# STRUCTURED TOTAL LEAST SQUARES

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### Overview

 Least Squares
 Gauss
 1820

 Total Least Squares
 Golub and Van Loam
 1980

 Structured Total Least Squares
 Abatzoglou, De Moor
 1990

### Structured Total Least Squares:

 $\min_{\Delta A, \Delta B, X} \left\| \left[ \Delta A \ \Delta B \right] \right\|_{\mathrm{F}}^2 \qquad \text{s.t.} \qquad (A - \Delta A) X = B - \Delta B \quad \text{and} \quad$  $[\Delta A \ \Delta B]$  has the same structure as  $[A \ B]$ 

(Abatzoglou *et al.*, 1991) (De Moor, 1993) CTLS constraint TLS RiSVD Riemannian SVD STLN structured total least norm (Rosen et al., 1996)

### Types of structure:

 $\begin{array}{ll} {\rm nonlinear} & {\rm (Rosen~\it{et~al.}, 1998; Lemmerling~\it{et~al.}, 2002)} \\ {\rm affine} & {\rm CTLS, RiSVD, STLN} \\ {\rm Toeplitz/Hankel~\it{all}~methods} \end{array}$ 

 $\begin{array}{l} d \geq 1 \;\; \text{multivariate problems} \quad \text{(Van Huffel $et$ al., 1996)} \\ d = 1 \;\; \text{univariate problems} \qquad \text{all methods} \end{array}$ 

Efficiency of the algorithms:

affine structure  $O(m^3)$  CTLS, RiSVD, STLN Toeplitz/Hankel  $O(m^2)$  STLN Toeplitz/Hankel O(m) (Mastronardi  $et\ al.,\ 2000;$  Lemmerling  $et\ al.,\ 2000)$ 

direction of arrival (DOA) nuclear magnetic resonance (NMR) image deblurring system identification

(De Moor and Roorda, 1994)

No software available

## New results

STLS problem: 
$$\min_{\Delta p, X} ||\Delta p||^2$$
 s.t.  $S(p - \Delta p) \begin{bmatrix} X \\ -I_d \end{bmatrix} = 0$ 

 $\mathcal{S}$  —structure specification, e.g., affine  $\mathcal{S}(p) = S_0 + \sum_{i=1}^{n_p} S_i p_i$ 

Define  $X_{\text{ext}} := \begin{bmatrix} X \\ -I \end{bmatrix}$ ,  $r := \text{vec} \left( \left( \begin{bmatrix} A & B \end{bmatrix} X_{\text{ext}} \right)^{\top} \right)$ , and  $G := \left[ \text{vec} \left( \left( S_{1} X_{\text{ext}} \right)^{\top} \right) \cdots \text{vec} \left( \left( S_{n_{p}} X_{\text{ext}} \right)^{\top} \right) \right]$ .

### Equivalent problem:

 $\min_{\mathbf{V}} r(X) \Gamma^{-1}(X) r(X), \qquad \text{where} \quad \Gamma(X) := G(X) G^\top(X) \qquad (*)$ 

Theorem: Under the assumption that

$$C = \left[C^{(1)} \ \cdots \ C^{(q)}\right], \quad \text{where } C^{(i)} \text{ is } \begin{cases} \text{block-Hankel/Toeplitz,} \\ \text{unstructured, or} \\ \text{exact} \end{cases}$$

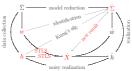
 $\Gamma$  is block-Toeplitz and block-banded with block size dK, where K is the row dimension of a block in block-Hankel/Toeplitz structured block  $C^{(\ell)}$  and with half block bandwidth, the maximum number of block columns in a block-Hankel/Toeplitz structured block  $C^{(\ell)}$ .

the structure of  $\Gamma$  allows computation of  $f_0(X)$  and  $f_0'(X)$  in O(m) flops ⇒ fast algorithms for (\*)

software: http://www.esat.kuleuven.ac.be/~imarkovs/stls.html

# Application for MIMO system

# Approximate modeling problems



(low complexity) approximating model - (high complexity) "true" model

observed general response observed impulse respons

Kernel subproblem: find a block-Hankel rank deficient matrix  $\mathcal{H}(\hat{w})$  approximating a given full rank matrix  $\mathcal{H}(w)$ 

Non-iterative methods like balanced model reduction, subspace identification, Kung's algorithm solve the kernel problem via SVD, which is suboptimal with respect to  $\|w-\hat{w}\|_{\ell_2}^2$ .

 $\mathcal{L}_{m,l}$  — the set of all LTI systems with m inputs and lag at most l (m and l specify the complexity of the model class  $\mathcal{L}_{m,l}$ )

 $\text{Identification problem:} \quad \min_{\mathcal{B} \in \mathcal{L}_{mi}} \left( \min_{\hat{w}} \ \| w - \hat{w} \|_{\ell_2}^2 \ \text{ s.t. } \ \hat{w} \in \mathcal{B} \right)$ 

STLS problem: 
$$\min_{X} \left( \min_{\hat{w}} \|w - \hat{w}\|_{\ell_{2}}^{2} \text{ s.t. } S(\hat{w}) \begin{bmatrix} X \\ -I \end{bmatrix} = 0 \right)$$

**Theorem:** Assume that  $\mathcal{B} \in \mathcal{L}_{m,l}$  admits a kernel representation

$$\mathcal{B} = \ker \left( \sum_{i=0}^{l} R_i \sigma^i \right), \quad \text{ with } \quad R_l =: \left[ Q_l \ P_l \right], \quad \quad P_l \in \mathbb{R}^{p \times p} \text{ full rank}$$

and let  $X^{\top} := -P_l^{-1} [R_0 \cdots R_{l-1} Q_l]$ 

$$X^{+} := -P_{l}^{-1} \left[ R_{0} \cdots R_{l-1} Q_{l} \right].$$
 Then 
$$w \in \mathcal{B}|_{[1,T]} \iff \mathcal{H}_{l+1}^{\top}(w) \begin{bmatrix} X \\ -I \end{bmatrix} = 0.$$

The extensions of the identification problem: multiple time series

latent inputs model reduction

and the special identification problems: noisy realization autonomous systems identification

are also solved as STLS problems.

### Results on data sets from DAISY

DAISY (DAta base for Identification of SYstems), (De Moor, 1998)

# Compared methods

 subid
 — robust combined subspace algorithm

 detss
 — deterministic balanced subspace algorithm

 pem
 — the prediction error method of the Identification Toolbox

 stls
 — the proposed method based on STLS

The order specified for the methods  $\mathtt{subid}, \mathtt{detss},$  and  $\mathtt{pem}$  is n=pl.  $\hat{\mathcal{B}}$  for  $\mathtt{detss}$  and  $\mathtt{pem}$  is the  $\underline{\mathtt{deterministic}}$  part of the identified system.

		parameters				scaled misfit			
#	Data set name	T	m	p	l	subid	detss	pem	
1	Destillation column	90	5	3	1	2.8	9.6	15.9	
2	Destillation column n10	90	5	3	1	2.8	9.6	15.9	
3	Destillation column n20	90		3		8.3	2.3	36.1	
4	Destillation column n30	90	5	3	1	7.8	3.3	132.2	
5	Glass furnace (Philips)	1247	3	6	1	2.9	2.5	2.7	
6	120 MW power plant	200	5	3	2	7.2	3.4	28.5	
7	pH process	2001	2	1	6	1.3	1.3	3.0	
8	Hair dryer	1000	1	1	5	1.2	1.2	1.0	
9	Winding process	2500	5	2	2	1.5	1.4	2.8	
10	Ball-and-beam setup	1000	1	1	2	1.0	10.6	1.0	
11	Industrial dryer	867	3	3	1	1.2	1.1	1.1	
12	CD-player arm	2048	2	2	1	1.2	1.1	1.4	
13	Wing flutter	1024	1	1	5	1.6	1.7	2.8	
14	Robot arm	1024	1	1	4	2.7	18.7	26.0	
15	Lake Erie	57	5	2	1	1.5	2.3	23.1	
16	Lake Erie n10	57	5	2	1	2.1	2.2	8.4	
17	Lake Erie n20	57	5	2	1	2.2	2.4	9.8	
18	Lake Erie n30	57	5	2	1	2.4	1.6	5.6	
19	Heat flow density	1680	2	1	2	1.8	1.3	9.8	
20	Heating system	801	1	1	2	1.3	1.2	1.3	
21	Steam heat exchanger	4000	1	1	2	1.8	1.8	8.1	
22	Industrial evaporator	6305	3	3	1	1.5	1.1	1.6	
23	Tank reactor	7500	1	2	1	2.3	2.1	52.9	
24	Steam generator	9600	4	4	1	2.4	3.1	3.3	

## Comparison of the execution time scaled by $M(w, \hat{B}_{subid})$ .

Destillation column   90   5   3   1   33   64   11.1	comparison of the execution time scaled by M (at, Dsubla).									
Destillation column   90   5   3   33   64   111			parameters				scaled exec. time			
2 Destillation column n10						l	detss	stls		
3 Destillation column n20	1	Destillation column	90	5	3	1	3.3	6.4	11.1	
A Destillation column n30	2	Destillation column n10	90	5	3	1	7.3	12.5	23.1	
5 Glass furnace (Philips)         1247         3 6 1         13.5 361.2 373.3         6 12 0 373.3         6 12 0 373.3         6 12 0 373.3         6 12 0 373.3         6 12 0 373.3         2 6.3 15.5 273.3         7 14 5 273.3         2 16 2 1 5 2 73.3         8 Hair dryer         1000 1 1 5 5 15.5 5.8 36.4         9 Winding process         2500 5 2 2 4 4.4 37.1 74.8         1000 1 1 2 19.9 4.1 7.2         1 1.5 15.5 5.8 36.4         3 3 1 6.6 25.5 27.3         1 1 1 2 19.9 4.1 7.2         1 1 74.1 74.8         1 1 1 2 19.9 4.1 7.2         1 1 7 4.7 33.5         1 1 7 4.7 33.5         1 1 7 4.7 3.3         3 0.7 17 4.7 3.3         1 1 8 3.8 30.7         1 1 4 6.1 19.5 4 94.1         1 1 1 2 1 4 1.8 3.8 30.7         3 0.7 17 4.7 3.5         1 1 4 6.7 1.4 4 6.1 1.4         1 1 1 2 1 4 1.8 3.8 30.7         1 1 1 2 1 4 1.8 3.8 30.7         1 1 1 2 1 4 1.8 3.8 30.7         1 1 1 2 1 4 1.8 3.8 30.7         1 1 1 2 1 4 1.8 3.8 30.7         1 1 1 2 1 4 1.8 3.8 30.7         1 1 1 2 1 4 1.8 3.8 30.7         1 1 1 2 1 4 1.8 3.8 30.7         1 1 1 2 1 4 1.8 3.8 30.7         1 1 1 2 1 4 1.8 3.8 30.7         1 1 1 2 1 4 1.8 3.8 30.7         1 1 1 2 1 4 1.8 3.8 30.7         1 1 1 2 1 4 1.8 3.8 30.7         1 1 1 2 1 4 1.8 3.8 30.7         1 1 1 2 1 4 1.8 3.8 30.7         1 1 1 2 1 3 1 3 1 3 1 3 1 3 1 3 1 3 1 3	3	Destillation column n20	90	5	3	1	7.2	12.8		
6 120 MW power plant 7 pH process 8 Hair dryer 9 Winding process 12 000 1 2 1 6 2.9 7.4 32.3 8 Hair dryer 19 Winding process 12 000 5 2 2 4.4 37.1 74.8 10 Ball-and-beam setup 11 Industrial dryer 12 CD-player arm 12 Wing flutter 12 Lize brie 20 13 Wing flutter 1024 1 1 1 5 1.7 4.7 33.1 14 Robot arm 1024 1 1 4 18 3.8 30.7 15 Lake Erie 10 16 Lake Erie 10 17 Lake Erie 10 18 Lake Erie 6 5 7 5 2 1 1.4 4.6 11.4 17 Lake Erie 10 19 Heating system 19 Heating system 19 Heating system 22 Industrial evaporator 22 Industrial evaporator 33 1 1 105 599 131.4 23 Tank reactor 7500 1 2 1 1.1 1.0 25.2 146.0	4	Destillation column n30	90	5	3	1	7.0	12.1	7.2	
T pH process	5	Glass furnace (Philips)	1247	3	6	1	13.5	361.2	373.3	
8 Hair dryer         1000         1 1 5         1.5         5.8         36.4           9 Winding process         2500         5 2 2         4.4         37.1         74.8           10 Ball-and-beam setup         11 Industrial dryer         867         3 3 1         6.6         25.5         27.3           11 Industrial dryer         2048         2 2 1         1.7         4.7         33.5           13 Wing flutter         1024         1 1 5         1.7         4.7         33.5           14 Robot arm         1024         1 1 4         1.8         3.8         30.7           15 Lake Erie         57         5 2 1         1.4         4.6         1.0           16 Lake Erie n10         57         5 2 1         1.4         4.6         1.1           18 Lake Erie n20         57         5 2 1         1.7         4.8         9.1           19 Heat flow density         1680         2 1 2         2.6         6.3         39.7           20 Heating system         24         1 1 2         4.3         8.4         31.1           21 Industrial evaporator         603         3 3 1         10.5         5.99         13.4           23 Tank reactor         7500 <td>6</td> <td>120 MW power plant</td> <td>200</td> <td>5</td> <td>3</td> <td>2</td> <td>6.3</td> <td>15.5</td> <td>27.3</td>	6	120 MW power plant	200	5	3	2	6.3	15.5	27.3	
9 Winding process         2500         5 2         2         4.4         37.1         7.48           10 Ball-and-beam setup         1000         1 1 2         1.9         4.1         7.2           11 Industrial dryer         867         3 3 1         66         25.5         27.3           12 CD-player arm         2048         2 2 1         64         19.5         49.4           13 Wing flutter         1024         1 1 4         18         3.8         30.7           14 Robot arm         1024         1 1 4         18         3.8         30.7           15 Lake Erie         57         5 2 1         1.4         4.6         7.0           16 Lake Erie n20         57         5 2 1         1.4         4.6         11.4           12 Lake Erie n30         57         5 2 1         1.6         4.8         9.1           19 Heat flow density         1680         2 1 2         2.6         6.3         39.7           20 Heating system         20         1 1 2         4.3         8.4         31.1           21 Tank reactor         700         1 2 1         1.10         2 9.9         13.4         2.5         14.6	7	pH process	2001	2	1	6	2.9	7.4	32.3	
10 Ball-and-beam setup   1000   11   2   199   41   72   11   11   11   12   199   41   72   12   12   12   12   14   14   15   14   13   13   16   16   15   14   13   13   13   16   16   15   14   13   13   13   15   14   15   14   14   15   14   15   14   15   14   15   15	8	Hair dryer	1000	1	1	5	1.5	5.8	36.4	
11 Industrial dryer	9	Winding process	2500	5	2	2	4.4	37.1	74.8	
12 CD-player arm   2048   2 2 1   6.4   19.5   49.4   19.5   17.   47   33.5   14   Robot arm   1024   1   1 5   17.   4.7   33.5   14   Robot arm   1024   1   1 4   18   3.8   30.7   15   Lake Eric 10   57   5 2   1   1.4   4.6   11.4   17   Lake Eric n20   57   5 2   1   1.6   4.8   9.1   18   Lake Eric n30   57   5 2   1   1.7   4.8   7.0   19   Heat flow density   1800   2   1 2   2.6   6.3   39.7   20   Heating system   801   1   1   2   1.7   3.7   12.4   21   22   Industrial evaporator   305   3 31   105   599   134.4   32   Tank reactor   305   1   21   11.0   252   146.0   32   Tank reactor   305   1   21   11.0   252   14.6   32   33   34   34   34   34   34   34	10	Ball-and-beam setup	1000	1	1	2	1.9	4.1	7.2	
13 Wing flutter         1024         1 1 1 5         1.7         4.7         33.5           14 Robot arm         1024         1 1 4         1.8         38         30.7           15 Lake Erie         5 7 5 2 1         1.4         4.6         1.7           16 Lake Erie n10         57 5 2 1         1.4         4.6         1.1           18 Lake Erie n20         57 5 2 1         1.7         4.8         9.1           19 Heat flow density         1680         2 1 2         2.6         3.9.7           20 Heating system         21         1 1 2         1.7         3.7         12.4           21 Industrial evaporator         630         3 3 3         3.3         3.8         3.1         3.5         5.99         3.4         3.1         3.5         5.99         3.4         3.1         3.5         5.99         3.4         3.1         3.5         5.99         3.4         3.1         3.5         3.2         3.3         3.3         3.3         3.3         3.3         3.3         3.3         3.3         3.3         3.5         3.8         3.2         3.8         3.2         3.3         3.3         3.3         3.3         3.3         3.3         3.3         3	11	Industrial dryer	867	3	3	1	6.6	25.5	27.3	
14 Robot arm         1024         1 1 4         1.8         3.8         30.7           15 Lake Erie 10         57         5 2 1         1.4         46         7.0           16 Lake Erie n20         57         5 2 1         1.6         4.8         9.1           15 Lake Erie n30         57         5 2 1         1.6         4.8         9.1           19 Heat flow density         1800         2 1 2         2.6         6.3         39.7           20 Heating system         801         1 1 2         2 2         6.3         34.         31.           22 Industrial evaporator         600         1 2 1         2 4.3         8.4         31.           23 Tank reactor         700         1 2 1         11.0         252         146.	12	CD-player arm	2048	2	2	1	6.4	19.5	49.4	
15 Lake Erie         57 5 2 1         1.4 4.6 7.0           16 Lake Erie n10         57 5 2 1         1.4 4.6 11.4           17 Lake Erie n20         57 5 2 1         1.6 4.8 9.1           18 Lake Erie n30         57 5 2 1         1.7 4.8 7.0           19 Heat flow density         1680 2 1 2         2.6 6.3 39.7           20 Heating system         801 1 1 2         1.7 3.7 12.4           21 Steam heat exchanger         4000 1 1 2         1.4 3 8.4 31.1           22 Industrial evaporator         6305 3 3 1         10.5 59.9 134.4           23 Tank reactor         7500 1 2 1         11.0 25.2 146.0	13	Wing flutter	1024	1	1	5	1.7	4.7	33.5	
16 Lake Erie n10     57     5 2     1     1.4     4.6     1.4       17 Lake Erie n20     57     5 2     1     1.6     4.8     9.1       18 Lake Erie n30     57     5 2     1     1.7     4.8     7.0       19 Heat flow density     1680     2     1.2     2.6     6.3     39.7       20 Heating system     80     1     1     2     3     4.4     3.1       21 Steam heat exchanger     4000     1     1     2     4.3     8.4     3.1       22 Indivistrial evaporator     7500     1     2     1     1.0     2     2.1     1.0     2     2.1     1.0     2     2.1     1.0     2     2.1     1.0     2     2.1     1.0     2     2.1     1.0     2     2.1     1.0     2     2.1     1.0     2     2.1     1.0     2     2.1     1.0     2     2.1     1.0     2     2     2.1     1.0     2     2     2     1.0     2     2     2     2     2     2     2     2     3     3     3     3     3     3     3     3     3     3     3     3     3     3     3	14	Robot arm	1024	1	1	4	1.8	3.8	30.7	
17 Lake Erie n20     57     5 2 1     1.6     4.8     9.1       18 Lake Erie n30     57     5 2 1     1.7     4.8     7.0       19 Heat flow density     180     2 1 2     2.6     6.3     3.7     12.4       20 Heating system     801     1 1 2     1.7     3.7     12.4       21 Steam heat exchanger     400     1 1 2     4.3     8.4     3.1       22 Industrial evaporator     630     3 3 1     10.5     59.9     13.4       23 Tank reactor     7500     1 2 1     11.0     25.2     1460	15	Lake Erie	57	5	2	1	1.4	4.6	7.0	
18 Lake Erie n30     57     5     2     1     1.7     4.8     7.0       19 Heat flow density     1680     2     1.2     2.6     6.3     39.7       20 Heating system     80     1     1     2     1.7     3.7     12.4       21 Steam heat exchanger     4000     1     1     1     2     4.3     3.4     3.1       22 Industrial evaporator     7500     1     2     1     1.0     2     2     1.0	16	Lake Erie n10	57	5	2	1	1.4	4.6	11.4	
19 Heat flow density     1680     2 1 2     2.6     6.3     39.7       20 Heating system     801     1 1 2     1.7     37     12.4       21 Steam heat exchanger     400     1 1 2     4.3     8.4     31.1       22 Industrial evaporator     6305     3 3 1     10.5     5.99     134.4       23 Tank reactor     7500     1 2 1     11.0     25.2     146.0	17	Lake Erie n20	57	5	2	1	1.6	4.8	9.1	
20 Heating system     801     1     1     2     1.7     3.7     12.4       21 Steam heat exchanger     4000     1     1     2     4.3     8.4     31.1       22 Industrial evaporator     6305     3     3     1     10.5     59.9     134.4       23 Tank reactor     7500     1     2     1     11.0     25.2     146.0	18	Lake Erie n30	57	5	2	1	1.7	4.8	7.0	
21 Steam heat exchanger     4000     1     1     2     4.3     8.4     31.1       22 Industrial evaporator     6305     3     3     1     10.5     59.9     134.4       23 Tank reactor     7500     1     2     1     11.0     25.2     146.0	19	Heat flow density	1680	2	1	2	2.6	6.3	39.7	
22 Industrial evaporator 6305 3 3 1 10.5 59.9 134.4 23 Tank reactor 7500 1 2 1 11.0 25.2 146.0	20	Heating system	801	1	1	2	1.7	3.7	12.4	
23 Tank reactor 7500 1 2 1 11.0 25.2 146.0	21	Steam heat exchanger	4000	1	1	2	4.3	8.4	31.1	
	22	Industrial evaporator	6305	3	3	1	10.5	59.9	134.4	
24 Steam generator 9600 4 4 1 13.6 192.0 220.1	23	Tank reactor	7500	1	2	1	11.0	25.2	146.0	
	24	Steam generator	9600	4	4	1	13.6	192.0	220.1	

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