Data-driven simulation and control

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Outline

LTI system representations

- Difference equation

$$R_0 w(t) + R_1 w(t+1) + \cdots + R_\ell w(t+\ell) = 0$$

- Convolution

$$w = \Pi \operatorname{col}(u, y), \quad y(t) = \textstyle\sum_{\tau=-\infty}^{t} h(\tau) u(t-\tau)$$

- Input/state/output equations

$$w = \Pi \operatorname{col}(u, y), \quad \begin{array}{rcl} x(t+1) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t) \end{array} \quad \text{(I/S/O)}$$

$\mathscr{B}(A, B, C, D)$ — the system defined by (I/S/O)
we will assume that $\Pi = I$

Behavior of a system = solution set of an equation

We identify the system with its behavior $\mathscr{B}$,

$$\mathscr{B} := \{\, w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{N}} \mid \text{representation eqns holds} \,\}, \;\; \textit{e.g.}$$

$$\mathscr{B}(A,B,C,D) = \{\, w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{N}} \mid \exists\, x, \text{ such that (I/S/O) holds} \,\}$$

$\mathtt{w}$ is the number of variables, $\mathbb{N} := \{\, 1, 2, \dots \,\}$ is the time axis

Restriction of the behavior to the interval $\{\, 1, 2, \dots t \,\}$

$$\mathscr{B}_t := \{\, w_{\mathrm{p}} \in (\mathbb{R}^{\mathtt{w}})^t \mid \exists\, w_{\mathrm{f}} \text{ such that } (w_{\mathrm{p}}, w_{\mathrm{f}}) \in \mathscr{B} \,\}$$

$\mathrm{lag}(\mathscr{B})$	—	lag of $\mathscr{B}$	(observability index of I/S/O repr.)
$\mathrm{order}(\mathscr{B})$	—	order of $\mathscr{B}$	

Notation for Hankel matrices

Given a signal $w = \big(w(1), \ldots, w(T)\big)$ and $t \leq T$, define

$$\mathscr{H}_t(w) := \begin{bmatrix} w(1) & w(2) & w(3) & \cdots & w(T-t+1) \\ w(2) & w(3) & w(4) & \cdots & w(T-t+2) \\ w(3) & w(4) & w(5) & \cdots & w(T-t+3) \\ \vdots & \vdots & \vdots & & \vdots \\ w(t) & w(t+1) & w(t+2) & \cdots & w(T) \end{bmatrix}$$

block-Hankel matrix with t block-rows, composed of w

Outline

The simulation problem

Classical simulation problem: Given

- system $\mathscr{B} := \mathscr{B}(A, B, C, D)$,
- input $u \in (\mathbb{R}^{\mathrm{m}})^t$, and
- initial conditions $x_{\mathrm{ini}} \in \mathbb{R}^{\mathrm{n}}$,

find the response y of $\mathscr{B}$ to u and ini. cond. x_{ini}.

Data-driven simulation problem: Given

- trajectory $w_{\mathrm{d}} \in (\mathbb{R}^{\mathrm{w}})^T$ of $\mathscr{B}$,
- input $u \in (\mathbb{R}^{\mathrm{m}})^t$, and
- initial trajectory $w_{\mathrm{ini}} \in (\mathbb{R}^{\mathrm{w}})^{T_{\mathrm{ini}}}$, $w_{\mathrm{ini}} \in \mathscr{B}_{T_{\mathrm{ini}}}$,

find the response y of $\mathscr{B}$ to u, such that $\big(w_{\mathrm{ini}}, (u, y)\big) \in \mathscr{B}_{T_{\mathrm{ini}}+t}$.

Notes:

- $\mathscr{B}$ is specified implicitly by w_d,
- the ini. cond. x_ini is specified implicitly by w_ini.

Algorithm 1: data-driven simulation, using I/S/O repr.

1. identification $\quad w_\text{d} \mapsto (A, B, C, D)$
2. observer $\quad \big(w_\text{ini}, (A, B, C, D)\big) \mapsto x_\text{ini}$
3. classical simulation $\quad \big(u, x_\text{ini}, (A, B, C, D)\big) \mapsto y$

Can we find y without deriving an explicit representation of $\mathscr{B}$?

Basic idea

Assuming that w_d is a trajectory of $\mathscr{B}$ (exact data),

lin. comb. of the columns of $\mathscr{H}_t(w_\mathrm{d})$ are trajectories of $\mathscr{B}$, *i.e.*,

$$\text{for all } g, \quad \mathscr{H}_t(w_\mathrm{d})g \in \mathscr{B}_t$$

$\Longrightarrow$ computing the response of $\mathscr{B}$ to given input and initial conditions from data w_d, requires choosing a suitable g

Under what conditions is every trajectory generated that way?

The fundamental lemma

$u_{\rm d}$ is persistently exciting of order L if $\mathscr{H}_L(u_{\rm d})$ is of full row rank.

Fundamental Lemma: Assume that

- the LTI system $\mathscr{B}$ is controllable,
- $u_{\rm d}$ is persistently exciting of order $L + \text{order}(\mathscr{B})$, and
- $w_{\rm d} := (u_{\rm d}, y_{\rm d})$ is a trajectory of $\mathscr{B}$, *i.e.*, $w_{\rm d} \in \mathscr{B}_T$.

Then

$$\text{image}\left(\mathscr{H}_L(w_{\rm d})\right) = \mathscr{B}_L.$$

Construction of responses from data

Problem: Find y, such that $\big(w_{\text{ini}}, (u, y)\big) \in \mathscr{B}$, where w_{ini}, u are given, and $\mathscr{B}$ is implicitly defined by w_d.

Under the conditions of the FL, there is g, such that

$$\mathscr{H}_{T_{\text{ini}}+t}(w_\text{d})g = \big(w_{\text{ini}}, (u, y)\big).$$

The eqns with RHS y, define y, for given g. The others restrict g.

Generic data-driven simulation algorithm:

1. compute any solution g of the equations with RHS w_{ini}, u
2. substitute g in the equations for y

Define

$$U := \mathscr{H}_{T_{\mathrm{ini}}+t}(u_{\mathrm{d}}), \qquad Y := \mathscr{H}_{T_{\mathrm{ini}}+t}(y_{\mathrm{d}})$$

and the partitionings

$$U =: \begin{bmatrix} U_{\mathrm{p}} \\ U_{\mathrm{f}} \end{bmatrix}, \qquad Y =: \begin{bmatrix} Y_{\mathrm{p}} \\ Y_{\mathrm{f}} \end{bmatrix}.$$

Algorithm 2: data-driven simulation

1. compute the least norm solution of

$$\begin{bmatrix} U_{\mathrm{p}} \\ Y_{\mathrm{p}} \\ U_{\mathrm{f}} \end{bmatrix} g = \begin{bmatrix} u_{\mathrm{ini}} \\ y_{\mathrm{ini}} \\ u \end{bmatrix}.$$

2. compute $y := Y_{\mathrm{f}} g$.

Special case $u = 0$: free response

Allows to compute an observability matrix $\mathscr{O}$ of $\mathscr{B}$ from data, by finding $\mathtt{n} \geq \text{order}(\mathscr{B})$ linearly indep. free responses.

Let ℓ_{max} be an upper bound for the lag of $\mathscr{B}$ and take $T_{\text{ini}} = \ell_{\text{max}}$.

Algorithm 3: compute an observability matrix $\mathscr{O}$

1. compute the least norm solution of

$$\begin{bmatrix} U_\text{p} \\ Y_\text{p} \\ U_\text{f} \end{bmatrix} G = \begin{bmatrix} U_\text{p} \\ Y_\text{p} \\ 0 \end{bmatrix}$$

2. compute $Y := Y_\text{f} G$
3. compute a rank revealing factorization $Y = \mathscr{O} X_{\text{ini}}$

Special case $w_{\text{ini}} = 0$: zero initial cond. response

Let h be the impulse response of $\mathscr{B}$, and define

$$\mathscr{T}_t(h) := \begin{bmatrix} h(0) & & & & \\ h(1) & h(0) & & & \\ h(2) & h(1) & h(0) & & \\ \vdots & \ddots & \ddots & \ddots & \\ h(t-1) & \cdots & \cdots & h(1) & h(0) \end{bmatrix}$$

For any $w = \operatorname{col}(u, y) \in \mathscr{B}_t$,

$$y = \mathscr{O} x_{\text{ini}} + \mathscr{T}_t(h) u$$

We can compute a basis for $\mathscr{B}_{0,t} := \operatorname{image}\left(\mathscr{T}_t(h)\right)$ from data, by finding $t\mathrm{m}$ lin. indep. zero initial cond. responses.

Algorithm 4: compute a basis of $\mathscr{B}_{0,t}$

1. compute the least norm solution of

$$\begin{bmatrix} U_\mathrm{p} \\ Y_\mathrm{p} \\ U_\mathrm{f} \end{bmatrix} G = \begin{bmatrix} 0 \\ 0 \\ \mathscr{H}_{t,t_\mathrm{m}}(u_\mathrm{d}) \end{bmatrix}$$

2. compute $Y_0 := Y_\mathrm{f} G$

Then $\text{image}(Y_0) = \text{image}\left(\mathscr{T}_t(h)\right) = \mathscr{B}_{0,t}$.

Special case $w_{\mathrm{ini}} = 0$, $u = I\delta$: impulse response

With the same construction we can find the first t Markov parameters of $\mathscr{B}$, which is a system identification method.

Algorithm 5: compute the impulse response

1. compute the least norm solution of

$$\begin{bmatrix} U_{\mathrm{p}} \\ Y_{\mathrm{p}} \\ U_{\mathrm{f}} \end{bmatrix} G = \begin{bmatrix} 0 \\ 0 \\ \mathrm{col}(I_{\mathrm{m}}, 0) \end{bmatrix}$$

2. compute $h := Y_{\mathrm{f}} G$

Simulation example $w_{\mathrm{d}} \mapsto h$

Simulation setup

- $\mathscr{B}$ is of order $\mathtt{n} = 4$, with $\mathtt{m} = 2$ inputs, and $\mathtt{p} = 2$ outputs
- w_{d} is a trajectory of $\mathscr{B}$ with length $T = 500$

Simulation example $w_{\mathrm{d}} \mapsto h$

Simulation setup

- $\mathscr{B}$ is of order $\mathtt{n} = 4$, with $\mathtt{m} = 2$ inputs, and $\mathtt{p} = 2$ outputs
- w_{d} is a trajectory of $\mathscr{B}$ with length $T = 500$

Compared algorithms

- Algorithm 5 (block computation)
- iterative version of Algorithm 5
- `impulse` from the Identification Toolbox of MATLAB

Simulation example $w_\mathrm{d} \mapsto h$

Compared algorithms

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Approximation error $e = \|h - \widehat{h}\|_\mathrm{F}$ and execution time

method	error, e	time, sec.
Algorithm 5	10^{-14}	0.293
iterative algorithm	10^{-14}	0.066
`impulse`	0.059	0.584

Outline

The output matching problem

"Classical" output matching problem: Given

- system $\mathscr{B} = \mathscr{B}(A, B, C, D)$,
- initial condition $x_{\text{ini}} \in \mathbb{R}^{\mathtt{n}}$, and
- reference response $y_{\text{r}} \in (\mathbb{R}^{\mathtt{p}})^{T_{\text{r}}}$

find an input $u_{\text{f}} \in (\mathbb{R}^{\mathtt{m}})^{T_{\text{r}}}$, such that the response of $\mathscr{B}$ to u_{f} and ini. cond. x_{ini} is y_{r}.

Data-driven output matching problem: Given

- trajectory $w_{\text{d}} \in (\mathbb{R}^{\mathtt{w}})^{T}$ of $\mathscr{B}$,
- initial trajectory $w_{\text{ini}} \in (\mathbb{R}^{\mathtt{w}})^{T_{\text{ini}}}$, $w_{\text{ini}} \in \mathscr{B}_{T_{\text{ini}}}$, and
- reference response $y_{\text{r}} \in (\mathbb{R}^{\mathtt{p}})^{T_{\text{r}}}$

find an input $u_{\text{f}} \in (\mathbb{R}^{\mathtt{m}})^{T_{\text{r}}}$, such that $\big(w_{\text{ini}}, (u_{\text{f}}, y_{\text{r}})\big) \in \mathscr{B}_{T_{\text{ini}}+T_{\text{r}}}$.

Output matching = “inverse simulation”

Note: simulation can be viewed as an “input matching” problem.

Algorithm 6: data-driven output matching

1. compute the least norm solution of

$$\begin{bmatrix} U_\mathrm{p} \\ Y_\mathrm{p} \\ Y_\mathrm{f} \end{bmatrix} g = \begin{bmatrix} u_\mathrm{ini} \\ y_\mathrm{ini} \\ y_\mathrm{r} \end{bmatrix}$$

2. compute $u_\mathrm{f} := U_\mathrm{f} g$

An arbitrary input is allowed.

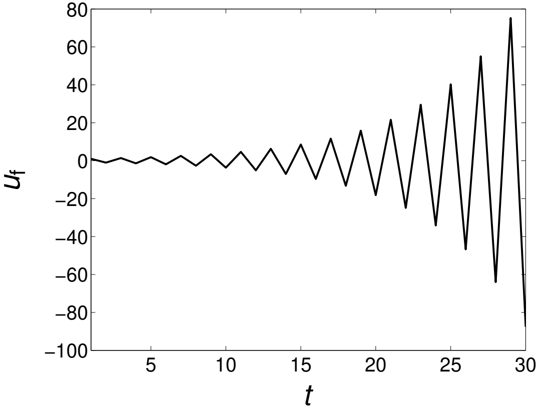
Simulation example

$\mathscr{B}$ — 2nd order, $\mathtt{m} = 1$ input, $\mathtt{p} = 1$ output
w_{d} — random trajectory of $\mathscr{B}$ with $T = 200$ samples

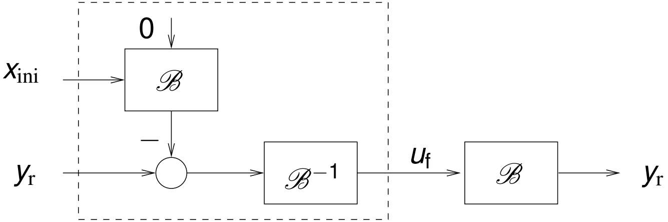
$y_{\mathrm{r}} = 0$

$T_{\mathrm{r}} = 30$

$w_{\mathrm{ini}} = (1, 1)$



Structure of the output matching controller



Feedforward controller

In the example, $\mathscr{B}$ is non-minimum phase $\implies$ $\mathscr{B}^{-1}$ unstable.

More general tracking problem:

follow a reference traj. w_{r} by trading-off errors in both u and y

Outline

Linear quadratic tracking problem

Given

- a trajectory $w_\mathrm{d} \in (\mathbb{R}^\mathtt{w})^T$ of $\mathscr{B}$,
- an initial trajectory $w_\mathrm{ini} \in (\mathbb{R}^\mathtt{w})^{T_\mathrm{ini}}$, $w_\mathrm{ini} \in \mathscr{B}_{T_\mathrm{ini}}$,
- a reference trajectory $w_\mathrm{r} \in (\mathbb{R}^\mathtt{w})^{T_\mathrm{r}}$, and
- a positive definite matrix $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$,

find a trajectory of $\mathscr{B}$ that is optimal with respect to the criterion

$$J(w_\mathrm{r}, w) := (w_\mathrm{r} - w)^\top \Phi (w_\mathrm{r} - w)$$

and has as a prefix the initial trajectory w_ini, *i.e.*, find

$$w_\mathrm{f}^* := \arg\min_{w_\mathrm{f}} J(w_\mathrm{r}, w_\mathrm{f}) \quad \text{subject to} \quad (w_\mathrm{ini}, w_\mathrm{f}) \in \mathscr{B}_{T_\mathrm{ini}+T_\mathrm{f}}.$$

Solution using an I/S/O representation

The classical but indirect solution is:

Algorithm 7: data-driven LQ tracking, using I/S/O repr.

1. $w_\mathrm{d} \xrightarrow{\text{Identification}} (A, B, C, D)$
2. $\big(w_\mathrm{ini}, (A, B, C, D)\big) \xrightarrow{\text{Observer (1,2)}} x_\mathrm{ini}$
3. $\big(\Phi, w_\mathrm{r}, x_\mathrm{ini}, (A, B, C, D)\big) \xrightarrow{\text{Synthesis (3,4,5)}} w_\mathrm{f}^*$

We aim to find algorithms that do not derive a repr. of $\mathscr{B}$.

Observer design

Let h be the impulse response of $\mathscr{B}$. We have,

$$y_{\text{ini}} = \mathscr{O}(A, C)x(1) + \mathscr{T}_{T_{\text{ini}}}(h)u_{\text{ini}}, \tag{1}$$

where

$$\mathscr{O}(A, C) := \text{col}(C, CA, \dots, CA^{T_{\text{ini}}-1})$$

defines a system of equations for the initial state $x(1)$.

$$\begin{array}{lcl} w_{\text{ini}} \in \mathscr{B}_{T_{\text{ini}}} & \implies & \text{existence of solution} \\ (A, B, C, D) \text{ minimal} & \implies & \text{uniqueness} \end{array}$$

$$x_{\text{ini}} = x(T_{\text{ini}} + 1) = CA^{T_{\text{ini}}}x(1) + \begin{bmatrix} h(T_{\text{ini}} - 1) & h(T_{\text{ini}} - 2) & \cdots & h(0) \end{bmatrix} u_{\text{ini}}. \tag{2}$$

Regulator synthesis

LQ tracking problem:

$$\min_{x,u,y} \; (w_{\mathrm{r}} - \mathrm{col}(u,y))^\top \Phi (w_{\mathrm{r}} - \mathrm{col}(u,y))$$

$$\text{subject to} \quad \begin{aligned} x(t+1) &= Ax(t) + Bu(t), \quad x(1) = x_{\mathrm{ini}} \\ y(t) &= Cx(t) + Du(t), \quad \text{for } t = 1, \dots, T_{\mathrm{r}}. \end{aligned}$$

The solution for the $w_{\mathrm{r}} = 0$ case (regulation problem) is

$$\begin{aligned} x^*(t+1) &= (A - BL_t)x^*(t), \quad x(1) = x_{\mathrm{ini}} \\ w_{\mathrm{f}}^*(t) &= \begin{bmatrix} -L_t \\ C - DL_t \end{bmatrix} x^*(t) \end{aligned} \qquad (3)$$

a state feedback.

Define

$$\Phi =: \begin{bmatrix} \Phi_u & \Phi_{uy} \\ \Phi_{yu} & \Phi_y \end{bmatrix}.$$

The optimal input is a state feedback with time-varying gain

$$L_t := \left(B^\top S_{t+1} B + \Phi_u + \Phi_{uy} D + D^\top \Phi_{uy}^\top + D^\top \Phi_y D\right)^{-1} \times \left(B^\top S_{t+1} A + \Phi_{uy} C + D^\top \Phi_y C\right) \quad (4)$$

where S is given by the Riccati difference equation

$$S_t = A^\top S_{t+1} A + C^\top \Phi_y C - \left(B^\top S_{t+1} A + \Phi_{uy} C + D^\top \Phi_y C\right)^\top \times \left(B^\top S_{t+1} B + \Phi_u + \Phi_{uy} D + D^\top \Phi_{uy}^\top + D^\top \Phi_y D\right)^{-1} \times \left(B^\top S_{t+1} A + \Phi_{uy} C + D^\top \Phi_y C\right), \qquad S_{T_\mathrm{r}} = 0. \quad (5)$$

Solution using the impulse response representation

LQ tracking problem:

$$\min_{w_\mathrm{f}} (w_\mathrm{r} - w_\mathrm{f})^\top \Phi (w_\mathrm{r} - w_\mathrm{f}) \quad \text{subject to} \quad (w_\mathrm{ini}, w_\mathrm{f}) \in \mathscr{B}_{T_\mathrm{ini}+T_\mathrm{f}}$$

Let $y_{\mathrm{f},0}$ be the free response of $\mathscr{B}$ initiated by w_ini.

$$y_\mathrm{f} = y_{\mathrm{f},0} + \mathscr{T}_{T_\mathrm{r}}(h) u_\mathrm{f}$$

so that the tracking problem becomes

$$\min_{u_\mathrm{f}} (w_\mathrm{r} - w_\mathrm{f})^\top \Phi (w_\mathrm{r} - w_\mathrm{f}) \quad \text{subject to} \quad y_\mathrm{f} = \mathscr{T}_{T_\mathrm{r}}(h) u_\mathrm{f} + y_{\mathrm{f},0}$$

a weighted least squares problem.

With $\widetilde{h} := \operatorname{col}(I\delta, h)$,

$$w_{\mathrm{f}} := \Big(\operatorname{col}\left(u_{\mathrm{f}}(1), y_{\mathrm{f}}(1)\right), \ldots, \operatorname{col}\left(u_{\mathrm{f}}(T_{\mathrm{r}}), y_{\mathrm{f}}(T_{\mathrm{r}})\right)\Big) = \mathscr{T}_{T_{\mathrm{r}}}(\widetilde{h})u_{\mathrm{f}}$$

Then

$$w_{\mathrm{f}} = \mathscr{T}_{T_{\mathrm{r}}}(\widetilde{h})u_{\mathrm{f}} + w_{\mathrm{f},0}, \quad \text{where} \quad w_{\mathrm{f},0} := \operatorname{col}(0, y_{\mathrm{f},0})$$

The tracking problem becomes

$$\min_{u_{\mathrm{f}}} \left(w_{\mathrm{r}} - w_{\mathrm{f},0} - \mathscr{T}_{T_{\mathrm{r}}}(\widetilde{h})\right)^{\top} \Phi \left(w_{\mathrm{r}} - w_{\mathrm{f},0} - \mathscr{T}_{T_{\mathrm{r}}}(\widetilde{h})\right)$$

and the solution is

$$\begin{aligned} u_{\mathrm{f}}^{*} &= \left(\mathscr{T}_{T_{\mathrm{r}}}^{\top}(\widetilde{h})\Phi\mathscr{T}_{T_{\mathrm{r}}}(\widetilde{h})\right)^{-1}\mathscr{T}_{T_{\mathrm{r}}}^{\top}(\widetilde{h})\Phi(w_{\mathrm{r}} - w_{\mathrm{f},1}) \\ y_{\mathrm{f}}^{*} &= \mathscr{T}_{T_{\mathrm{r}}}(h)u_{\mathrm{f}}^{*} + y_{\mathrm{f},1} \end{aligned} \tag{6}$$

Ingredients of the solution:

- the free response $y_{\mathrm{f},0}$ and
- the impulse response h.

We can compute them directly from w_{d}.

Algorithm 8: data-driven LQ tracking, using impulse resp. repr.

1. $(w_{\mathrm{ini}}, w_{\mathrm{d}}, T_{\mathrm{r}}) \xrightarrow{\text{Algorithm 2}} y_{\mathrm{f},0}$
2. $(w_{\mathrm{d}}, T_{\mathrm{r}}) \xrightarrow{\text{Algorithm 6}} h$
3. $(\Phi, w_{\mathrm{r}}, w_{\mathrm{f},0}, h) \xrightarrow{(6)} w_{\mathrm{f}}^{*}$

Data-driven solution

Define the zero initial conditions subbehavior of $\mathscr{B}$

$$\mathscr{B}_{0,T_\mathrm{r}} := \{\, w \in (\mathbb{R}^\mathtt{w})^{T_\mathrm{r}} \mid (\underbrace{0,\dots,0}_{\mathrm{lag}(\mathscr{B})}, w) \in \mathscr{B}_{\mathrm{lag}(\mathscr{B})+T_\mathrm{r}} \,\}$$

Theorem: Let $W_0 \in \mathbb{R}^{T_\mathrm{r}\mathtt{w}\times\bullet}$ be a matrix, such that

$$\operatorname{image}(W_0) = \mathscr{B}_{0,T_\mathrm{r}}$$

Then the LQ optimal trajectory is

$$w_\mathrm{f}^* = W_0\left(W_0^\top \Phi W_0\right)^+ W_0^\top \Phi (w_\mathrm{r} - w_{\mathrm{f},0}) + w_{\mathrm{f},0} \qquad (7)$$

where $w_{\mathrm{f},0}$ is the free response of $\mathscr{B}$, caused by w_ini.

Proof

Any zero initial conditions trajectory $w = \operatorname{col}(u, y) \in (\mathbb{R}^{\mathtt{w}})^{T_{\mathrm{r}}}$ is of the form $w = \mathscr{T}_{T_{\mathrm{r}}}(\widetilde{h})u$. Therefore,

$$\mathscr{B}_{0,T_{\mathrm{r}}} = \operatorname{image}\left(\mathscr{T}_{T_{\mathrm{r}}}(\widetilde{h})\right) = \operatorname{image}\left(W_0\right)$$

Consider the space $\mathscr{W} = (\mathbb{R}^{\mathtt{w}})^{T_{\mathrm{r}}}$ with inner product defined by $\langle w_1, w_2 \rangle = w_1^{\top} \Phi w_2$. The projector on $\mathscr{B}_{0,T_{\mathrm{r}}}$ in $\mathscr{W}$ is

$$\mathscr{T}_{T_{\mathrm{r}}}(\widetilde{h})\left(\mathscr{T}_{T_{\mathrm{r}}}^{\top}(\widetilde{h})\Phi\mathscr{T}_{T_{\mathrm{r}}}(\widetilde{h})\right)^{-1}\mathscr{T}_{T_{\mathrm{r}}}^{\top}(\widetilde{h})\Phi = W_0\left(W_0^{\top}\Phi W_0\right)^{+}W_0^{\top}\Phi$$

Then the data-driven solution (7) follows from the solution (6), using the impulse response representation.

Algorithm 9: data-driven LQ tracking

1. $(w_{\rm ini}, w_{\rm d}, T_{\rm r}) \xrightarrow{\text{Algorithm 2}} y_{\rm f,0}$
2. $(w_{\rm d}, T_{\rm r}) \xrightarrow{\text{Algorithm 4}} W_0$
3. $(\Phi, w_{\rm r}, y_{\rm f,0}, W_0) \xrightarrow{(7)} w_{\rm f}^*$

Simulation example

Aim: illustrate numerically the equivalence of the three methods.

$\mathscr{B}$	—	2nd order, $\mathtt{m} = 1$ input, $\mathtt{p} = 1$ output (the same system as in the output matching example)
w_d	—	random trajectory of $\mathscr{B}$ with $T = 200$ samples
Φ	—	identity (assign equal weights to the variables)

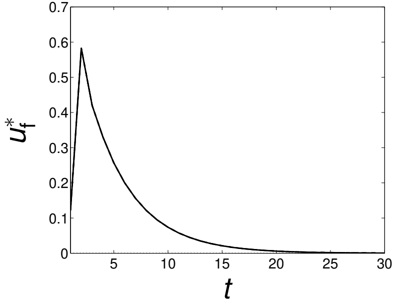
- Experiment 1: data-driven regulation

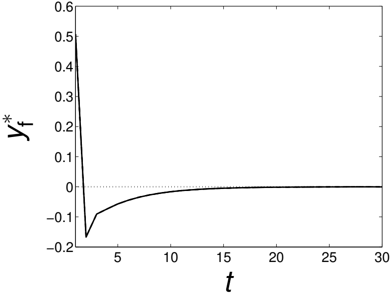
$$w_{\mathrm{r}} = 0, \quad T_{\mathrm{r}} = 30, \quad \text{and} \quad w_{\mathrm{ini}} = (1,1)$$

- Experiment 2: data-driven step tracking

$$u_{\mathrm{r}} = 0, \quad y_{\mathrm{r}}(t) = \begin{cases} 0, & \text{for } t = 1,2\ldots,30 \\ 1, & \text{for } t = 31,52,\ldots,60 \end{cases}, \quad w_{\mathrm{ini}} = (1,1)$$

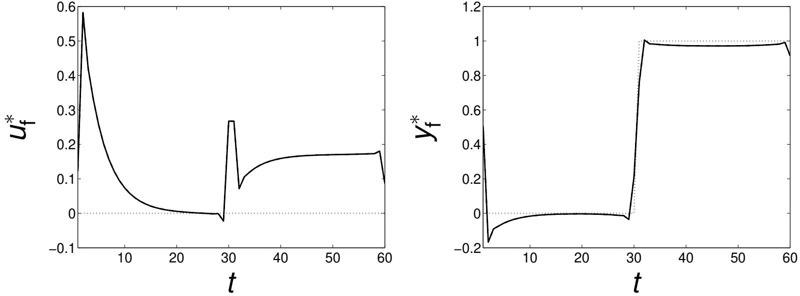
Result for Experiment 1





w_{r} — dotted line, w_f^* — solid line, $J(0, w_{\mathrm{f}}^*) = 1.1139$

Result for Experiment 2



w_r — dotted line, w_f^* — solid line, $J(w_\mathrm{r}, w_\mathrm{f}^*) = 2.1034$

Conclusions

- Given $w_d \in \mathscr{B}_T$, we can compute feedforward LQ tracking control without deriving a repr. of $\mathscr{B}$. (data-driven control)
- For doing this we need
  - $\mathscr{B}$ to be controllable,
  - u_d to be persistently exciting of sufficient order.
- The construction of the optimal control is based on
  - free response $y_{f,0}$ of $\mathscr{B}$ under w_{ini}, and
  - zero ini. cond. trajectories W_0 (a basis for $\mathscr{B}_{0,T_r}$).

Perspectives for future work

- Find data-driven solutions to other control problems
- Derivation of a feedback controller directly from data
- Recursive algorithms
- Dealing with perturbed data

Final goal:

approximate recursive algorithms for data-driven control.

Thank you