Notes signal theory: Part 1, Lecuture 2

Transition from lecture 1

- questions and feedback
- homework and introduction
- recup of lecture 1
- size of a signal

New topic: systems

- behavioral approach
- subspace and basis
- representations
- autonomus systems

Solution to homework

Problems

Sensor speed-up

A thermometer reading 21°C, which has been inside a house for a long time, is taken outside. After one minute the thermometer reads 15°C; after two minutes it reads 11°C. What is the outside temperature? (According to Newton's law of cooling, an object of higher temperature than its environment cools at a rate that is proportional to the difference in temperature.)

Solution

Let $y(t)$ be the reading of the thermometer at time t and let $\bar{u}$ be the environmental temperature. From Newton's law of cooling, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t} y = a\big(\bar{u}s - y\big)$$

for some unknown constant $a \in \mathbb{R}$, $a > 0$, which describes the cooling process. Integrating the differential equation, we obtain an explicit formula for y in terms of the constant a, the environmental temperature $\bar{u}$, and the initial condition $y(0)$

$$y(t) = e^{-at} y(0) + (1 - e^{-at})\bar{u}, \qquad \text{for } t \geq 0 \tag{1}$$

The problem is to find $\bar{u}$ from (1) given that $y(0) = 21$, $y(1) = 15$, and $y(2) = 11$. Substituting the data in (1), we obtain a nonlinear system of two equations in the unknowns $\bar{u}$ and $f := e^{-a}$

$$\begin{cases} y(1) = f y(0) + (1 - f)\bar{u} \\ y(2) = f^2 y(0) + (1 - f^2)\bar{u} \end{cases} \tag{2}$$

We may stop here and declare that the solution can be computed by a method for solving numerically a general nonlinear system of equations. (Such methods and software are available, see, *e.g.*, [**?**].)

System (2), however, can be solved without using "nonlinear" methods. Define Δy to be the temperature increment from one measurement to the next, *i.e.*, $\Delta y(t) := y(t) - y(t-1)$, for all t. The increments satisfy the homogeneous differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\Delta y(t) = a\Delta y(t)$, so that

$$\Delta y(t+1) = e^{-a}\Delta y(t) \qquad \text{for } t = 0, 1, \ldots \tag{3}$$

From the given data we evaluate

$$\Delta y(0) = y(1) - y(0) = 15 - 21 = -6, \qquad \Delta y(1) = y(2) - y(1) = 11 - 15 = -4.$$

Substituting in (3), we find the constant $f = e^{-a} = 2/3$. With f known, the problem of solving (2) in $\bar{u}$ is linear, and the solution is found to be $\bar{u} = 3°\mathrm{C}$.

[?, Chapter 2, Problem 2]

assignment

A bank offers 7% annual interest. What would be the overall annual rate if the 7% interest were compounded quarterly?

[?, Chapter 2, Problem 5]

assignment

Find the second order linear homogeneous difference equation which generates the sequence 1, 2, 5, 12, 29, 70, 169. What is the limiting ratio of consecutive terms?

[?, Chapter 2, Problem 10]

assignment

Consider the second order difference equation

$$y(k+2) - 2ay(k+1) + a^2 y(k) = 0.$$

Its characteristic polynomial has both roots equal to $\lambda = a$.

1. Show that both $y(k) = a^k$ and $y(k) = ka^k$ are solutions.
2. Find the solutions of this equation that satisfies the auxiliary conditions $y(0) = 1$ and $y(1) = 0$.