ELEC 3035, Lecture 8: Polynomial approach to pole placement

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1. Review of the state space approach
2. Polynomial approach and the Diophantine equation
3. Example

Review of the state space pole placement approach $(A, B, C, D) \mapsto (A_c, B_c, C_c, D_c)$

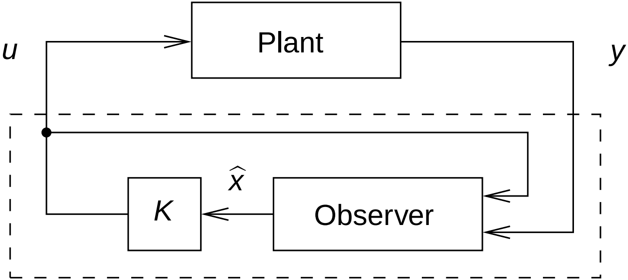
1. State feedback pole placement: compute the controller gain K
2. Pole placement observer design: compute the observer gain L
3. Dynamic output feedback controller:

$$\begin{aligned} \sigma\widehat{x} &= A_c\widehat{x} + B_c y \\ u &= C_c\widehat{x} + D_c y \end{aligned} \qquad \text{where}$$

$$A_c = A + LC + BK + LDK\,, \quad B_c = -L, \quad C_c = K\,, \quad D_c = 0$$

Computational tool: transition to controller/observer canonical forms

Involves computing controllability/observe. matrices and their inverses.



Output feedback controller

Main results for output feedback pole placement of $\mathscr{B}_{\mathrm{i/s/o}}(A, B, C, D)$:

The poles of the closed loop system can be assigned arbitrarily by dynamic output feedback if A, B is controllable and A, C is observable.

Suppose that the plant is given by an I/O representation $\mathscr{B}_{\mathrm{i/o}}(P, Q)$. Then it is natural to ask for an I/O repr. $\mathscr{B}_{\mathrm{i/o}}(P_\mathrm{c}, Q_\mathrm{c})$ of the controller.

An approach to obtain a pole placement controller $\mathscr{B}_{\mathrm{i/o}}(P_\mathrm{c}, Q_\mathrm{c})$ is:

- Derive a state space repr. $\mathscr{B}_{\mathrm{i/s/o}}(A, B, C, D)$ of $\mathscr{B}_{\mathrm{i/o}}(P, Q)$

$$(P, Q) \mapsto (A, B, C, D)$$

- Apply the state space output feedback pole placement approach

$$(A, B, C, D) \mapsto (A_\mathrm{c}, B_\mathrm{c}, C_\mathrm{c}, D_\mathrm{c})$$

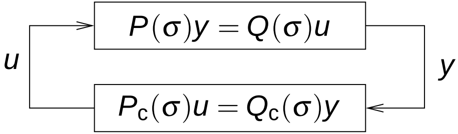
- Derive an I/O repr. $\mathscr{B}_{\mathrm{i/o}}(P_\mathrm{c}, Q_\mathrm{c})$ of $\mathscr{B}_{\mathrm{i/s/o}}(A_\mathrm{c}, B_\mathrm{c}, C_\mathrm{c}, D_\mathrm{c})$

$$(A_\mathrm{c}, B_\mathrm{c}, C_\mathrm{c}, D_\mathrm{c}) \mapsto (P_\mathrm{c}, Q_\mathrm{c})$$

We will study direct approach $(P, Q) \mapsto (P_\mathrm{c}, Q_\mathrm{c})$ as an alternative to

$$(P, Q) \mapsto (A, B, C, D) \mapsto (A_\mathrm{c}, B_\mathrm{c}, C_\mathrm{c}, D_\mathrm{c}) \mapsto (P_\mathrm{c}, Q_\mathrm{c})$$

Polynomial approach to pole placement



$$\text{Plant:} \quad P(\sigma)y = Q(\sigma)u$$

$$\text{Controller:} \quad P_c(\sigma)u = Q_c(\sigma)y$$

The closed-loop system is autonomous. In the SISO case

$$\text{Closed-loop system:} \quad \big(p_c(\sigma)p(\sigma) - q_c(\sigma)q(\sigma)\big)y = 0$$

and the closed-loop characteristic polynomial is

$$p_{cl}(z) := p_c(z)p(z) - q_c(z)q(z)$$

Diophantine equation

For SISO pole placement we need to solve the polynomial equation

$$p_c(z)p(z) - q_c(z)q(z) = p_{des}(z) \tag{D}$$

in p_c, q_c with $\text{degree}(p_c) \geq \text{degree}(q_c)$ (for causality of the controller).

Notes:

- p_{des} is the desired char. polynomial of the closed-loop system
- $\underbrace{\text{degree}(p_{des})}_{\text{CL sys's order } n_{cl}} = \underbrace{\text{degree}(p)}_{\text{plant order } n} + \underbrace{\text{degree}(p_c)}_{\text{controller order } n_c}$
- In state space, p_{des} includes plant and observer's desired poles.

The equation (D) is called Diophantine equation (also Bezout eqn).

polynomial $\times$ polynomial $\iff$ Toeplitz matrix $\times$ vector

$$c(z) = a(z)b(z) \quad \iff \quad \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{\ell_c} \end{bmatrix} = \begin{bmatrix} a_0 & & & \\ a_1 & a_0 & & \\ \vdots & a_1 & \ddots & \\ a_{\ell_a} & \vdots & \ddots & a_0 \\ & a_{\ell_a} & & a_1 \\ & & \ddots & \vdots \\ & & & a_{\ell_a} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{\ell_b} \end{bmatrix}$$

$$\iff: \quad c = S_{\ell_b}(a)b \quad \iff \quad c = S_{\ell_a}(b)a$$

polynomial $c(z) \in \mathbb{R}[z]$, $\deg(c) = \ell_c$ $\quad\longleftrightarrow\quad$ vector $c \in \mathbb{R}^{\ell_c+1}$

polynomial operations $\quad\longleftrightarrow\quad$ structured matrix operations

Diophantine equation

With $n_\mathrm{c} := \mathrm{degree}(p_\mathrm{c})$ and $m_\mathrm{c} := \mathrm{degree}(q_\mathrm{c})$ given,

$$p_\mathrm{c}(z)p(z) - q_\mathrm{c}(z)q(z) = p_\mathrm{des}(z)$$

can be written as

$$\begin{bmatrix} S_{n_\mathrm{c}}(p) & S_{m_\mathrm{c}}(q) \end{bmatrix} \begin{bmatrix} p_\mathrm{c} \\ q_\mathrm{c} \end{bmatrix} = p_\mathrm{des} \qquad \text{(D')}$$

where

$$p_\mathrm{c} = \mathrm{col}(p_{\mathrm{c},0}, p_{\mathrm{c},1}, \ldots, p_{\mathrm{c},n_\mathrm{c}}) \quad , \quad q_\mathrm{c} = \mathrm{col}(q_{\mathrm{c},0}, q_{\mathrm{c},1}, \ldots, q_{\mathrm{c},m_\mathrm{c}}),$$

$$p_\mathrm{des} = \mathrm{col}(p_{\mathrm{des},0}, p_{\mathrm{des},1}, \ldots, p_{\mathrm{des},n_\mathrm{cl}})$$

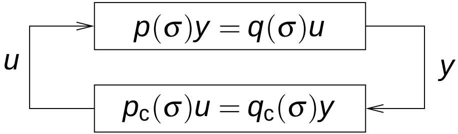
$\Longrightarrow$ solving (D) (with n_c, m_c given) is a standard linear algebra problem

Main result in polynomial approach for pole placement

Controller $\mathscr{B}_{\mathrm{i/o}}(p_{\mathrm{c}}, q_{\mathrm{c}})$ solving the pole placement problem exists if and only if the Diophantine equation (D) has solution.

Fact: (D) has solution if and only if the GCD of p and q divides p_{des}.

Corollary: The poles of the closed-loop system



can be assigned to arbitrary locations iff $\mathscr{B}_{\mathrm{i/o}}(p, q)$ is controllable.

(Recall that $\mathscr{B}_{\mathrm{i/o}}(p, q)$ is controllable iff p, q have no common factor.)

Main computational problem in the polynomial approach is solving (D).

Example

Consider the plant defined by the difference equation

$$(2-3\sigma+\sigma^2)y = (-3+2\sigma)u$$

Our goal is to design a feedback deadbeat controller for this plant.

We need to solve the Diophantine equation

$$(2-3z+z^2)p_c(z) - (-3+2z)q_c(z) = z^{2+n_c} \tag{$*$}$$

however, we do not know the controller order n_c.

We know that a controller of order $n_c = 2$ exists, however, it turns out that there is a controller of lower order.

Try $n_\mathrm{c} = 0$ (static controller). Then $(*)$ can be written (see (D')) as

$$\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} p_{\mathrm{c},0} + \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} q_{\mathrm{c},0} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This system has no solution, so a controller of order 0 does not exist.

Try $n_\mathrm{c} = 1$. Then $(*)$ can be written (see (D')) as

$$\begin{bmatrix} 2 & 0 & 3 & 0 \\ -3 & 2 & -2 & 3 \\ 1 & -3 & 0 & -2 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{\mathrm{c},0} \\ p_{\mathrm{c},1} \\ q_{\mathrm{c},0} \\ q_{\mathrm{c},1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

This system has solution, so there is a controller of order 1.

$$\begin{bmatrix} p_{c,0} \\ p_{c,1} \\ q_{c,0} \\ q_{c,0} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 & 0 \\ -3 & 2 & -2 & 3 \\ 1 & -3 & 0 & -2 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -15 \\ 1 \\ 10 \\ -9 \end{bmatrix}$$

Therefore the controller is given by the difference equation

$$(-15+\sigma)u = (10-9\sigma)y$$

and the closed loop system is $\sigma^3 y = 0$.

Comments:

- In general, the minimal order of the controller is $n_c = n-1$ (count the # of equations and unknowns in (D'))
- Reduced order observer design in the state space approach would give us the same result
- State controllability $+$ observability vs I/O controllability