Signal theory, part 1, test 1 **Solutions**

1. *Step function output* (1 points)
   Write down a representation of an autonomous linear time-invariant system that can give a step function output.

   Solution: The continuous-time system, defined by
   $$\frac{\mathrm{d}}{\mathrm{d}t} y = 0$$
   and the discrete-time system, defined by
   $$\sigma y = y$$
   have constant output $y(0)e^{0t}$ and $y(0)1^t$, for $t \geq 0$. □

2. *Sine function output* (1 points)
   Write down a representation of an autonomous LTI system that can give a sine with frequency ω output.

   Solution: A sine with frequency ω can be written as $ce^{\mathbf{i}\omega t} + \bar{c}e^{-\mathbf{i}\omega t}$, for some $c \in \mathbb{C}$. For the continuous-time system, we have
   $$p(s) = (s - \mathbf{i}\omega)(s + \mathbf{i}\omega) = s^2 + \omega^2 \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} y + \omega^2 y = 0.$$
   For the discrete-time system, we have
   $$p(z) = (z - e^{\mathbf{i}\omega})(z + e^{-\mathbf{i}\omega}) = z^2 - 2\cos(\omega)z + 1 \quad \Longrightarrow \quad \sigma^2 y - 2\cos(\omega)\sigma y + y = 0.$$
   □

3. *System's order* (2 points)
   The signal $y(t) = e^t + e^{2t} + e^{3t}$ is a response of an autonomous LTI system. What can you say about the order $\mathtt{n}$ of the system?

   (a) $\mathtt{n} = 1$ (b) $\mathtt{n} = 2$ (c) $\mathtt{n} = 3$ (d) $\mathtt{n} \geq 3$

   Solution: (d) □

4. $y \overset{?}{\in} \mathscr{B}(A, C)$ (2 points)
   Check if $y_\mathrm{d} = (2, 3, 5, 9, 17, 33, 64)$ is a possible output of the system defined by the state space representation
   $$x(t+1) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x(t), \quad y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t). \tag{SS}$$
   If y_d is not a possible output of the system, suggest a way of correcting it, so that the corrected signal is.

   Solution: $y_\mathrm{d} = \big(y_\mathrm{d}(1), \dots, y_\mathrm{d}(T)\big) \in \mathscr{B}(A, C)$ if and only if there is $x(0)$, such that $y_\mathrm{d}(t) = CA^t x(0)$, for $t = 1, \dots, T$. Written in a matrix form, this condition is the following system of linear equations
   $$\begin{bmatrix} y_\mathrm{d}(1) \\ y_\mathrm{d}(2) \vdots \\ y_\mathrm{d}(T) \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ \\ CA^{T-1} \end{bmatrix}}_{\mathscr{O}} x(0). \tag{$*$}$$

   For the given example, we have
   $$\begin{bmatrix} 2 \\ 3 \\ 5 \\ 9 \\ 17 \\ 33 \\ 64 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 8 \\ 1 & 16 \\ 1 & 32 \\ 1 & 64 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix},$$

which has no solution, so that $y_{\mathrm{d}} \notin \mathscr{B}(A,C)$. A way of correcting y_{d}, so that it becomes an output of $\mathscr{B}(A,C)$, is to solve $(*)$ in a least-squares sense, and define the corrected output $\widehat{y} = \mathscr{O}\widehat{x}(0)$, where $\widehat{x}(0) = (\mathscr{O}^\top \mathscr{O})^{-1}\mathscr{O}^\top y_{\mathrm{d}}$ is the least-squares approximate solution of $(*)$. $\widehat{y}$ has the property that $\|y_{\mathrm{d}} - \widehat{y}\|_2$ is minimized over all outputs $\widehat{y}$ of $\mathscr{B}(A,C)$. □

5. $y \overset{?}{\in} \ker(P(z))$ (2 points)
Check if $y_{\mathrm{d}} = (2,3,5,9,17,33,64)$ is a possible output of the system defined by the difference equation

$$2y(t) - 3y(t+1) + y(t+2) = 0. \tag{KER}$$

If y_{d} is not a possible output of the system, suggest a way of correcting it, so that the corrected signal is.

Solution: $y_{\mathrm{d}} = \big(y_{\mathrm{d}}(1),\dots,y_{\mathrm{d}}(T)\big) \in \ker(P(z))$ if and only if

$$P_0 y(t) + P_1 y(t+1) + \cdots + P_n y(t+n) = 0, \qquad \text{for } t = 1,\dots,T-n.$$

Written in a matrix form, this condition is the following system of linear equations

$$\begin{bmatrix} P_0 & P_1 & \cdots & P_n \end{bmatrix} \begin{bmatrix} y(1) & y(2) & \cdots & y(T-n) \\ y(2) & y(3) & \cdots & y(T-n+1) \\ \vdots & \vdots & \cdots & \vdots \\ y(n+1) & y(n+2) & \cdots & y(T) \end{bmatrix} = 0. \tag{$**$}$$

For the given example, we have

$$\begin{bmatrix} 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 & 9 & 17 \\ 3 & 5 & 9 & 17 & 33 \\ 5 & 9 & 17 & 33 & 64 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \end{bmatrix} \neq 0,$$

so that $y_{\mathrm{d}} \notin \ker(P(z))$. An ad hock way of correcting y_{d} in the specific example is to set $y_{\mathrm{d}}(7) = 65$. □

6. $\mathscr{B}(A,C) \overset{?}{=} \ker(P(z))$ (2 points)
Check if the system defined by (SS) is the same system as the one defined by (KER).

Solution: We need to verify that the eigenvalues $z_1 = 1$ and $z_2 = 2$ of A are the same as the roots of $P(z)$. Indeed,

$$(z-1)(z-2) = 2 - 3z + z^2,$$

so that the two representations define the same system. □

7. $\mathscr{B}(A_1,C_1) + \mathscr{B}(A_2,C_2)$ (2 points)
Let $\mathscr{B}$ be the system obtained by adding the outputs of two autonomous LTI systems $\mathscr{B}_1$ and $\mathscr{B}_2$ of orders n_1 and n_2. Is $\mathscr{B}$ linear time-invariant? What is its order?

Solution: $\mathscr{B}$ is linear because the sum of two subspaces is a subspace. $\mathscr{B}$ is also time-invariant because, for any $y \in \mathscr{B}$, it follows that $y = y_1 + y_2$, where $y_1 \in \mathscr{B}_1$ and $y_2 \in \mathscr{B}_2$, but $\sigma^\tau y_1 \in \mathscr{B}_1$ and $\sigma^\tau y_2 \in \mathscr{B}_2$ for all τ, so that

$$\sigma^\tau y = \sigma^\tau y_1 + \sigma^\tau y_2 \in \mathscr{B}.$$

The order of $\mathscr{B}$ is

$$n = n_1 + n_2 - \text{"\# of common poles of } \mathscr{B}_1 \text{ and } \mathscr{B}_2\text{"}.$$

□

8. *Fast method for computing A^{100}* (4 points)
How many scalar multiplications requires the direct computation of A^{100} as $\underbrace{A\cdots A}_{100}$ for a 2×2 matrix A?

Suggest a faster method. Using the method, find a good approximation of $\begin{bmatrix} -1/4 & 1/4 \\ -3/2 & 1 \end{bmatrix}^{100}$.

Solution: The matrix product AB, for $A,B \in \mathbb{R}^{2\times 2}$, requires 8 multiplications, so that A^{100} requires 99×8 multiplications. A fast method is obtained by the eigenvalue decomposition $A = V\Lambda V^{-1}$ of A, because

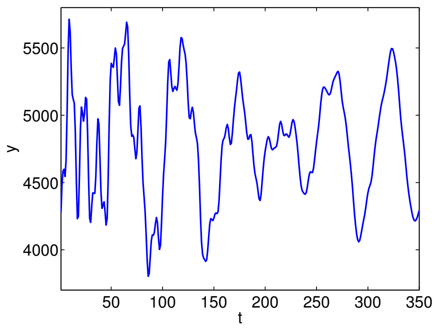
$$A^{100} = V\Lambda^{100}V^{-1} = V\begin{bmatrix} \lambda_1^{100} & \\ & \lambda_2^{100} \end{bmatrix} V^{-1},$$

requires only $2\times 99+12$ multiplications once the eigenvalue decomposition is computed. In the example the eigenvalues of A are $1/2$ and $1/4$. We have $(1/2)^{100} < 10^{-30}$ and $(1/4)^{100} < 10^{-60}$, so that $A^{100} \approx 0$. □

9. *Is the data generating system LTI?* (4 points)

A colleague of yours shows you the signal on the right and says:

> "I think the data generating system is not linear time-invariant, because the response of such a system is a sum of terms that are exponentially decaying, exponentially growing, or periodic while the behavior of the given signal is more complicated: it is obviously not periodic and it is neither exponentially decaying nor exponentially growing."



Do you agree? If so, how would you make the argument rigorous? If not, what is wrong with the argument and how would you prove that it is wrong? You can assume that you have the observed signal numerically and a computer available to process the data.

Solution: The argument is wrong because

(a) the LTI system may have an (unobserved) output, in which case the output may not be a sum-of-exponentials.

(b) Even for an autonomous LTI system, however, the order of the system is not bounded in the argument so that any finite signal can be made an exact output of an autonomous LTI system by increasing the order.

The minimum order can be found computationally from the signal by a realization algorithm. □