more recent references in this area, see Brillinger (1981), Priestley (1982), Ljung (1985b, 1987), Hannan (1970), Wellstead (1981), and Bendat and Piersol (1980). Kay (1988) also presents many parametric methods for spectral analysis. The FFT algorithm for efficiently computing the discrete Fourier transforms is due to Cooley and Tukey (1965). See also Bergland (1969) for a tutorial description.

Appendix A3.1
Covariance functions, spectral densities and linear filtering

Let $u(t)$ be an nu-dimensional stationary stochastic process. Assume that its mean value is m_u and its covariance function is

$$r_u(\tau) = E[u(t + \tau) - m_u][u(t) - m_u]^{\mathrm{T}} \tag{A3.1.1}$$

Its spectral density is then, by definition,

$$\phi_u(\omega) \triangleq \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} r_u(\tau)e^{-i\tau\omega} \tag{A3.1.2}$$

The inverse relation to (A3.1.2) describes how the covariance function can be found from the spectral density. This relation is given by

$$r_u(\tau) = \int_{-\pi}^{\pi} \phi_u(\omega)e^{i\tau\omega}d\omega \tag{A3.1.3}$$

As a verification, the right-hand side of (A3.1.3) can be evaluated using (A3.1.2), giving

$$\int_{-\pi}^{\pi} \frac{1}{2\pi} \sum_{\tau'=-\infty}^{\infty} r_u(\tau')e^{-i\tau'\omega}e^{i\tau\omega}d\omega = \frac{1}{2\pi} \sum_{\tau'=-\infty}^{\infty} r_u(\tau') \int_{-\pi}^{\pi} e^{i(\tau-\tau')\omega}d\omega$$

$$= \sum_{\tau'=-\infty}^{\infty} r_u(\tau')\delta_{\tau,\tau'} = r_u(\tau)$$

which proves the relation (A3.1.3).

Now consider a linear filtering of $u(t)$, that is

$$y(t) = \sum_{k=0}^{\infty} h(k)u(t - k) \tag{A3.1.4}$$

where $y(t)$ is an ny-dimensional signal and $\{h(k)\}$ a sequence of $(ny|nu)$-dimensional matrices. We assume that the filter in (A3.1.4) is stable, which implies that $\|h(k)\| \to 0$ as $k \to \infty$.

Under the given conditions the signal $y(t)$ is stationary. The aim of this appendix is to derive its mean value m_y, covariance function $r_y(\tau)$ and spectral density $\phi_y(\omega)$; and in addition the cross-covariance function $r_{yu}(\tau)$ and the cross-spectral density $\phi_{yu}(\omega)$. It will be convenient to introduce the filter, or transfer function operator

$$H(q^{-1}) = \sum_{k=0}^{\infty} h(k)q^{-k} \tag{A3.1.5}$$

where q^{-1} is the backward shift operator. Using $H(q^{-1})$, the filtering (A3.1.4) can be rewritten as

$$y(t) = H(q^{-1})u(t) \tag{A3.1.6}$$

The mean value of $y(t)$ is easily found from (A3.1.4):

$$m_y = Ey(t) = E \sum_{k=0}^{\infty} h(k)u(t-k) = \sum_{k=0}^{\infty} h(k)m_u = H(1)m_u \tag{A3.1.7}$$

Note that $H(1)$ can be interpreted as the static (dc) gain of the filter.

Now consider how the deviations from the mean values $\tilde{y}(t) \triangleq y(t) - m_y$, $\tilde{u}(t) \triangleq u(t) - m_u$ are related. From (A3.1.4) and (A3.1.7),

$$\begin{aligned}\tilde{y}(t) &= \sum_{k=0}^{\infty} h(k)u(t-k) - \sum_{k=0}^{\infty} h(k)m_u = \sum_{k=0}^{\infty} h(k)[u(t-k) - m_u]\\ &= \sum_{k=0}^{\infty} h(k)\tilde{u}(t-k) = H(q^{-1})\tilde{u}(t)\end{aligned} \tag{A3.1.8}$$

Thus $(\tilde{u}(t), \tilde{y}(t))$ are related in the same way as $(u(t), y(t))$. When analyzing the covariance functions, strictly speaking we should deal with $\tilde{u}(t)$, $\tilde{y}(t)$. For simplicity we drop the $\tilde{}$ notation. This means formally that $u(t)$ is assumed to have zero mean. Note, however, that the following results are true also for $m_u \neq 0$.

Consider first the covariance function of $y(t)$. Some straightforward calculations give

$$\begin{aligned}r_y(\tau) &= Ey(t+\tau)y^{\mathrm{T}}(t)\\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} h(j)Eu(t+\tau-j)u^{\mathrm{T}}(t-k)h^{\mathrm{T}}(k)\\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} h(j)r_u(\tau - j + k)h^{\mathrm{T}}(k)\end{aligned} \tag{A3.1.9}$$

In most situations this relation is not very useful, but its counterpart for the spectral densities has an attractive form. Applying the definition (A3.1.2),

$$\begin{aligned}\phi_y(\omega) &= \frac{1}{2\pi}\sum_{\tau=-\infty}^{\infty} r_y(\tau)e^{-i\tau\omega}\\ &= \frac{1}{2\pi}\sum_{\tau=-\infty}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} h(j)e^{-ij\omega}r_u(\tau - j + \mathrm{k})e^{-i(\tau-j+k)\omega}h^{\mathrm{T}}(k)e^{ik\omega}\\ &= \frac{1}{2\pi}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} h(j)e^{-ij\omega}\left[\sum_{\tau'=-\infty}^{\infty} r_u(\tau')e^{-i\tau'\omega}\right]h^{\mathrm{T}}(k)e^{ik\omega}\\ &= \left[\sum_{j=0}^{\infty} h(j)e^{-ij\omega}\right]\phi_u(\omega)\left[\sum_{k=0}^{\infty} h^{\mathrm{T}}(k)e^{ik\omega}\right]\end{aligned}$$

or

$$\boxed{\phi_y(\omega) = H(e^{-i\omega})\phi_u(\omega)H^T(e^{i\omega}) \qquad \text{(A3.1.10)}}$$

This is a useful relation. It describes how the frequency content of the output depends on the input spectral density $\phi_u(\omega)$ and on the transfer function $H(e^{i\omega})$. For example, suppose the system has a weakly damped resonance frequency ω_0. Then $|H(e^{i\omega_0})|$ will be large and so will $\phi_y(\omega_0)$ (assuming $\phi_u(\omega_0) \neq 0$).

Next consider the cross-covariance function. For this case

$$\begin{aligned} r_{yu}(\tau) &= Ey(t + \tau)u^T(t) \\ &= \sum_{j=0}^{\infty} h(j)Eu(t + \tau - j)u^T(t) \\ &= \sum_{j=0}^{\infty} h(j)r_u(\tau - j) \end{aligned} \qquad \text{(A3.1.11)}$$

For the cross-spectral density, some simple calculations give

$$\begin{aligned} \phi_{yu}(\omega) &= \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} r_{yu}(\tau)e^{-i\tau\omega} \\ &= \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} \sum_{j=0}^{\infty} h(j)e^{-ij\omega}r_u(\tau - j)e^{-i(\tau-j)\omega} \\ &= \sum_{j=0}^{\infty} h(j)e^{-ij\omega}\left[\frac{1}{2\pi} \sum_{\tau'=-\infty}^{\infty} r_u(\tau')e^{-i\tau'\omega}\right] \end{aligned}$$

or

$$\boxed{\phi_{yu}(\omega) = H(e^{-i\omega})\phi_u(\omega) \qquad \text{(A3.1.12)}}$$

The results of this appendix were derived for stationary processes. Ljung (1985c) has shown that they remain valid, with appropriate interpretations, for quasi-stationary signals. Such signals are stochastic processes with deterministic components. In analogy with (A3.1.1), mean and covariance functions are then defined as

$$m_u = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} Eu(t) \qquad \text{(A3.1.13a)}$$

$$r_u(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} E(u(t + \tau) - m_u)(u(t) - m_u)^T \qquad \text{(A3.1.13b)}$$

assuming the limits above exist. Once the covariance function is defined, the spectral density can be introduced as in (A3.1.2). As mentioned above, the general results (A3.1.10) and (A3.1.12) for linear filtering also hold for quasi-stationary signals.