In Figure 6.1 (d) it shows a similar sequence after band-pass filtering. The autocorrelation function is now spread considerably to either side of $m = 0$, reflecting correlation between adjacent sample values. In Figure (e), white noise is passed through a low pass filter and a repetitive impulse train with a period of 10 sampling intervals was added to the signal. Notice how the impulse train is not recognisable in the signal $x[n]$, but the autocorrelation brings it out quite clearly. Also notice how the filtered noise only contributes to the autocorrelation function around $m = 0$, whereas the pulse train, being strictly repetitive contributes over the complete range of time shift. This suggests an important practical application for autocorrelation – the detection of a repetitive signal in the presence of unwanted noise.

The central value of an autocorrelation function equals the *mean square value* of the sequence, and is therefore a measure of its *total power*. The central value is always the maximum value. It may be equalled at other values of time shift but it can never be exceeded. Thus by setting $m = 0$ in Equation 6.6 we obtain the following measure.

$$\phi_{xx}[0] = E\{(x[n]^2\} \tag{6.9}$$

Similarly, by setting $m = 0$ in Equation 6.7, the central value of an autocovariance function equals the *variance* of the corresponding sequence – equivalent to its *ac power*.

6.4.1 Signals in noise

One of the most important topics in digital signal processing concerns the extraction of wanted signals from unwanted noise. When a signal, contaminated by noise, is to be recovered or detected, a useful way of detecting it is by using autocorrelation. An expression for the autocorrelation function is derived below.

$$y[n] = s[n] + q[n]$$

$$\phi_{yy}[m] = E\{[s[n] + q[n]][s[n+m] + q[n+m]]\}$$

$$= E\{s[n]\cdot s[n+m]\} + E\{s[n]\cdot q[n+m]\} + E\{q[n]\cdot s[n+m]\} + E\{q[n]\cdot q[n+m]\}$$

$$= \phi_{ss}[m] + \phi_{qq}[m] + 2E\{s[n]\cdot q[n+m]\}$$

The periodic signal $s[n]$ and noise $q[n]$ are completely uncorrelated to each other. Another way of stating this is that the probability of any particular value of $q[n]$ is totally independent of the value of $s[n]$ (i.e. P[q/s] = P[q], or in words, the probability of q given s is identically equal to the probability of s). Under these circumstances the last term in the equation above is determined by the following:

$$E\{s[n]\cdot q[n+m]\} = E\{s[n]\}\cdot E\{q[n+m]\} = 0$$

It is identically equal to zero because $E\{q[n+m]\}$ is the mean of the noise which is by definition itself zero. Hence the auto-correlation function of a signal with white noise of zero mean is found by replacing the last term with zero, hence

$$\phi_{yy}[m] = \phi_{ss}[m] + \phi_{qq}[m]$$

This is the Principle of Superposition that states the ACF is composed of the individual ACF's of both the signal and noise, *providing that signal and noise are uncorrelated*. This is an extremely important relationship, which is often used to detect the signal from the unwanted noise.

6.5 Wiener-Khintchine Power Theorem

The Wiener-Khintchine theorem states that the power spectrum of a signal can be expressed as the Fourier transform of its *ACF*. The power spectrum of a signal can be defined by:

$$P_{xx}(\Omega) = |X(\Omega)|^2 = X(\Omega)\cdot X^*(\Omega) = \left\{\sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}\right\}\left\{\sum_{n=-\infty}^{\infty} x[n]e^{j\Omega n}\right\} \tag{6.10}$$

Where $X(\Omega)$ and $X^*(\Omega)$ are the Fourier transform of $x[n]$ and its complex conjugate respectively. However, the complex conjugate of the Fourier transform of $x[n]$ is identical to the Fourier transform of $x[-n]$, as we can see below by a simple replacement of n by $-n$ in the second summation:

$$P_{xx}(\Omega) = \left\{ \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \right\} \left\{ \sum_{n=-\infty}^{\infty} x[n] e^{j\Omega n} \right\} = \left\{ \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \right\} \left\{ \sum_{n=-\infty}^{\infty} x[-n] e^{-j\Omega n} \right\} \tag{6.11}$$

Since Equation 6.11 describes the product of the Fourier Transforms of $x[n]$ and $x[-n]$, it can also be expressed as the Fourier Transform of the convolution of $x[n]$ and $x[-n]$.

$$P_{xx}(\Omega) = X(\Omega) \cdot X^*(\Omega) = \sum_{n=-\infty}^{\infty} \{x[n] * x[-n]\} \cdot e^{-j\Omega n} \tag{6.12}$$

Furthermore, the convolution of $x[n]$ with $x[-n]$ is identical to the correlation of $x[n]$ and $x[n]$ (i.e. the ACF ϕ_{xx}). Therefore the power spectrum can be obtained by correlating the signal $x[n]$ with $x[-n]$ and then taking the Fourier transform.

$$P_{xx}(\Omega) = \sum_{m=-\infty}^{\infty} \phi_{xx}[m] \cdot e^{-j\Omega m} \tag{6.13}$$

There also exists the inverse Fourier transform expressing ϕ_{xx} as a function of $P_{xx}(\Omega)$:

$$\phi_{xx}[m] = \frac{1}{2\pi} \int_{2\pi} P_{xx}(\Omega) \cdot e^{j\Omega m} d\Omega = \int_{-0.5}^{0.5} P_{xx}(F) \cdot e^{j2\pi Fm} dF \tag{6.14}$$

When we calculate the autocorrelation function, it is customary to work with a finite portion of the sequence and estimate the ACF for a limited set of time shift values. Figure 6.2 illustrates the estimated autocorrelation functions and power spectra for a band-pass filtered noise sequence. The ACF has been estimated by cross-multiplying 512 values of an infinite sequence with time shift values in the range $-128 \le m \le 128$, and the corresponding estimated power spectrum was calculated using the Fourier transform.

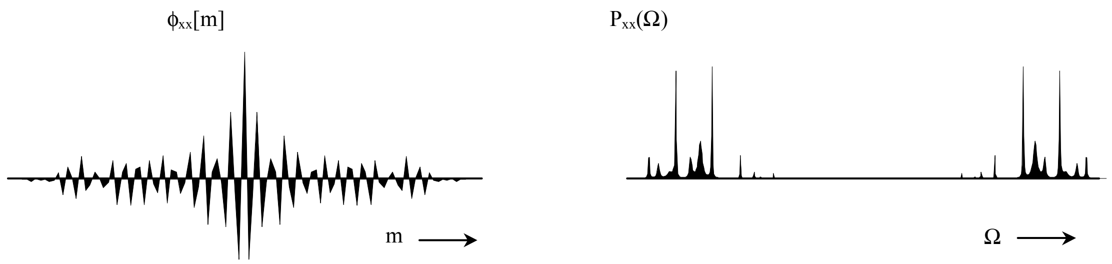

Figure 6.2: Autocorrelation and power spectral estimates for a band-pass filtered noise sequence.

If we assume the ACF to be zero outside the estimated range, we are effectively truncating it with a rectangular window. This tends to produce substantial sidelobes in the power spectrum, owing to the Gibb’s phenomenon (directly comparable to the *spectral leakage* problem described in the chapter on the DFT).

6.6 Cross-Correlation

The autocorrelation function can be used to characterise a sequence’s time domain structure. Cross-correlation is essentially the same process but instead of comparing a sequence with a time shifted version of itself, it compares two different sequences. The *cross-correlation function* (CCF) of two sequences $x[n]$ and $y[n]$, and the *cross-covariance* function are defined in terms of time averages by Equation 6.14 and Equation 6.15 respectively.

$$\phi_{xy}[m] = E\{x[n]y[n+m]\} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x[n]y[n+m] \tag{6.15}$$

$$\gamma_{xy}[m] = E\left\{(x[n] - \overline{x[n]})(y[n+m] - \overline{y[n]})\right\} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (x[n] - \overline{x[n]})(y[n+m] - \overline{y[n]}) \tag{6.16}$$

Both of these functions are second-order measures, with the CCF providing a statistical comparison of two sequences as a function of the time-shift between them. Cross-covariance is the same as the CCF, except that the mean values of the two sequences are removed. The CCF reflects the various frequency components *held* in *common* between the two sequences $x[n]$ and $y[n]$. In addition, it also holds vital information about the relative phases of shared frequency components. Unfortunately, when the cross-correlation of two sequences is performed, sometimes the fine detail of the shared frequency components is hard to interpret. If a detailed spectral analysis of the signals is required then it is better to use the cross-spectrum approach. However from a practical point of view there is one situation where the CCF is useful – namely when there are *timing differences* between two sequences. For example, suppose that $x[n]$ and $y[n]$ are identical white noise sequences which differ only in the time origin. Their CCF will then be zero for all values of m, except the one which corresponds to the timing difference.

Now let us suppose that the two signals $x[n]$ and $y[n]$, are completely *uncorrelated* with each other. From Equation 6.14, it can be shown that their CCF is a product of the expectation of each signal, as illustrated below.

$$\phi_{xy}[m] = E\{x[n]\} \cdot E\{y[n+m]\}$$

6.7 Cross-Correlation Coefficient

Sometimes it is preferable to express the cross correlation of two signals in terms of the *cross-correlation coefficient*. It is calculated by normalising the cross-correlation of the two signals with the power of the two signals i.e. by setting m = 0, as illustrated in Equation 6.16 below. The cross-correlation coefficient lies between -1 and +1, with zero indicating no correlation between the two signals.

$$\ell_{xy}[m] = \frac{\phi_{xy}[m]}{\left[\phi_{xx}[0] \cdot \phi_{yy}[0]\right]^{\frac{1}{2}}} \tag{6.17}$$

6.8 Cross Spectrum

The frequency domain counterpart of a cross-correlation function relating two sequences is known as the *cross-spectral density* or the *cross-spectrum*. It is an indication of the frequencies held in common between $x[n]$ and $y[n]$. If the two sequences have no shared frequencies, or frequency ranges, then their cross-spectrum (like their CCF) is zero. These types of sequences are said to be *linearly independent*, or *orthogonal* to each other. By definition the cross-correlation function and cross-spectrum of a digital sequence are related as a Fourier transform pair, given by Equation 6.17 and Equation 6.18 respectively.

$$\phi_{xy}[m] = \frac{1}{2\pi} \int_{2\pi} P_{xy}(\Omega) \cdot e^{j\Omega m} d\Omega \tag{6.18}$$

$$P_{xy}(\Omega) = \sum_{m=-\infty}^{\infty} \phi_{xy}[m] \cdot e^{-j\Omega m} \tag{6.19}$$

In section 6.5, we described the difficulty of obtaining a reliable estimate for the power spectrum of a time-limited random sequence. The same problems also apply herein the cross-spectrum estimation.

6.9 Examples

It is shown below how to calculate the auto correlation functions for a number of signals directly from their power spectra. This can be achieved by using the relationship between the power spectrum and the auto-correlation function given by Equation 6.13 and Equation 6.14. Essentially they are both related by a Fourier transform pair, so the auto-correlation function can be found by integrating the power spectrum.

$$\phi_{xx}[m] = \int_{-0.5}^{0.5} P_{xx}(F) \cdot e^{j2\pi Fm} dF$$

1.) Auto-correlation of a DC signal.

The power spectrum of a DC signal is given by the following: $P_{xx}(F) = A^2\delta(F)$

$$\phi_{xx}[m] = A^2 \int_{-0.5}^{0.5} \delta(F) \cdot e^{j2\pi Fm} dF \tag{6.20}$$

$$\phi_{xx}[m] = A^2 \tag{6.21}$$

2.) Auto-correlation of a sine wave signal

A sine wave function $x[n]$ is depicted in Figure 6.1 (a), with its auto-correlation function depicted on the right hand side. However, an alternative mathematical proof can be shown by using the Fourier transform relationship.

The power spectrum of a sine wave is given by: $P_{xx}(F) = \frac{A^2}{4}\left[\delta(F+F_c) + \delta(F-F_c)\right]$

Hence the auto-correlation function is found by integration:

$$\phi_{xx}[m] = \frac{A^2}{4} \int_{-0.5}^{0.5} [\delta(F+F_c) + \delta(F-F_c)] \cdot e^{j2\pi Fm} dF$$

$$\phi_{xx}[m] = \frac{A^2}{4}\left[e^{j2\pi F_c m} + e^{-j2\pi F_c m}\right]$$

$$\phi_{xx}[m] = \frac{A^2}{2}\cos(2\pi F_c m) \tag{6.22}$$

Notice how the expression is the same as the diagram on the right side of Figure 6.1 (a).

3.) Auto-correlation of white noise

A white noise sequence $x[n]$ is depicted in Figure 6.1 (c), with its auto correlation function depicted on the right hand side. We shall now derive the auto correlation function directly from its power spectrum.

The power spectrum of white noise is given by the following: $P_{xx}(F) = S$

Hence the auto-correlation function is found by integration:

$$\phi_{xx}[m] = S \int_{-0.5}^{0.5} e^{j2\pi Fm} dF$$

$$\phi_{xx}[m] = \begin{cases} S & \text{for } m = 0 \\ 0 & \text{for } m \neq 0 \end{cases} \tag{6.23}$$

Notice how the expression above is the same as the diagram on the right hand side of Figure 6.1 (c).

4.) Auto-correlation of a low-pass filtered white noise sequence

A low-pass filtered white noise sequence was one of the components depicted Figure 6.2. Its power spectrum is:

$$P_{xx}(F) = \begin{cases} S & \text{for } -F_1 \leq F \leq F_1 \\ 0 & \text{otherwise} \end{cases}$$

Hence:

$$\phi_{xx}[m] = S \int_{-F_1}^{F_1} e^{j2\pi Fm} dF$$

$$= \frac{S}{j2\pi m}\left[e^{jm2\pi F_1} - e^{-jm2\pi F_1}\right]$$

$$= S\frac{\sin(m\Omega_1)}{\pi m}$$

$$= S \cdot 2F_1 \frac{\sin(m\Omega_1)}{(m\Omega_1)} \tag{6.24}$$

The expression Equation 6.23 is the well known *sinc* function. Note that the width of the main peak of the *sinc* (i.e. between the first two zero crossings) is equal to $1/F_1$ samples (i.e. first zero crossings at $m = \pm\pi/\Omega_1 = \pm 1/2F_1$).

5.) Auto-correlation of a band-pass filtered white noise sequence

A band-pass filtered white noise sequence $x[n]$ is depicted in Figure 6.1 (d), with its auto correlation function depicted on the right hand side. We shall now once again derive the auto correlation function directly from the power spectrum. The power spectrum of a band-pass filtered white noise sequence is:

$$P_{xx}(F) = \begin{cases} S & \text{for } F_1 \le |F| \le F_2 \\ 0 & \text{otherwise} \end{cases}$$

Hence:

$$\phi_{xx}[m] = S\int_{-F_2}^{-F_1} e^{j2\pi Fm}\,dF + S\int_{F_1}^{F_2} e^{j2\pi Fm}\,dF \tag{6.25}$$

$$= \frac{S}{j2\pi m}\left[e^{-jm2\pi F_1} - e^{-jm2\pi F_2} + e^{jm2\pi F_2} - e^{jm2\pi F_1}\right]$$

$$= \frac{S}{j2\pi m}\left[e^{jm2\pi F_2} - e^{-jm2\pi F_2}\right] - \frac{S}{j2\pi m}\left[e^{jm2\pi F_1} - e^{-jm2\pi F_1}\right]$$

$$= S\frac{\sin(m\Omega_2)}{\pi m} - S\frac{\sin(m\Omega_1)}{\pi m}$$

$$= S2F_2\frac{\sin(m\Omega_2)}{(m\Omega_2)} - S2F_1\frac{\sin(m\Omega_1)}{(m\Omega_1)} \tag{6.26}$$

The expression above is the sum of two *sinc* functions and is exactly the same as the diagram on the right hand side of Figure 6.1 (d).