Welcome to ELEC2021

Signal Processing and Communications

The teaching staff

Section A:

Ivan Markovsky
ISIS group, Building 1, room 2029
Tel. 8059 8715, im@ecs.soton.ac.uk
Office hours: Wednesday 17:00-18:00

Section B:

Michael Ng
Comms Group, Building 53, Level 4, Room 4007
Tel. 023 8059 3376, sxn@ecs

Course Leader:

Lajos Hanzo
Comms Group, Building 53, Level 4, Room 4004
Tel. 023 8059 3125, lh@ecs

Topics covered in the course

- Part A: Signal processing
  1. Overview (1 lecture)
  2. Time-domain representations of signals and systems (2 lectures)
  3. Fourier series (2 lectures)
  4. Fourier transform (2 lectures)
  5. Discrete Fourier transform (2 lectures)
  6. Random signals (2 lectures)
  7. Revision (1 lecture)
- Part B: Communications
  1. Sampling and quantization
  2. Analogue modulation
  3. Digital modulation and detection
  4. Base-band channel and filtering

Links to other courses

- ELEC1011 Communications and control

  Linear time-invariant (LTI) system, Transfer function, Filtering
- MATH1013 and MATH2021 Mathematics for electronic & electrical engineering

  Differential equations, Fourier series, Random variables
- ELEC2019 Control and systems engineering

  LTI system, Transfer function, Stability, Frequency response, Bode characteristics
- ELEC 3035 Control system design
- ELEC 3026 Digital control system design

Webpage and materials

- Lecture notes from the course webpage

  `https://secure.ecs.soton.ac.uk/notes/elec2021/`

- For additional reading

  Signals & systems by A. Oppenheim and A. Willsky (TK 5102.S5 OPP)

  Textbook and video lectures on "The Fourier Transform and its Applications"

  `http://see.stanford.edu/see/courses.aspx`

- Communication engineering principles by Otung

  is useful for part B but not part A

Labs and assessment

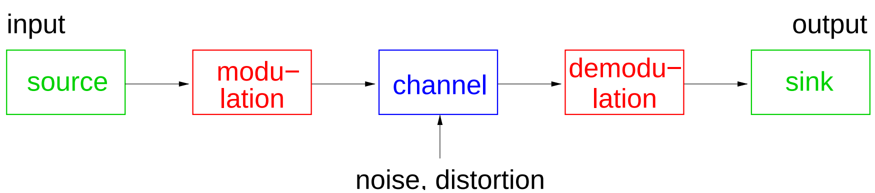
- C?: Signal processing with Matlab
- C5: Fourier transform and frequency domain representation of waveforms
- C8: Modulation and detection
- C2: Digital filter simulation

Matlab is used extensively in C? and C5.

Assessment: 80% Examination, 20% Laboratories

Overview of a communication system

- We are concerned with the understanding and analysis of a communications system:



- the components of such a communications system involve signals and their processing in system blocks;
- we will need to review and study suitable techniques for the representation of signals and systems in the above block diagram, as well as their analysis and evaluation.

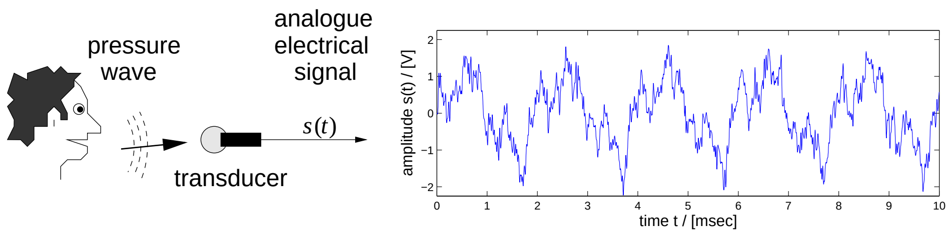
Signals

- Signals can have various properties:
  - continuous-time or discrete-time
  - continuous-valued or discrete-valued

    analog ↔ continuous digital ↔ discrete
  - one-variable (scalar) or multi-variable (vector)
  - one-dimensional or multi-dimensional (a function of time, space, . . .);
- the properties have an impact on how a signal is acquired and processed.

Example: Speech signal

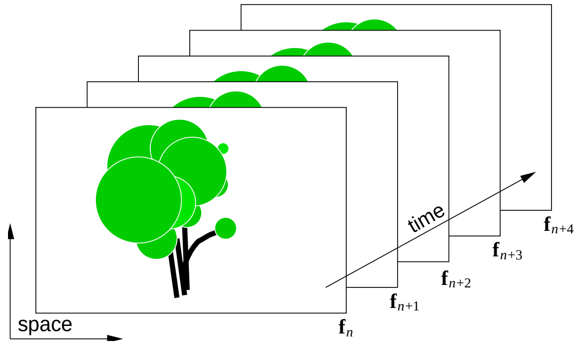
- A speech signal can be an analogue electric signal, which has been converter by a microphone from an acoustic pressure wave:



- the resulting signal is continuous in time and amplitude, and only has a temporal dimension;

Example: Video signal

- A video signal consists of a series of consecutive frames:



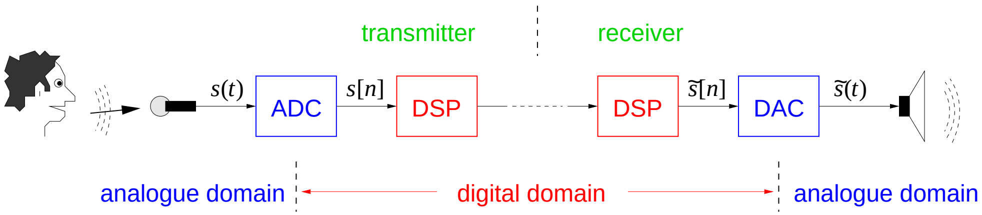
- a frame is taken at a discrete time and contains a two-dimensional array of pixels (usually discrete luminescence values);

- this signal is 3-d (1 temp. & 2 spatial dimensions) and discrete in time and values.

Possible Conversions in Transmitter and Receiver

- Digital signal processing (DSP) is flexible, robust to noise, and insensitive to environmental changes;

- therefore analogue sink and source signals often require conversions (ADC, DAC):



- ADC involves sampling and quantisation:
  (i) how fast do we need to sample (frequency content / bandwidth of signal)?
  (ii) can we quantify the distortion of the quantiser (clipping, quantisation noise)?

Signal Analysis

- Time domain
  - Differential/difference equations
  - Convolution

- Frequency domain / spectral analysis of signals:
  - Fourier series for periodic signals
  - Fourier transform for aperiodic signals
  - discrete Fourier transform for practical calculations on digital data

- characterisation of stochastic signals:
  - histogram, probability and cumulative density functions
  - mean, variance, and correlation

Signal Operations

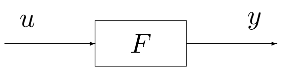
- Conversion from analogue to digital and vice verse: ADC and DAC;
  - sampling
  - quantisation
- filtering of signals
  - anti-alias and reconstruction filtering
- modulation of signals onto a radio frequency carrier; demodulation;
  - analogue modulation (amplitude/ frequency / phase modulation)
  - digital (baseband) modulation schemes
- signal detection in the receiver

Overview of time-domain analysis

- Linear time-invariant (LTI) filters
- Example: moving average (MA) filter
- Finite impulse response (FIR) filters
- Difference and differential equations representation of LTI filter
- Convolution and causality
- Continuous-time case

Filters

- A filter (or system) F transforms an input signal u into an output signal y



$$y = F(u)$$

- Communication channels can be modelled as filters and therefore analysed
- We need filters to shape communications signals appropriately (synthesis)
- Filters are mathematical objects but they can be realized numerically and simulated
- Filters can also be realized in analog electronics or by mechanical devices, in which case they become physical devices

Linear time-invariant (LTI) filters

- The filter F is linear if

$$F(a_1u_1 + a_2u_2) = a_1F(u_1) + a_2F(u_2), \quad \text{for all inputs } u_1,\ u_2, \text{ and scalars } a_1,\ a_2$$

- Define the backwards time-shift operator σ^τ by

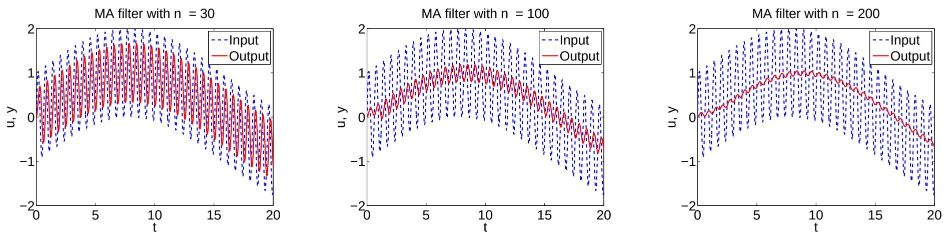
$$\big(\sigma^\tau(u)\big)(t) = u(t+\tau)$$

- The filter F is time-invariant if

$$F(\sigma^\tau u) = \sigma^\tau F(u), \quad \text{for all input } u \text{ and time shifts } \tau$$

Example: moving average (MA) filter

$$y(t) = \frac{1}{m+1}\big(u(t) + u(t-1) + \cdots + u(t-m)\big), \qquad \text{for all } t \qquad \text{(MA)}$$



Exercise: Show that (MA) defines an LTI filter.

Initial conditions

In order to compute the response

$$y = \big(y(0), y(1), \dots\big)$$

of an MA filter (MA) to an input

$$u = \big(u(0), u(1), \dots\big)$$

we need to know m values of the input in the "past"

$$\big(u(-m), \dots, u(-2), u(-1)\big)$$

these are called initial conditions of the MA filter

Finite impulse response (FIR) filter

MA filter is a special case of an FIR filter

$$y(t) = a_0 u(t) + a_1 u(t-1) + \cdots + a_m u(t-m), \qquad \text{for all } t \qquad \text{(FIR)}$$

The response of an FIR filter to a unit pulse input

$$\delta(t) = \begin{cases} 1, & \text{when } t = 0 \\ 0, & \text{otherwise.} \end{cases}$$

under zero initial conditions is

$$\big(a_0, a_1, \dots, a_m, 0, 0, \dots\big)$$

thus the name—finite impulse response.

Difference equation representation of LTI filters

- (FIR) defines $y(t)$ in terms of $u(t)$ and a finite number of past input values.
- This implies that the filter has memory (it "remembers" past values of u).
- Memory is a characteristic property of all dynamical systems.
- More generally, $y(t)$ may depend on $u(t)$ and a finite number of past inputs and outputs

$$\begin{aligned} y(t) &+ b_1 y(t-1) + \cdots + b_n y(t-n) \\ &= a_0 u(t) + a_1 u(t-1) + \cdots + a_m u(t-m), \qquad \text{for all } t \end{aligned}$$

- This is a linear constant coefficients difference equation.

Example

Consider the homogeneous difference equation

$$y(t) = y(t-1) + y(t-2), \qquad \text{for all } t > 1$$

with initial conditions

$$y(0) = y(1) = 1$$

(This equation defines a dynamical system without input.)

Iterating by hand the equation, we find

$$y(2) = 2, \quad y(3) = 3, \quad y(4) = 5, \quad y(5) = 8, \quad y(6) = 13, \quad \ldots$$

These numbers are called Fibonacci numbers, see
`http://en.wikipedia.org/wiki/Fibonacci_number`

Another example

Consider the non-homogeneous difference equation

$$y(t) - y(t-1) - y(t-2) = u(t), \qquad \text{for all } t \geq 0, \text{ with } y(-2) = y(-1) = 0$$

which defines an LTI filter. (Show this.)

The impulse response of this filter can be computed by hand:

$$y(0) = 1, \quad y(1) = 1, \quad y(2) = 2, \quad y(3) = 3, \quad y(4) = 5, \quad y(5) = 8, \quad y(6) = 13, \quad \ldots \tag{1}$$

Again the Fibonacci numbers.

Note the impulse response is infinite $\rightsquigarrow$ infinite impulse response (IIR) filter.

Solving linear homogeneous difference equations

Given the linear, constant coefficients, homogeneous difference equation

$$y(t) + b_1 y(t-1) + \cdots + b_n y(t-n) = 0, \qquad \text{for all } t \geq 0 \tag{HDE}$$

Form the polynomial equation (called characteristic equation)

$$1 + b_1 z^{-1} + \cdots + b_n z^{-n} = 0 \qquad \Longleftrightarrow \qquad z^n + b_1 z^{n-1} + \cdots + b_n = 0$$

Find the roots $z_1, \ldots, z_n$ of this polynomial (this is the hard part).

Any solution of (HDE) is of the form

$$y(t) = c_1 z_1^t + c_2 z_2^t + \cdots + c_3 z_3^t, \qquad \text{for all } t \geq 0$$

The numbers $c_1, \ldots, c_n$ are determined from the initial conditions $y(-1), \ldots, y(-n)$.

Example

Consider again the homogeneous difference equation

$$y(t) = y(t-1) + y(t-2), \qquad \text{for all } t > 1, \text{ with } y(0) = y(1) = 1$$

The characteristic equation is

$$z^2 - z - 1 = 0$$

Its roots are

$$z_1 = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad z_2 = \frac{1-\sqrt{5}}{2},$$

so that

$$y(t) = c_1 z_1^t + c_2 z_2^t$$

Example

In order to find c_1 and c_2, we solve the system

$$\begin{aligned} f(0) &= c_1 z_1^0 + c_2 z_2^0 \\ f(1) &= c_1 z_1^1 + c_2 z_2^1 \end{aligned} \qquad \Longleftrightarrow \qquad \begin{bmatrix} 1 & 1 \\ z_1 & z_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

From where we find

$$c_1 = \frac{z_2 - 1}{z_2 - z_1}, \qquad c_2 = \frac{1 - z_1}{z_2 - z_1}$$

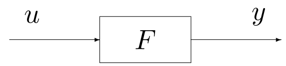
so that

$$f(t) = \frac{z_2 - 1}{z_2 - z_1} z_1^t + \frac{1 - z_1}{z_2 - z_1} z_2^t$$

$\rightsquigarrow$ closed form solution (known as Binet's or Moivre's formula).

Convolution

- Consider a filter F with input u, impulse response h, and output y:



- Represent the input as a sum of shifted delta functions

$$u = \sum_{\tau=-\infty}^{\infty} u(\tau)\sigma^{-\tau}(\delta)$$

- Now using the linearity and time-invariance properties of the filter, we have

$$\begin{aligned} y &= F(u) \\ &= F\left(\sum_{\tau=-\infty}^{\infty} u(\tau)\sigma^{-\tau}(\delta)\right) \\ &= \sum_{\tau=-\infty}^{\infty} u(\tau)\sigma^{-\tau}\big(F(\delta)\big) \\ &= \sum_{\tau=-\infty}^{\infty} u(\tau)\sigma^{-\tau}(h) =: h \star u \end{aligned}$$

- Therefore, the relation between input and output is:

$$y(t) = \sum_{\tau=-\infty}^{\infty} u(\tau)h(t-\tau), \qquad \text{for all } t \tag{2}$$

- Property of convolution

$$y = h \star u = u \star h$$

  (show this)

- Special case: Finite Impulse Response (FIR) filter

$$y(t) = \sum_{\tau=0}^{n} h(\tau)u(t-\tau) \tag{3}$$

- Nonzero values of the inputs response in the past, i.e., $h(t) \neq 0$ for some $t < 0$ implies that the response of the filter precedes the action of the input.

- Such systems are called noncausal.

- In order to operate in real-time, the filter must be causal.

Continuous-time case

- Shifts in time become derivatives: linear constant coeff. differential equation

$$y(t) + b_1 \frac{\mathrm{d}}{\mathrm{d}t} y(t) + \cdots + b_n \frac{\mathrm{d}^n}{\mathrm{d}t^n} y(t)$$
$$= a_0 u(t) + a_1 \frac{\mathrm{d}}{\mathrm{d}t} u(t) + \cdots + a_m \frac{\mathrm{d}^m}{\mathrm{d}t^m} u(t), \qquad \text{for all } t > 0$$

- The initial conditions are

$$y(0), \quad \frac{\mathrm{d}}{\mathrm{d}t} y(0), \quad \ldots \quad \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} y(0)$$

- Sums over time become integrals: continuous-time convolution

$$y(t) = (h \star u)(t) = \int_{-\infty}^{\infty} h(\tau) u(t-\tau)\, d\tau, \qquad \text{for all } t \tag{4}$$

Fourier Techniques — Structure / Overview

- Fourier Series (applicable to periodic signals)
- Fourier Transform (applicable to non-periodic signals)
- Digital Implementation: from the Fourier transform to the DFT
- Properties of the DFT, windowing techniques
- Applications of the DFT

Importance of Fourier Transform Techniques

The Fourier transform analyses signals (or systems) with respect to *sinusoids*. What makes sinusoids so special?

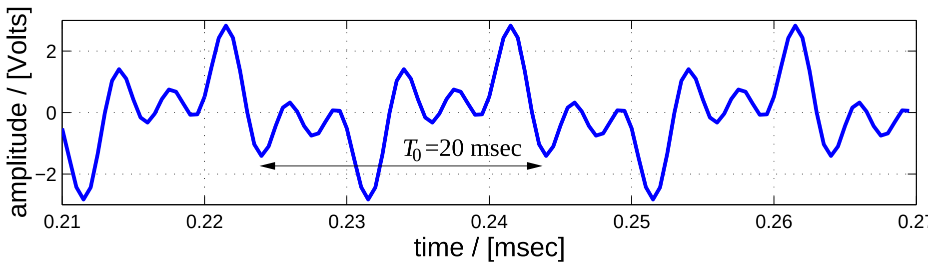
- many natural processes produce sinusoidal behaviour (e.g. rotating machinery)
- When giving a sinusoid as input to a stable LTI system, the steady-state output will also be a sinusoid

  At a given frequency, the response of the system can then be described by the change in amplitude and phase that it imposes.

- Periodic signals can be repressnted or approximated by series or linear combinations of sinusoidal signals.

Example: Signal Analysis

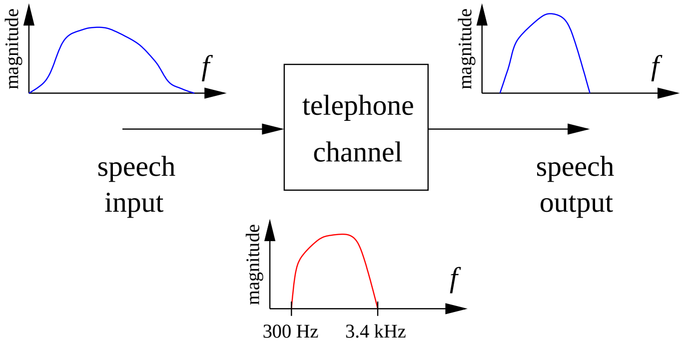
- We therefore want to know which sinusoids are “present” in a signal;
- consider the vibration signal from an electrical transformer, which gives engineers information of the health of the transformer (loose or broken parts, etc.):



- it is impossible to tell the *harmonic content* of the signal above the mains frequency of 50 Hz (i.e. 100 Hz, 150 Hz, 200 Hz, etc contribution).

Example: Frequency Response

- Comparing the input and output spectra of a system, the system's behaviour can be described by a frequency response:



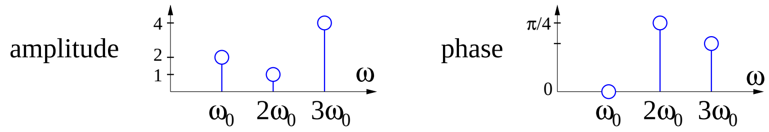
- e.g. a standard telephone system passes only frequencies between approximately 300 Hz and 3.4 kHz.

Sum of Tones

- A signal composed of a sum of sine and cosine waveforms can be represented in the time domain, or with the aid of line frequency amplitude and phase plots;
- for example, the waveform

$$x(t) = 2\cos(\omega_0 t) + \cos(2\omega_0 t + \frac{\pi}{4}) + 4\cos(3\omega_0 t + \frac{\pi}{5})$$

is completely represented by frequency amplitude and phase plots / spectra:



Fundamental Fourier Series Theorem

Any periodic signal $x(\cdot)$ with period T_0 (i.e. $x(t+T_0) = x(t)$ for all t) admits a series expansion of the form:

$$x(t) \;=\; \frac{A_0}{2} + \sum_{k=1}^{\infty} (A_k \cos(k\omega_0 t) \;+\; B_k \sin(k\omega_0 t)) \tag{5}$$

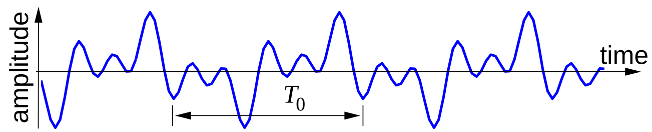
where $\omega_0 := 2\pi/T_0$ is the fundamental frequency of the signal $x(\cdot)$, and the Fourier coefficients are:

$$A_k := \frac{\omega_0}{\pi} \int_{-T_0/2}^{T_0/2} x(t)\cos(k\omega_0 t)dt \qquad B_k := \frac{\omega_0}{\pi} \int_{-T_0/2}^{T_0/2} x(t)\sin(k\omega_0 t)dt \tag{6}$$

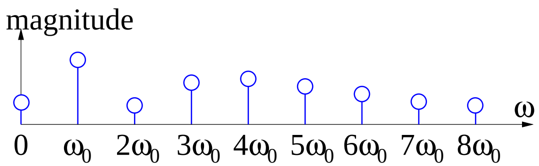
Fourier Series Theorem

Therefore, the fundamental statement of the Fourier Series Theorem is:

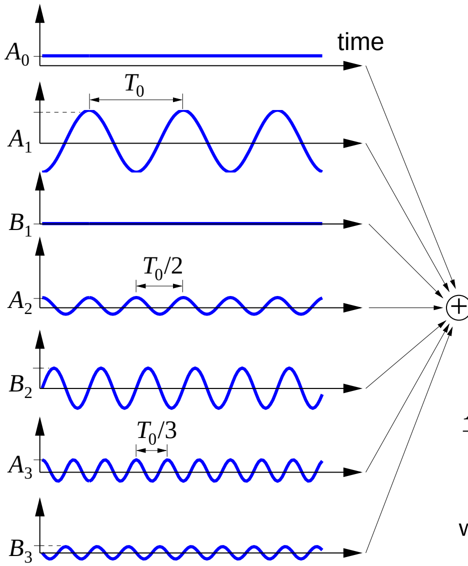
- If a waveform is periodic with period $T_0 = 2\pi/\omega_0$



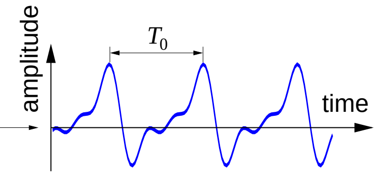
- then it can be represented by a series of harmonically related sine and cosine waves at angular frequency ω_0 and harmonics thereof, i.e. $2\omega_0$, $3\omega_0$, $4\omega_0$ etc.



Fourier Series Example



- a periodic signal can be realised by sine and cosine contributions weighted in amplitude by the Fourier coefficients A_k and B_k.



$$\frac{A_0}{2} + \sum_{k=1}^{3} (A_k \cos(k\omega_0 t) + B_k \sin(k\omega_0 t))$$

with $\omega_0 = 2\pi/T_0$

Amplitude / Phase Fourier Series

- The Fourier series for a real valued periodic signal $y(t)$ can also be expressed in terms of amplitude $\hat{C}_k$ and phase ϕ_k of a sinusoid:

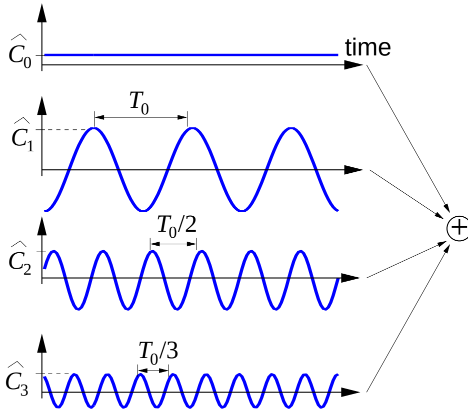
$$x(t) \;=\; \sum_{k=0}^{\infty} \hat{C}_k \cos(k\omega_0 t - \phi_k) \qquad (7)$$

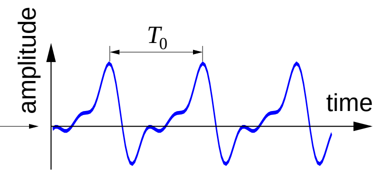
- this is related to (5) and (6) by

$$\hat{C}_0 = A_0/2 \qquad \hat{C}_k = \sqrt{A_k^2 + B_k^2} \qquad \phi_k = \tan^{-1}\left(\frac{B_k}{A_k}\right) \qquad (8)$$

Fourier Series Example Revisited

- we construct the expansion of slide 37 using amplitude/phase representation:





$$\sum_{k=0}^{3} \hat{C}_k \cos(k\omega_0 t - \phi_k) \qquad , \quad \omega_0 = \frac{2\pi}{T_0}$$

- each sinusoidal constribution has an adjusted phase ϕ_k and is weighted by an amplitude value $\hat{C}_k$.

Complex Fourier Series

- A convenient mathematical form of the Fourier series is possible by exploiting Euler's formula, $\cos\psi = \frac{1}{2}(e^{j\psi} + e^{-j\psi})$:

$$x(t) \;=\; \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t} \qquad (9)$$

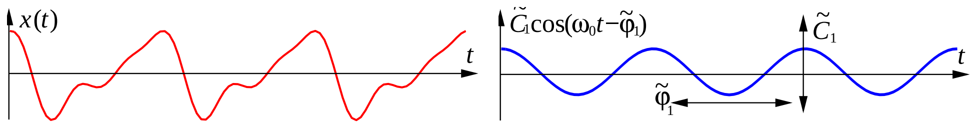
- the complex Fourier coefficient C_k contains both amplitude and phase:

$$C_k \;=\; \frac{\omega_0}{2\pi} \int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t} dt \;=\; \begin{cases} \frac{1}{2}(A_k + jB_k), & k \geq 0 \\ \frac{1}{2}(A_{-k} - jB_{-k}), & k < 0 \end{cases} \qquad (10)$$

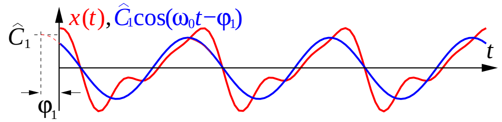
- note: (i) a "negative frequency" is introduced; (ii) $x(t)$ can be complex valued;

Calculation of a Fourier Coefficient

- A Fourier coefficient represents the best least-squares fit of a sinusoid or complex exponential at a given frequency to the signal to be analysed;



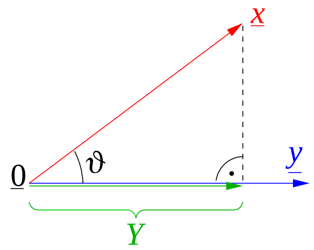
- in (7), the amplitude and phase parameters have to be adjusted to fit the waveform in the least-squares sense: the optimum is given for a specific set $\hat{C}_k$ and φ_k;



- how does this relate to the analytic formulae for determining Fourier series coefficients (6) and (10)?

Geometry – Least–Squares Fit

- Consider finding the best representation of a vector x in terms of a vector y:



- the best representation in the least squares sense is an othogonal projection of x onto y;

- we want to determine Y — mathematically, this is performed by a scalar or inner product.

Scalar Product in a Geometric Space I

- Consider the vectors x and y containing N elements:

$$x = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \quad ; \quad y = \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix} \tag{11}$$

- using the complex conjugate transpose $(\cdot)^*$, the scalar product is defined as

$$A = y^* x = \begin{bmatrix} y^*(0) & y^*(1) & \cdots & y^*(N-1) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \tag{12}$$

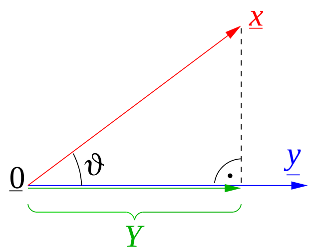
$$= \sum_{n=0}^{N-1} x(n) y^*(n) \tag{13}$$

- note: complex conjugation is not standard procedure, but will help later.

Scalar Product in a Geometric Space II

- The length of a vector y is given by $\|y\| = \sqrt{y^* y}$ — compare to Pythagoras for the 2-dimensional case $N = 2$;

- if in the example on slide 42, $\|y\| = 1$, then

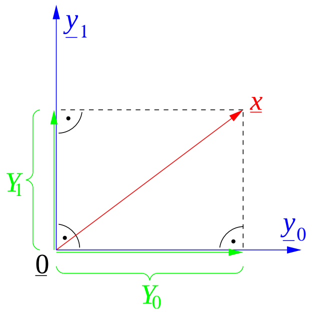


$$Y = y^* x = \sum_{n=0}^{N-1} x(n) y^*(n) \tag{14}$$

- Note that the best representation of x in terms of y is given by Yy.

Basis of a Geometric Space

- We ideally want an orthonormal basis of a geometric space; we consider $N = 2$:



- basis vectors must be orthogonal:
  $y_0^* y_1 = 0$;
- basis vectors must have unit length:
  $y_k^* y_k = 1$ for $k = 0, 1$;
- representation of a vector within the basis:

$$x = Y_0 y_0 \; + \; Y_1 y_1 \tag{15}$$

where $\quad Y_k = y_k^* x \; , \qquad k = 0, 1$

- the y_k from a basis if they are dense in space (need N y_k for N-dim space); x can be represented in this coordinate system by its coordinates Y_k.

Analogy between Function Space and Geometric Space

- Signal $x(t)$ and $y(t)$ can be interpreted as "vectors" lying in a signal or function space;
- in the Fourier series, we want to represent the signal $x(t)$ as best as possible by a signal $y(t)$ being a sinusoid;
- a scalar product exists also for continuous time (analogue) signals, whereby we only consider the fundamental period T_0:

$$A = \int_{-T_0/2}^{T_0/2} x(t) y^*(t) \; dt \tag{16}$$

- this looks very similar to (6) and (10)!

Complex Fourier Series — Basis and Coefficients I

- Comparing the complex Fourier series coefficients in (10) with (16), we identify

$$y_k(t) = \frac{\omega_0}{2\pi} e^{jk\omega_0 t} = \frac{1}{T_0} e^{jk\omega_0 t} \tag{17}$$

- "length" of basis (it turns out to be not orthonormal):

$$\int_{-T_0/2}^{T_0/2} y_k(t) y_k^*(t) dt \;=\; \frac{1}{T_0^2} \int_{-T_0/2}^{T_0/2} e^0 dt \;=\; \frac{1}{T_0} \tag{18}$$

- orthogonality: $k - l = m \neq 0$

$$\int_{-T_0/2}^{T_0/2} y_k(t) y_l^*(t) dt \;=\; \frac{1}{T_0^2} \int_{-T_0/2}^{T_0/2} e^{jm\omega_0 t} dt \;=\; 0 \tag{19}$$

as we integrate over an integer multiple of 2π phasor rotations.

Complex Fourier Series — Basis and Coefficients II

- the length of a basis function in the Fourier series is unequal unity;
- as result, the new representation is scaled; this is compensated by the modification of $y_k(t)$ in the representations (5), (7), and (9);
- this scaling has historical reasons; a basis-oriented formulation would define

$$y_k(t) \;=\; \frac{1}{\sqrt{T_0}} e^{jk\omega_0 T} \tag{20}$$

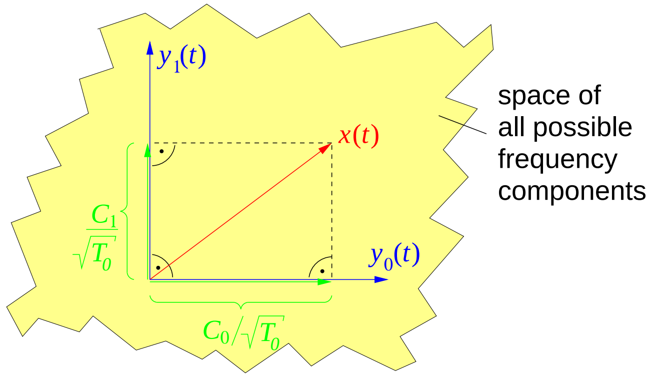
$$C_k \;=\; \int_{-T_0/2}^{T_0/2} x(t) y^*(t) \; dt \tag{21}$$

$$x(t) \;=\; \sum_{k=0}^{\infty} C_k y_k(t) \tag{22}$$

instead of (10) and (9).

Fourier Series Interpretation

- Fourier series theorem: any periodic $x(t)$ can be represented in a basis of (an infinite number of) sinusoids at the fundamental frequency and harmonics thereof;
- each Fourier coefficient is the result of an orthogonal projection (orthonormal except for a factor of $1/\sqrt{T_0}$) from the signal $x(t)$ onto a basis function $y_k(t)$:

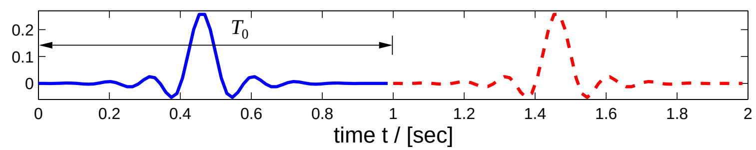


Aperiodic Signals

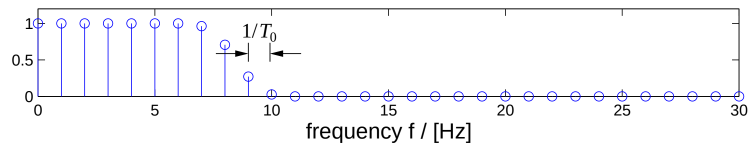
- The Fourier series is limited to periodic signals — most real world signals do not satisfy this assumption;
- any signal with transitory or random components is *aperiodic*;
- music or speech signals are often referred to as *quasi-periodic*, i.e. they can be considered as approximately periodic only over a short time interval;
- a mathematical tool to extend the Fourier series to the aperiodic case is desirable and known as the Fourier transform.

From Fourier Series to Fourier Transform

- Given an aperiodic signal, the fundamental period T_0 for the Fourier series cannot be determined;
- to obtain some sort of answer, we could assume that the signal repeats after the entire signal duration;

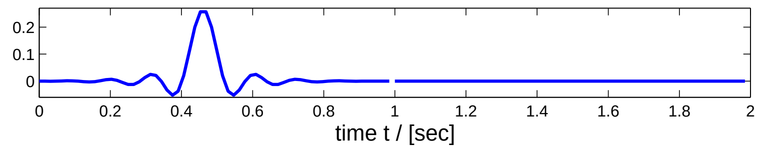


- the result of choosing a large period T_0 is that the harmonics of ω_0 are very closely spaced.

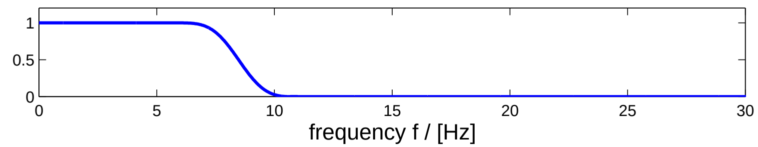


Fourier Transform

- We now make the transition $T_0 \longrightarrow \infty$:

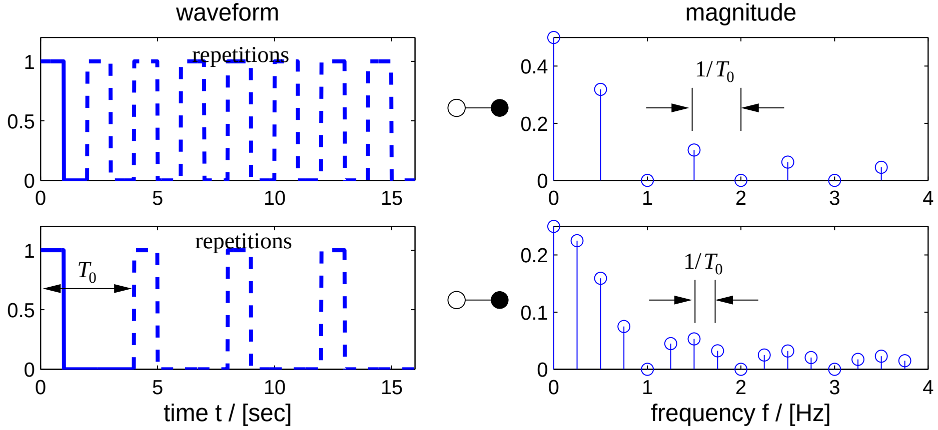


- hence $\omega_0 \longrightarrow 0$ and the spacing between the harmonics becomes infinitesimally small, and we obtain a continuous function for the Fourier transform:



Square Wave Fourier Series

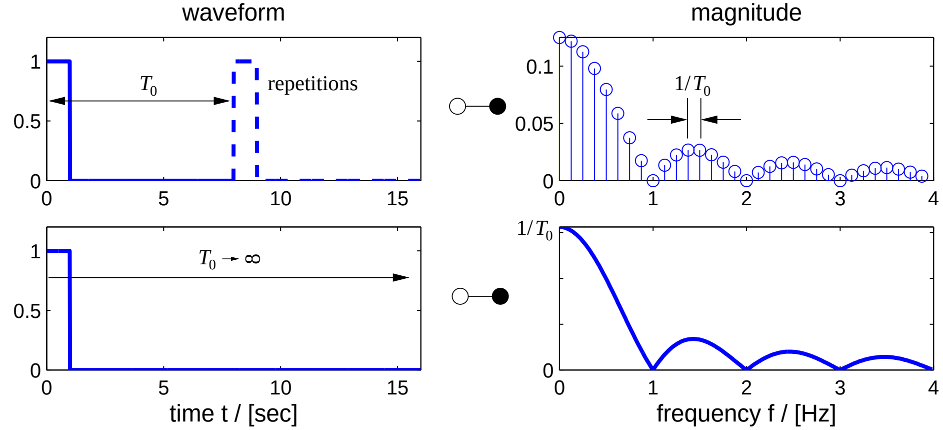
- as an example, we enforce a periodicity T_0 on a rectangular pulse:



- by increasing the enforced period T_0, the spectral lines become spaced more closely.

Square Wave Fourier Transform

- we further enlarge T_0 and make the transition to infinity;



- the magnitude spectrum becomes higher in resolution and finally continuous.

Fourier Transform

- Compared to (10), the Fourier transform formula is gained by substituting $\omega = k\omega_0$, $T_0 \rightarrow \infty$, making the transition from a sum to an integral, and dropping the scaling factor, hence:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \tag{23}$$

- analogous to (9), a series expansion can be built:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \tag{24}$$

- this defines the inverse Fourier transform; we denote the Fourier pair as $x(t) \circ\!\!-\!\!\bullet X(j\omega)$.

Fourier Transform — Basic Properties

- **Linearity**. If $x_1(t) \circ\!\!-\!\!\bullet X_1(j\omega)$ and $x_2(t) \circ\!\!-\!\!\bullet X_2(j\omega)$, then

$$a_1 x_1(t) + a_2 x_2(t) \quad \circ\!\!-\!\!\bullet \quad a_1 X_1(j\omega) + a_2 X_2(j\omega) \tag{25}$$

- **Time Shift**. ($\rightarrow$ phase shift!)

$$x(t - t_0) \quad \circ\!\!-\!\!\bullet \quad X(j\omega) e^{-j\omega t_0} \tag{26}$$

- **Frequency Shift**. ($\rightarrow$ modulation!)

$$x(t)\, e^{j\omega_0 t} \quad \circ\!\!-\!\!\bullet \quad X(j(\omega - \omega_0)) \tag{27}$$

- Time shift:

$$x(t-t_0) \quad \circ\!\!-\!\!\bullet \quad \int\limits_{-\infty}^{\infty} x(t-t_0)e^{-j\omega t}dt = \int\limits_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau+t_0)}d\tau \qquad (28)$$

$$\text{substitution: } \tau = t - t_0 \quad \frac{d\tau}{dt} = 1 \qquad (29)$$

$$= e^{-j\omega t_0} \underbrace{\int\limits_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau}_{X(j\omega)} \qquad (30)$$

- Modulation:

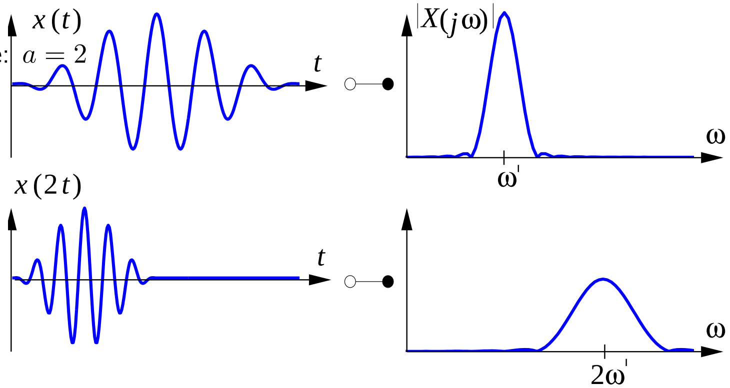
$$x(t)e^{j\omega_0 t} \ \circ\!\!-\!\!\bullet \int\limits_{-\infty}^{\infty} x(t)e^{j\omega_0 t}e^{-j\omega t}dt = \underbrace{\int\limits_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt}_{X(j(\omega-\omega_0))} \qquad (31)$$

Fourier Transform — Scaling

- **Scaling.** If $x(t) \circ\!\!-\!\!\bullet X(j\omega)$, then

$$x(at) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{|a|}X(j\omega/a) \qquad (32)$$

- example: $a = 2$



$$x(at) \circ\!\!-\!\!\bullet \int\limits_{-\infty}^{\infty} x(at)e^{-j\omega t}dt \qquad \text{substitution: } \tau = at \quad \frac{d\tau}{dt} = a \qquad (33)$$

- case $a > 0$:

$$\int\limits_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau/a}\frac{d\tau}{a} = \frac{1}{a}\underbrace{\int\limits_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau/a}\,d\tau}_{X(j\omega/a)} \qquad (34)$$

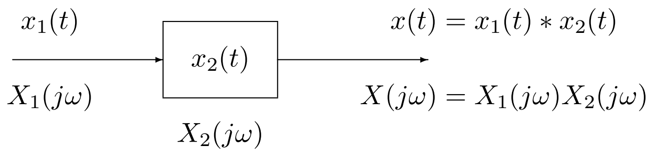
- case $a < 0$:

$$\int\limits_{\infty}^{-\infty} x(\tau)e^{-j\omega\tau/a}\frac{d\tau}{a} = \frac{1}{-a}\underbrace{\int\limits_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau/a}\,d\tau}_{X(j\omega/a)} \qquad (35)$$

Fourier Transform — Convolution

- An important equivalence is between a time domain convolution '$*$' and its Fourier transform:

$$x(t) = x_1(t)*x_2(t) = \int\limits_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau \ \circ\!\!-\!\!\bullet \ X(j\omega) = X_1(j\omega)X_2(j\omega) \quad (36)$$



- this property allows to perform the convolution of two signals/systems via simpler multiplication of their Fourier transforms.

$$x_1(t) * x_2(t) \quad \circ\!\!-\!\!\bullet \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau) d\tau \;\; e^{-j\omega t} dt = \tag{37}$$

swapping integrations:

$$= \int_{-\infty}^{\infty} x_1(\tau) \underbrace{\int_{-\infty}^{\infty} x_2(t-\tau) e^{-j\omega t}\, dt}_{X_2(j\omega)e^{-j\omega\tau}} \; d\tau \tag{38}$$

exploiting time shift property:

$$= X_2(j\omega) \int_{-\infty}^{\infty} x_1(\tau) e^{-j\omega\tau} d\tau = X_1(j\omega) X_2(j\omega) \tag{39}$$

. . . yippie!

Fourier Transform — Parseval, Differentiation

- Parseval's theorem establishes a link between the energy of the time domain waveform and the energy of the spectrum:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega \tag{40}$$

- differentiation in the time domain is equivalent to a simple multiplication by $j\omega$ in the frequency domain;

$$\frac{d^n x(t)}{dt^n} \;\circ\!\!-\!\!\bullet\; (j\omega)^n X(j\omega) \tag{41}$$

- the latter allows us to transform *differential equations* into *polynomials*, which are mathematically easier to solve.

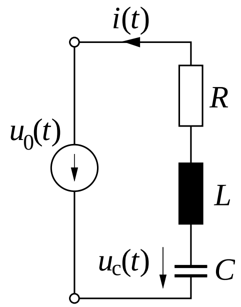
For Parseval, consider $|x(t)|^2 = x(t)x^*(t)$. Hence:

$$\begin{aligned}
\int_{-\infty}^{\infty} x(t)\,(x(t))^*\, dt &= \int_{-\infty}^{\infty} x(t) \Big(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t} d\omega\Big)^* dt && (42)\\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt\; d\omega && (43)\\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\omega) X(j\omega)\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega && (44)
\end{aligned}$$

Differentation:

$$\frac{dx(t)}{dt} = \frac{d}{dt} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) \underbrace{\left(\frac{d}{dt} e^{j\omega t} \right)}_{j\omega e^{j\omega t}} d\omega \tag{45}$$

- Time domain analysis of the following circuit:



$$\begin{aligned}
u_0(t) &= Ri(t) + L\frac{di(t)}{dt} + u_c(t) && (46)\\
i(t) &= C\frac{du_c(t)}{dt} && (47)
\end{aligned}$$

hence

$$u_0(t) = LC\frac{d^2 u_c(t)}{dt} + RC\frac{du_c(t)}{dt} + u_c(t) \tag{48}$$

- frequency domain analysis (assuming steady state excitation of circuit):

$$\begin{aligned}
U_0(j\omega) &= -\omega^2 LCU_c(j\omega) + j\omega RCU_c(j\omega) + U_c(j\omega) && (49)\\
&= (1 + j\omega RC - \omega^2 LC)U_c(j\omega) && (50)
\end{aligned}$$

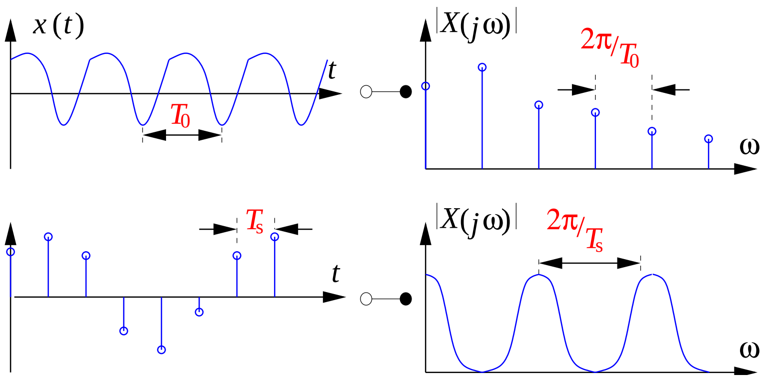
Fourier Transform — Duality

- A Fourier transform pair $x(t) \circ\!\!-\!\!\bullet\, X(j\omega)$ is usually seen as connecting a time-domain quantity $x(t)$ and a frequency-domain quantity $X(j\omega)$;
- however, a duality between the two domains exists:

$$\text{if} \quad x(t) \circ\!\!-\!\!\bullet\, X(j\omega) \qquad \text{then} \quad X(jt) \circ\!\!-\!\!\bullet\, 2\pi x(-\omega) \tag{51}$$

- this has already been noted for the time-shift / frequency-shift properties (slide 56);
- other important dualities are:
  - ◦ periodic waveform $\circ\!\!-\!\!\bullet$ discrete spectrum (Fourier series!)
    $\longrightarrow$ discrete waveform $\circ\!\!-\!\!\bullet$ periodic spectrum;
  - ◦ convolution $\circ\!\!-\!\!\bullet$ multiplication
    $\longrightarrow$ multiplication $\circ\!\!-\!\!\bullet$ convolution.

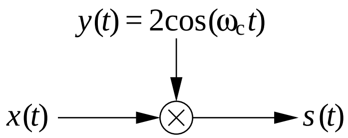
- Consider the following relations due to the duality:



- What would the spectrum of a discrete periodic signal $x(t)$ look like?

Tutorial Q1: Modulation

- A signal $x(t)$ is modulated onto a carrier $y(t) = 2\ \cos(\omega_c t)$, resulting in the transmitted signal $s(t) = x(t) \cdot y(t)$.



1. Show that $\qquad S(j\omega) = X(j(w - w_c)) + X(j(w + w_c))$;
2. Sketch $|S(j\omega)|$ for a suitably chosen $|X(j\omega)|$;
3. What effect has a time delay t_0 in the input signal, $s(t) = x(t - t_0) \cdot y(t)$ onto the magnitude $|S(j\omega)|$; justify your answer.

Fourier Transform — An Analytic Tool Only

- The Fourier transform is a mainly analytical tool;
- the spectra of *simple* signals and systems can be evaluated from transform tables and by using the properties of the transform;
- for more complex signals and systems, and as an on-line numerical tool, the Fourier transform itself is unsuitable;
- here, the discrete Fourier transform (DFT) is of interest, which consists of a number of simplifications of the Fourier transform;

DFT — Discretisation in Time

- We consider a sampled version $x_s(t)$ of $x(t)$ with sampling period T_s, and apply the Fourier transform:

$$x_s(t) \;=\; x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) \tag{52}$$

$$X_s(j\omega) \;=\; \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(n)\delta(t - nT_s)e^{-j\omega t}dt = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n T_s} \tag{53}$$

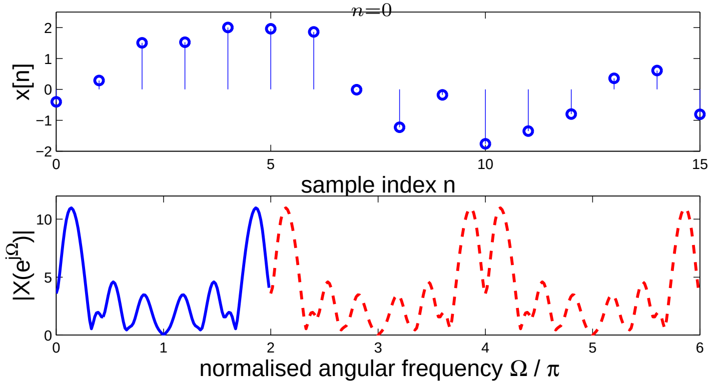
- this gives the Fourier transform for a discrete-time signal $x(n)$; we further introduce a normalised angular frequency $\Omega = \omega T_s$ to express the periodicity of the spectrum:

$$X(e^{j\Omega}) = X_s(j\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{j\Omega n} = X(e^{j(\Omega+2\pi)}) \tag{54}$$

DFT — Limitation in Time

- It is assumed that the signal $x(n)$ is causal and finite, hence only defined for $0 \leq n < N$

$$X(e^{j\Omega}) = \sum_{n=0}^{N-1} x(n)e^{-j\Omega n} \tag{55}$$



DFT — Discretisation in Frequency

- We evaluate the Fourier transform only for a number of discrete frequency points ("bins") $\Omega = 0, \Omega_0, 2\Omega_0, 3\Omega_0$, etc.:

$$X(e^{j\Omega})|_{\Omega=\Omega_0 k} = \sum_{n=0}^{N-1} x(n)e^{-j\Omega_0 kn} \tag{56}$$

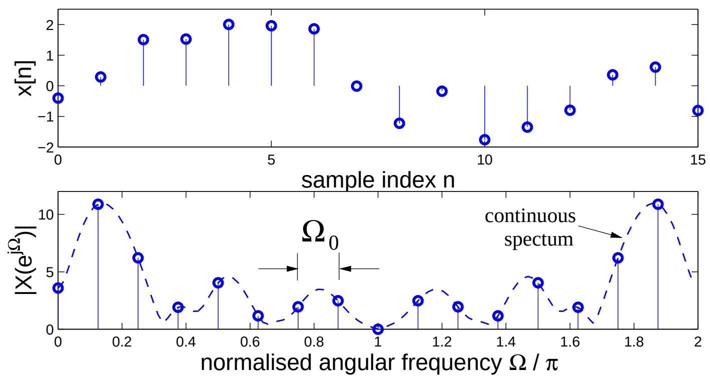
- as a standard, N such frequency bins are evaluated, such that $\Omega_0 = 2\pi/N$; in terms of absolute frequency, this means that the bin separation is

$$f_0 \;=\; \frac{1}{NT_s} \tag{57}$$

- this "bin width" f_0 determines the *frequency resolution* of the DFT; hence the higher N, the higher the resolution of the DFT.

Discretisation in Frequency — Example

- Given: a signal $x(n)$ with $N = 16$ samples; the DFT will only evaluate N discrete frequency points in the interval $\Omega = [0; 2\pi]$;



- the frequency sample points are $\Omega = 0,\ \Omega_0,\ 2\Omega_0,\ \ldots (N-1)\Omega_0$, with $\Omega_0 = \frac{2\pi}{N}$.

Implementation: DFT - Matrix

- If k is the index into the frequency bins, we can also write

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\Omega_0 kn} \tag{58}$$

- this can be brought into matrix notation $X = Tx$ with DFT matrix T

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-j\Omega_0} & & e^{-j\Omega_0(N-1)} \\ 1 & e^{-j\Omega_0 2} & \ddots & e^{-j\Omega_0 2(N-1)} \\ \vdots & & & \vdots \\ 1 & e^{-j\Omega_0(N-1)} & \cdots & e^{-j\Omega_0(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

- complexity of an N-point DFT: N^2 complex multiply-accumulates (MACs);

Inverse DFT

- For the discrete-time Fourier transform, the inverse transform is given by

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega})\, e^{j\Omega n}\, d\Omega \tag{59}$$

- the inverse DFT can be reached by discretising (59)

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{jk\Omega_0 n} \tag{60}$$

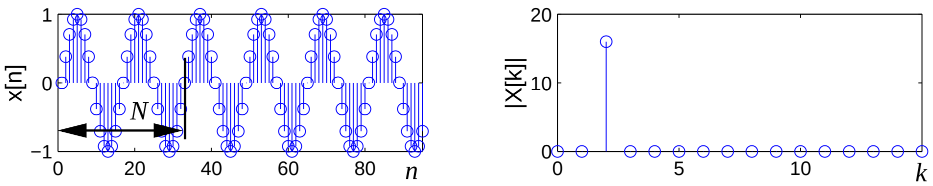
- alternatively, from $X = Tx$, we can deduce the inversion as $x = T^{-1}X$, whereby $T^{-1} = 1/NT^* = 1/NT^*$ due to special properties of T.

- note: analogous to Parseval, with $X = Tx$ we have $\|X\|_2^2 = \frac{1}{\sqrt{N}}\|x\|_2^2$.

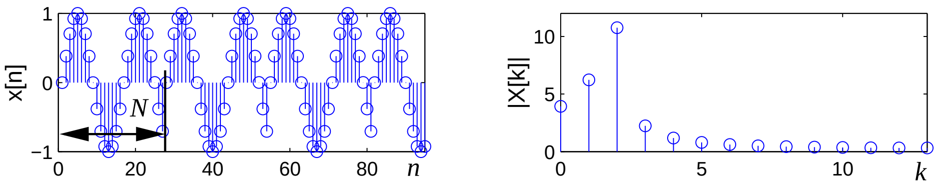
Discontinuities — Spectral Leakage

- The DFT "periodises" the data, which is likely to create aberrations;

- $N = 32$, by chance the window ends fit:



- $N = 27$, discontinuties arise at the window edges, causing the main peak to "leak":

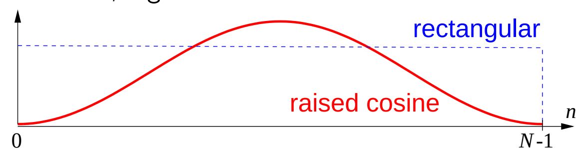


Windowing

- Periodisation in the time domain is due to the DFT enforcing a discretised frequency domain (see FT properties, slide 65)

- the N data points can be considered as extracted from a longer data stream by multiplication with a rectangular window; spectral leakage occurs due to discontinuities at the ends of the extracted data interval;

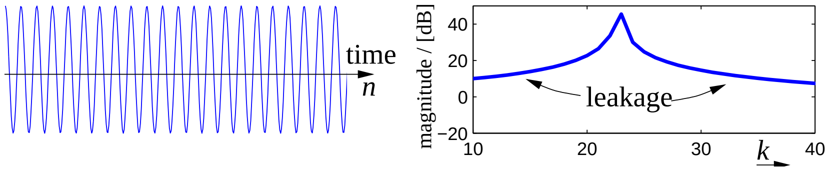
- discontinuities can be avoided by dis-emphasising the ends of the interval with a non-rectangular window, e.g.



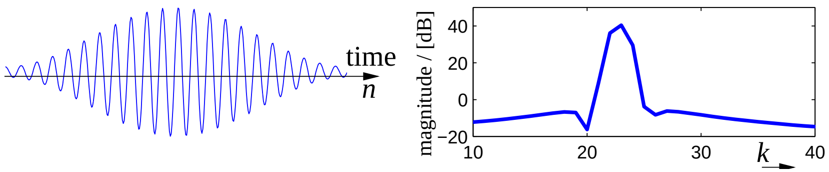
- popular window choices are Hamming, Hann, Blackman-Harris, Bartlett, etc.

Windowing — Example

- DFT applied to a segment of rectangularly windowed sinusoidal data:



- DFT after application of a Hamming window to the same data:



- with windowing, the spectral leakage is reduced at the cost of a widened main lobe.

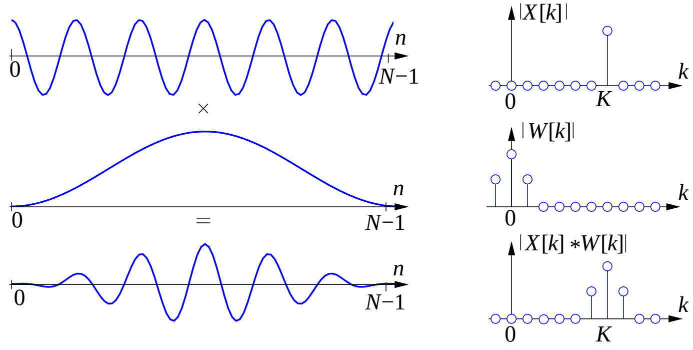
Windowing — Widening of Main Lobe

- Consider windowing of a sinusoid $y(n)$ with a raised cosine window $w(n)$:

$x(n) = \cos(\frac{2\pi}{N}Kn)$

$w(n) = 1 - \cos(\frac{2\pi}{N}n)$

$x(n)w(n)$



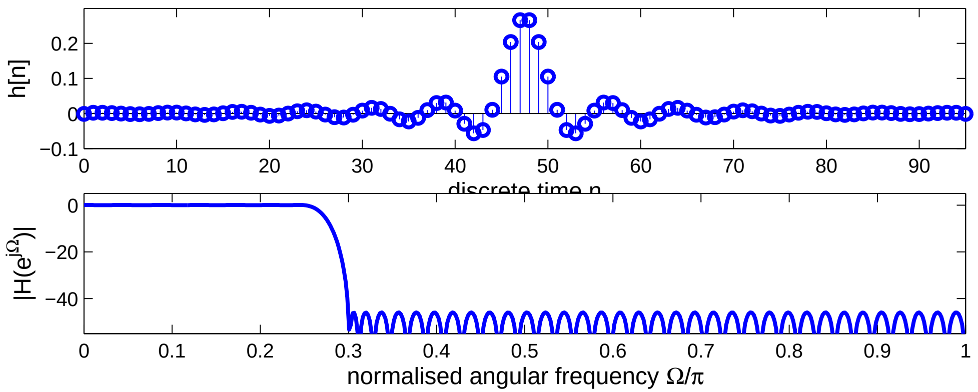
- Reason for widening of the "main lobe": in the frequency domain, a convolution with the lowpass window blurs the peak.

Windowing — Summary

- windowing was introduced to combat spectral leakage;
- tapered windows can reduce spectral leakage but cause a wider main lobe in the spectrum;
- thus, windowing can obscure the presence of closely spaced sinusoids and reduce the resolution;
- therefore, usually a trade-off has to be made between spectral leakage and the achievable spectral resolution.

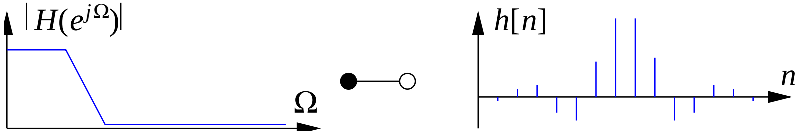
DFT Applications — Frequency Response

- Impulse response $h(n)$ and frequency response $H(e^{j\Omega})$ of a discrete time system are related by the DFT/IDFT:



DFT Applications — FIR Filter Design

- In filter design, we often have an idea what the magnitude response $|H(e^{j\Omega})|$ of the desired filter should look like;
- adding appropriate phase values to the magnitude response, we obtain a frequency response;
- this frequency response if inversely transformed;

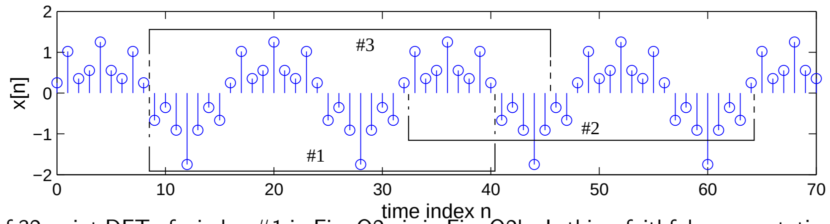


- the resulting time domain response is an approximation of the desired impulse response (holding the filter coefficients);
- generally, some more refinement is required, but the principle is based on the IDFT.

Tutorial Q2: Fourier Series and DFT

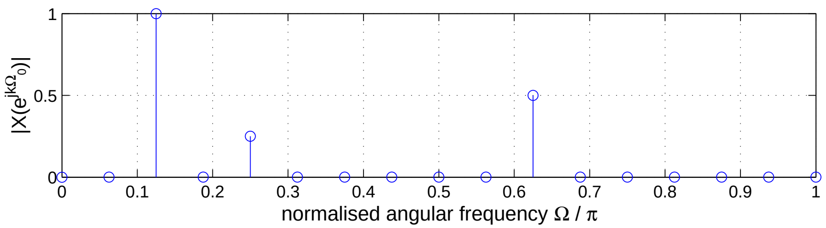
1. Why is the Fourier series applicable to $x(n)$ in Fig. Q2a and at which frequencies are spectral components expected? (*7 marks*)

Fig. Q2a



2. Magnitude of 32-point DFT of window#1 in Fig. Q2a is in Fig. Q2b. Is this a faithful representation of the data? (*5 marks*)

Fig. Q2b



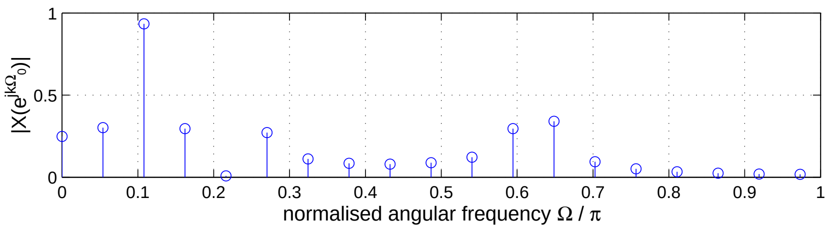
3. Consider the DFT of data segment #2 in Figure Q2a. This segment is shifted by $L = 24$ samples with respect to the one analysed in Question (2). If we have $\hat{x}(n) = x(n-L]$, derive the relation between the DFT $\hat{X}(e^{jk\Omega_0}) \bullet\!\!-\!\!\circ\ \hat{x}(n)$ and $X(e^{jk\Omega_0}) \bullet\!\!-\!\!\circ\ x(n)$,

$$X(e^{jk\Omega_0}) = \sum_{n=0}^{N-1} x(n) e^{-jk\Omega_0 n}$$

with $N = 32$ and $\Omega_0 = 2\pi/(NT_s)$ whereby T_s is the sampling period. (*9 marks*)

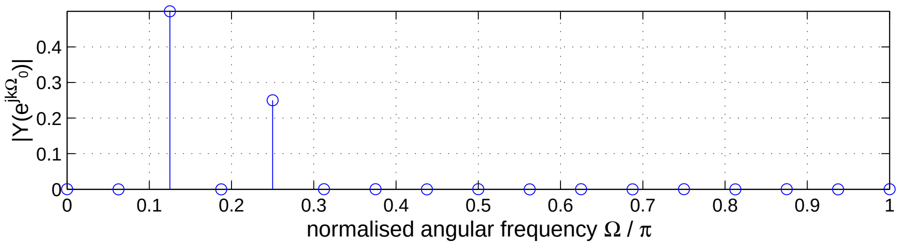
4. Applying a DFT to the data segment #3 in Figure Q2a, the magnitude in Figure Q2c results. Describe why , compared to Figure Q2b, additional non-zero Fourier coefficients appear in the spectrum, and how you could mitigate this effect by windowing. Also briefly comment on any trade-offs involved in windowing. (*8 marks*)

Fig. Q2c



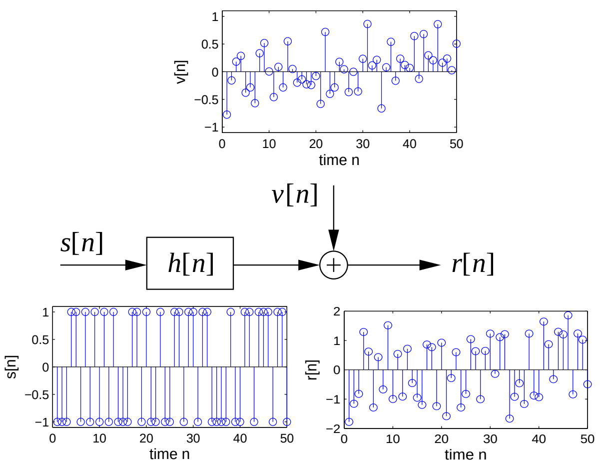
5. The waveform $x(n)$ has been used to excite a linear time-invariant filter $h(n)$ for a long time, and the 32-point DFT $Y(e^{jk\Omega_0})$ of the filter output $y(n)$ is given in Figure Q2d. What quantitative statements can you make about the frequency response of $h(n)$? (*4 marks*)

Figure Q2d:



Random Signals

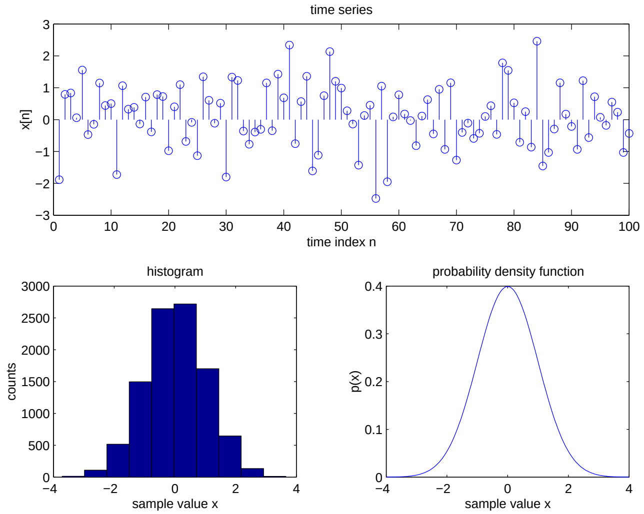
- Most communications signals are non-periodic and non-deterministic:



- we need some statistics to describe random signals;
- parameters that characterise a random signal are:
  - probability and cumulative density function (PDF / CDF);
  - mean, variance, etc.
  - auto-correlation function and power spectral density (PSD)

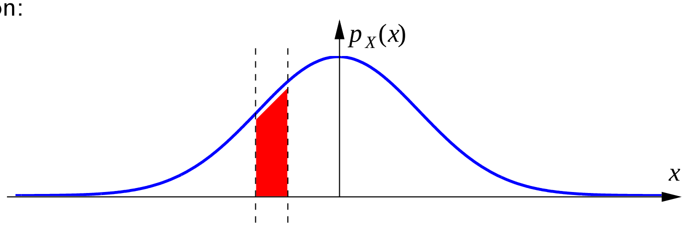
Probability Density Function (PDF)

- Consider a random, discrete time signal $x(n)$;
- observation of signal's histogram: how often do certain amplitude values occur?
- taking a potentially infinite number of samples and by normalisation, the probability density function (PDF) of $x(n)$ emerges.



Interpretation of the PDF

- The PDF $p_X(x)$ of a signal $x(n)$ (or $x(t)$) gives information on its amplitude distribution:



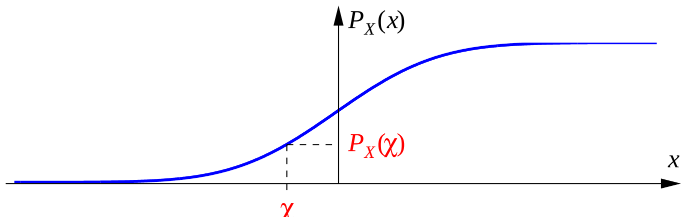
- the area $p_X(x)\ dx$ represents the probability for the amplitude of $x(n)$ to fall into the interval dx $\longrightarrow$ $\int_{-\infty}^{\infty} p_X(x)dx = 1$;
- analytic description for e.g. Gaussian or normal PDF:

$$p_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad \text{with parameters} \quad \mu, \sigma. \qquad (61)$$

Cumulative Density Function (CDF)

- The cumulative density function (CDF) $P_X(x)$ of a signal $x(n)$ (or $x(t)$) is generated by the integration of the PDF across the signal's entire dynamic range:

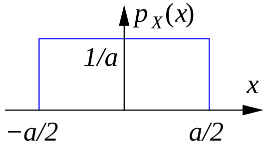
$$P_X(x) = \int_{-\infty}^{x} p_X(\chi)d\chi$$



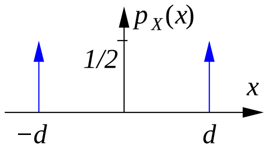
- the value $P_X(\chi)$ represents the probability of $x(n)$ having an amplitude smaller than or equal to χ. Clear: as $\chi \longrightarrow \infty$, $P_X(-\chi) = 0$ and $P_X(\chi) = 1$.

Different Distributions

- The normal or Gaussian distribution $\mathcal{N}(\mu, \sigma^2)$ was introduced in (61);
- uniform distribution:
  e.g. random phase of a sinusoid



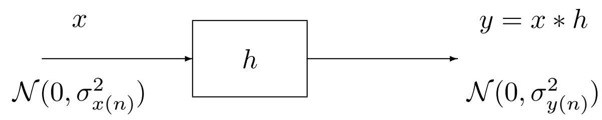
- binary distribution:
  e.g. binary communications signal



- Central Limit Theorem: the superposition of an infinite number of arbitrarily distributed random variables (rv) results in a Gaussian distributed rv.

PDF and LTI systems

- Filtering a random signal $x(n)$ by a Linear Time-Invariant system having an impulse response (IR) $h(n)$, in general the output $y(n)$ has a PDF different from that of the input;



- by contrast, when passing a Gaussian signal through a LTI system, the PDF remains Gaussian, but its mean and variance changes;
- for other distributions, the output $y(n)$ becomes more reminiscent of a Gaussian distribution owing to the Central Limit Theorem (CLT), since convolution with the IR results in the superposition of the differently weighted delayed original samples;

Mean, Variance and Expected Value

- The expectation operator $\mathcal{E}\{\cdot\}$ evaluates the average over an ensemble of rvs (i.e. parallel realisations of random processes obeying the same PDF);
- mean of a distribution:

$$\mu = \mathcal{E}\{x\} = \int_{-\infty}^{\infty} x \; p_X(x) \; dx \tag{62}$$

- variance of a distribution:

$$\sigma_x^2 = \mathcal{E}\left\{(x-\mu)^2\right\} = \int_{-\infty}^{\infty} (x-\mu)^2 \; p_X(x) \; dx \tag{63}$$

- we may interpret the *mean* as the 'centre of gravity' and the *variance* as a PDF-width-related characteristic of a distribution, while σ is the *standard deviation*

Determining the Mean and the Variance

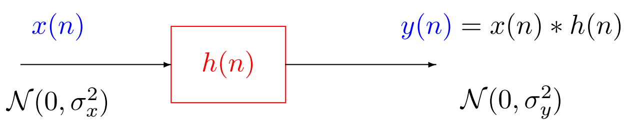
- To practically calculate the mean and the variance, we assume *ergodicity*, namely that the *ensemble-average* is identical to the *time-average*.
- example: throwing 10 000 dices to calculate the mean is replaced by throwing a single dice 10 000 times. Provided that the dices are identical, the result will be the same;
- time averages instead of expectations:

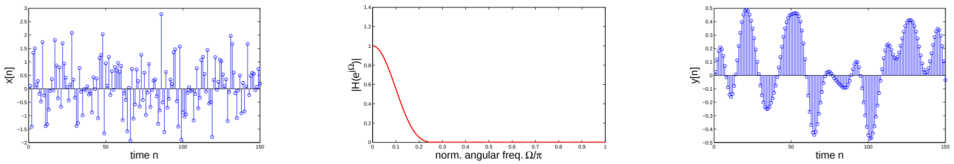
$$\mu_x = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} x(n) \qquad ; \qquad \sigma_x^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (x(n) - \mu_x)^2 \tag{64}$$

- for continuous-time variables the summations are replaced by integration;
- note that for-zero mean signals ($\mu_x = 0$), the variance physically represents the power of the signal $x(n)$.

Filtering a Random Signal

- Consider lowpass filtering a Gaussian signal $x(n)$, which is "completely random":





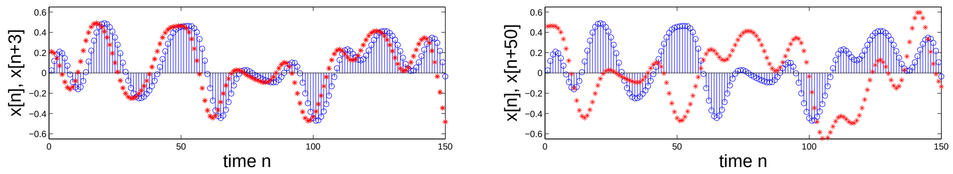
- the output will have a Gaussian distribution, but the signal now changes more smoothly: neighbouring samples become "correlated" - we need a measure.

Auto-Correlation Function I

- The correlation between a sample $x(n)$ and a neighbouring value $x(n+\tau)$ is given by

$$r_{xx}(\tau) \quad = \quad \mathcal{E}\{x(n)x^*(n+\tau)\} \quad = \quad \lim_{N\rightarrow\infty} \frac{1}{N} \sum_{n=0}^{N-1} x(n)x^*(n+\tau) \qquad (65)$$

- For the specific time-lags $\tau = 3$ (left) and $\tau = 50$ samples (right), consider:



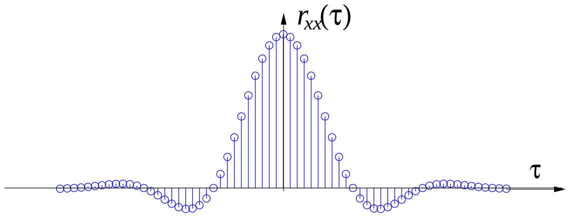
- the curves on the left appear "similar", the ones on the right "dissimilar".

Auto-Correlation Function II

- For a time-lag of zero we have:

$$r_{xx}(0) \quad = \quad \lim_{N\rightarrow\infty} \frac{1}{N} \sum_{n=0}^{N-1} x(n)x^*(n) \qquad (66)$$

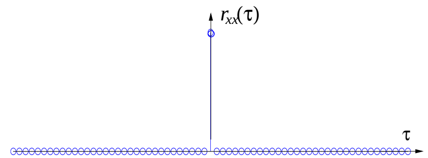
- This value for $\tau = 0$ is the maximum of the auto-correlation function $r_{xx}(\tau)$;



- large values in the ACF indicate strong correlation, small values weak correlation;

Auto-Correlation Function III

- If a signal has no self-similarity, i.e. it is "completely random", the ACF obeys the following form:



- **Wiener-Khintshine Theorem:** The ACF and the PSD are Fourier transform pairs;
- If we take the Fourier transform of $r_{xx}(\tau)$, we obtain a flat spectrum (or a lowpass spectrum for the ACF on slide 95);
- due to the presence of all frequency components in a flat spectrum, a completely random signal is often referred to as "white noise".

Power Spectral Density

- Again, according to the Wiener-Khintshine theorem, the PSD and ACF constitute a Fourier-pair, $r_{xx}(\tau) \circ\!\!-\!\!\bullet\, R_{xx}(e^{j\Omega})$, therefore

$$r_{xx}(\tau) = FFT^{-1}(R_{xx}(e^{j\Omega})) = \frac{1}{2\pi}\int_{-\pi}^{\pi} R_{xx}(e^{j\Omega})\, e^{-j\Omega\tau}\, d\Omega \qquad (67)$$

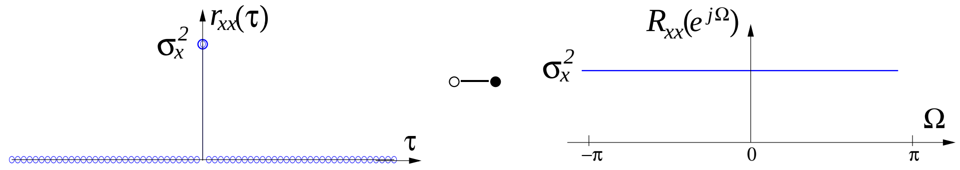
- note that the power of $x(n)$ is given by

$$r_{xx}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} R_{xx}(e^{j\Omega})\, d\Omega \qquad (= \text{scaled area under PSD}) \qquad (68)$$

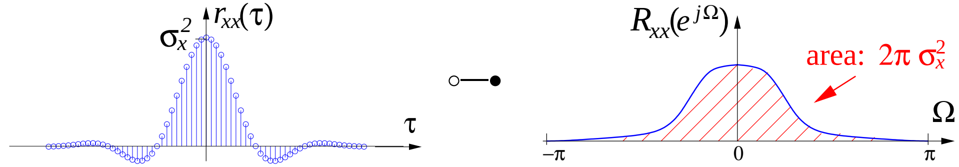
- This is a manifestation of the Parseval theorem, stating that the power is the same in the TD and FD.

PSD – Examples

- PSD for uncorrelated ("white") zero-mean noise:



- PSD for correlated zero-mean noise:



Cross-Correlation

- The cross-correlation function of the signals $x(n)$ and $y(n)$ is defined analogously to (65):

$$r_{xy}(\tau) \;=\; \mathcal{E}\{x(n)y^*(n+\tau)\} \;=\; \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} x(n)y^*(n+\tau) \qquad (69)$$

- note: $\quad r_{yx}(\tau) \;=\; r_{xy}^*(-\tau)$; by contrast: $\quad r_{xx}(\tau) \;=\; r_{xx}(-\tau)$

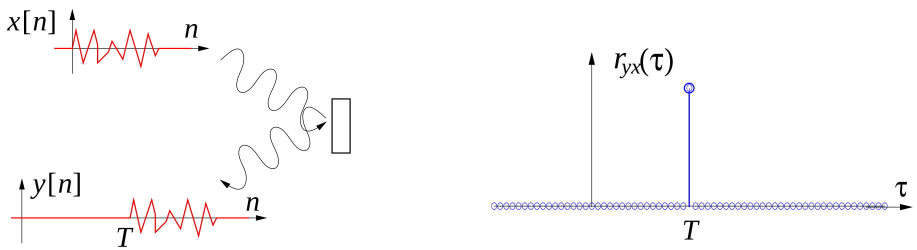
  i.e. the auto-correlation function is symmetric, while the cross-correlation function is 'conjugate-complex symmetric';

- for uncorrelated signals:

$$r_{xy}(\tau) \;=\; \mathcal{E}\{x(n)y^*(n-\tau)\} \;=\; \mathcal{E}\{x(n)\}\,\mathcal{E}\{y^*(n-\tau)\} \;=\; \mu_x\,\mu_y^* \qquad (70)$$

Examples of Applying Cross-Correlation Techniques

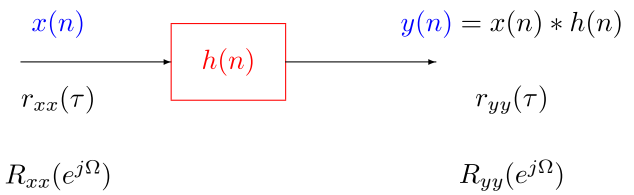
- Delay estimation: assume that we transmit a random pulse $x(n)$ and we detect the delayed, reflected signal $y(n)$:



- "Matched filtering". Compare the received signal to a legitimate transmitted waveform; the cross-correlation will be maximum, if the noise-contaminated received signal matches the hypothesized sequence.

Filtering of Random Signals Revisited I

- Consider again filtering a random signal $x(n)$ with a filter having an impulse response $h(n)$:



- relation between $x(n)$ and $y(n)$ is given by convolution: $y(n) = \sum_{\nu=-\infty}^{\infty} h(\nu)\, x(n-\nu)$;
- we are looking for the relations between $r_{xx}(\tau)$ and $r_{yy}(\tau)$ and between $R_{xx}(e^{j\Omega})$ and $R_{yy}(e^{j\Omega})$.

- The cross-correlation is:

$$\begin{aligned}
r_{yx}(\tau) &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} y(n) x^*(n-\tau) && (71)\\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \left(\sum_{\nu=-\infty}^{\infty} h(\nu)\, x(n-\nu) \right) x^*(n-\tau) && (72)\\
&= \sum_{\nu=-\infty}^{\infty} h(\nu) \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} x(n-\nu)\, x^*(n-\tau) && (73)\\
&= \sum_{\nu=-\infty}^{\infty} h(\nu) \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, x^*(n-\tau+\nu) && (74)\\
&= \sum_{\nu=-\infty}^{\infty} h(\nu)\, r_{xx}(\tau-\nu) \;=\; h(\tau) * r_{xx}(\tau) && (75)
\end{aligned}$$

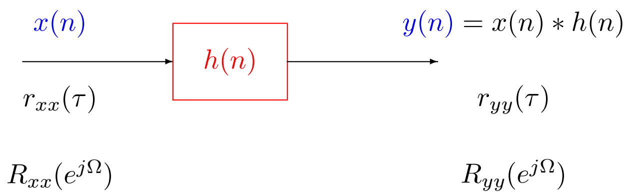
- note: $\quad r_{xy}(\tau) = r^*_{yx}(-\tau) = h^*(-\tau) * r_{xx}(\tau)$

- Going further:

$$\begin{aligned}
r_{yy}(\tau) &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} y(n) y^*(n-\tau) && (76)\\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} y(n) \sum_{\nu=-\infty}^{\infty} h^*(\nu)\, x^*(n-\nu) && (77)\\
&= \sum_{\nu=-\infty}^{\infty} h^*(\nu) \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} y(n)\, x^*(\nu-n+\tau) && (78)\\
&= \sum_{\nu=-\infty}^{\infty} h^*(\nu)\, r_{yx}(\nu+\tau) \;=\; h^*(-\tau) * r_{yx}(\tau) && (79)\\
&= h^*(-\tau) * h(\tau) * r_{xx}(\tau) && (80)
\end{aligned}$$

Filtering of Random Signals Revisited II

- Hence, if a random system is filtered:



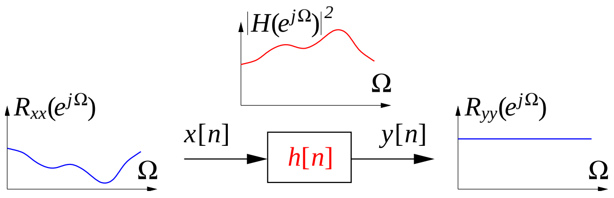
- we have

$$\begin{aligned}
r_{yy}(\tau) &= h^*(-\tau) * h(\tau) * r_{xx}(\tau) && (81)\\
R_{yy}(e^{j\Omega}) &= H^*(e^{j\Omega})\, H(e^{j\Omega})\, R_{xx}(e^{j\Omega}) && (82)\\
&= |H(e^{j\Omega})|^2\, R_{xx}(e^{j\Omega}) && (83)
\end{aligned}$$

- note that a filter $h(n)$ correlates an originally white signal $x(n)$.

Application of the PSD: "Whitening" for Source Coding

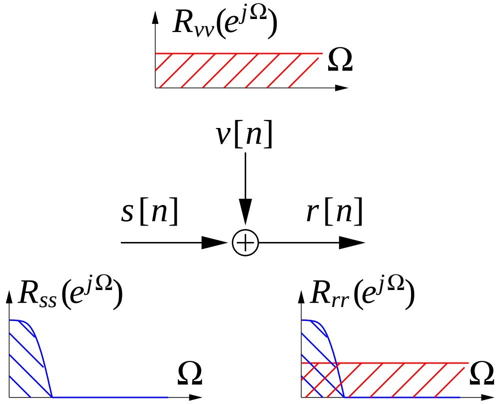
- Any signal $x(n)$ with a non-flat PSD exhibits more or less strong correlation between at least adjacent signal samples;
- this makes successive samples "predictable" to some extend; $x(n)$ therefore carries redundancy;
- this redundancy is undesired in source coding (contrary to willingly injected redundancy for channel coding!) and should be removed from the signal;



- $h(n)$ can be designed as a "whitening filter";
- $y(n)$ with flat PSD is ideal for source coding.

Additive Noise in the Frequency Domain

- Recall from slide 85 that we assume the model of additive noise;



- $s(n)$ is the transmitted signal, $r(n)$ the received signal;
- the additive noise $v(n)$ distorts the received signal;
- measure signal quality: signal-to-noise ratio (SNR);
- signal and noise power can be determined from the PSDs.
- Assumption: $s(n)$ and $v(n)$ are independent $\longrightarrow$ $\sigma_r^2 = \sigma_s^2 + \sigma_v^2$ and $P_{rr}(e^{j\Omega}) = P_{ss}(e^{j\Omega}) + P_{vv}(e^{j\Omega})$.

Signal-to-Noise Ratio

- The signal to noise ratio is a power ratio:

$$\text{SNR} = \frac{\text{signal power}}{\text{noise power}} \tag{84}$$

- for zero-mean signals: $\text{SNR} = \sigma^2_{\rm signal}/\sigma^2_{\rm noise}$;
- the range of values to be measured may span several orders of magnitude (such as the human hearing); therefore a logarithmic scale has been introduced:

$$\text{SNR}_{\rm dB} = 10\log_{10}\frac{\sigma^2_{\rm signal}}{\sigma^2_{\rm noise}} = 20\log_{10}\frac{\sigma_{\rm signal}}{\sigma_{\rm noise}} \quad \text{[deciBel, dB]} \tag{85}$$

- Examples: $\sigma^2_{\rm signal} = 1,000 \cdot \sigma^2_{\rm noise} \longrightarrow \text{SNR}_{\rm dB} = 30$ dB;
  $\sigma^2_{\rm signal} = 1,000,000 \cdot \sigma^2_{\rm noise} \longrightarrow \text{SNR}_{\rm dB} = 60$ dB.