ELEC 3035, Lecture 5: Observability and state estimation

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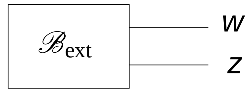
- Definition of observability
- State reconstruction and observability matrix
- Least squares observer

General observability problem

A dynamical system $\mathscr{B}_{\text{ext}}$ with two types of external variables:

- observed variables w
- to-be-estimated variables z



is observable if z can be recovered from w and the model parameters.

System accepting w and producing z is called an observer.

Our goal is to construct an observer for a given system $\mathscr{B}_{\text{ext}}$.

State observability

We will consider the special case:

$$\mathscr{B} = \mathscr{B}_{\text{i/s/o}}(A,B,C,D), \qquad w = (u,y), \qquad z = x$$

(Recall problem 1 from tutorial 2: $\mathscr{B} = \mathscr{B}_{\text{ss}}(A,C)$, $w = y$, $z = x$.)

State observability — a property of the system ensuring that

the state can be recovered uniquely from the input and the output

- How to check observability in terms of A,B,C,D?
- How to reconstruct x (observer design)?
- How to approximate x when the measurements are noisy?

Output trajectories

The trajectories of the system

$$\mathscr{B}_{\text{i/s/o}}(A,B,C,D) = \{\, (u,x) \mid \sigma x = Ax + Bu,\ y = Cx + Du \,\}$$

are in the DT case

$$y(t) = CA^t x(0) + C \sum_{\tau=0}^{t-1} A^{t-1-\tau} Bu(\tau) + Du(t)$$

and in the CT case

$$y(t) = Ce^{At} x(0) + C \int_0^t e^{A(t-\tau)} Bu(\tau)\mathrm{d}\tau + Du(t)$$

DT–CT analogy: $\quad A^t \leftrightarrow e^{At} \quad$ and $\quad \sum_{\tau=0}^{t-1} (\cdot) \leftrightarrow \int_0^t (\cdot)\mathrm{d}\tau$

Observability of DT systems

Suppose we have observed u and y over the period $[0, t-1]$.

The system of equations

$$y(\tau) = CA^{\tau}x(0) + C\sum_{s=0}^{\tau-1} A^{\tau-1-s}Bu(s) + Du(\tau), \qquad \text{for } \tau = 0, 1, \ldots, t-1$$

written in a matrix form is

$$\underbrace{\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(t-1) \end{bmatrix}}_{Y_t} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{t-2} \end{bmatrix}}_{\mathscr{O}_t} x(0) + \underbrace{\begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ CA^{t-1}B & \cdots & CB & D \end{bmatrix}}_{\mathscr{T}_t} \underbrace{\begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t-1) \end{bmatrix}}_{U_t}$$

$$Y_t = \mathscr{O}_t x(0) + \mathscr{T}_t U_t$$

- $\mathscr{O}_t$ maps the initial state to the output over $[0, t-1]$
- $\mathscr{T}_t$ maps the input to the output over $[0, t-1]$

Estimating the initial state requires to solve for x_0

$$\mathscr{O}_t x(0) = Y_t - \mathscr{T}_t U_t$$

Therefore,

$$\boxed{x_0 \text{ can be reconstructed uniquely if and only if } \ker(\mathscr{O}_t) = \{0\}}$$

Note: the input plays no role in the state estimation problem.

Observability matrix of the system $\mathscr{B}_{\mathrm{ss}}(A, C)$

By the Caley-Hamilton theorem A^t, for $t \geq n$, can be expressed as linear combination of $A^0, A^1, \ldots, A^{n-1}$.

Therefore,

$$\ker(\mathscr{O}_t) = \ker(\mathscr{O}_n), \qquad \text{for } t \geq n$$

$\Longrightarrow$ the initial state can be reconstructed from n output samples

The matrix

$$\mathscr{O} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is called the observability matrix of the system $\mathscr{B}_{\mathrm{ss}}(A, C)$.

Observability test and state observer

$$\boxed{\mathscr{B}_{\mathrm{i/s/o}}(A, B, C, D) \text{ is observable if and only if } \ker(\mathscr{O}) = \{0\}}$$

As for controllability, the observability test is the same in CT as in DT.

An observer F for the ini. state $x(0)$ is a linear function of U_t and Y_t

$$x_0 = F(Y_t - \mathscr{T}_t U_t)$$

satisfying the condition

$$F\mathscr{O}_t = I$$

Least squares observer

Assume that the output is observed with measurement noise v, *i.e.*,

$$y = Cx + Du + v$$

Then the system of equations for x_0

$$\mathscr{O}_t x(0) = Y_t - \mathscr{T}_t U_t$$

(generically) has no exact solution. The least-squares observer is

$$\widehat{x}_{\mathrm{ls}}(0) = \underbrace{(\mathscr{O}_t^\top \mathscr{O}_t)^{-1} \mathscr{O}_t^\top}_{F_{\mathrm{ls}}} (Y_t - \mathscr{T}_t U_t)$$

It minimizes the output estimation error $\|Y - \widehat{Y}\|_2$, where

$$\widehat{Y} := \mathscr{O}_t \widehat{x}_{\mathrm{ls}}(0) + \mathscr{T}_t U_t$$

Observability Gramian

Consider the case $u = 0$,

$$\widehat{x}_{\mathrm{ls}}(0) = (\mathscr{O}_t^\top \mathscr{O}_t)^{-1} \mathscr{O}_t^\top Y_t = \underbrace{\left(\sum_{\tau=0}^{t-1} (A^\tau)^\top C^\top C A^\tau \right)}_{G_{\mathrm{o},t} := \mathscr{O}_t^\top \mathscr{O}_t}^{-1} \underbrace{\sum_{\tau=0}^{t-1} (A^\tau)^\top C^\top y(t)}_{\mathscr{O}_t^\top Y_t}.$$

The initial state estimation error is

$$x(0) - \widehat{x}_{\mathrm{ls}}(0) = -G_{\mathrm{o},t}^{-1} \mathscr{O}_t^\top V_t, \quad \text{where } V_t := \begin{bmatrix} v(0) \\ \vdots \\ v(t-1) \end{bmatrix}.$$

If V_t has bounded norm, then the error $x(0) - \widehat{x}_{\mathrm{ls}}(0)$ is also bounded

$$x(0) - \widehat{x}_{\mathrm{ls}}(0) \in \{ G_{\mathrm{o},t}^{-1} \mathscr{O}_t^\top V_t \mid \|V_t\| \le \sigma \} \qquad \text{uncertainty ellipsoid}$$

Infinite horizon state estimation

$G_{\mathrm{o},t}^{-1}$ characterizes the uncertainty in the estimate $\widehat{x}_{\mathrm{ls}}(0)$

For $t \to \infty$, the uncertainty ellipsoid is given by the matrix

$$G_{\mathrm{o}} := \lim_{t \to \infty} G_{\mathrm{o},t} \qquad \text{observability Gramian}$$

For stable systems, G_{o} exists and satisfies the Lyapunov equation

$$A^\top G_{\mathrm{o}} A - G_{\mathrm{o}} = -C^\top C$$

Even for infinite number of measurements the initial condition estimate is not perfect.

Duality between observability and controllability

The system $\mathscr{B}(A^\top, C^\top, B^\top, D^\top)$ is called the dual of $\mathscr{B}_{\mathrm{i/s/o}}(A, B, C, D)$.

The observability matrix of $\mathscr{B}_{\mathrm{i/s/o}}(A, B, C, D)$ is

$$\begin{aligned} \mathscr{O}(A, C) &= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \\ &= \begin{bmatrix} C^\top & A^\top C^\top & \cdots & (A^{n-1})^\top C^\top \end{bmatrix}^\top \\ &= \mathscr{C}^\top (A^\top, C^\top) \end{aligned}$$

equal to the transposed of the controllability matrix of

$$\mathscr{B}(A^\top, C^\top, B^\top, D^\top)$$

Example

Consider the second order system

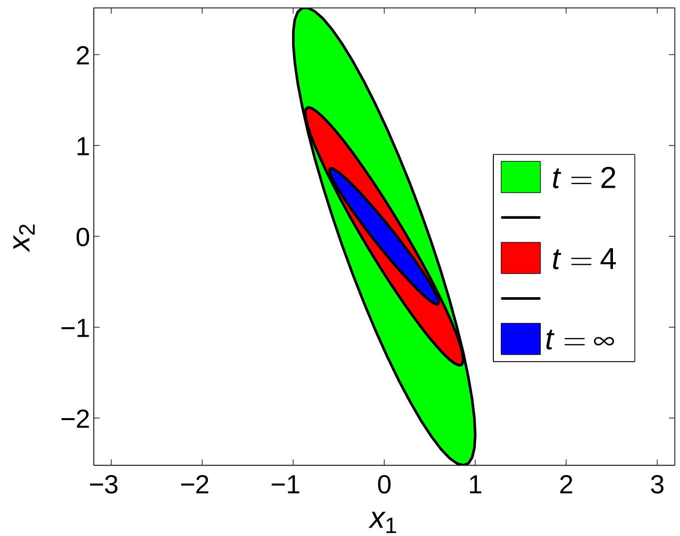
$$\mathscr{B}_{\text{ss}}(A,c) = \left\{ (u,y) \mid \sigma x = \underbrace{\begin{bmatrix} -1.75 & -0.8 \\ 1 & 0 \end{bmatrix}}_{A} x,\ y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{c} x \right\}$$

The observability matrix is

$$\mathscr{O} = \begin{bmatrix} c \\ cA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1.75 & -0.8 \end{bmatrix}$$

Is this system observable?

Estimation uncertainty regions



Estimation uncertainty vs observation time