

ELEC 3035, Lecture 5:

Observability and state estimation

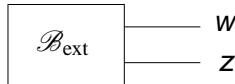
Ivan Markovsky

- Definition of observability
- State reconstruction and observability matrix
- Least squares observer

General observability problem

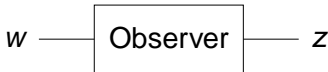
A dynamical system \mathcal{B}_{ext} with two types of external variables:

- observed variables w
- to-be-estimated variables z



is observable if z can be recovered from w and the model parameters.

System accepting w and producing z is called an observer.



Our goal is to construct an observer for a given system \mathcal{B}_{ext} .

State observability

We will consider the special case:

$$\mathcal{B} = \mathcal{B}_{i/s/o}(A, B, C, D), \quad w = (u, y), \quad z = x$$

(Recall problem 1 from tutorial 2: $\mathcal{B} = \mathcal{B}_{ss}(A, C)$, $w = y$, $z = x$.)

State observability — a property of the system ensuring that

the state can be recovered uniquely from the input and the output

- How to **check observability** in terms of A, B, C, D ?
- How to **reconstruct x** (observer design)?
- How to **approximate x** when the measurements are noisy?

Output trajectories

The trajectories of the system

$$\mathcal{B}_{i/s/o}(A, B, C, D) = \{ (u, x) \mid \sigma x = Ax + Bu, y = Cx + Du \}$$

are in the DT case

$$y(t) = CA^t x(0) + C \sum_{\tau=0}^{t-1} A^{t-1-\tau} Bu(\tau) + Du(t)$$

and in the CT case

$$y(t) = Ce^{At} x(0) + C \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau + Du(t)$$

DT-CT analogy: $A^t \leftrightarrow e^{At}$ and $\sum_{\tau=0}^{t-1} (\cdot) \leftrightarrow \int_0^t (\cdot) d\tau$

Observability of DT systems

Suppose we have observed u and y over the period $[0, t-1]$.

The system of equations

$$y(\tau) = CA^\tau x(0) + C \sum_{s=0}^{\tau-1} A^{\tau-1-s} Bu(s) + Du(\tau), \quad \text{for } \tau = 0, 1, \dots, t-1$$

written in a matrix form is

$$\underbrace{\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(t-1) \end{bmatrix}}_{Y_t} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{t-2} \end{bmatrix}}_{\mathcal{O}_t} x(0) + \underbrace{\begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ CA^{t-1}B & \cdots & CB & D \end{bmatrix}}_{\mathcal{I}_t} \underbrace{\begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(t-1) \end{bmatrix}}_{U_t}$$

$$Y_t = \mathcal{O}_t x(0) + \mathcal{I}_t U_t$$

- \mathcal{O}_t maps the initial state to the output over $[0, t-1]$
- \mathcal{I}_t maps the input to the output over $[0, t-1]$

Estimating the initial state requires to solve for x_0

$$\mathcal{O}_t x(0) = Y_t - \mathcal{I}_t U_t$$

Therefore,

x_0 can be reconstructed uniquely if and only if $\ker(\mathcal{O}_t) = \{0\}$

Note: the input plays no role in the state estimation problem.

Observability matrix of the system $\mathcal{B}_{ss}(A, C)$

By the Caley-Hamilton theorem A^t , for $t \geq n$, can be expressed as linear combination of A^0, A^1, \dots, A^{n-1} .

Therefore,

$$\ker(\mathcal{O}_t) = \ker(\mathcal{O}_n), \quad \text{for } t \geq n$$

\Rightarrow the initial state can be reconstructed from n output samples

The matrix

$$\mathcal{O} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is called the **observability matrix** of the system $\mathcal{B}_{ss}(A, C)$.

Observability test and state observer

$\mathcal{B}_{i/s/o}(A, B, C, D)$ is observable if and only if $\ker(\mathcal{O}) = \{0\}$

As for controllability, the observability test is the same in CT as in DT.

An observer F for the ini. state $x(0)$ is a linear function of U_t and Y_t

$$x_0 = F(Y_t - \mathcal{T}_t U_t)$$

satisfying the condition

$$F\mathcal{O}_t = I$$

Least squares observer

Assume that the output is observed with **measurement noise** v , i.e.,

$$y = Cx + Du + v$$

Then the system of equations for x_0

$$\mathcal{O}_t x(0) = Y_t - \mathcal{I}_t U_t$$

(generically) has no exact solution. The least-squares observer is

$$\hat{x}_{ls}(0) = \underbrace{(\mathcal{O}_t^\top \mathcal{O}_t)^{-1} \mathcal{O}_t^\top}_{F_{ls}} (Y_t - \mathcal{I}_t U_t)$$

It minimizes the **output estimation error** $\|Y - \hat{Y}\|_2$, where

$$\hat{Y} := \mathcal{O}_t \hat{x}_{ls}(0) + \mathcal{I}_t U_t$$

Observability Gramian

Consider the case $u = 0$,

$$\hat{x}_{ls}(0) = (\mathcal{O}_t^\top \mathcal{O}_t)^{-1} \mathcal{O}_t^\top Y_t = \underbrace{\left(\sum_{\tau=0}^{t-1} (A^\tau)^\top C^\top C A^\tau \right)^{-1}}_{\mathbf{G}_{0,t} := \mathcal{O}_t^\top \mathcal{O}_t} \underbrace{\sum_{\tau=0}^{t-1} (A^\tau)^\top C^\top y(t)}_{\mathcal{O}_t^\top Y_t}.$$

The initial state estimation error is

$$x(0) - \hat{x}_{ls}(0) = -\mathbf{G}_{0,t}^{-1} \mathcal{O}_t^\top V_t, \quad \text{where } V_t := \begin{bmatrix} v(0) \\ \vdots \\ v(t-1) \end{bmatrix}.$$

If V_t has bounded norm, then the error $x(0) - \hat{x}_{ls}(0)$ is also bounded

$$x(0) - \hat{x}_{ls}(0) \in \{ \mathbf{G}_{0,t}^{-1} \mathcal{O}_t^\top V_t \mid \|V_t\| \leq \sigma \} \quad \text{uncertainty ellipsoid}$$

Infinite horizon state estimation

$G_{0,t}^{-1}$ characterizes the uncertainty in the estimate $\hat{x}_{ls}(0)$

For $t \rightarrow \infty$, the uncertainty ellipsoid is given by the matrix

$$G_0 := \lim_{t \rightarrow \infty} G_{0,t} \quad \text{observability Gramian}$$

For stable systems, G_0 exists and satisfies the Lyapunov equation

$$A^\top G_0 A - G_0 = -C^\top C$$

Even for infinite number of measurements the initial condition estimate is not perfect.

Duality between observability and controllability

The system $\mathcal{B}(A^\top, C^\top, B^\top, D^\top)$ is called the dual of $\mathcal{B}_{i/s/o}(A, B, C, D)$.

The observability matrix of $\mathcal{B}_{i/s/o}(A, B, C, D)$ is

$$\begin{aligned}\mathcal{O}(A, C) &= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \\ &= [C^\top \quad A^\top C^\top \quad \dots \quad (A^{n-1})^\top C^\top]^\top \\ &= \mathcal{C}^\top(A^\top, C^\top)\end{aligned}$$

equal to the transposed of the controllability matrix of

$$\mathcal{B}(A^\top, C^\top, B^\top, D^\top)$$

Example

Consider the second order system

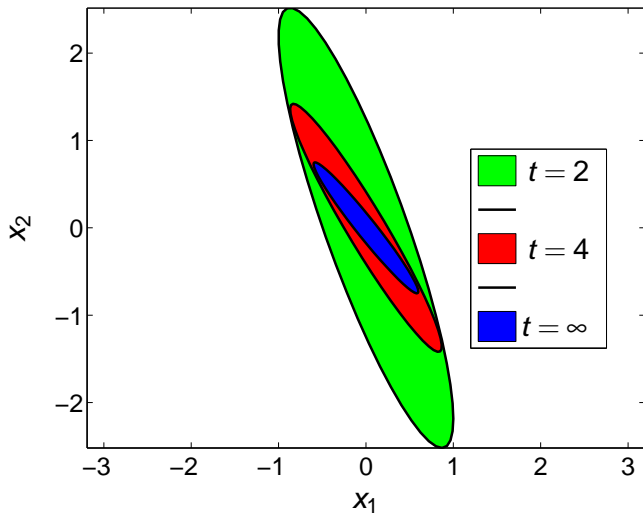
$$\mathcal{B}_{ss}(A, c) = \left\{ (u, y) \mid \sigma x = \underbrace{\begin{bmatrix} -1.75 & -0.8 \\ 1 & 0 \end{bmatrix}}_A x, y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_c x \right\}$$

The observability matrix is

$$\mathcal{O} = \begin{bmatrix} c \\ cA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1.75 & -0.8 \end{bmatrix}$$

Is this system observable?

Estimation uncertainty regions



Estimation uncertainty vs observation time

