ELEC 3035: Control systems design, Exam part I solutions

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1. *Short questions on basic concepts in systems and control*

   (a) [3 marks] List at least five applications of control. Be as specific as possible.

   (b) [3 marks] Sketch the block diagrams of the feedback and feedforward control schemes.

   (c) [3 marks] List the main advantages and disadvantages of feedback control over feedforward control.

   (d) [3 marks] List and give short description of the control methods that you know.

   (e) [3 marks] Most control methods need a model of the plant. Moreover, most model based methods require a linear time-invariant model. Indicate in your list of control methods which ones are model based and which ones require linear time-invariant model.

   (f) [3 marks] How are models specified? List and give short description of the model representations that you know. Indicate which ones are limited to linear time-invariant models.

   (g) [3 marks] What are the main characteristics of continuous-time and discrete-time dynamical models? When is the model static?

   (h) [4 marks] When is a model linear? Give a definition and examples (defined by representations) of i) static linear, ii) static nonlinear, iii) dynamic linear, and iv) dynamic nonlinear models.

Solution:

(a) i. Stabilization of operational amplifiers.
    ii. Thermostat — temperature regulation to a desired set point.
    iii. Cruise control — a system that regulates the speed of a motor vehicle to a desired set point.
    iv. Autopilot — a system that guides a vehicle.
    v. Control of the electrical energy production to match the consumption.

(b) The feedforward and feedback interconnections block diagrams are shown in Figure 1.

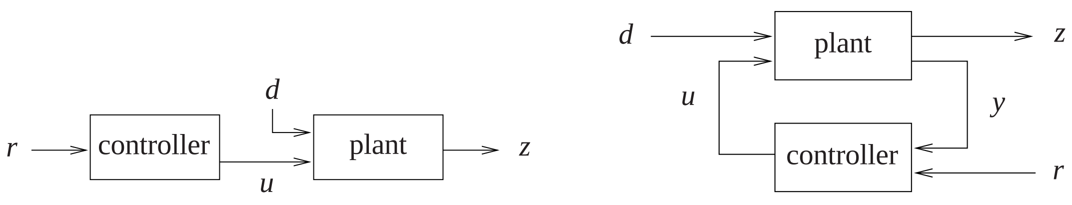

Figure 1: Feedforward (left) and feedback (right) interconnections; r is the reference signal, u is the control signal, d is a disturbance, z is a performance signal, y is the measurement signal.

(c) Advantages of feedback over feedforward:

- Improved robustness to model uncertainty, disturbances, and unknown initial condition.
- Economy: specified by the precomputed feedback gain, rather than a precomputed signal.

Disadvantages of feedback compared to feedforward:

- Danger of instability.
- More complex theory for controller synthesis.

(d) Control methods:

- Lead-lag compensation. It is a frequency domain method, based on LTI model. (Studied in ELEC 2019.)

- PID control, It is a time domain method. Can be LTI model based or model free. The synthesis is often based on heuristics. (Studied in ELEC 2019.)
- State transfer, state space, LTI model based,
- Pole placement, LTI model based,
- Adaptive control, model based, can be used for nonlinear and time-varying systems.

(e) Models are specified by equations. The most common representations are:

- convolution (the parameter is the impulse response), limited to linear systems
- transfer function, limited to linear time-invariant systems
- differential/difference equation
- state space

(f) Dynamical systems are defined by a relation of the variables and their derivatives (in continuous-time) or time shifts (in discrete-time). The model is static when the relation is algebraic, *i.e.*, it does not involve derivatives or time shifts.

(g) A model $\mathscr{B}$ is linear when

$$w_1, w_2 \in \mathscr{B} \text{ implies that } a_1 w_1 + a_2 w_2 \in \mathscr{B}, \text{ for all scalars } a_1 \text{ and } a_2.$$

Examples

i. static linear model: $\mathscr{B} = \{\, w = (u,y) \mid y = u \,\}$
ii. static nonlinear model: $\mathscr{B} = \{\, w = (u,y) \mid y = u^2 \,\}$
iii. dynamic linear model: $\mathscr{B} = \{\, w = (u,y) \mid y = \sigma u \,\}$
iv. dynamic nonlinear model: $\mathscr{B} = \{\, w = (u,y) \mid y = (\sigma u)^2 \,\}$

□

3. *Fibonacci numbers* The Fibonacci numbers $f(0), f(1), f(2), \dots$ are defined by the recursion

$$f(0) = 1, \quad f(1) = 1, \quad \text{and} \quad f(t) = f(t-1) + f(t-2), \quad \text{for } t = 2, 3, \dots. \tag{FN}$$

(a) [5 marks] What kind of system generates as its output the sequence of the Fibonacci numbers $f := \big(f(0), f(1), f(2), \dots\big)$?

(b) [10 marks] Find a state space representation of the system that, under suitable choice of the initial conditions, generates as its output f. What is the initial condition, under which the output of the system is f?

(c) [10 marks] Find the 39th Fibonacci number $f(39)$.

Solution:

(a) The system that generates as its output f is defined by (FN), which is a homogeneous, linear, constant coefficients, difference equation with two time shifts. Therefore, the system is autonomous, linear, time-invariant, discrete-time, of order 2.

(b) A state representation of a general autonomous, linear, time-invariant, discrete-time is

$$\sigma x = Ax, \qquad y = Cx.$$

Define the state vector in the usual way as the output and its $n-1$ shifts, where n is the order of the system. In this case, $n = 2$ and a state is $x = \operatorname{col}(f, \sigma f)$. The corresponding state space representation is

$$x(t+1) = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}_{A} x(t), \qquad y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{c} x(t).$$

By the definition of the state vector, we have that for

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

the system generates f, *i.e.*, $y = f$.

(c) Using the formula $y(t) = cA^t x(0)$ for the response of a state space model, we have

$$f(39) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^{38} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

This is an explicit expression, so it is a valid answer.

Another approach for finding $f(39)$ is based on the polynomial representation

$$r(\sigma) f = 0, \quad \text{where } r(z) = z^2 - z - 1.$$

The general solution is of the form

$$f(t) = c_1 z_1^t + c_2 z_2^t$$

where

$$z_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad z_2 = \frac{1-\sqrt{5}}{2}$$

are the roots of $r(z)$ (the poles of the system) and c_1 and c_2 are constants depending on the initial conditions. In order to find c_1 and c_2, we solve the system

$$\begin{array}{l} f(0) = c_1 z_1^0 + c_2 z_2^0 \\ f(1) = c_1 z_1^1 + c_2 z_2^1 \end{array} \quad \iff \quad \begin{bmatrix} 1 & 1 \\ z_1 & z_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \iff \quad \begin{array}{l} c_1 = \frac{z_2 - 1}{z_2 - z_1} \\ c_2 = \frac{1 - z_1}{z_2 - z_1} \end{array}$$

Finally, we have

$$f(39) = \frac{z_2 - 1}{z_2 - z_1} z_1^{39} + \frac{1 - z_1}{z_2 - z_1} z_2^{39},$$

which is again an explicit expression (since z_1 and z_2 are known numbers) and can be evaluated to find that

$$f(39) = 102334155.$$

□

4. *Controllability and state transfer* Consider the system defined by the state space representation

$$x(t+1) = Ax(t) + Bu(t).$$

(a) [5 marks] Give a definition of state controllability.

(b) [5 marks] Write Matlab code to construct the controllability matrix.

(c) [5 marks] Write Matlab function that given a state space representation of a discrete-time linear time-invariant system, an initial state x_{ini}, a final state x_{des}, and a transfer time t, computes the minimum energy input that transfers the system from state x_{ini} to state x_{des} in t time-samples.

(d) [10 marks] Compute by hand the minimum energy input that transfers the system

$$x(t+1) = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b} u(t). \tag{1}$$

from initial state $x_{\text{ini}} = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top$ to final state $x_{\text{des}} = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$, in $t = 3$ time steps.

Solution:

(a) A system $\mathscr{B}$ is controllable if for any given trajectories $w_{\text{ini}}, w_{\text{des}}$ of $\mathscr{B}$, there is a trajectory w_{crt} of $\mathscr{B}$, such that $w_{\text{crt}}(t) = w_{\text{ini}}(t)$, for $t < 0$ and $w_{\text{crt}}(t) = w_{\text{des}}(t)$, for $t \geq \tau$, for some τ. Informally, any given trajectory $w_{\text{ini}} \in \mathscr{B}$ can be steered to any desired trajectory $w_{\text{des}} \in \mathscr{B}$.

(b)
```
% Controllability matrix C_t = [b Ab ... A^{t-1}b]
C = b;
for i = 1:t-1
    C = [C a^(i-1)*b];
end
```

(c)
```
% LN_STATE_TRANSFER - Minimum energy state transfer
function [u,e] = ln_state_transfer(sys,xini,xdes,t)

% Controllability matrix
C = sys.b;
for i = 1:t-1
    C = [C sys.a*C(:,end)];
end

if rank(C) < size(sys,'order')
   disp('The system is not controllabale ...
         or the transfer time is less than the order')
   u = NaN; e = NaN;
else
   u = flipud(C' * inv(C*C') * (xdes - a^(t)*xini)); e = norm(u);
end
```

(d) The extended controllability matrix

$$\mathscr{C} = \begin{bmatrix} b & Ab & A^2 b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

and the minimum energy control is

$$\begin{aligned} u_{\text{ln}} &= \mathscr{C}^\top (\mathscr{C}\mathscr{C}^\top)^{-1} (x_{\text{des}} - A^t * x_{\text{ini}}) \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{6} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

so that the optimal control is, $u_{\text{ln}}(1) = u_{\text{ln}}(2) = u_{\text{ln}}(3) = 1/3$.

□