ELEC 3035 — Control Systems Design 2008/9 — Solutions

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Question 4(a)

$$\dot{x}_1 = x_2$$

and the state-space model follows by direct substitution.

Equilibrium points — set $\dot{x}_1 = 0$, $\dot{x}_2 = 0$, $\dot{x}_3 = 0$ and ignore the control input terms. It then follows immediately (from the first state equation) that the only possible equilibrium point must have $x_2 = 0$. The third state equation shows that at any equilibrium point $x_3 = 0$. Finally, from the second state equation we have

$$\sin x_1 = 0$$

at any equilibrium point. Hence the (distinct) equilibrium points are $(0, 0, 0)$, $(\pi, 0, 0)$. [9 marks]

Question 4(b)

Lyapunov's first theorem – Jacobian matrix (for three state equations)

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix}$$

Next compute the eigenvalues of A and then i) stable if all have strictly negative real parts, ii) unstable if at least one has a positive real part, and iii) inconclusive if there are any on the imaginary axis in the complex plane.

In this case

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{g}{l}\cos x_1 & -\frac{b}{ml^2} - \frac{c}{ml^2} & \frac{c}{ml^2} \\ 0 & \frac{c}{J} & -\frac{c}{J} \end{bmatrix}$$

[9 marks]

Question 4(b)

Equilibrium point $(0, 0, 0)$

$$\det(sI_3 - A) = s^2 + \alpha_0 s^2 + \alpha_1 s + \alpha_2$$

where

$$\alpha_1 = \frac{c}{J} + \frac{(b+c)}{ml^2},\ \alpha_2 = \frac{g}{l} + \frac{bc}{Jml^2},\ \alpha_3 = \frac{gc}{lJ}$$

All coefficients of the polynomial in this case are positive and hence by the given fact on third order polynomials all of its roots, and hence the eigenvalues of A, have strictly negative real parts. Hence this is a stable equilibrium point.

Equilibrium point $(\pi, 0, 0)$

Repeating the above analysis we find that one of the coefficients in the polynomial is negative — either case ii) or case iii) applies. Polynomial has no root at $s = 0$ but could have a factor of the form $s^2 + p^2$ where p is a positive real number. If such exists then it is case iii).

[7 marks]

Question 5(a)

Simply substitute the given control law into the system which cancels the nonlinear terms and leaves the linear system given in the question which is clearly stable.

Alternative controller — $x_1 e^{x_1}$ term is not needed. [8 marks]

Question 5(b)

$$\dot{V} = x_1^3 x_1 + x_2 \dot{x}_2$$

and hence

$$\dot{V} = -(3x_1^4 + x_2^2)f(x)$$

Case i) — Can guarantee that $\dot{V} < 0$ except at $(0,0)$ for $\sqrt{x_1^2 + x_2^2} \leq 3$ — local asymptotic stability holds.

Case ii) — Can guarantee that $\dot{V} < 0$ except at $(0,0)$ — global asymptotic stability holds.

Case iii) — more detailed analysis of $3x_1^4 - x_2^2$ is required as this term determines where $\dot{V} < 0$. [12 marks]

Question 5(c)

La Salles invariant principle in application can be applied if it can be shown that $\dot{V}(x) = 0$ only holds for the origin. Hence use the system equations to see if this property holds. If it does then by this principle asymptotic stability holds either locally or globally depending on whether or not $V(x) > 0$ locally or globally and the same for $\dot{V}(x)$. [5 marks]

Question 6(a)

Using the formula given and Figure 1, we have

$$N(A) = \frac{4}{\pi A}\left[d\int_0^{\alpha} \sin t dt + 2d\int_{\alpha}^{\frac{\pi}{2}} \sin t dt\right]$$

where if $h < A$ the second integral is omitted and also

$$\sin\alpha = \frac{h}{A}$$

Evaluating the integrals in each case now gives the formulas stated in the question. [8 marks]

Question 6(b)

The equation defining a limit cycle (it its exists) is

$$G(j\omega) = -\frac{1}{N(A)}$$

A solution on the negative real axis requires $\omega = \sqrt{3}$ rad/sec. At this frequency

$$8 = kN(A)$$

and with $A = \frac{5}{4}h$

$$k = \frac{10}{7}\frac{\pi h}{d}$$

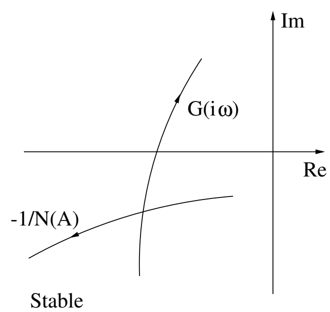

Figure 1

[10 marks]

Question 6(c)

Figures 1 and 2

Limit profile is stable.

1. The amplitude and frequency of the predicted limit cycle are not accurate.

2. A predicted limit cycle does not actually exist.

3. An existing limit cycle is not predicted.

[7 Marks]

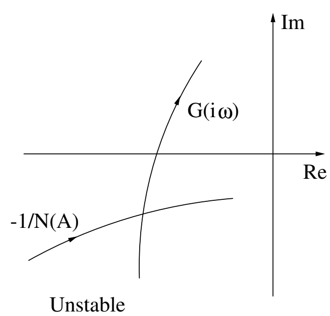

Figure 2