

ELEC 3035, Lecture 9:

Polynomial approach to pole placement

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1. Review of the state space approach
2. Polynomial approach and the Diophantine equation
3. Example

Review of the state space pole placement approach

$$(A, B, C, D) \mapsto (A_c, B_c, C_c, D_c)$$

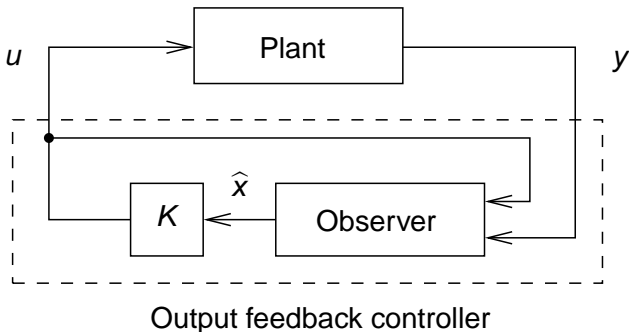
1. **State feedback pole placement:** compute the controller gain K
2. **Pole placement observer design:** compute the observer gain L
3. **Dynamic output feedback controller:**

$$\begin{aligned}\sigma \hat{x} &= A_c \hat{x} + B_c y \\ u &= C_c \hat{x} + D_c y\end{aligned}\quad \text{where}$$

$$A_c = A + LC + BK + LDK, \quad B_c = -L, \quad C_c = K, \quad D_c = 0$$

Computational tool: transition to controller/observer canonical forms

Involves computing controllability/observe. matrices and their inverses.



Main results for output feedback pole placement of $\mathcal{B}_{i/s/o}(A, B, C, D)$:

The poles of the closed loop system can be assigned arbitrarily by dynamic output feedback if A, B is controllable and A, C is observable.

Suppose that the plant is given by an I/O representation $\mathcal{B}_{i/o}(P, Q)$. Then it is natural to ask for an **I/O repr. $\mathcal{B}_{i/o}(P_c, Q_c)$ of the controller.**

An approach to obtain a pole placement controller $\mathcal{B}_{i/o}(P_c, Q_c)$ is:

- Derive a state space repr. $\mathcal{B}_{i/s/o}(A, B, C, D)$ of $\mathcal{B}_{i/o}(P, Q)$

$$(P, Q) \mapsto (A, B, C, D)$$

- Apply the state space output feedback pole placement approach

$$(A, B, C, D) \mapsto (A_c, B_c, C_c, D_c)$$

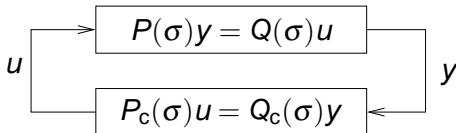
- Derive an I/O repr. $\mathcal{B}_{i/o}(P_c, Q_c)$ of $\mathcal{B}_{i/s/o}(A_c, B_c, C_c, D_c)$

$$(A_c, B_c, C_c, D_c) \mapsto (P_c, Q_c)$$

We will study **direct approach $(P, Q) \mapsto (P_c, Q_c)$** as an alternative to

$$(P, Q) \mapsto (A, B, C, D) \mapsto (A_c, B_c, C_c, D_c) \mapsto (P_c, Q_c)$$

Polynomial approach to pole placement



Plant: $P(\sigma)y = Q(\sigma)u$

Controller: $P_c(\sigma)u = Q_c(\sigma)y$

The closed-loop system is autonomous. In the SISO case

Closed-loop system: $(p_c(\sigma)p(\sigma) - q_c(\sigma)q(\sigma))y = 0$

and the closed-loop characteristic polynomial is

$$p_{cl}(z) := p_c(z)p(z) - q_c(z)q(z)$$

Diophantine equation

For SISO pole placement we need to solve the polynomial equation

$$p_c(z)p(z) - q_c(z)q(z) = p_{\text{des}}(z) \quad (\text{D})$$

in p_c, q_c with $\text{degree}(p_c) \geq \text{degree}(q_c)$ (for causality of the controller).

Notes:

- p_{des} is the desired char. polynomial of the closed-loop system
- $\underbrace{\text{degree}(p_{\text{des}})}_{\text{CL sys's order } n_{\text{cl}}} = \underbrace{\text{degree}(p)}_{\text{plant order } n} + \underbrace{\text{degree}(p_c)}_{\text{controller order } n_c}$
- In state space, p_{des} includes plant and observer's desired poles.

The equation (D) is called **Diophantine equation** (also Bezout eqn).

polynomial \times polynomial \iff Toeplitz matrix \times vector

$$c(z) = a(z)b(z) \iff \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{\ell_c} \end{bmatrix} = \begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ \vdots & a_1 & \ddots & & \\ a_{\ell_a} & \vdots & \ddots & a_0 & \\ & a_{\ell_a} & & a_1 & \\ & & \ddots & \vdots & a_{\ell_a} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{\ell_b} \end{bmatrix}$$

$$\iff : \quad c = S_{\ell_b}(a)b \iff c = S_{\ell_a}(b)a$$

polynomial $c(z) \in \mathbb{R}[z]$, $\deg(c) = \ell_c \iff$ vector $c \in \mathbb{R}^{\ell_c+1}$

polynomial operations \iff structured matrix operations

Diophantine equation

With $n_c := \text{degree}(p_c)$ and $m_c := \text{degree}(q_c)$ given,

$$p_c(z)p(z) - q_c(z)q(z) = p_{\text{des}}(z)$$

can be written as

$$\begin{bmatrix} S_{n_c}(p) & S_{m_c}(q) \end{bmatrix} \begin{bmatrix} p_c \\ q_c \end{bmatrix} = p_{\text{des}} \quad (\text{D}')$$

where

$$p_c = \text{col}(p_{c,0}, p_{c,1}, \dots, p_{c,n_c}) \quad , \quad q_c = \text{col}(q_{c,0}, q_{c,1}, \dots, q_{c,m_c}),$$

$$p_{\text{des}} = \text{col}(p_{\text{des},0}, p_{\text{des},1}, \dots, p_{\text{des},n_{cl}})$$

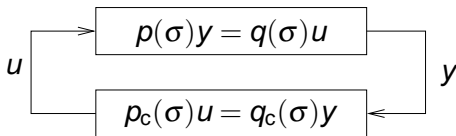
\implies solving (D) (with n_c , m_c given) is a standard linear algebra problem

Main result in polynomial approach for pole placement

Controller $\mathcal{B}_{i/o}(p_c, q_c)$ solving the pole placement problem exists if and only if the Diophantine equation (D) has solution.

Fact: (D) has solution if and only if the GCD of p and q divides p_{des} .

Corollary: The poles of the closed-loop system



can be assigned to arbitrary locations iff $\mathcal{B}_{i/o}(p, q)$ is controllable.

(Recall that $\mathcal{B}_{i/o}(p, q)$ is controllable iff p, q have no common factor.)

Main computational problem in the polynomial approach is solving (D).

Example

Consider the plant defined by the difference equation

$$(2 - 3\sigma + \sigma^2)y = (-3 + 2\sigma)u$$

Our goal is to design a feedback deadbeat controller for this plant.

We need to solve the Diophantine equation

$$(2 - 3z + z^2)p_c(z) - (-3 + 2z)q_c(z) = z^{2+n_c} \quad (*)$$

however, we do not know the controller order n_c .

We know that a controller of order $n_c = 2$ exists, however, it turns out that there is a controller of lower order.

Try $n_c = 0$ (static controller). Then (*) can be written (see (D')) as

$$\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} p_{c,0} + \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} q_{c,0} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This system has no solution, so a controller of order 0 does not exist.

Try $n_c = 1$. Then (*) can be written (see (D')) as

$$\begin{bmatrix} 2 & 0 & 3 & 0 \\ -3 & 2 & -2 & 3 \\ 1 & -3 & 0 & -2 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{c,0} \\ p_{c,1} \\ q_{c,0} \\ q_{c,1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

This system has solution, so there is a controller of order 1.

$$\begin{bmatrix} p_{c,0} \\ p_{c,1} \\ q_{c,0} \\ q_{c,1} \end{bmatrix} = \begin{bmatrix} 2 & 0 & -3 & 0 \\ -3 & 2 & 2 & -3 \\ 1 & -3 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -15 \\ 1 \\ 10 \\ -9 \end{bmatrix}$$

Therefore the controller is given by the difference equation

$$(-15 + \sigma)u = (10 - 9\sigma)y$$

and the closed loop system is $\sigma^3 y = 0$.

Comments:

- In general, the minimal order of the controller is $n_c = n - 1$ (count the # of equations and unknowns in (D'))
- **Reduced order observer design** in the state space approach would give us the same result
- State controllability + observability vs I/O controllability