Exam questions for "Signal theory: Part 1"

Work alone. You can not use printed materials and electronic devices. Time allowed: 40 minutes.

1 Prediction using a model (4 points)

1.1 *State space approach* (2 points)

Given a state space representation of a discrete-time autonomous system $\mathscr{B}(A,C)$ of order $\mathtt{n}$ and a finite, $T \geq \mathtt{n}$ samples long, trajectory

$$y_{\mathrm{p}} := \big(y(1),\ldots,y(T)\big)$$

of that system, find the next T_{f} samples

$$y_{\mathrm{f}} := \big(y(T+1),\ldots,y(T+T_{\mathrm{f}})\big)$$

of the given trajectory, *i.e.*, find $y(T+1),\ldots,y(T+T_{\mathrm{f}})$ such that

$$y := \big(y(1),\ldots,y(T),y(T+1),\ldots,y(T+T_{\mathrm{f}})\big)$$

is a trajectory of $\mathscr{B}(A,C)$.

SOLUTION

A trajectory y of an autonomous system $\mathscr{B}(A,C)$ is completely specified by an initial condition $x(1)$, so the problem of predicting the future part y_{f} of the trajectory y from its given past y_{p} is equivalent to the problem of determining the initial condition $x(1)$ of y from y_{p}.

From the general expression of a response of an autonomous system

$$y(t_1) = CA^{t_1-t_2}x(t_2)$$

we have a system of equations for the unknown initial condition $x(1)$

$$\underbrace{\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(T) \end{bmatrix}}_{y_{\mathrm{p}}} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{T-1} \end{bmatrix}}_{\mathscr{O}_{\mathrm{p}}} x(1). \tag{1}$$

In order to be able to determine $x(1)$ uniquely from y_{p}, the matrix $\mathscr{O}_{\mathrm{p}}$ should have full column rank. Note that $\mathscr{O}_{\mathrm{p}}$ is $T\mathtt{p} \times \mathtt{n}$, where $\mathtt{p} = \operatorname{row dim}(C)$ is the number of outputs. Under the assumption $T \geq \mathtt{n}$ (*i.e.*, "enough data is given") the matrix $\mathscr{O}_{\mathrm{p}}$ has the right dimension for being full column rank.

The condition $T\mathtt{p} \geq \mathtt{n}$ is necessary but not sufficient for $\mathscr{O}_{\mathrm{p}}$ to be full column rank. The extended observability matrix $\mathscr{O}_{\mathrm{p}}$ depends on the system parameters A and C, so an extra condition is needed on the matrices A and C, *i.e.*, on the state space representation of the system. This condition is so important that it is given the name *observability*.

Provided that we have enough data $T \geq \mathtt{n}/\mathtt{p}$ and the representation $\mathscr{B}(A,C)$ is observable, we can uniquely determine $x(1)$ from y_{f} via

$$x(1) = (\mathscr{O}_{\mathrm{p}}^{\top}\mathscr{O}_{\mathrm{p}})^{-1}\mathscr{O}_{\mathrm{p}}y_{\mathrm{p}}.$$

Then we predict y_f using $y(t) = CA^{t-1}x(1)$:

$$\underbrace{\begin{bmatrix} y(T+1) \\ \vdots \\ y(T_{\mathrm{f}}) \end{bmatrix}}_{y_{\mathrm{f}}} = \underbrace{\begin{bmatrix} CA^{T} \\ \vdots \\ CA^{T+T_{\mathrm{f}}-1} \end{bmatrix}}_{\mathscr{O}_{\mathrm{f}}} x(1).$$

The final answer is

$$y_{\mathrm{f}} = \mathscr{O}_{\mathrm{f}}(\mathscr{O}_{\mathrm{p}}^{\top}\mathscr{O}_{\mathrm{p}})^{-1}\mathscr{O}_{\mathrm{p}}y_{\mathrm{p}} = \mathscr{O}_{\mathrm{f}}\left(\sum_{\tau=0}^{T-1} CA^{\tau}(A^{\tau})^{\top}C^{\top}\right)^{-1}\mathscr{O}_{\mathrm{p}}y_{\mathrm{p}}.$$

1.2 *Polynomial approach* (2 points)

Solve the problem of 1.1 using a polynomial representation of the system $\mathscr{B}(P) = \mathscr{B}(A,C)$. Assume that the highest power coefficient of P is I.

SOLUTION

In this case we consider the polynomial representation

$$P_0 y(t) + P_1 y(t+1) + \cdots + P_{\ell-1} y(t+\ell-1) + y(t+\ell) = 0, \qquad \text{for all } t \in \mathbb{Z}.$$

Because of the assumption that the highest power coefficient P_ℓ is I, we can find for each t, $y(t)$ as a linear combination of $y(t-1), \dots, y(t-\ell)$

$$y(t) = -\big(P_0 y(t-\ell) + P_1 y(t-\ell+1) + \cdots + P_{\ell-1} y(t-1)\big). \tag{2}$$

Assuming that $T \geq \ell$, we can apply this formula recursively and "extend" y_p to y_f, *i.e.*, we simulate the response y_f corresponding to the initial conditions $y(T), y(T-1) \dots, y(T-\ell+1)$, which is the end part of y_f. For this to be possible, we need $T \geq \ell$. It can be shown that this condition follows from the assumption $T \geq \mathtt{n}$.

2 Wiener-Khintchine theorem (3 points)

For a discrete-time signal y, let

- $\phi_y := |F(y)|^2$, where $F(y)$ be a Fourier transform of y, and
- $r_y := \sum_{t=1}^{T} y(t) y(t-\tau)$.

Show that $\phi_y = F(r_y)$.

SOLUTION

The proof

$$\phi_y = F(y)F^*(y) = F(y)F\big(\mathrm{rev}(y)\big) = F(y \star \mathrm{rev}(y)) = F(r_y)$$

is based on the following properties of the Fourier transform

- $F(y \star y) = F(y)F(y)$,
- $F\big(\mathrm{rev}(y)\big) = F^*(y)$,
- $y \star \mathrm{rev}(y) = r_y$.

3 Weighted least-squares approximate solution (3 points)

For a given positive definite matrix $W \in \mathbb{R}^{m \times m}$, define the weighted 2-norm

$$\|e\|_W = e^\top W e.$$

The weighted least-squares approximation problem is

$$\text{minimize} \quad \text{over } \widehat{x} \in \mathbb{R}^n \quad \|A\widehat{x} - b\|_W. \tag{WLS}$$

When does a solution exist and when is it unique? Under the assumptions of existence and uniqueness, derive a closed form expression for the least squares approximate solution.

SOLUTION

Since W is a symmetric positive definite matrix, it has a factorization $W = CC^\top$, where C is an $m \times m$ full rank matrix. We can re-write the weighted least-squares approximation problem as an equivalent standard least-squares approximation problem for a system of linear equations $A'x = b'$, where

$$A' = CA \qquad \text{and} \qquad b' = Cb.$$

At this point we can use existing results: 1) a solution always exists, 2) it is unique if and only if the matrix is full column rank (f.c.r.). Since C is full rank, A' is f.c.r. if and only if A is f.c.r. In this case the unique weighted least-squares approximate solution is

$$\widehat{x} = (A^\top W A)^{-1} A^\top W b.$$