When is a pole spurious?

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joint work with
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Modal analysis $\leftrightarrow$ System identification

Modal analysis: Given data from a mechanical structure, find eigen frequencies and dampings.

LTI system identification: Given observed data, find an LTI system that fits that data.

The frequencies and dampings are parameters of an LTI system.

$\Downarrow$

Modal analysis is a special linear system identification problem.

operational modal analysis $\leftrightarrow$ ARMA or autonomous SYSID

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Detect physical poles $\leftrightarrow$ Choose model order

Important questions:

- in modal analysis: How to detect physical poles?
- in system identification: How to choose the model order?

Tools used to answer these questions:

- in modal analysis: stabilization diagram
- in system identification:
  - balanced model reduction
  - decay of singular values (in subspace identification)
  - Kumaresan–Tufts's method
  - Akaike's information criterion
  - Cross-validation, ...

How do they compare on simulation examples?

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Representations and parameters of LTI systems

$\mathscr{L}$ — discrete-time LTI model class

σ — shift operator $(\sigma w)(t) := w(t+1)$

Any $\mathscr{B} \in \mathscr{L}$ can be represented as a solution set of a constant coefficients difference equation (kernel representation)

$$R_0\sigma^0 w + R_1\sigma^1 w + \cdots + R_l\sigma^l w = 0 \qquad \text{(DE)}$$

or a first order system of equations (input/state/output repr.)

$$w = \Pi \operatorname{col}(u,y), \quad \sigma x = Ax + Bu, \quad y = Cx + Du, \qquad \text{(SS)}$$

where Π is a permutation matrix.

The polynomial matrix $R(\xi) := R_0\xi^0 + R_1\xi^1 + \cdots + R_l\xi^l$ and the matrices (A,B,C,D,Π) are parameters of $\mathscr{B}$.

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Representations and parameters of LTI systems

Notation:

$$\mathscr{B}(R) := \{ w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{N}} \mid \text{(DE) holds} \}$$

$$\mathscr{B}_{\mathrm{i/s/o}}(A,B,C,D,\Pi) := \{ w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{N}} \mid \text{(SS) holds} \}$$

$\mathtt{l}$ — lag of (DE)
$\mathtt{n}$ — order of (SS)

Given $\mathscr{B} \in \mathscr{L}$, the smallest lag $\mathbf{l}(\mathscr{B})$ and order $\mathbf{n}(\mathscr{B})$ are independent of the representations.

$\Downarrow$

minimal kernel and state space representations

Autonomous systems

$\mathscr{L}_0$ — model class of autonomous systems

Kernel representation

$$P_0\sigma^0 y + P_1\sigma^1 y + \cdots + P_{\mathtt{l}}\sigma^{\mathtt{l}} y = 0, \qquad \det\big(P(\xi)\big) \neq 0 \quad \text{(DE AUT)}$$

State space representation

$$\sigma x = Ax, \quad y = Cx \qquad \text{(SS AUT)}$$

$\mathscr{B}(P)$, $\mathscr{B}(A,C)$ — systems defined by (DE AUT) and (SS AUT).

$\lambda(\mathscr{B})$ — the poles of $\mathscr{B}$ (zeros of $\det(P(\xi)) = 0$)

If $\mathscr{B}(A,C)$ is minimal, the set of eigenvalues $\lambda(A) \equiv \lambda(\mathscr{B})$.

"Physical" poles $\leftrightarrow$ "True" data generating system

Consider given: y_{d} — data and $\mathscr{M}$ — model class

The intuition behind "physical" pole is:

λ is a physical pole for y_{d}
$\Updownarrow$
λ is a pole of a true data generating system $\bar{\mathscr{B}} \in \mathscr{M}$ for y_{d}

The question "when is a pole spurious" is actually the question:

What does it mean "$\bar{\mathscr{B}} \in \mathscr{M}$ is a data generating system for y_{d}"?

Classical approaches to define "true" system

Let $\mathscr{M}$ be the class of autonomous LTI systems $\mathscr{L}_0$.

Output error: y_{d} is a noise corrupted trajectory of $\bar{\mathscr{B}}$

$$y_{\mathrm{d}} = \bar{y} + \tilde{y}, \quad \text{where} \quad \bar{y} \in \bar{\mathscr{B}} \in \mathscr{L}_0 \quad \text{and} \quad \tilde{y} \sim \mathrm{N}(0, \sigma^2 I) \quad \text{(OE)}$$

ARMA: $\operatorname{col}(y_{\mathrm{d}}, e_{\mathrm{d}}) \in \bar{\mathscr{B}}$, where $e_{\mathrm{d}} \sim \mathrm{N}(0, \sigma^2 I)$

These classical approaches are stochastic, they

- assume that $\bar{\mathscr{B}}$ exists in the model class, but
- even if $\bar{\mathscr{B}} \in \mathscr{M}$ exists, it is not computable from y_{d}

Alternative deterministic approach

Definition (Most powerful unfalsified model (MPUM))

$\mathscr{B}_{\text{mpum}}(y_{\text{d}})$ is the MPUM of $y_{\text{d}} \in (\mathbb{R}^{\text{p}})^T$ in the model class $\mathscr{L}^{\text{p}}$ if

1. $\mathscr{B}_{\text{mpum}}(y_{\text{d}})$ is unfalsified by y_{d}, *i.e.*, $y_{\text{d}} \in \mathscr{B}_{\text{mpum}}(y_{\text{d}})$,
2. $\mathscr{B}_{\text{mpum}}(y_{\text{d}})$ is in the model class, *i.e.*, $\mathscr{B}_{\text{mpum}}(y_{\text{d}}) \in \mathscr{L}^{\text{p}}$, and
3. any other unfalsified model in $\mathscr{L}^{\text{p}}$ is less powerful, *i.e.*,

$$y_{\text{d}} \in \mathscr{B} \in \mathscr{L}^{\text{p}} \implies \mathscr{B}_{\text{mpum}}(y_{\text{d}}) \subseteq \mathscr{B}.$$

Notes:

- the MPUM exists and is unique (no prior assumptions)
- there are algorithms for computing it from y_{d}
- $\mathscr{B}_{\text{mpum}}(y_{\text{d}})$ is an exact model for y_{d}

Alternative deterministic approach

Definition (Physical poles)

The poles $\lambda(\mathscr{B}_{\text{mpum}}(y_{\text{d}}))$ of the MPUM of $y_{\text{d}} \in (\mathbb{R}^{\text{p}})^T$ are called physical poles w.r.t. the data y_{d}. Any $z \in \mathbb{C}$, such that $z \notin \lambda(\mathscr{B}_{\text{mpum}}(y_{\text{d}}))$, is called a spurious pole.

$\implies$ By computing the MPUM, we compute the physical poles.

Question:

> *Assuming there is a "true" data generating system $\bar{\mathscr{B}}$ for y_{d},* i.e., $y_{\text{d}} \in \bar{\mathscr{B}}$*, under want conditions $\mathscr{B}_{\text{mpum}}(y_{\text{d}})$ coincides with $\bar{\mathscr{B}}$?*

Recovering the data generating system

Theorem (Identifiability)

If $\mathscr{B} \in \mathscr{L}^{\text{p,n}}_{0,\text{l}}$ and $y_{\text{d}} \in (\mathbb{R}^{\text{p}})^T$ satisfy the following conditions:

1. *y_{d} is an exact trajectory of $\mathscr{B}$,* i.e., $y_{\text{d}} \in \mathscr{B}|_{[1,T]}$, *and*
2. *y_{d} is persistently exciting of order $\mathbf{l}(\mathscr{B})$,*

then $\mathscr{B} = \mathscr{B}_{\text{mpum}}(y_{\text{d}})$.

Condition 1 is restrictive for practical applications
$\rightsquigarrow$ need of approximation

Simulation setup

The data generating system is $\bar{\mathscr{B}} \in \mathscr{L}^{1}_{0,4}$, with physical poles

$$\lambda(\bar{\mathscr{B}}) = \{0.8556 \pm 0.4674j, 0.8980 \pm 0.3797j\}.$$

The trajectory $y_{\text{d}} \in \mathbb{R}^{250}$ is exact, persistently exciting of order 4.

The MPUM of y_{d} is $\mathscr{B}(P)$, where

$$P(\xi) = 0.1278\xi^0 - 0.4716\xi^1 + 0.7037\xi^2 - 0.4961\xi^3 + 0.1414\xi^4$$

$$\mathscr{B}(P) = \bar{\mathscr{B}} \quad \text{and} \quad \lambda(\mathscr{B}(P)) = \lambda(\bar{\mathscr{B}})$$

Exact data $\implies$ exact model

Noisy data

Perturb the exact data y_{d} by a zero mean white Gaussian noise.

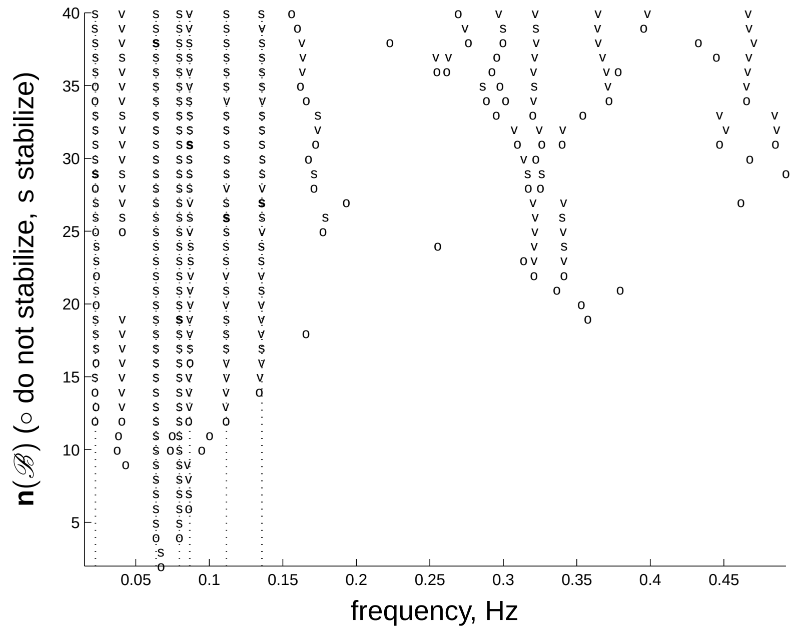
Consider a bounded complexity model class $\mathscr{L}_{0,\mathtt{l}}$.

For given $\mathtt{l}$, we can find an approximation $\widehat{\mathscr{B}} \in \mathscr{L}_{0,\mathtt{l}}$ of $\mathscr{B}_{\mathrm{mpum}}(y_{\mathrm{d}})$.

Main question: How to find the correct $\mathtt{l}$? Compared methods:

1. stabilization diagram
2. method of Kumaresan and Tufts
3. singular value analysis in Kung's algorithm
4. principal angle analysis in subspace identification
5. trade-off curve, used with a maximum likelihood method

Stabilization diagram



Method of Kumaresan and Tufts

Assuming exact data, specifying a higher order $L > \mathtt{l}$ implies

$$\lambda\big(\mathscr{B}(\widehat{P})\big) = \lambda\big(\mathscr{B}(P)\big) \cup \lambda\big(\mathscr{B}(S)\big)$$

where P, $\mathrm{degree}(P) = L - \mathtt{l} - 1$ is arbitrary

$\Longrightarrow$ $\widehat{\mathscr{B}}$ contains all physical poles and $L - \mathtt{l} - 1$ spurious poles

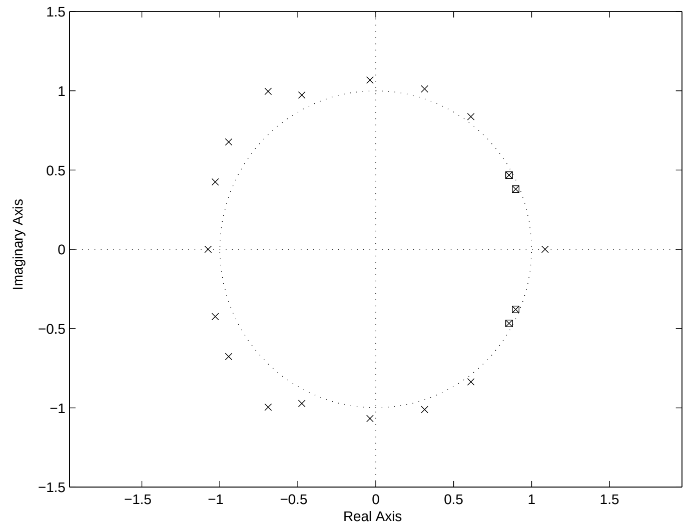
S depends on the identification method

Kumaresan–Tufts method — force the spurious poles to be outside the unit circle

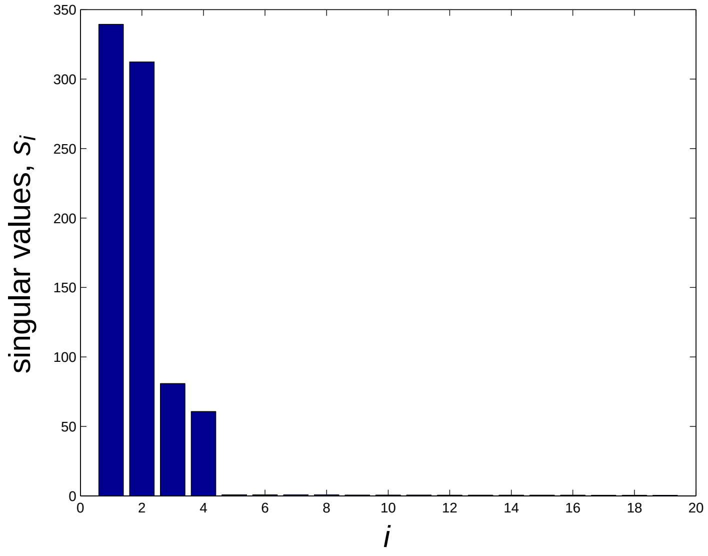
$\Longrightarrow$ the stable physical poles can be extracted

Method of Kumaresan and Tufts $L = 20$

$\times$ — poles of $\widehat{\mathscr{B}}$, $\square$ — physical poles



Singular value analysis in Kung's algorithm



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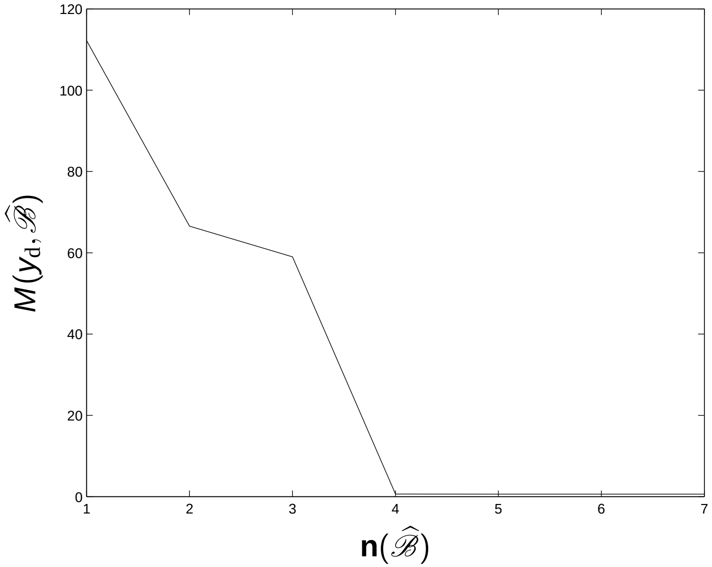
Misfit–complexity trade-off



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Conclusions

- Detecting spurious poles $\leftrightarrow$ order estimation.
- Can be done in many ways.
  (one of which is the stabilization diagram)
- Kumaresan–Tufts', Kung's, and subspace methods
  1. compute a model of complexity higher than intended
  2. perform model reduction

  They have tests that indicate an appropriate low order.

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