1. Internal model: the model run by u, gives an estimate $\hat{x}$ for x
2. Feedback: correct the estimate $\hat{x}$, so that the error

$$x(t) - \hat{x}(t) =: e(t) \to 0 \qquad \text{as} \quad t \to \infty$$

Let the feedback be a linear function of the output error

$$\text{feedback correction} = L(y - \hat{y})$$

Then the observer for the model $\mathscr{B}_{\mathrm{i/s/o}}(A, B, C, D)$ is

$$\begin{aligned} \sigma\hat{x} &= A\hat{x} + Bu - L(y - \hat{y}) \\ \hat{y} &= C\hat{x} + Du \end{aligned}$$

Observer design by pole placement

The condition $e(t) \to 0$ as $t \to \infty$ is a minimum requirement.

In fact we want $e(t) \to 0$ fast

(possibly in a finite (small) number of steps $\rightsquigarrow$ deadbeat observer)

The error dynamics is governed by the poles of the matrix

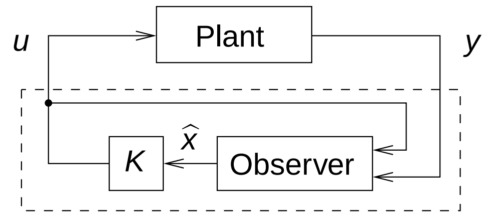
$$A_{\mathrm{o}} := A + LC$$

so for desired error dynamics we can

select desired pole locations of A_{o} and choose L to achieve them.

Closed-loop system with output feedback controller

Consider the closed loop system



Output feedback controller

where

Plant: $\sigma x = Ax + Bu, \quad y = Cx + Du$

Observer: $\sigma\hat{x} = A\hat{x} + Bu - L(y - C\hat{x} - Du)$

State feedback controller: $u = K\hat{x}$

Feedback controller:

$$\sigma\widehat{x} = (A+LC)\widehat{x} + (B+LD)u - Ly, \qquad u = K\widehat{x}$$
$$= (A+LC+BK+LDK)\widehat{x} - Ly$$

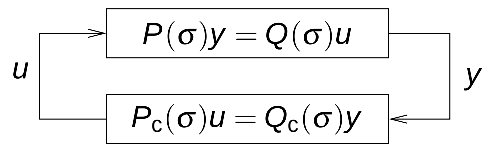
Note: the feedback controller is a dynamical system

Closed-loop system: $$\sigma\begin{bmatrix} x \\ \widehat{x}\end{bmatrix} = \begin{bmatrix} A & BK \\ -LC & A+LC+BK\end{bmatrix}\begin{bmatrix} x \\ \widehat{x}\end{bmatrix}$$

Note: closed-loop system order $=$ plant order + controller order

Error equation: $$\sigma\begin{bmatrix} x \\ e\end{bmatrix} = \begin{bmatrix} A+BK & -BK \\ 0 & A+LC\end{bmatrix}\begin{bmatrix} x \\ e\end{bmatrix}$$

Polynomial approach to pole placement



Plant: $P(\sigma)y = Q(\sigma)u$

Controller: $P_c(\sigma)u = Q_c(\sigma)y$

The closed-loop system is autonomous. In the SISO case

Closed-loop system: $(p_c(\sigma)p(\sigma) - q_c(\sigma)q(\sigma))y = 0$

and the closed-loop characteristic polynomial is

$$p_{cl}(z) := p_c(z)p(z) - q_c(z)q(z)$$

Diophantine equation

For SISO pole placement we need to solve the polynomial equation

$$p_c(z)p(z) - q_c(z)q(z) = p_{des}(z) \qquad \text{(D)}$$

in p_c, q_c with $\text{degree}(p_c) \geq \text{degree}(q_c)$ (for causality of the controller).

Notes:

- p_{des} is the desired char. polynomial of the closed-loop system
- $\underbrace{\text{degree}(p_{des})}_{\text{CL sys's order } n_{cl}} = \underbrace{\text{degree}(p)}_{\text{plant order } n} + \underbrace{\text{degree}(p_c)}_{\text{controller order } n_c}$
- In state space, p_{des} includes plant and observer's desired poles.

The equation (D) is called Diophantine equation (also Bezout eqn).

Diophantine equation

With $n_c := \text{degree}(p_c)$ and $m_c := \text{degree}(q_c)$ given,

$$p_c(z)p(z) - q_c(z)q(z) = p_{des}(z)$$

can be written as

$$\begin{bmatrix} S_{n_c}(p) & S_{m_c}(q)\end{bmatrix}\begin{bmatrix} p_c \\ q_c\end{bmatrix} = p_{des} \qquad \text{(D')}$$

where

$$p_c = \text{col}(p_{c,0}, p_{c,1}, \ldots, p_{c,n_c}) \quad , \quad q_c = \text{col}(q_{c,0}, q_{c,1}, \ldots, q_{c,m_c}),$$
$$p_{des} = \text{col}(p_{des,0}, p_{des,1}, \ldots, p_{des,n_{cl}})$$

$\Longrightarrow$ solving (D) (with n_c, m_c given) is a standard linear algebra problem

Part I: Linear system design

1. Review of linear algebra
2. Introduction to state space and polynomial methods
3. Autonomous systems and stability
4. Controllability and observability
5. Design by pole placement and observer design
6. Linear quadratic control and Kalman filter
7. System identification

Part II: Nonlinear system design

- Mathematical modelling of nonlinear systems
- Lyapunov stability analysis
- Describing functions
- Feedback linearization
- Adaptive control