ELEC 3035: Control systems design, Exam part I solutions

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1. *Static multi-input muli-output controller* This problem involves three static linear systems

Plant:	$\mathscr{P} = \{ (u,y) \mid y = Pu \}$,	where	$P \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$,
Controller:	$\mathscr{C} = \{ (r,u) \mid u = Cr \}$,	where	$C \in \mathbb{R}^{\mathtt{m}\times\mathtt{r}}$,
Reference model:	$\mathscr{M} = \{ (r,y) \mid y = Mr \}$,	where	$M \in \mathbb{R}^{\mathtt{p}\times\mathtt{r}}$.

The controller and the plant are interconnected in the feedforward configuration, shown in Figure 1.

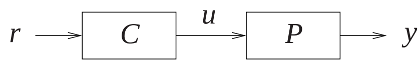

Figure 1: Interconnection of the plant and the controller.

Given a plant and a reference model, the aim is to find a controller that makes the controlled system as close to the reference model M as possible.

(a) [5 marks] Define the controlled system $\mathscr{P}_{\mathscr{C}}$, corresponding to the feedforward interconnection of Figure 1. Find an input/output representation of $\mathscr{P}_{\mathscr{C}}$. Is $\mathscr{P}_{\mathscr{C}}$ a static linear system? Justify your answer.

(b) [10 marks] Under what conditions on $\mathscr{P}$, for any choice of $\mathscr{M}$, there is a controller $\mathscr{C}$, such that $\mathscr{P}_{\mathscr{C}} = \mathscr{M}$. Under what conditions on $\mathscr{P}$, the controller is moreover unique? When the controller is unique, show how one can obtain its input/output representation C from the given P and M matrices. When the controller is not unique, display one particular solution.

(c) [10 marks] Consider the case when there is no controller $\mathscr{C}$ that achieves the reference model $\mathscr{M}$ exactly. Define a problem of finding a controller that is as close to achieving the specification as possible. In what sense a controlled system $\mathscr{P}_{\mathscr{C}}$ with a controller solving this problem is "close" to the reference model $\mathscr{M}$? Is your controller design problem equivalent to a standard approximation problem. If so, give a solution.

Solution:

(a) The controlled system is

$$\begin{aligned}\mathscr{P}_{\mathscr{C}} &= \{ (r,y) \mid \text{ there is } u \text{ such that } (r,u) \in \mathscr{C} \text{ and } (u,y) \in \mathscr{P} \} \\ &= \{ (r,y) \mid \text{ there is } u \text{ such that } u = Cr \text{ and } y = Pu \} \\ &= \{ (r,y) \mid y = \underbrace{PC}_{P_{\mathrm{c}}} r \} = \{ (r,y) \mid y = P_{\mathrm{c}} r \}.\end{aligned}$$

(b) Since $\mathscr{P}_{\mathscr{C}}$ admits an input/output representation $y = P_{\mathrm{c}} r$ that is a linear function, $\mathscr{P}_{\mathscr{C}}$ is a static linear system.

(c) The specification $\mathscr{P}_{\mathscr{C}} = \mathscr{M}$ is equivalent to $P_{\mathrm{c}} = M$, *i.e.*,

$$PC = M. \tag{1}$$

(1) is a linear system of equations in the unknown P. There is a controller that achieves the specification if and only if (1) has a solution. Moreover, a solution should exist for any $M \in \mathbb{R}^{\mathtt{p}\times\mathtt{r}}$. A necessary and sufficient condition for this is that $\operatorname{image}(P) = \mathbb{R}^{\mathtt{p}}$ or equivalently that P is full row rank.

The solution of the control problem is unique if the solution of the system (1) is unique. Necessary and sufficient condition for uniqueness of solution is that P is square and invertible, in which case the unique solution is given by

$$C = P^{-1} M.$$

If a solution exists but is not unique, *i.e.*, P is full row rank but not square, a particular solution is the least norm one

$$C_{\mathrm{ln}} = P^{\top} (PP^{\top})^{-1} M.$$

(d) Now we consider the case when P is not full row rank. Assuming that P is full column rank, a least squares approximate solution is unique and is given by

$$C_{\text{ls}} = (P^\top P)^{-1} P^\top M.$$

The controlled system with this controller has an input/output representation with a parameter

$$P_{\text{c,ls}} = PC_{\text{ls}} = P(P^\top P)^{-1} P^\top M.$$

The matrix $P_{\text{c,ls}}$ approximates the matrix M in the sense that $\|P_{\text{c}} - M\|_{\text{F}}$ is minimized. (This is the least squares cost function.) Unfortunately there is no nice interpretation of this optimality criterion in term of the systems $\mathscr{P}_{\mathscr{C}}$ and $\mathscr{M}$. Therefore, in this case, the least squares approximate solution is not meaningful.

□

2. *Approximate state transfer of uncontrollable system* Consider the system defined by the state space representation

$$x(t+1) = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{b} u(t).$$

(a) [5 marks] Show that this system is state uncontrollable. Sketch the controllable subspace.

(b) [10 marks] Consider an initial state $x(0) = 0$ and a target state $x_{\text{target}} = \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$. Show that the target state is not reachable from the initial state in two time steps. Determine a state x^* that is as close (in the 2-norm distance) to x_{target} as possible and is reachable from $x(0)$ in two time steps.

(c) [10 marks] Find all inputs $u = \big(u(0), u(1)\big)$ that transfers the system from the initial state $x(0) = 0$ to a final state $x(2) = x^*$ and select the minimum 2-norm input.

Solution:

(a) The controllability matrix of the the system

$$\begin{bmatrix} b & Ab \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

is rank deficient (the second column is equal to three times the first column), so that the system is not state controllable. The controllable subspace is the image of the controllability matrix, which in this case is

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \alpha \mid \alpha \in \mathbb{R} \right\}.$$

(b) Since the system is of order 2 and the initial state $x(0) = 0$ is in the controllable subspace, the point x_{target} is reachable from $x(0)$ in two time steps if and only if it is in the controllable subspace. The condition that x_{target} is in the controllable subspace is equivalent to the condition that there is $\alpha \in \mathbb{R}$, such that

$$x_{\text{target}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \alpha.$$

For $x_{\text{target}} = \text{col}(2,1)$ this system is incompatible, so that the target state $\text{col}(2,1)$ is not reachable from the origin in two time steps (and therefore in any number of steps).

The closest point x^* in the controllable subspace to x_{target} is the projection of x_{target} on the controllable subspace, see Figure 2. This point can be found by solving the least squares problem

$$\text{minimize} \quad \text{over } \alpha \in \mathbb{R} \quad \left\| \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} \alpha \right\|_2 .$$

We have

$$\alpha^* = \left(\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4/5,$$

so that $x^* = 4/5\,\text{col}(1,2)$.

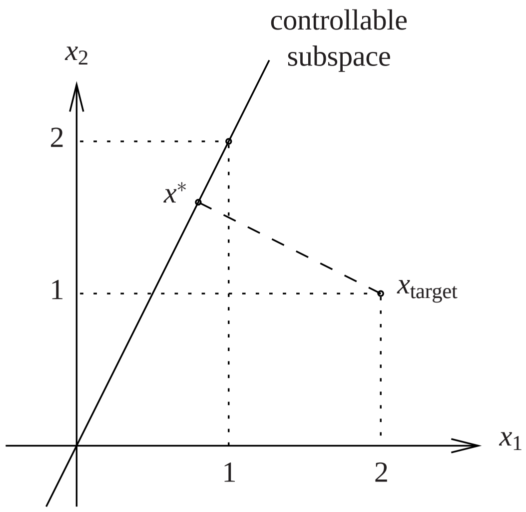

Figure 2: Controllable subspace for the system in Problem 2.

(c) Any input $u = \big(u(0), u(1)\big)$ that transfers the state from $x(0)$ to $x(2) = x^*$ is a solution of the system

$$\begin{bmatrix} b & Ab \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = x_{\text{target}} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} u(1) \\ u(0) \end{bmatrix} = 4/5 \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The matrix $\left[\begin{smallmatrix} 1 & 3 \\ 2 & 6 \end{smallmatrix}\right]$, however, is singular so that there is no unique solution. (Also a solution can not be found by elimination!) A nonunique solution, however, exists: choosing an arbitrary value $\beta \in \mathbb{R}$ for $u(0)$ the resulting system in $u(1)$ is

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} u(1) = \big(4/5 - 3\beta\big) \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

This system is compatible and has a solution $u(1) = 4/5 - 3\beta$. Therefore, the general solution is of the form

$$\begin{bmatrix} u(1) \\ u(2) \end{bmatrix} = \begin{bmatrix} 4/5 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \end{bmatrix} \beta.$$

The minimum norm input corresponds to the β parameter that solves the problem

$$\text{minimize} \quad \text{over } \beta \in \mathbb{R} \quad \left\| \begin{bmatrix} 4/5 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \end{bmatrix} \beta \right\|_2.$$

This is a least squares problem and has solution

$$\beta^* = \left(\begin{bmatrix} -3 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} -3 & 1 \end{bmatrix} \begin{bmatrix} -4/5 \\ 0 \end{bmatrix} = 6/25.$$

Therefore, the 2-norm optimal input that transfers the state from $x(0) = 0$ to $x(2) = x^*$ is

$$u^*(0) = 6/25 \quad \text{and} \quad u^*(1) 2/25.$$

□

3. *A state estimation problem* Consider the system defined by the state space representation

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \qquad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t) + e(t), \end{aligned} \tag{2}$$

where e is measurement noise. This problem is concerned with the estimation of the initial state x_0 from given input/output observations.

(a) [10 marks] Suppose that the observations of the input are

$$\big(u(1), u(2), \dots, u(T)\big) \tag{3}$$

and the observations of the output are

$$\big(y(t_1), y(t_2), \dots, y(t_K)\big), \qquad \text{where } t_1 < t_2 < \cdots < t_K \text{ are integers in the interval } [1, T]. \tag{4}$$

Give verbal interpretation of the data (3–4). Is your interpretation practically relevant? If so, give (engineering) examples where it may occur.

(b) [10 marks] Assuming that the system is observable and its order is smaller than the number of output samples K, find the least squares estimate of the initial state x_0 from the observations (3–4).

(c) [5 marks] Assuming that $t_K < T$, predict the output at $t_K + 1, \dots, T$ from the observations (3–4). Is your predicted output optimal in some sense? If so, in what sense?

Solution:

(a) The discrete-time system (2) can be thought of as obtained from a continuous-time system after *uniform* sampling. Let the sampling time be T_s. The inputs (3) are uniformly sampled in the interval $[T_s, T_s T]$ with a sampling time T_s. The outputs (4), however, are either non uniformly sampled in the same interval or are sampled at a sampling time that is an integer multiple of T_s.

Both nonuniform sampling and sampling at a different rate are realistic situations that happen in practice. For example, sporadic failures of the output sensors results in missing output observations. This is a special case of (4). It is not difficult to imagine a practical situation when a sensor malfunctions and there is some monitoring device that gives information when the sensor is not working.

(b) Consider the general equation relating the response of the system to the initial condition and the input

$$\underbrace{\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(T) \end{bmatrix}}_{Y_T} = \underbrace{\begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^T \end{bmatrix}}_{\mathscr{O}_T} x_0 + \underbrace{\begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ CA^{T-2}B & \cdots & CB & D \end{bmatrix}}_{\mathscr{T}_T} \underbrace{\begin{bmatrix} u(1) \\ u(2) \\ \vdots \\ u(T) \end{bmatrix}}_{U_T}. \tag{5}$$

In the case when the output is observed at the samples $1 \le t_1 < t_2 < \cdots < t_K \le T$, we set the estimation problem for x_0 selecting the rows of (5) with indexes $t_1, t_2, \dots, t_K$, *i.e.*, the ones for which we have given output data. Let Y_d, $\mathscr{O}_\mathrm{d}$, and $\mathscr{T}_\mathrm{d}$ be the matrices obtained by choosing the t_1 ,t_2, ..., t_K rows of Y_T, $\mathscr{O}_T$, and $\mathscr{T}_T$, respectively. Then the state estimation problem is to solve

$$Y_\mathrm{d} = \mathscr{O}_\mathrm{d} x_0 + \mathscr{T}_\mathrm{d} U_T$$

for x_0. Under the assumption that $\dim(x_0) < \dim(Y_\mathrm{d}) = K$ and the system is controllable, the system is overdetermined and the unique least squares estimate is given by

$$\widehat{x}_0 = (\mathscr{O}_\mathrm{d}^\top \mathscr{O}_\mathrm{d})^{-1}(Y_\mathrm{d} - \mathscr{T}_\mathrm{d} U_T).$$

(c) The least squares prediction of the output, given the data (3–4) is

$$\begin{bmatrix} \widehat{y}(t_K+1) \\ \vdots \\ \widehat{y}(T) \end{bmatrix} = \begin{bmatrix} CA^{t_K+1} \\ \vdots \\ CA^T \end{bmatrix} \widehat{x}_0 + \mathscr{T}_T(t_K+1{:}T, :) \begin{bmatrix} u(t_K+1) \\ \vdots \\ u(T) \end{bmatrix},$$

where $\mathscr{T}_T(t_K+1{:}T,:)$ is the matrix formed from the $t_K+1, \dots, T$ rows of $\mathscr{T}_T$. It is optimal in the sense that it minimizes the prediction error $\|Y_\mathrm{d} - \widehat{Y}\|_2$ over all predictions $\widehat{Y}$ that are responses of the system (2) under the input $\big(u(t_K+1), \dots, u(T)\big)$.

$\square$