ELEC 3035, Lecture 3: Autonomous systems

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- Equilibrium points and linearization
- Eigenvalue decomposition and modal form
- State transition matrix and matrix exponential
- Stability

Autonomous system = system without inputs

State space representation

$$\mathscr{B}(A,C) = \{\, y \mid \text{there is } x,\ \text{such that } \sigma x = Ax,\ y = Cx \,\}$$

x is the state, $n := \dim(x)$ is the "state dimension", y is the output

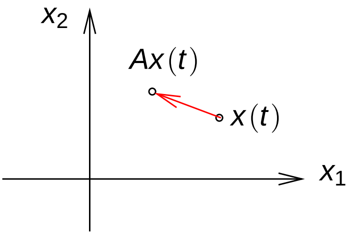
Polynomial representation

$$\mathscr{B}(P) = \{\, y \mid P(\sigma)y = 0 \,\}$$

where $P \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[z]$ and $\det(P) \neq 0$.

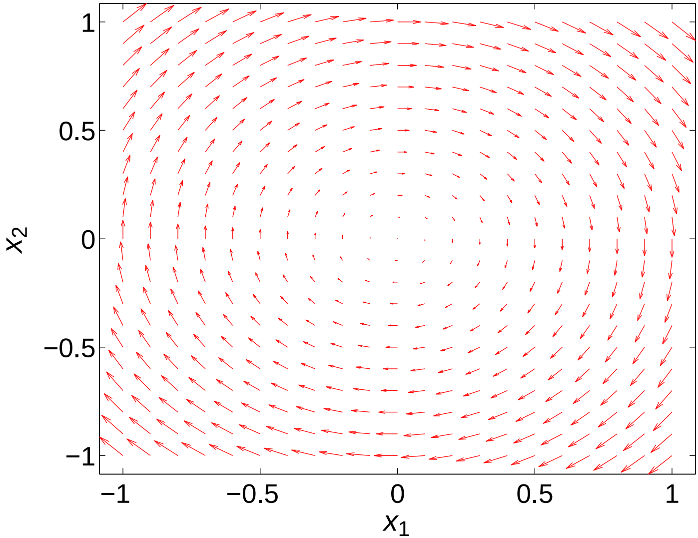
Phase plane

In $\sigma x = Ax$, Ax is a "velocity" vector — it shows how x changes in time.



For $n = 2$, the plot of Ax over $x \in \mathbb{R}^n$ is called phase plane.

Example: harmonic oscillator $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$



Equilibrium point of a dynamical system

Consider a nonlinear autonomous system

$$\mathscr{B} = \{ x \mid \sigma x = f(x) \}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and suppose that $f(x_e) = x_e$, for some $x_e \in \mathbb{R}^n$.

x_e is called an equilibrium point of $\mathscr{B}$

If $x(t_1) = x_e$ for some t_1, $x(t) = x_e$, for all $t > t_1$.

The set of equilibrium points of and LTI autonomous system

$$\mathscr{B} = \{ x \mid \sigma x = Ax \}$$

is $\ker(A - I)$ — the nullspace of $A - I$.

Linearization around an equilibrium point

Suppose that $x(t)$ is near an equilibrium point x_e. Then

$$\sigma x = f(x) \approx f(x_\mathrm{e}) + A(x - x_\mathrm{e}),$$

where

$$A = \left[a_{ij}\right] = \left[\left.\frac{\partial f_i}{\partial x_j}\right|_{x_{\mathrm{e},j}} \right].$$

The dynamics of the deviation from x_e

$$\widetilde{x} = x - x_\mathrm{e}$$

is described approximately be a linear system

$$\mathscr{B} = \{\widetilde{x} \mid \sigma\widetilde{x} = A\widetilde{x}\}$$

(Linearlization of a nonlinear system will be covered in part 2.)

Initial conditions

A trajectory of an autonomous system is uniquely determined by the initial state $x(0)$ or initial conditions:

- in discrete-time (DT) $y(-\ell+1), y(-\ell+2), \cdots y(0)$
- in continuous-time (CT) $\left(\frac{d}{dt}\right)^{-\ell+1} y(0), \left(\frac{d}{dt}\right)^{-\ell+2} y(0), \ldots \left(\frac{d}{dt}\right)^{0} y(0)$.

In the DT case

$$y(t) = CA^t x(0), \qquad t > 0.$$

In the CT case

the matrix power A^t is replaced by the matrix exponential e^{At}.

Modal form

Assume that there is a nonsingular matrix V, such that

$$V^{-1}AV = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} =: \Lambda.$$

- $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A
- the columns of V are the corresponding eigenvectors.

Then $\mathscr{B}(A, C) = \mathscr{B}(\Lambda, \widetilde{C})$, where $\widetilde{C} := CV$.

The state equation of $\sigma x = \Lambda x$ is a set of n decoupled equations.

- λ_i — pole of the system
- $e^{\lambda_i t}$ (in CT) or λ_i^t (in DT) — mode of the system

Eigenvalues and eigenvectors of a matrix

Consider a square matrix $A \in \mathbb{R}^{n \times n}$. $v \in \mathbb{C}^n$ is an eigenvectors of A if

$$Av = \lambda v, \qquad \text{for some } \lambda \in \mathbb{C}$$

λ is called an eigenvalue of A, corresponding to v.

Computing λ and v for given A involves solving a nonlinear equation.

Suppose that A has n linearly independent eigenvectors $v_1, \ldots, v_n$, then

$$Av_i = \lambda_i v_i, \quad i = 1, \ldots, n$$

$$\implies \quad A \underbrace{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}}_{\Lambda}$$

Let $\widetilde{x}$ be the state vector of $\mathscr{B}(\Lambda, \widetilde{C})$. In the DT case,

$$\widetilde{x}(t) = \Lambda^t \widetilde{x}(0) = \begin{bmatrix} \lambda_1^t & & \\ & \ddots & \\ & & \lambda_n^t \end{bmatrix} \widetilde{x}(0)$$

so that

$$\widetilde{x}_i(t) = \lambda_i^t \widetilde{x}_i(0)$$

and therefore

$$y = Cx(t) = \widetilde{C}\widetilde{x}(t) = \widetilde{c}_1 \widetilde{x}_1(t) + \cdots \widetilde{c}_n \widetilde{x}_n(t) = \alpha_1 \widetilde{\lambda}_1^t + \cdots \alpha_n \widetilde{\lambda}_n^t, \qquad \alpha_i = \widetilde{c}_i \widetilde{x}_i(0)$$

$\boxed{\mathscr{B}(A, C) = \mathscr{B}(\Lambda, \widetilde{C}) \text{ is a linear combination of its modes } \lambda_1, \ldots, \lambda_n.}$

Complex poles

The complex eigenvalues of $A \in \mathbb{R}^{n \times n}$ can always be grouped in complex conjugate pairs

$$\lambda_i = a + b\mathbf{i} = \alpha e^{\mathbf{i}\omega}, \qquad \lambda_j = a - b\mathbf{i} = \alpha e^{-\mathbf{i}\omega} \qquad (\mathbf{i} := \sqrt{-1})$$

so the sum of the two complex modes λ_i^t and λ_j^t gives one real mode

$$\lambda_i^t + \lambda_j^t = \alpha^t e^{\mathbf{i}\omega t} + \alpha^t e^{-\mathbf{i}\omega t} = 2\alpha^t \cos(\omega t)$$

α — damping factor

ω — frequency

A real mode is of the form λ_i^t — exponential

Matrix exponential

If the system is in a modal form $\mathscr{B}(\Lambda, CV)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x} = \Lambda\widetilde{x} \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{x}_i = \lambda_i\widetilde{x}_i, \quad \text{for } i = 1, \ldots, n.$$

so that

$$\widetilde{x}_i(t) = e^{\lambda_i t}\widetilde{x}_i(0) \quad \Longrightarrow \quad \widetilde{x}(t) = \underbrace{\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}}_{e^{\Lambda t}} \widetilde{x}(0)$$

Going back to the original basis we have

$$x(t) = \underbrace{V e^{\Lambda t} V^{-1}}_{e^{At}} x(0).$$

State transition matrix

The dynamics of the sate vector x is given by the equation

$$x(t) = \Phi(t)x(0)$$

where $\Phi(t) = A^t$ in DT and $\Phi(t) = e^{At}$ in CT.

The matrix $\Phi(t)$ is called state transition matrix.

$\Phi(t)$ shows how the initial state $x(0)$ is propagated in t time steps

Note: if $t < 0$, $\Phi(t)$ propagates backwards in time.

State construction

Consider a scalar autonomous system $\mathscr{B}(P)$, where

$$P(z) = P_0 z^0 + P_1 z^1 + \cdots + P_{n-1} z^{n-1} + I z^n.$$

How can we represent this system in a state space form $\mathscr{B}(A, C)$?

Choose $x(t) = \text{col}\big(y(t-1), \ldots, y(t-n)\big)$. Then

$$A = \begin{bmatrix} -P_{n-1} & -P_{n-2} & \cdots & -P_1 & -P_0 \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & I & 0 \end{bmatrix} \qquad \text{companion matrix of } P$$

$$C = \begin{bmatrix} -P_{n-1} & -P_{n-2} & \cdots & -P_1 & -P_0 \end{bmatrix}$$

Characteristic polynomial of a matrix

The polynomial equation

$$\det(\lambda I_n - A) = c_0\lambda^0 + c_1\lambda^1 + \cdots + c_n\lambda^n = 0$$

is called the characteristic equation of the matrix $A \in \mathbb{R}^{n\times n}$.

The roots of the characteristic polynomial

$$c(z) = c_0 z^0 + c_1 z^1 + \cdots + c_n z^n$$

are equal to the eigenvalues of A.

Cayley-Hamilton thm: Every matrix satisfies its own char. polynomial

$$c_0 A^0 + c_1 A^1 + \cdots + c_n A^n = 0.$$

Example: harmonic oscillator $A = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$

Characteristic equation

$$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix}\right) = \lambda^2 + 1 = 0$$

Eigenvalues and eigenvectors

$$\lambda_{1,2} = \pm i, \qquad v_{1,2} = \begin{bmatrix} 1 \\ \pm i \end{bmatrix}.$$

Matrix exponential

$$e^{At} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} e^{i} & \\ & e^{-i} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}^{-1} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}.$$

Sampling a continuous-time system

x — CT trajectory, $\quad x_{\mathrm{d}}$ — DT trajectory

$$x : \mathbb{R} \to \mathbb{R}^{\mathrm{n}} \quad \mapsto \quad x_{\mathrm{d}} : \mathbb{Z} \to \mathbb{R}^{\mathrm{n}}$$

Let $x_{\mathrm{d}}(t) := x(ht)$, h is the sampling time. Then

$$x_{\mathrm{d}}(t) = e^{Aht} x(0) = A_{\mathrm{d}}^{t} x(0), \qquad A_{\mathrm{d}} := e^{Ah}.$$

Stability

An autonomous system

$$\mathscr{B} = \{\, x \mid \sigma x = f(x) \,\}$$

is stable if $x \in \mathscr{B}$ implies $x(t) \to 0$ as $t \to \infty$.

For a linear time-invariant system,

$$\mathscr{B} = \{\, x \mid \sigma x = Ax \,\}$$

the eigenvalues of A determine the stability property of the system.

CT LTI system is stable iff all eigenvalues have negative real parts.

DT LTI system is stable iff all eigenvalues have absolute value < 1.

Qualitative behaviour of the system

If the eigenvalues are distinct

$$y_i = \alpha_{i1} e^{\lambda_1 t} + \cdots + \alpha_{in} e^{\lambda_n t}$$

where α_{ij} depend on the initial condition $x(0)$

- real λ_j — exponentially decaying or growing term
- complex λ_j — exponentially decaying or growing sinusoidal terms

In CT

- $\Re(\lambda_j) > 0$ — exponentially growing mode
- $\Re(\lambda_j) < 0$ — exponentially decaying mode
- $\Re(\lambda_j) = 0$ — a periodic or constant mode

Repeated eigenvalues give rise to polynomial terms in the solution.

Qualitative behaviour of the system

In DT

- $|\lambda_j| > 1$ — exponentially growing mode
- $|\lambda_j| < 1$ — exponentially decaying mode
- $|\lambda_j| = 1$ — a periodic or constant mode

Repeated eigenvalues give rise to polynomial terms in the solution.