Data driven simulation with applications to system identification

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Outline

Notation

- discrete-time LTI system Σ

$$\sigma x = Ax + Bu, \quad y = Cx + Du,$$

where $x(t) \in \mathbb{R}^{\mathtt{n}}$, $u(t) \in \mathbb{R}^{\mathtt{m}}$, $y(t) \in \mathbb{R}^{\mathtt{p}}$, and $(\sigma f)(t) = f(t+1)$

- $\mathscr{B}_T$ — set of all T samples long trajectories $w := (u,y)$ of Σ,

$$\mathscr{B}_T := \Big\{ w := (u,y) \in (\mathbb{R}^{\mathtt{m}})^T \times (\mathbb{R}^{\mathtt{p}})^T \mid \exists\, x \in (\mathbb{R}^{\mathtt{n}})^T \text{ s.t. } \sigma x = Ax + Bu,\ y = Cx + Du \Big\}$$

- lag $\mathtt{l}$ of Σ — observability index of (A,C)

$\mathtt{n}$, $\mathtt{m}$, $\mathtt{p}$, $\mathtt{l}$, $\mathscr{B}$ have the same meaning throughout

Problem formulation

Problem (Data driven simulation)

given:

1. *trajectory* $(\tilde{u},\tilde{y}) \in \mathscr{B}_T$ *of an LTI system* Σ
2. *upper bound* $\mathtt{n}_{\max}$ *of the order* $\mathtt{n}$
3. *upper bound* $\mathtt{l}_{\max}$ *of the lag* $\mathtt{l}$
4. *time series* $u_{\mathrm{f}} \in (\mathbb{R}^{\mathtt{m}})^L$, *where* $L \in \mathbb{N}$

find: the response y_{f} *of* Σ *under zero initial conditions and input* u_{f}

Initial conditions

Setting zero initial conditions

Let

$$\left(\begin{bmatrix} u_\mathrm{p} \\ u_\mathrm{f} \end{bmatrix}, \begin{bmatrix} y_\mathrm{p} \\ y_\mathrm{f} \end{bmatrix} \right) \in \mathscr{B}_T$$

where $(u_\mathrm{p}, y_\mathrm{p}) = 0$ is at least $\mathtt{l}$ samples long zero sequence.

Then y_f is a zero initial conditions response due to u_f.

Fundamental lemma

Let

$$\mathscr{H}_\Delta(\tilde{u}) = \begin{bmatrix} \tilde{u}(1) & \tilde{u}(2) & \cdots & \tilde{u}(T-\Delta+1) \\ \tilde{u}(2) & \tilde{u}(3) & \cdots & \tilde{u}(T-\Delta+2) \\ \vdots & \vdots & & \vdots \\ \tilde{u}(\Delta) & \tilde{u}(\Delta+1) & \cdots & \tilde{u}(T). \end{bmatrix}$$

$\tilde{u}$ is persistently exciting of order Δ if $\mathscr{H}_\Delta(\tilde{u})$ is of full row rank.

Fundamental Lemma

Assume that

1. the LTI system Σ is controllable,
2. $\tilde{u}$ is persistently exciting of order $\Delta + \mathtt{n}$, and
3. $(\tilde{u}, \tilde{y})$ is a trajectory of Σ, *i.e.*, $(\tilde{u}, \tilde{y}) \in \mathscr{B}_T$.

Then

$$\operatorname{image}\left(\begin{bmatrix} \mathscr{H}_\Delta(\tilde{u}) \\ \mathscr{H}_\Delta(\tilde{y}) \end{bmatrix} \right) = \mathscr{B}_\Delta.$$

Fundamental Lemma

$$\left[\begin{array}{cccc} \tilde{u}(1) & \tilde{u}(2) & \cdots & \tilde{u}(T-\Delta+1) \\ \tilde{u}(2) & \tilde{u}(3) & \cdots & \tilde{u}(T-\Delta+2) \\ \vdots & \vdots & & \vdots \\ \tilde{u}(\Delta) & \tilde{u}(\Delta+1) & \cdots & \tilde{u}(T) \\ \hline \tilde{y}(1) & \tilde{y}(2) & \cdots & \tilde{y}(T-\Delta+1) \\ \tilde{y}(2) & \tilde{y}(3) & \cdots & \tilde{y}(T-\Delta+2) \\ \vdots & \vdots & & \vdots \\ \tilde{y}(\Delta) & \tilde{y}(\Delta+1) & \cdots & \tilde{y}(T) \end{array}\right]$$

Every linear combination of the columns of $\mathscr{H}_\Delta(\tilde{w})$ is a response.

Under what conditions is every response generated that way?

Algorithms for data driven simulation

Define: $\begin{bmatrix} \mathscr{H}_{n+t}(u) \\ \mathscr{H}_{n+t}(y) \end{bmatrix} =: \begin{bmatrix} U_\mathrm{p} \\ U_\mathrm{f} \\ Y_\mathrm{p} \\ Y_\mathrm{f} \end{bmatrix}$ $\quad \begin{array}{rclcl} \operatorname{row\,dim}(U_\mathrm{p}) & = & \operatorname{row\,dim}(Y_\mathrm{p}) & = & n \\ \operatorname{row\,dim}(U_\mathrm{f}) & = & \operatorname{row\,dim}(Y_\mathrm{f}) & = & t \end{array}$

Theorem

Let Σ be controllable, $(\tilde{u}, \tilde{y}) \in \Sigma$, and $\tilde{u}$ be persistently exciting of order $\Delta + \mathtt{l}_\mathrm{max} + \mathtt{n}_\mathrm{max}$. Then the system of equations

$$\begin{bmatrix} U_\mathrm{p} \\ U_\mathrm{f} \\ Y_\mathrm{p} \end{bmatrix} g = \begin{bmatrix} 0 \\ u_\mathrm{f} \\ 0 \end{bmatrix}, \tag{1}$$

is solvable for any u_f and

$$y_\mathrm{f} = Y_\mathrm{f}\bar{g}$$

generates any zero initial conditions response.

Block algorithm

Algorithm

Input: $\tilde{u}, \tilde{y}, \mathtt{n}_{\max}, \mathtt{l}_{\max}$, and u_{f}.

1: Solve the system of equations (1) and let $\bar{g}$ be a solution.

2: Compute $y_{\mathrm{f}} = Y_{\mathrm{f}}\bar{g}$.

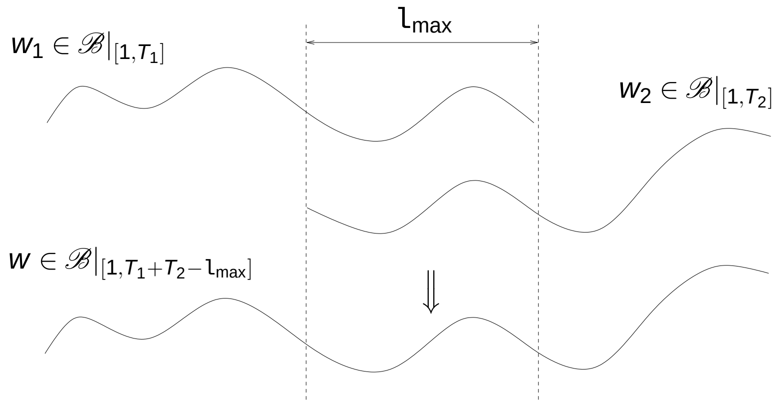
Output: the response y_{f} of Σ to zero initial conditions and input u_{f}.

Limitation

$$t \leq \frac{T+1}{\mathtt{m}+1} - \mathtt{l}_{\max} - \mathtt{n}_{\max}.$$

This can be avoided by “weaving” responses.

Weaving responses



Iterative algorithm

Algorithm

Input: $\tilde{u}, \tilde{y}, \mathtt{n}_{\max}, \mathtt{l}_{\max}, u_{\mathrm{f}}$, and Δ satisfying the conditions of Theorem 2.

1: Set $k := 0$, $f_{\mathrm{u}}^{(0)} := \begin{bmatrix} 0_{\mathtt{l}_{\max}\mathtt{m}\times 1} \\ u_{\mathrm{f}}(1:\Delta) \end{bmatrix}$ and $f_{\mathrm{y,p}}^{(0)} := 0_{\mathtt{l}_{\max}\mathtt{p}\times 1}$.

2: **repeat**

3: Solve $\begin{bmatrix} U_{\mathrm{p}} \\ U_{\mathrm{f}} \\ Y_{\mathrm{p}} \end{bmatrix} g^{(k)} = \begin{bmatrix} f_{\mathrm{u}}^{(k)} \\ f_{\mathrm{y,p}}^{(k)} \end{bmatrix}$ and let $\bar{g}^{(k)}$ be the solution found.

4: Compute the response $y_{\mathrm{f}}^{(k)} := Y_{\mathrm{f}}\bar{g}^{(k)}$.

5: $f_{\mathrm{u}}^{(k+1)} := \begin{bmatrix} \sigma^{\Delta} f_{\mathrm{u}}^{(k)} \\ u(k\Delta+1:(k+1)\Delta) \end{bmatrix}$, $f_{\mathrm{y,p}}^{(k+1)} := \sigma^{\Delta} \begin{bmatrix} f_{\mathrm{y,p}}^{(k)} \\ y_{\mathrm{f}}^{(k)} \end{bmatrix}$.

6: $k := k+1$

7: **until** $t < k\Delta$

Output: $y_{\mathrm{f}} := \mathrm{col}\,(y_{\mathrm{f}}^{(0)}, \ldots, y_{\mathrm{f}}^{(k-1)})$.

Simulation example $\tilde{w} \mapsto$ impulse response H

Simulation setup

- $\mathscr{B}$ is of order $\mathtt{n} = 4$, lag $\mathtt{l} = 2$, with $\mathtt{m} = 2$ inputs, and $\mathtt{p} = 2$ outputs
- $\tilde{w}$ is a trajectory of $\mathscr{B}$ with length $T = 500$

Compared algorithms

- the block algorithm
- an iterative refinement of the block algorithm
- the function `impulse` from the Identification Toolbox of MATLAB

Approximation error $e = \|H - \hat{H}\|_{\mathrm{F}}$ and execution time

method	error, e	time, sec.
block algorithm	10^{-14}	0.293
iterative algorithm	10^{-14}	0.066
`impulse`	0.059	0.584

Computation of free responses

The orthogonal projection, used in the MOESP algorithms,

$$Y_0 := \mathscr{H}_{n_{\max}}(\tilde{y})\Pi_{\tilde{u}}^{\perp}, \tag{2}$$

where

$$\Pi_{\tilde{u}}^{\perp} := I - \mathscr{H}_{n_{\max}}^{\top}(\tilde{u})\big(\mathscr{H}_{n_{\max}}(\tilde{u})\mathscr{H}_{n_{\max}}^{\top}(\tilde{u})\big)^{-1}\mathscr{H}_{n_{\max}}(\tilde{u}),$$

is a way to compute free responses from data. Observe that

$$\begin{bmatrix}\mathscr{H}_{n_{\max}}(\tilde{u})\\ \mathscr{H}_{n_{\max}}(\tilde{y})\end{bmatrix}\Pi_{\tilde{u}}^{\perp} = \begin{bmatrix}0\\ Y_0\end{bmatrix}.$$

Computation of sequential free responses

The oblique projection, used in the N4SID algorithms,

$$Y_0 := Y_{\mathrm{f}}/_{U_{\mathrm{f}}}W_{\mathrm{p}} := Y_{\mathrm{f}}\Pi_{\mathrm{obl}}, \tag{3}$$

where

$$\Pi_{\mathrm{obl}} := \begin{bmatrix}W_{\mathrm{p}}^{\top} & U_{\mathrm{f}}^{\top}\end{bmatrix}\begin{bmatrix}W_{\mathrm{p}}W_{\mathrm{p}}^{\top} & W_{\mathrm{p}}U_{\mathrm{f}}^{\top}\\ U_{\mathrm{f}}W_{\mathrm{p}}^{\top} & U_{\mathrm{f}}U_{\mathrm{f}}^{\top}\end{bmatrix}^{+}\begin{bmatrix}W_{\mathrm{p}}\\ 0\end{bmatrix},$$

is a way to compute free responses, which initial conditions form a state sequence. Observe that

$$\begin{bmatrix}W_{\mathrm{p}}\\ U_{\mathrm{f}}\\ Y_{\mathrm{f}}\end{bmatrix}\Pi_{\mathrm{obl}} = \begin{bmatrix}W_{\mathrm{p}}\\ 0\\ Y_0\end{bmatrix}.$$

Conclusions

- system responses can be computed directly from (exact) data
- the algorithms require solving a linear system of equations
- the iterative algorithm weaves consecutive pieces of the response
- data driven simulation is relevant for system identification
- the orthogonal projection is a tool for computing free responses
- the oblique projection is a tool for computing sequential free responses