Lectures notes "Signal theory: Part 1"

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1 Introduction

General information about the course

- **Lecturers:**
  - Leo Van Biesen (part 2, weeks 5, 6, 7)
  - Ivan Markovsky (part 1, weeks 2, 3, 4)
- **Homework for part 1:**
  - on Tuesday due Friday, the same week
  - on Friday due Tuesday, the week after
- **MATLAB:**
  - homework involve reading assignment, exercise problems, and numerical experiments with MATLAB
  - if you are not familiar with MATLAB, follow the optional course given by Peter Zoltan
- **Evaluation:**
  - 25% homework
  - 75% open-book exam on problems similar to the exercises

Part 1 topics

1. Signals and systems (mostly review from a new perspective)
   - expansion of a signal in a basis; orthonormal basis
   - linear time-invariant systems; behavioral approach
   - representations of LTI systems
     - convolution
     - differential/difference equation
     - transfer function
     - state-space representation
   - the realization problem
2. Random signals
   - covariance function and spectrum
   - Wiener-Khintchine theorem
   - stochastic systems
   - stochastic realization

3. Least-squares estimation
    - underdetermined and overdetermined systems of linear equations
    - least-norm solution and least-squares approximation
    - recursive least-squares approximation
    - Kalman filtering
4. Subspace methods
    - deterministic subspace algorithms
    - stochastic subspace algorithms
5. Compressive sensing
    - ℓ_0 vs ℓ_1-norm minimization
    - LASSO

Teaching materials for Part 1

- in pointcarre: `notes`, homework, lecture slides
- library references

Demo noise filtering

Demo singular spectrum analysis

2 Signals and systems

Classification of signals

- examples of "physical" signals
    - sound (speech, music, noise, ...)
    - image (black/white, gray scale — integer 0–255, color)
    - video (sequence of images)
    - daily exchange rates of one currency into another
- abstract representation as a function
    - notation $f : \mathscr{X} \to \mathscr{Y}$ (function from $\mathscr{X}$ to $\mathscr{Y}$)
        - $\mathscr{X}$ — domain (where the function argument, say x, takes its values)
        - $\mathscr{Y}$ — image (where the function values belong)
    - $y = f(x)$ — value of the function at the point $x \in \mathscr{X}$
    - note that the function f is not defined for values of x outside the domain, *i.e.*, for $x \notin \mathscr{X}$
- scalar (single-channel) vs vector (multi-channel)
- real-valued vs complex-valued
- continuous/analog (the domain is $\mathbb{R}$) vs discrete/sampled (the domain is $\mathbb{Z}$)
- periodic vs non-periodic
- one-dimensional (*e.g.*, sound) vs multi-dimensional (*e.g.*, image and video)

Basic signals and operations with them

- Basic signals
  - Dirak (continuous-time) and Kroneker (discrete-time) delta

$$\delta(t) = \begin{cases} 1, & t = 0 \\ 0, & \text{otherwise} \end{cases}$$

  - unit step

$$s(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

  - constant (or DC) signal: a, where $a \in \mathbb{R}$
  - rectangular pulse:

$$p_{t_1,t_2}(t) = \begin{cases} 1, & t_1 \leq t \leq t_2 \\ 0, & \text{otherwise} \end{cases}$$

    (Represent the pulse p_{t_1,t_2} by a linear combination of shifted steps s.)
  - sinusoidal signal: $a\cos(\omega t + \phi)$
  - complex exponentials

$$e^{(\alpha+\beta t)+\mathbf{i}(\omega t+\phi)} = e^{\alpha} e^{\beta t}\big(\cos(\omega t + \phi) + \mathbf{i}\sin(\omega t + \phi)\big)$$

$$\begin{array}{lll} A := e^{\alpha} & \text{—} & \text{amplitude} \\ \beta & \text{—} & \text{damping coefficient} \\ \omega & \text{—} & \text{frequency} \\ \phi & \text{—} & \text{initial phase} \\ \omega t + \phi & \text{—} & \text{instantaneous phase} \end{array}$$

    (link to a response of a second order system)

- Signal transformations
  - addition $x + y$
  - multiplication (modulation) xy
  - amplitude scaling ax, $a \in \mathbb{R}$
    * $|a| > 1$ — gain factor (ideal linear amplifier)
    * $|a| < 1$ — attenuation factor
  - time scaling $x(at)$, $a \in \mathbb{R}$
    (Exercise: sketch by hand and plot with MATLAB)
  - shift in time

$$x(t+\tau) =: (\sigma^{\tau} x)(t), \quad \tau \in \mathbb{R}$$

  - sampling (discretization in time, analog to digital conversion)
    * uniform sampling

$$\text{sample}_{t_s} : x_c \mapsto x_d, \quad x_d(t) = x_c(t_s t), \quad t \in \mathbb{Z}$$

      · t_s — sampling time
    * loss of information (aliasing)
    * Nyquist–Shannon sampling theorem (for band limited signals, critical frequency)

* reconstruction of the continuous-time signal from the discrete-time samples via sinc interpolation is a convolution of of x_{d} with sinc function

$$f_{\mathrm{c}}(t) = \left(\sum_{\tau=-\infty}^{\infty} f_{\mathrm{d}}(\tau)\delta(t - \tau t_{\mathrm{s}}) \right) \star \mathrm{sinc}(t)$$

* non-uniform sampling (sparsity, compressive sampling)

- interpolation $x_{\mathrm{d}} \mapsto x_{\mathrm{c}}$
  * digital to analog conversion
  * zero order hold
  * linear interpolation
  * polynomial interpolation
  * sinc interpolation (for band limited signals)
- extrapolation: predict the signal in the future
- quantization (discretization in value) $\mathrm{quant}_{\Delta} : x_{\mathrm{c}} \mapsto x_{\mathrm{q}},\ x_{\mathrm{q}}(t) = \left\lceil \frac{x_{\mathrm{c}}(t)}{\Delta} \right\rceil \Delta$
  * Δ — quantization level
  * loss of information
  * "quantization noise"

• Signal expansion in a basis
  - spaces and subspaces
  - basis of a subspace

• Transform techniques
  - integral/sum of Dirak/Kroneker deltas

$$x(t) = \cdots + x(-1)\delta(t+1) + x(0)\delta(t) + x(1)\delta(t-1) + \cdots = \sum_{\tau=-\infty}^{\infty} x(\tau)\delta(t-\tau) \qquad (\delta\text{-train})$$

  No computation required!
  - (discrete) Fourier transform

$$x(t) = \frac{1}{T}\sum_{k=0}^{T-1} X(k)e^{\mathbf{i}\frac{2\pi k}{T}t} \qquad \text{(DFT)}$$

  Computation is required.

• Measuring the size of a signal (x with domain $[0,\infty)$)
  - total energy: $\int_0^{\infty} x^2(\tau)\,d\tau$
  - peak value: $\max_{t\geq 0} |x(t)|$
  - root-mean-square (RMS) value

$$\sqrt{\lim_{t\to\infty} \frac{1}{t} \int_0^t x^2(\tau)\,d\tau}$$

Classification of systems

- examples of "physical" systems
  - electrical circuits
  - chemical processes
  - stock markets
  - the solar system
  - behavioral approach
    - manifest variables w
    - the universal set $\mathscr{U}$
    - behavior: the set of (allowed) signals $\mathscr{B}$; the system is a prohibition rule
    - example: the solar system, the prohibition rule are Kepler's laws
      (gravitational law + Newton's laws $\leftrightarrow$ Kepler's laws $\leftarrow$ observations)
  - a "signal processor" view of a system: map from an input signal to an output signal (an operator)
    - notation: $y = S(u)$

$$\begin{array}{ccl} u & — & \text{input} \\ y & — & \text{output} \\ S & — & \text{system} \end{array}$$

    - block diagram
      - building complicated systems with interconnections (summation and multiplication)
    - examples
      - scaling system: $y(t) = au(t)$, a is called the gain
      - amplifier if $|a| > 1$
      - attenuator if $|a| < 1$
      - inverter if $a < 0$
      - differentiator: $y(t) = u'(t)$
      - integrator: $y(t) = \int_0^t u(\tau)d\tau$
      - time delay: $y(t) = u(t - T)$
      - convolution system: $y(t) = \int u(t-\tau)h(\tau)d\tau$, h is a given function

- static (memory-less) vs dynamic (with memory)
- causal vs non-causal
- linear vs nonlinear
  - classical definition of a linear system:
    1. homogeneity: $S(au) = aS(u)$ (scaling before or after is the same, illustrate in block diagram)
    2. superposition: $S(u_1 + u_2) = S(u_1) + S(u_2)$ (summing before or after is the same, illustrate)
- time-invariant vs time-varying
- inputs and outputs (filters)
- response of a system

$$y = S(u) = S\left(\sum_{\tau=-\infty}^{\infty} x(\tau)\delta(t-\tau)\right) = \sum_{\tau=-\infty}^{\infty} x(\tau)S\big(\delta(t-\tau)\big) = \sum_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)$$

Homework **HW**

Optional reading assignments

- notes Leo part 1, sections 1.1–4
- section 1.1 (classification of signals), 1.2 (classification of systems), and chapter 2 (representation of signals and systems) from A. Oppenheim and A. Willsky, *Signals and Systems*, Prentice Hall, 1996

Problems

- **System classification**
  - Give specific examples of:
    - linear static system
    - nonlinear static system
    - linear time-invariant dynamical systems
      - finite impulse response (FIR)
      - infinite impulse response (IIR)
      - scalar
      - multivariable
    - linear time-varying dynamical systems
    - nonlinear time-invariant dynamical systems
    - nonlinear time-varying dynamical systems
- **Response of an LTI system**
  1. Find analytically the response of 1st and 2nd order continuous-time linear time-invariant autonomous systems.
  2. Write a function that computes the response.
  3. Test the function on a numerical example and compare the result with the one obtained via the function `lsim` from Control Toolbox of Matlab.
  4. Generalize 1–3 for a general nth order linear time-invariant autonomous systems.

3 Representations of LTI systems

Review of matrix algebra

Representation of static systems

- function vs relation
- image representation
- kernel representation
- input/output representation