A new measure for distance to uncontrollability

Ivan Markovsky

K.U.Leuven, ESAT-SISTA

Outline

Controllability test

Consider the system

$$\mathscr{B}: \quad \sigma x = Ax + Bu, \quad y = Cx + Du,$$

where $A \in \mathbb{R}^{n \times n}$, $D \in \mathbb{R}^{p \times m}$, and σ is the

- shift operator $(\sigma x)(t) = x(t+1)$ (in discrete-time) or
- derivative operator $\sigma x = dx/dt$ (in continuous-time)

$$\boxed{\mathscr{B} \text{ is controllable iff } \mathscr{C} := \begin{bmatrix} A & BA & \cdots & BA^{n-1} \end{bmatrix} \text{ is full rank.}}$$

$\Longrightarrow$ checking controllability is a rank test problem for a structured matrix, which is a nonlinear transformation of A, B

Distance to rank deficiency

In numerical linear algebra, yes/no questions ($\mathscr{B}$ contr./uncontr.) are replaced by quantitative measures (distance of $\mathscr{B}$ to uncontr.)

Checking whether $\mathscr{C}$ is full rank is a yes/no question.

A corresponding quantitative measure is distance of $\mathscr{C}$ to rank deficiency: smallest $\|\Delta\mathscr{C}\|$, such that $\widehat{\mathscr{C}} := \mathscr{C} + \Delta\mathscr{C}$ is rank def.

However for $\|\Delta\mathscr{C}\|$ to be a meaningful measure for distance to uncontr., $\widehat{\mathscr{C}}$ has to be a controllability matrix for some system $\widehat{\mathscr{B}}$.

$\Longrightarrow$ $\Delta\mathscr{C}$ should have the same structured as $\mathscr{C}$.

Unstructured/structured low rank approximation

Consider a set of structured matrices $\mathbb{M}$ and define

$$d_r(A) := \min_{\Delta A \in \mathbb{M}} \|\Delta A\| \quad \text{subject to} \quad A + \Delta A \text{ has rank } r.$$

With $\mathbb{M} = \mathbb{R}^{m \times n}$, $d_r(A)$ is unstructured distance to rank-r matrices.

In special cases, unstructured $d_r(A)$ can be computed from the SVD of A

$$A = U \operatorname{diag}(\sigma_1, \ldots, \sigma_{\min(m,n)}) V^\top$$

- spectral norms: $d_r(A) = \sigma_{r+1}$
- Frobenius norm: $d_r(A) = \sqrt{\sigma_{r+1}^2 + \cdots + \sigma_{\min(m,n)}^2}$.

In general, $d_r(A)$ is difficult to compute.

Paige's distance to uncontrollability

C. C. Paige defined in

Properties of numerical algorithms related to computing controllability, IEEE-AC, vol. 26, 1981

the following measure for distance of $\mathscr{B}$ to uncontrollability

$$d(A,B) := \text{minimize}_{\widehat{A},\widehat{B}} \left\| \begin{bmatrix} A & B \end{bmatrix} - \begin{bmatrix} \widehat{A} & \widehat{B} \end{bmatrix} \right\|_{\mathrm{F}}$$
$$\text{subject to} \quad (\widehat{A}, \widehat{B}) \text{ is uncontrollable}$$

many papers on computing $d(A,B)$ (98 citations in WoS)

However, $d(A,B)$ depends on the choice of the state space basis!

$\Longrightarrow$ $d(A,B)$ not a genuine property of the pair of systems $(\mathscr{B}, \widehat{\mathscr{B}})$

More general definition

$$d(\mathscr{B}) := \min_{\widehat{\mathscr{B}} \in \overline{\mathscr{L}_{\text{ctrb}}}} \operatorname{dist}(\mathscr{B}, \widehat{\mathscr{B}}) \qquad (*)$$

where

- $\overline{\mathscr{L}_{\text{ctrb}}}$ is the set of uncontrollable LTI systems
- $\operatorname{dist}(\mathscr{B}, \widehat{\mathscr{B}})$ is a measure for the distance from $\mathscr{B}$ to $\widehat{\mathscr{B}}$

Note: $d(A,B)$ is formally a special case of $d(\mathscr{B})$ with

$$\operatorname{dist}(\mathscr{B}, \widehat{\mathscr{B}}) = \left\| \begin{bmatrix} A & B \end{bmatrix} - \begin{bmatrix} \widehat{A} & \widehat{B} \end{bmatrix} \right\|_{\mathrm{F}} \qquad \text{(Paige)}$$

however given $\mathscr{B}$ and $\widehat{\mathscr{B}}$, A, B and $\widehat{A}, \widehat{B}$ are not uniquely defined

$\Longrightarrow$ (Paige) is not well defined

Special case: I/O representation of a SISO system

Consider a SISO system $\mathscr{B}$ with an input/output representation

$$p(\sigma)y = q(\sigma)u$$

With p monic, p, q are unique and

$$\operatorname{dist}(\mathscr{B}, \widehat{\mathscr{B}}) := \sqrt{\|p - \widehat{p}\|_2^2 + \|q - \widehat{q}\|_2^2}$$

becomes a property of the pair of systems $(\mathscr{B}, \widehat{\mathscr{B}})$.

Equivalent problem: structured low-rank approx.

$\widehat{\mathscr{B}} \in \overline{\mathscr{L}_{\text{ctrb}}}$ is equivalent to rank deficiency of the Sylvester matrix

$$S(\widehat{p},\widehat{q}) := \begin{bmatrix} \widehat{p}_0 & & & & \widehat{q}_0 & & & \\ \widehat{p}_1 & \widehat{p}_0 & & & \widehat{q}_1 & \widehat{q}_0 & & \\ \vdots & \widehat{p}_1 & \ddots & & \vdots & \widehat{q}_1 & \ddots & \\ \widehat{p}_n & \vdots & \ddots & \widehat{p}_0 & \widehat{q}_n & \vdots & \ddots & \widehat{q}_0 \\ & \widehat{p}_n & & \widehat{p}_1 & & \widehat{q}_n & & \widehat{q}_1 \\ & & \ddots & \vdots & & & \ddots & \vdots \\ & & & \widehat{p}_n & & & & \widehat{q}_n \end{bmatrix}$$

problem $(*)$ is a Sylvester structured low-rank approximation

$$\min_{\widehat{p},\widehat{q},w} \left\| \begin{bmatrix} p \\ q \end{bmatrix} - \begin{bmatrix} \widehat{p} \\ \widehat{q} \end{bmatrix} \right\|_2 \quad \text{subject to} \quad S(\widehat{p},\widehat{q}) \begin{bmatrix} w \\ 1 \end{bmatrix} = 0 \qquad (**)$$

Algorithm

Theorem The optimization problem $(**)$ is equivalent to

$$\min_{c} \operatorname{trace}\Big(\begin{bmatrix} p & q \end{bmatrix}^\top \Big(I - T(c)\big(T^\top(c)T(c)\big)^{-1} T^\top(c) \Big) \begin{bmatrix} p & q \end{bmatrix} \Big),$$

where $T(c) \in \mathbb{R}^{(n+1)\times n}$ is a lower triangular banded Toeplitz matrix with first column equal to $\operatorname{col}(c,1,0,\dots,0)$.

Notes:

- $\widehat{p}$, $\widehat{q}$, and the constraint are eliminated
- nonconvex nonlinear least squares problem
- solved numerically using local optimization methods
- cost function evaluations: solve a structured LS problem
- exploiting structure, comput. complexity per iteration $O(n)$

Suboptimal initial approximations

Can be computed from unstructured low rank approximation (SVD) of

1. Sylvester matrix $S(p,q)$
2. Bezout matrix $B(p,q)$
3. Hankel matrix $H(h)$

$$B(p,q) := \begin{bmatrix} p_1 & \cdots & p_n \\ \vdots & \iddots & \\ p_n & & 0 \end{bmatrix} \begin{bmatrix} q_0 & \cdots & q_{n-1} \\ & \ddots & \vdots \\ 0 & & q_{n-1} \end{bmatrix} - \begin{bmatrix} q_1 & \cdots & q_n \\ \vdots & \iddots & \\ q_n & & 0 \end{bmatrix} \begin{bmatrix} p_0 & \cdots & p_{n-1} \\ & \ddots & \vdots \\ 0 & & p_{n-1} \end{bmatrix}$$

$$H(h) := \begin{bmatrix} h_1 & h_2 & \cdots & h_n \\ h_2 & h_3 & \iddots & h_{n+1} \\ \vdots & \iddots & \iddots & \vdots \\ h_n & h_{n+1} & \cdots & h_{2n} \end{bmatrix}, \qquad \frac{q(z)}{p(z)} = \sum_{t=0}^{\infty} h_t z^{-t-1}$$

Using the Sylvester matrix

$$\begin{aligned} \mathscr{B}(p,q) \in \overline{\mathscr{L}_{\text{ctrb}}} &\iff p,q \text{ have common divisor } c,\ \deg(c) \geq 1 \\ &\iff \exists\, u,v \text{ such that } p(\xi)v(\xi) = q(\xi)u(\xi) \\ &\iff \exists\, u,v \text{ such that } S(p,q)\begin{bmatrix} v \\ -u \end{bmatrix} = 0 \\ &\iff S(p,q) \text{ is low-rank} \end{aligned}$$

After computing u,v from the SVD, we solve the LS problem

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} T(u) \\ T(v) \end{bmatrix} c \qquad \left(\begin{array}{l} T(\cdot) \text{ is lower triangular} \\ \text{banded Toeplitz matrix} \end{array} \right)$$

and define $\widehat{p}(\xi) = u(\xi)c_{\text{ls}}(\xi)$ and $\widehat{q}(\xi) = v(\xi)c_{\text{ls}}(\xi)$. Then

$$d\big(\mathscr{B}(p,q)\big) \leq \|\operatorname{col}(p,q) - \operatorname{col}(\widehat{p},\widehat{q})\|_2$$

Simulation example

Given $\mathscr{B}(p,q)$, where

$$p(\xi) = \quad 0.058 + 0.684\xi + 2.745\xi^2 + 4.751\xi^3 + 3.622\xi^4 + 1.000\xi^5$$
$$q(\xi) = -0.134 - 1.408\xi - 5.149\xi^2 - 8.381\xi^3 - 6.092\xi^4 - 1.604\xi^5$$

we compute $d(\mathscr{B}) = 0.0025$, with the certificate $\mathscr{B}(\widehat{p},\widehat{q})$, where

$$\widehat{p}(\xi) = \quad 0.057 + 0.684\xi + 2.744\xi^2 + 4.751\xi^3 + 3.622\xi^4 + 1.000\xi^5$$
$$\widehat{q}(\xi) = -0.135 - 1.407\xi - 5.150\xi^2 - 8.381\xi^3 - 6.092\xi^4 - 1.604\xi^5$$

have a common factor $c(\xi) = 1.684 + \xi$.

Conclusions

- motivation: replace the statement "$\mathscr{B}$ contr./uncontr." with a quantitative one "distance of $\mathscr{B}$ to uncontrollability"
- the definition invariably considered in the literature is not representation invariant
- behavioral measure: $d(\mathscr{B}) := \min_{\widehat{\mathscr{B}} \in \overline{\mathscr{L}_{\text{ctrb}}}} \text{dist}(\mathscr{B}, \widehat{\mathscr{B}})$
- in the SISO case, $d(\mathscr{B})$ can be defined in terms of the normalized I/O representation $p(\sigma)y = q(\sigma)u$
- the computation of $d(\mathscr{B})$ leads to a nonlinear least squares problem, which cost function evaluation is $O(n)$
- SVD upper bounds, based on the Sylvester, Bezout, and Hankel matrices

Thank you