# Exercise 1: Misfit computation

- given data $w_\mathrm{d}$ and an LTI system $\mathscr{B}$, represented by
  - image $\big(P(\sigma)\big)$
  - $\mathscr{B}(A,B,C,D)$
- explain how to compute misfit$(w_\mathrm{d}, \mathscr{B})$ in 2-norm
- *i.e.*, find the orthogonal projection of $w_\mathrm{d}$ on $\mathscr{B}$
- HW: misfit computation using ker $\big(R(\sigma)\big)$

# $w \overset{?}{\in} \operatorname{image}\big(P(\sigma)\big)$

$\iff$ there is $v$, such that $w = P(\sigma)v$

$\iff$ there is $v$, such that for $t = 1, \dots, T$

$$w(t) = P_0 v(t) + P_1 v(t+1) + \cdots + P_\ell v(t+\ell)$$

$\iff$ there is solution $v$ of the system

$$\begin{bmatrix} w(1) \\ w(2) \\ \vdots \\ w(T) \end{bmatrix} = \underbrace{\begin{bmatrix} P_0 & P_1 & \cdots & P_\ell & & & \\ & P_0 & P_1 & \cdots & P_\ell & & \\ & & \ddots & \ddots & & \ddots & \\ & & & P_0 & P_1 & \cdots & P_\ell \end{bmatrix}}_{\mathscr{M}_{T+\ell}(P)} \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(T+\ell) \end{bmatrix}$$

# Using image $(P(\sigma))$

- we showed that

$$\widehat{w} \in \ker\big(R(\sigma)\big) \quad \iff \quad \widehat{w} = \mathscr{M}_T(P)v, \text{ for some } v$$

- then the misfit computation problem

$$\text{misfit}(w_\text{d}, \mathscr{B}) := \min_{\widehat{w} \in \mathscr{B}} \|w_\text{d} - \widehat{w}\|$$

becomes

$$\text{minimize} \quad \text{over } v \quad \|w_\text{d} - \mathscr{M}_T(P)v\|$$

- with $\|\cdot\| = \|\cdot\|_2$, the problem is standard least-norm
- projector on $\mathscr{B} = \text{image}(P)$

$$\Pi_{\text{image}(P)} := \mathscr{M}_T(P)\big(\mathscr{M}_T^\top(P)\mathscr{M}_T(P)\big)^{-1}\mathscr{M}_T^\top(P)$$

- misfit

$$\text{misfit}(w_\text{d}, \mathscr{B}) := \sqrt{w_\text{d}{}^\top\big(I - \Pi_{\text{image}(P)}\big)w_\text{d}}$$

and optimal approximation

$$\widehat{w} = \Pi_{\text{image}(P)} w_\text{d}$$

## $w \overset{?}{\in} \mathscr{B}(A,B,C,D)$

$$\mathscr{B}(A,B,C,D) = \{\, (u,y) \mid \sigma x = Ax + Bu,\; y = Cx + Du \,\}$$

$$(u_{\mathrm{d}},y_{\mathrm{d}}) \in \mathscr{B}(A,B,C,D) \quad \iff \quad \exists\, x_{\mathrm{ini}} \in \mathbb{R}^{\mathrm{n}}, \text{ such that}$$

$$y = \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{T-1} \end{bmatrix}}_{\mathscr{O}_T(A,C)} x_{\mathrm{ini}} + \begin{bmatrix} D & & & & \\ CB & D & & & \\ CAB & CB & D & & \\ \vdots & \ddots & \ddots & \ddots & \\ CA^{T-1}B & \cdots & CAB & CB & D \end{bmatrix} u$$

# Using $\mathscr{B}(A,B,C,D)$

- we showed that

$$\widehat{w} \in \mathscr{B}(A,B,C,D) \iff \widehat{y} = \mathscr{O}_T(A,C)\widehat{x}_{\text{ini}} + \mathscr{T}_T(H)\widehat{u}$$

- then the misfit computation problem

$$\min_{\widehat{x}_{\text{ini}},\widehat{u}} \left\| \begin{bmatrix} u_{\text{d}} \\ y_{\text{d}} \end{bmatrix} - \begin{bmatrix} 0 & I \\ \mathscr{O}_T(A,C) & \mathscr{T}_T(H) \end{bmatrix} \begin{bmatrix} \widehat{x}_{\text{ini}} \\ \widehat{u} \end{bmatrix} \right\|$$

# Exercise 2: Latency computation

- given data $w_\mathrm{d}$ and an LTI system $\mathscr{B} = \ker\big(R(\sigma)\big)$
- explain how to compute latency$(w_\mathrm{d}, \mathscr{B})$ in 2-norm
- HW: latency computation using $\mathscr{B}(A, B, C, D)$

- partition $R = \begin{bmatrix} R_e & R_w \end{bmatrix}$ conformably with $w_{\text{ext}} = \left[\begin{smallmatrix} e \\ w \end{smallmatrix}\right]$
- by analogy with the derivation on page 2, we have

$$\begin{bmatrix} e \\ w \end{bmatrix} \in \ker\big(R(\sigma)\big) \iff \begin{bmatrix} \mathscr{M}_T(R_e) & \mathscr{M}_T(R_w) \end{bmatrix} \begin{bmatrix} e \\ w \end{bmatrix} = 0$$

- the latency computation problem is

$$\min_{e} \quad \|e\|_2 \quad \text{subject to} \quad \mathscr{M}_T(R_e)e = -\mathscr{M}_T(R_w)w$$

- the solution is given by

$$\widehat{e} = -\big(\mathscr{M}_T(R_e)^\top \mathscr{M}_T(R_e)\big)^{-1} \mathscr{M}_T(R_e)^\top \mathscr{M}_T(R_w)w$$