Homework "Signal theory: Part 1" solutions

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1 Introduction

Homework

Reading assignment

- notes Leo part 1, sections 1.1–4
- If needed, follow this MATLAB tutorials.

2 Signals and systems

Homework

Reading assignment

- section 1.1 (classification of signals), 1.2 (classification of systems), and chapter 2 (representation of signals and systems) from A. Oppenheim and A. Willsky, *Signals and Systems*, Prentice Hall, 1996

Problems

- **Periodicity in discrete-time**

  1. When is the signal $f(t) = a\cos(\omega t + \phi)$, $t \in \mathbb{Z}$ periodic?
  2. Is $a\cos(\omega t + \phi)$, $t \in \mathbb{Z}$ periodic when $\phi = 0$ and
     (a) $\omega = 2\pi/12$,
     (b) $\omega = 8\pi/31$,
     (c) $\omega = 1/6$?

  – SOLUTION

  1. The discrete-time signal f is periodic if there is a natural number T such that $f(t) = f(t+T)$, *i.e.*,

     $$a\cos(\omega t + \phi) = a\cos(\omega(t+T) + \phi), \quad \text{for all } t \in \mathbb{Z}.$$

     This is the case if $\omega T = 2\pi k$ for some integer k. Therefore, $\omega = 2\pi k/T$, *i.e.*, the discrete-time signal $f(t) = a\cos(\omega t + \phi)$ is periodic if the frequency ω is π times a rational number.
  2. (a) Yes, because $\omega = 2\pi \times 1/12$.
     (b) Yes, because $\omega = 2\pi \times 4/31$.
     (c) No, because $\omega = 2\pi \times \frac{1}{12\pi}$ and $\frac{1}{12\pi}$ is not a rational number.

- **Relation between impulse and step functions**

  – Find relations between the impulse δ and step s functions.

– SOLUTION

  * in discrete-time

$$\delta(t) = s(t) - s(t-1), \qquad s(t) = \delta(t) + \delta(t-1) + \cdots = \sum_{\tau=-\infty}^{t} \delta(\tau)$$

  * in continuous-time

$$\delta(t) = \frac{d}{dt} s(t), \qquad s(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

- **System classification**

  – Give specific examples of:
    * linear static system
    * nonlinear static system
    * linear time-invariant dynamical systems
      · finite impulse response (FIR)
      · infinite impulse response (IIR)
      · scalar
      · multivariable
    * linear time-varying dynamical systems
    * nonlinear time-invariant dynamical systems
    * nonlinear time-varying dynamical systems

  – A solution is given in this document. SOLUTION

- **Peak and RMS values**

  Find the peak and RMS values of $x(t) := a\cos(\omega t + \phi)$, for $t \geq 0$.

  – SOLUTION

  The peak value is $\max_t |x(t)| = a$.
  The RMS value is by definition

$$\sqrt{\lim_{t\to\infty} \frac{1}{t} \int_{0}^{t} x^2(\tau)\, d\tau}.$$

  Since x is a periodic function, we need to compute the integral over one period $[0,T]$

$$\sqrt{\frac{1}{T} \int_{0}^{T} x^2(\tau)\, d\tau}.$$

  Using the identies $2\cos(t) = 1 + \cos(2t)$ and $\int \cos(t) = \sin(t)$, we have

$$\sqrt{\frac{1}{T} \int_{0}^{T} \big(a\cos(\omega t + \phi)\big)^2(t)} = \sqrt{\frac{a^2}{2} + \frac{1}{2T} \sin(2\omega t + \phi)\Big|_0^T} = \frac{a}{\sqrt{2}}.$$

- **Response of 1st and 2nd order LTI system**

  Find analytically the response of 1st and 2nd order linear time-invariant autonomous systems.

  – Solution using Laplace transform SOLUTION

  See, this document.

– Solution using a state space representation SOLUTION

An autonomous linear time-invariant system has a state space representation $\dot{x} = Ax$, $y = Cx$. In the 1st order case, $x(t) \in \mathbb{R}$, $A \in \mathbb{R}$, and $C \in \mathbb{R}$. In the 2nd order case, $x(t) \in \mathbb{R}^2$, $A \in \mathbb{R}^{2\times 2}$, and $C \in \mathbb{R}^{1\times 2}$. All responses y of the autonomous system can be parameterized by the initial state vector $x(0)$ (initial condition) by $y(t) = Ce^{At}x(0)$, where e^{At} is the matrix exponential. The computation of the matrix exponential (and therefore a specific response of the system) can be done by a change of bases transformation that brings the system in a modal form (diagonal A matrix). See pages 8–12 in this document for details.

- **Multiple poles**

Consider the autonomous system represented by a difference equation

$$y(t+2) - 2ay(t+1) + a^2 y(t) = 0.$$

(Its characteristic polynomial has both roots equal to $\lambda = a$.)

1. Show that both $y(t) = a^t$ and $y(t) = ta^t$ are solutions.
2. Find the trajectory generated from the initial conditions $y(0) = 1$ and $y(1) = 0$.

– SOLUTION

1. To check $y(t) = a^t$, we have

$$a^{t+2} - 2a \cdot a^{t+1} + a^2 \cdot a^t = a^{t+2} - 2a^{t+2} + a^{t+2} = 0.$$

Thus, $y(t) = a^t$ is a solution. Checking $y(t) = ta^t$ we have

$$\begin{aligned}(t+2)a^{t+2} - 2a(t+1)a^{t+1} + a^2 ta^t &= (t+2)a^{t+2} - 2(t+1)a^{t+2} + ta^{t+2} \\ &= (t+2-2t-2+t)a^{t+2} = 0.\end{aligned}$$

Thus, $y(t) = ta^t$ is also a solution.

2. A second-order linear time-invariant autonomous system has a two dimensional solution set (two degrees of freedom in its general solution). This implies that two linearly independent solutions can form a basis set for the solutions space. We have found two solutions a^t and ta^t. It is easy to prove that these two solutions are linear independent. Because we can't find two constant c_1 and c_2 at least one of which is nonzero to satisfy

$$c_1 a^t + c_2 t a^t = 0$$

for all $t = 0, 1, 2, \ldots$. Thus, any solution $y(t)$ can be expressed as a linear combination of the basis set of solutions a^t and ta^t.

$$y(t) = c_1 a^t + c_2 t a^t,$$

where c_1 and c_2 are constant. Using the conditions $y(0) = 1$ and $y(1) = 0$, we find c_1 and c_2

$$\begin{aligned} y(0) = c_1 a^0 = 1 \quad &\Longrightarrow \quad c_1 = 1 \\ y(1) = c_1 a + c_2 a = a + c_2 a = 0 \quad &\Longrightarrow \quad c_2 = -1. \end{aligned}$$

Thus, the trajectory is

$$y(t) = a^t - ta^t.$$

- $(A,B,C,D) \mapsto$ **impulse response**

Find the impulse response of the linear time-invariant system

$$\mathscr{B}(A,B,C,D) := \{\, (u,y) \mid \text{there is } x \text{, such that } \sigma x = Ax + Bu,\ y = Cx + Bu \,\}.$$

– SOLUTION

The impulse response h of a dynamical system is the response of the system under zero initial conditions and input that is a delta function. For a discrete-time linear time invariant system given by a state space represented, we have

$$\begin{array}{rclcrcl} h(0) & = & Cx(0)+D\delta(0)=D & \quad & x(1) & = & Ax(0)+B\delta(0)=B \\ h(1) & = & Cx(1)+D\delta(1)=CB & & x(2) & = & Ax(1)+B\delta(1)=AB \\ h(2) & = & Cx(2)+D\delta(2)=CAB & & x(3) & = & Ax(2)+B\delta(2)=A^2B \\ h(3) & = & Cx(3)+D\delta(3)=CA^2B & & x(4) & = & Ax(3)+B\delta(3)=A^3B \\ & \vdots & & & & \vdots & \\ h(t) & = & Cx(t)+D\delta(t)=CA^{t-1}B & & x(t+1) & = & Ax(t)+B\delta(t)=A^tB \end{array}$$

- $(A,B,C,D) \mapsto$ **transfer function**

  Find the transfer function of a linear time-invariant system given by a state space representation $\mathscr{B}(A,B,C,D)$.

  – SOLUTION

  Applying the Laplace transform $\mathscr{L}$ on the state equation, we have

  $$\mathscr{L}(\dot{x})=\mathscr{L}(Ax+Bu) \quad \Longrightarrow \quad sX=AX+BU \quad \Longrightarrow \quad (sI-A)X=BU \quad \Longrightarrow \quad X=(sI-A)^{-1}BU.$$

  Then, using the output equation

  $$\mathscr{L}(y)=\mathscr{L}(Cx+Du) \quad \Longrightarrow \quad Y=CX+DU \quad \Longrightarrow \quad Y=\big(\underbrace{C(sI-A)^{-1}B+D}_{H}\big)U.$$

  Therefore, the transfer fucntion of $\mathscr{B}(A,B,C,D)$ is

  $$H(s)=C(sI-A)^{-1}B+D.$$

3 Representations of LTI systems

Homework

Additional reading

Chapters 1 (behavioral models) and 4 (state-space representation) from

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http://wwwhome.math.utwente.nl/~poldermanjw/onderwijs/DISC/mathmod/book.
pdf
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Problems

- **Matrix representation of the convolution operation**

  Find a matrix representation of the discrete-time convolution operation.

  – SOLUTION

  The discrete-time convolution operation (with a kernel h) is the map $u \mapsto y$, defined by

  $$y(t)=\sum_{\tau=0}^{t} h(\tau)u(t-\tau).$$

Convolution is a linear operation, so that it has a matrix representation $y = Hu$, where the matrix H depends on the kernel h. In order to find explicitely the matrix H, we write the convolution formula as a system of linear equations:

$$\begin{array}{rcl} y(0) & = & h(0)u(0) \\ y(1) & = & h(0)u(1)+h(1)u(0) \\ y(2) & = & h(0)u(2)+h(1)u(1)+h(2)u(0) \\ & \vdots & \\ y(t) & = & h(0)u(t)+h(1)u(t-1)+\cdots+h(t)u(0) \end{array}$$

This shows that H is low-tringular with equal elements on the main and sub diagonals:

$$H = \begin{bmatrix} h(0) & & & & \\ h(1) & h(0) & & & \\ h(2) & h(1) & h(0) & & \\ \vdots & \vdots & \ddots & \ddots & \\ h(t) & h(t-1) & \cdots & h(1) & h(0) \end{bmatrix}.$$

- **Matrix representation of the discrete Fourier transform**

  Find a matrix representation of the discrete Fourier transform.

  – SOLUTION

    Consider a finite sequence

    $$u = \big(u(0),\dots,u(T-1)\big).$$

    The discrete Fourier transform is the map $u \mapsto U$, defined by

    $$U(k) = \sum_{\tau=0}^{T-1} w^{k\tau} u(\tau), \quad \text{for } k = 0,1,\dots,T-1, \qquad \text{where} \quad w := e^{-\mathbf{i}\frac{2\pi}{T}}.$$

    The discrete Fourier transform is a linear transformation, so that it has a matrix representation $U = Fu$, where

    $$F = \begin{bmatrix} w^0 & w^0 & \cdots & w^0 \\ w^0 & w^1 & \cdots & w^{T-1} \\ w^0 & w^2 & \cdots & w^{2(T-1)} \\ \vdots & \vdots & & \vdots \\ w^0 & w^{T-1} & \cdots & w^{(T-1)^2} \end{bmatrix}.$$

- **Prediction using a model**

  (separate document "exercise autonomous models" with problems and solutions)

4 Stochastic models

Homework

Reading assignment

- notes Leo part 1 sections 1.5–1.8 and notes Leo part 2

Problems

- **Wiener-Khintchine theorem**

  For a discrete-time signal y, let

  - $\phi_y := |F(y)|^2$, where $F(y)$ be a Fourier transform of y, and
  - $r_y := \sum_{t=1}^{T} y(t)y(t-\tau)$.

  Show that $\phi_y = F(r_y)$.

  - SOLUTION

    The proof
    $$\phi_y = F(y)F^*(y) = F(y)F\big(\mathrm{rev}(y)\big) = F(y \star \mathrm{rev}(y)) = F(r_y)$$
    is based on the following properties of the Fourier transform
    * $F(y \star y) = F(y)F(y)$,
    * $F\big(\mathrm{rev}(y)\big) = F^*(y)$,
    * $y \star \mathrm{rev}(y) = r_y$,

    Which are easy to show using the definitions of the Fourier transform and the convolution.

5 Least-squares estimation

Homework

Reading assignment

- notes Leo part 3, sections 3.1–3.3

Problems

- **Orthogonality principle for least-squares estimation**

  Show that

  1. $\widehat{x}$ being a least squares approximate solution of the system $Ax = b$, and
  2. $\widehat{x}$ being such that $b - A\widehat{x}$ is orthogonal to the span of the columns of A,

  are equivalent. (This result is known as the orthogonality principle for least squares approximation.)

  - SOLUTION
    * Let $\widehat{x}$ be a least squares approximate solution of the system $Ax = b$. Assuming that A is full column rank, $\widehat{x}$ is unique and is given by $\widehat{x} = (A^\top A)^{-1}A^\top b$. We have to show that
    $$A^\top(b - A\widehat{x}) = 0. \tag{$*$}$$
    Indeed,
    $$A^\top(b - A\widehat{x}) = A^\top(I - A(A^\top A)^{-1}A^\top)b = (A^\top - A^\top)b = 0.$$
    * Let $\widehat{x}$ being such that $b - A\widehat{x}$ is orthogonal to the span of the columns of A, *i.e.*, ($*$) holds. Then, assuming that A is full column rank, $A^\top A$ is invertible and $\widehat{x} = (A^\top A)^{-1}A^\top b$. This proves that $\widehat{x}$ is a least squares approximate solution of the system $Ax = b$.

- **Weighted least-squares approximate solution**

  For a given positive definite matrix $W \in \mathbb{R}^{m\times m}$, define the weighted 2-norm

  $$\|e\|_W = e^\top We.$$

  The weighted least-squares approximation problem is

  $$\text{minimize} \quad \text{over } \widehat{x} \in \mathbb{R}^n \quad \|A\widehat{x} - b\|_W. \tag{WLS}$$

  When does a solution exist and when is it unique? Under the assumptions of existence and uniqueness, derive a closed form expression for the least squares approximate solution.

  – SOLUTION

    Since W is a symmetric positive definite matrix, it has a factorization $W = CC^\top$, where C is an $m \times m$ full rank matrix. We can re-write the weighted least-squares approximation problem as an equivalent standard least-squares approximation problem for a system of linear equations $A'x = b'$, where

    $$A' = CA \qquad \text{and} \qquad b' = Cb.$$

    At this point we can use existing results: 1) a solution always exists, 2) it is unique if and only if the matrix is full column rank (f.c.r.). Since C is full rank, A' is f.c.r. if and only if A is f.c.r. In this case the unique weighted least-squares approximate solution is

    $$\widehat{x} = (A^\top WA)^{-1}A^\top Wb.$$