Solutions

1. Give specific examples of:

   - linear static system

     Solution: *e.g.*, ideal resistor $v = Ri$ (Ohm's law) $\square$

   - nonlinear static system

     Solution: *e.g.*, saturation $v = iR$, if $|iR| \leq v_{\max}$ and $v = v_{\max}$ otherwise $\square$

   - linear time-invariant dynamical systems

     – finite impulse response

       Solution: *e.g.*, moving average filter $y(t) = \frac{1}{n}\sum_{\tau=0}^{n-1} u(t-\tau)$ $\square$

     – infinite impulse response

       Solution: *e.g.*, mass-spring-damper system with external force $m\ddot{y} + d\dot{y} + ky = u$, y — body position, u — external force $\square$

   - linear time-varying dynamical systems

     Solution: *e.g.*, a rocket burning its fuel has decreasing mass $\square$

2. The sequence $y = (0,1,1,2,3,5,8,\dots)$ is generated by a second order linear time-invariant autonomous system. Explain what does this means. Extend the sequence, *i.e.*, give a formula for the general term $y(t)$.

   Solution: y is a trajectory of a second order LTI autonomous system if

   $$r_0 y + r_1 \sigma y + r_2 \sigma^2 y = 0$$

   for some r_0, r_1, and $r_2 \neq 0$. In order to extend the sequence, we need to determine the r's. We have

   $$\begin{bmatrix} r_0 & r_1 & r_2 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & 5 & 8 \end{bmatrix}}_{\mathscr{H}_3(y)} = 0$$

   Choosing $r_2 = -1$ (why we can do this?), we obtain a linear system of equations for r_1 and r_2

   $$\begin{bmatrix} r_0 & r_1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 5 & 8 \end{bmatrix}.$$

   The solution is $r_0 = r_1 = 1$. Therefore, the recursive formula is $y(t) = y(t-1) + y(t-2)$. $\square$

3. The sequence $y = (0,1,1,2,5,9,18,\dots)$ is generated by a linear time-invariant autonomous system. Explain how to find the order of the system. Extend the sequence, *i.e.*, give a formula for the general term $y(t)$. Describe an algorithm for solving any problem of this type.

   Solution: The order of the system is the smallest ℓ, for which the Hankel matrix $\mathscr{H}_{\ell+1}(y)$ is rank deficient. An algorithm for detecting the order is then sequential computation of the ranks of $\mathscr{H}_{\ell+1}(y)$ for $\ell = 1,2,\dots$ until the matrix becomes rank deficient, at which step ℓ is the order of the system. Moreover, any nonzero vector r in the left kernel of $\mathscr{H}_{\ell+1}(y)$ defines a representation $r(\sigma)y = 0$ of the system. The rest of the problem is identical to the previous one: the sequence is extended by recursive evaluation of $r(\sigma)y = 0$ starting from initial conditions that are the final values of the given sequence.

   Applying this procedure to the data in the problem, we find $n = 3$ and $y(t) = y(t-1) + y(t-2) + 2y(t-3)$. $\square$

4. A thermometer reading 21°C, which has been inside a house for a long time, is taken outside. After one minute the thermometer reads 15°C; after two minutes it reads 11°C. What is the outside temperature? (According to Newton's law of cooling, an object of higher temperature than its environment cools at a rate that is proportional to the difference in temperature.)

   Solution: Let $y(t)$ be the reading of the thermometer at time t and let $\bar{u}$ be the environmental temperature. From Newton's law of cooling, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}y = a\big(\bar{u}s - y\big)$$

   for some unknown constant $a \in \mathbb{R}$, $a > 0$, which describes the cooling process. Integrating the differential equation, we obtain an explicit formula for y in terms of the constant a, the environmental temperature $\bar{u}$, and the initial condition $y(0)$

$$y(t) = e^{-at}y(0) + (1 - e^{-at})\bar{u}, \qquad \text{for } t \geq 0 \tag{1}$$

   The problem is to find $\bar{u}$ from (1) given that $y(0) = 21$, $y(1) = 15$, and $y(2) = 11$. Substituting the data in (1), we obtain a nonlinear system of two equations in the unknowns $\bar{u}$ and $f := e^{-a}$

$$\begin{cases} y(1) = fy(0) + (1 - f)\bar{u} \\ y(2) = f^2y(0) + (1 - f^2)\bar{u} \end{cases} \tag{2}$$

   We may stop here and declare that the solution can be computed by a method for solving numerically a general nonlinear system of equations.

   System (2), however, can be solved without using "nonlinear" methods. Define Δy to be the temperature increment from one measurement to the next, *i.e.*, $\Delta y(t) := y(t) - y(t-1)$, for all t. The increments satisfy the homogeneous differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\Delta y(t) = a\Delta y(t)$, so that

$$\Delta y(t+1) = e^{-a}\Delta y(t) \qquad \text{for } t = 0, 1, \ldots \tag{3}$$

   From the given data we evaluate

$$\Delta y(0) = y(1) - y(0) = 15 - 21 = -6, \qquad \Delta y(1) = y(2) - y(1) = 11 - 15 = -4.$$

   Substituting in (3), we find the constant $f = e^{-a} = 2/3$. With f known, the problem of solving (2) in $\bar{u}$ is linear, and the solution is found to be $\bar{u} = 3$°C. □