ELEC 3035: Practice problems for part 1

Lecturer: Ivan Markovsky

1. *Angle between vectors and a length of a vectors* Verify that the triangle in $\mathbb{R}^2$ with vertexes $(1/2,1/2)$, $(2,-1)$, and $(4,4)$ is a right triangle and verify that the Pythagorean theorem holds for it.

   Solution: □

2. *Inverse of a* 2×2 *matrix and related graph* Find the inverse of the matrix $A = \left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]$. Visualize the mapping $u \overset{A}{\mapsto} y := Au \overset{A^{-1}}{\mapsto} z := A^{-1}y$ by drawing a graph from u_1, u_2 to y_1, y_2 and from y_1, y_2 to z_1, z_2. Since $z = u$, the graph from u_1, u_2 to z_1, z_2 is simple: $\begin{array}{c} u_1 \mapsto u_1 \\ u_2 \mapsto u_2 \end{array}$. Think of $y := Au$ as a *decoder* and of $u = A^{-1}y$ as an *encoder*. (For larger matrices you have to relay on a computer for doing the coding and decoding operations for you.)

   Solution: □

3. *Distance to a subspace* Find the distance from a given point $y \in \mathbb{R}^n$ to the line $\mathscr{L}_a := \{\, \alpha a \mid \alpha \in \mathbb{R} \,\}$, where $a \in \mathbb{R}^n$ is a given vector defining the line $\mathscr{L}_a$. Apply the general solution to the special case $y = (1,0)$, $a = (1,1)$.

   Solution: □

4. *Distance from the origin to an affine space* Find the distance from the point $0 \in \mathbb{R}^n$ to the line $\mathscr{A}_{a,y} := \{\, u \mid a^\top u = y \,\}$, where $a \in \mathbb{R}^n$ and $y \in \mathbb{R}$ are given vector and scalar that define the line $\mathscr{A}_{a,y}$. Apply the general solution to the special case $a = \begin{bmatrix} 1 & 1 \end{bmatrix}$, $y = 1$.

   Solution: □

5. *Hand computation of eigenvalues and eigenvectors of a* 2×2 *matrix* Find the eigenvalues and a set of linearly independent eigenvectors of the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right]$.

   Solution: □

6. *Fibonacci numbers* The Fibonacci numbers $F_1, F_2, \ldots$ are defined by

$$F_1 = 1, \quad F_2 = 2, \quad \text{and} \quad F_{t+1} = F_t + F_{t-1}, \quad \text{for } t = 3,4,\ldots. \tag{FN}$$

   Find the 50th Fibonacci numbers F_{50}.

   (Hint: (FN) defines an autonomous linear dynamical system. Find modal state space representation of (FN).)

   Solution: □

7. *Harmonic oscillator* The differential equation defining the behaviour of a harmonic oscillator is

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} y = -ky,$$

   where k is a positive constant. (A physical example of a harmonic oscillator is a unit mass attached to a spring, in which case k is the spring constant. You have seen harmonic oscillators also in circuit theory.) Find a state space representation of the harmonic oscillator. Give a formula for the trajectories of the system, starting from a given initial condition.

   Solution: □

8. *Feedforward control* Given a static system $\mathscr{B}_2 = \{\, (u,y) \mid y = A_2 u \,\}$ (the plant) explain how to find a system $\mathscr{B}_1 = \{\, (r,u) \mid u = A_1 r \,\}$ (feedforward controller), such that the series connection of $\mathscr{B}_1$ and $\mathscr{B}_2$ (the controlled system) matches or is as close as possible to a given system $\mathscr{B}_3 = \{\, (r,y) \mid y = A_3 u \,\}$ (reference model).

   Solution: □

9. *[Lue79, Chapter 2, Problem 2]* A bank offers 7% annual interest. What would be the overall annual rate if the 7% interest were compounded quarterly?

Solution: Let $y(k)$ denote the amount in the account at the beginning of season k and the bank pays interest at the end of each season. If the 7% interest were compounded quarterly, then the quarter interest is 7/4% and the account balance is governed by

$$y(k+1) = (1+7/4\%)y(k).$$

For an year, suppose $y(k)$ is the amount in the account at the beginning of this year, then $y(k+4)$ is the amount in the account at the beginning of following year. In order to know the overall annual rate, we should specify the relationship between $y(k)$ and $y(k+4)$. We have

$$\begin{aligned} y(k+4) &= (1+7/4\%)y(k+3) = (1+7/4\%)^2 y(k+2) \\ &= (1+7/4\%)^3 y(k+1) = (1+7/4\%)^4 y(k). \end{aligned}$$

Thus, the annual rate is

$$(1+7/4\%)^4 - 1 = 7.1859\%.$$

□

10. *[Lue79, Chapter 2, Problem 5]* Find the second order linear homogeneous difference equation which generates the sequence 1, 2, 5, 12, 29, 70, 169. What is the limiting ratio of consecutive terms?

Solution: By observing the sequence we find the relationship among three consecutive terms is

$$\begin{aligned} 5 &= 2\cdot 2+1 \\ 12 &= 2\cdot 5+2 \\ 29 &= 2\cdot 12+5 \\ 70 &= 2\cdot 29+12 \\ 169 &= 2\cdot 70+29 \end{aligned}$$

This relation can be written as a second-order linear homogeneous difference equation

$$y(k+2) = 2y(k+1)+y(k).$$

By dividing both sides of the equation by $y(k+1)$, the equation becomes

$$\frac{y(k+2)}{y(k+1)} = 2 + \frac{y(k)}{y(k+1)}.$$

Here $\frac{y(k+1)}{y(k)}$ is the ratio of two consecutive term. When $k \to \infty$, the ratio converges to a constant which defines as a. Therefore

$$\frac{y(k+2)}{y(k+1)} = \frac{y(k+1)}{y(k)} = a, \qquad k \to \infty.$$

The limiting ratio a satisfy an equation

$$a = 2+\frac{1}{a} \quad \Longrightarrow \quad a^2 = 2a+1 \quad \Longrightarrow \quad a = 1\pm\sqrt{2}.$$

However, all terms of the sequence are positive so that the ratio of consecutive terms is positive. Therefore, the limiting ratio $a = 1+\sqrt{2}$. □

11. *[Lue79, Chapter 2, Problem 10]* Consider the second order difference equation

$$y(k+2) - 2ay(k+1) + a^2 y(k) = 0.$$

Its characteristic polynomial has both roots equal to $\lambda = a$.

(a) Show that both

$$y(k) = a^k \qquad \text{and} \qquad y(k) = ka^k$$

are solutions.

(b) Find the solutions of this equation that satisfies the auxiliary conditions $y(0) = 1$ and $y(1) = 0$.

Solution:

(a) To check $y(k) = a^k$, we note that $y(k+2) = a^{k+2}$ and $y(k+1) = a^{k+1}$

$$a^{k+2} - 2a \cdot a^{k+1} + a^2 \cdot a^k = a^{k+2} - 2a^{k+2} + a^{k+2} = 0.$$

Thus, $y(k) = a^k$ is a solution.
Checking $y(k) = ka^k$ we have

$$\begin{aligned}(k+2)a^{k+2} - 2a(k+1)a^{k+1} + a^2ka^k &= (k+2)a^{k+2} - 2(k+1)a^{k+2} + ka^{k+2} \\ &= (k+2-2k-2+k)a^{k+2} = 0.\end{aligned}$$

Thus, $y(k) = ka^k$ is also a solution.

(b) A second-order linear difference equation has two degrees of freedom in its general solution, i.e., two linearly independent solutions can form a fundamental set of solutions. We have found two solutions a^k and ka^k. It is easy to prove that these two solutions are linear independent. Because we can't find two constant c_1 and c_2 at least one of which is nonzero to satisfy

$$c_1a^k + c_2ka^k = 0$$

for all $k = 0, 1, 2, \ldots, N$. Thus, any solution $y(k)$ can be expressed as a linear combination of the fundamental set of solutions a^k and ka^k.

$$y(k) = c_1a^k + c_2ka^k,$$

where c_1 and c_2 are constant. Using two auxiliary conditions $y(0) = 1$ and $y(1) = 0$ to find constant c_1 and c_2.

$$y(0) = c_1a^0 = 1 \qquad \Longrightarrow \qquad c_1 = 0$$
$$y(1) = c_1a + c_2a = a + c_2a = 0 \qquad \Longrightarrow \qquad c_2 = -1.$$

Thus, the solution to this equation is

$$y(k) = a^k - ka^k.$$

□

References

[Lue79] D. G. Luenberger. *Introduction to Dynamical Systems: Theory, Models and Applications*. John Wiley, 1979.