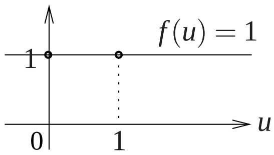
ELEC 3035: Solutions for the quiz on linear algebra

1. *Functions* Give a formula defining the function whose graph is a straight line passing through $(0,1)$ and $(1,1)$.

   The easiest way to derive the answer is to draw the function (the solid dots are the given points $(0,1)$ and $(1,1)$):



   Drawing the function, however, is possible only for the special case of single input single output functions. Here is an algebraic approach that works in general, see also problem 2 in the first tutorial.

   A function which graph is a straight line is affine (linear plus constant). Conversely, each straight line is a graph of an affine function. Let f be the function we are after. There are numbers $a, b \in \mathbb{R}$, such that $f(u) = au + b$. From the given data, $f(0) = b = 1$ and $f(1) = a + b = 1$, we have $b = 1$ and $a = 0$. Answer: $f(u) = 1$, for all u.

2. *Subspaces* Give an example of a subspace in $\mathbb{R}^2$. Describe all subspaces of $\mathbb{R}^2$.

   All subspaces of $\mathbb{R}^2$ are: $\{0\}$, $\mathbb{R}^2$, or any straight line passing through 0. Any one of these is a valid example.

3. *Rank, range, and kernel* What are the rank, range, and kernel of the matrix $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

   $\operatorname{rank}(A) = 1$ — follows from $\operatorname{rank}(A) \leq \min(2,1) = 1$ and $\operatorname{rank}(A) = 0 \iff A = 0$

   $\operatorname{image}(A) = \{\, \alpha \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] \mid \alpha \in \mathbb{R} \,\}$ — by the definition of $\operatorname{image}(A)$ (range $\equiv$ image $\equiv$ column span)

   $\ker(A) = \{\, 0 \,\}$ — derivation: $\ker(A) = \{\, u \in \mathbb{R} \mid \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] u = 0 \,\} = \{\, 0 \,\}$ (kernel $\equiv$ null space)

   State connections among rank, range, and kernel of a matrix.

   Facts stated in the lecture slides:

   - A full row rank $\iff$ $\operatorname{image}(A) = \mathbb{R}^m$ (Lecture 2, page 21)
   - A full column rank $\iff$ $\ker(A) = \{\, 0 \,\}$ (Lecture 2, page 20)

   More generally, $\dim\big(\operatorname{image}(A)\big) = \operatorname{rank}(A)$ and $\dim\big(\ker(A)\big) = \operatorname{coldim}(A) - \operatorname{rank}(A)$.

4. *Underdetermined system of linear equations* What is the solution set of $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 1$.

   $\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 1 \quad \Longrightarrow \quad u_1 + u_2 = 1 \quad \Longrightarrow \quad u_1 = 1 - u_2$. Therefore, the solution set is $\{\, \left[\begin{smallmatrix} 1-\alpha \\ \alpha \end{smallmatrix}\right] \mid \alpha \in \mathbb{R} \,\}$.

   What is the least norm solution?

   The system is of the standard form $Au = y$ with $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $y = 1$. From the general formula of a least norm solution, we have:

$$u_{\text{ln}} = A^\top (AA^\top)^{-1} y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)^{-1} 1 = 1/2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

5. *Special matrices* Explain in words what the following matrices do when multiplying a column vector and suggest self-explanatory names. (All missing elements are zeros and $\theta \in [0, 2\pi)$.)

   - $\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$ — does not change the vector; identity matrix

- $\begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}$ — flips the vector upside-down (reverses the elements' order); flip up-down (reversed identity)
- $\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$ — shifts all elements up and pads with zero; up-shift-zero-pad
- $\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix}$ — shifts all elements up and pads with the first element; circular-up-shift
- $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ — rotates the vector by θ rad anti-clock wise, rotation matrix