1. *Step function output* (1 points)
   Write down a representation of an autonomous linear time-invariant system that can give a step function output.

   Solution: The continuous-time system, defined by

   $$\frac{\mathrm{d}}{\mathrm{d}t}y = 0$$

   and the discrete-time system, defined by

   $$\sigma y = y$$

   have constant output $y(0)e^{0t}$ and $y(0)1^t$, for $t \geq 0$. □

2. *Sine function output* (1 points)
   Write down a representation of an autonomous LTI system that can give a sine with frequency ω output. #+BEGIN_LaTeX ´=´}{

   Solution: A sine with frequency ω can be written as $ce^{\mathbf{i}\omega t} + \bar{c}e^{-\mathbf{i}\omega t}$, for some $c \in \mathbb{C}$. For the continuous-time system, we have

   $$p(s) = (s - \mathbf{i}\omega)(s + \mathbf{i}\omega) = s^2 + \omega^2 \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}y + \omega^2 y = 0.$$

   For the discrete-time system, we have \[p(z) = (z - e$^{\mathbf{i}\omega}$)(z + e$^{-\mathbf{i}\omega}$) = z^2 -2cos(ω)z + 1 $\quad\Longrightarrow\quad$ σ^2 y - 2cos(ω)

3. $y \overset{?}{\in} \mathscr{B}(A,C)$ (2 points)
   Check if $y_\mathrm{d} = (2,3,5,9,17,33,64)$ is a possible output of the system defined by the state space representation

   $$x(t+1) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x(t), \quad y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t). \tag{SS}$$

   If y_d is not a possible output of the system, suggest a way of correcting it, so that the corrected signal is.

   Solution: $y_\mathrm{d} = \big(y_\mathrm{d}(1), \dots, y_\mathrm{d}(T)\big) \in \mathscr{B}(A,C)$ if and only if there is $x(0)$, such that $y_\mathrm{d}(t) = CA^t x(0)$, for $t = 1, \dots, T$. Written in a matrix form, this condition is the following system of linear equations

   $$\begin{bmatrix} y_\mathrm{d}(1) \\ y_\mathrm{d}(2) \\ \vdots \\ y_\mathrm{d}(T) \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{T-1} \end{bmatrix}}_{\mathscr{O}} x(0). \tag{$*$}$$

   For the given example, we have

   $$\begin{bmatrix} 2 \\ 3 \\ 5 \\ 9 \\ 17 \\ 33 \\ 64 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 8 \\ 1 & 16 \\ 1 & 32 \\ 1 & 64 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix},$$

   which has no solution, so that $y_\mathrm{d} \notin \mathscr{B}(A,C)$. A way of correcting y_d, so that it becomes an output of $\mathscr{B}(A,C)$, is to solve ($*$) in a least-squares sense, and define the corrected output $\widehat{y} = \mathscr{O}\widehat{x}(0)$, where $\widehat{x}(0) = (\mathscr{O}^\top \mathscr{O})^{-1}\mathscr{O}^\top y_\mathrm{d}$ is the least-squares approximate solution of ($*$). $\widehat{y}$ has the property that $\|y_\mathrm{d} - \widehat{y}\|_2$ is minimized over all outputs $\widehat{y}$ of $\mathscr{B}(A,C)$. □

4. $y \overset{?}{\in} \ker(P(z))$ (2 points)
Check if $y_{\mathrm{d}} = (2,3,5,9,17,33,64)$ is a possible output of the system defined by the difference equation

$$2y(t) - 3y(t+1) + y(t+2) = 0. \tag{KER}$$

If y_{d} is not a possible output of the system, suggest a way of correcting it, so that the corrected signal is.

Solution: $y_{\mathrm{d}} = \big(y_{\mathrm{d}}(1),\dots,y_{\mathrm{d}}(T)\big) \in \ker(P(z))$ if and only if

$$P_0 y(t) + P_1 y(t+1) + \cdots + P_n y(t+n) = 0, \qquad \text{for } t = 1,\dots,T-n.$$

Written in a matrix form, this condition is the following system of linear equations

$$\begin{bmatrix} P_0 & P_1 & \cdots & P_n \end{bmatrix} \begin{bmatrix} y(1) & y(2) & \cdots & y(T-n) \\ y(2) & y(3) & \cdots & y(T-n+1) \\ \vdots & \vdots & \cdots & \vdots \\ y(n+1) & y(n+2) & \cdots & y(T) \end{bmatrix} = 0. \tag{$**$}$$

For the given example, we have

$$\begin{bmatrix} 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 5 & 9 & 17 \\ 3 & 5 & 9 & 17 & 33 \\ 5 & 9 & 17 & 33 & 64 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \end{bmatrix} \neq 0,$$

so that $y_{\mathrm{d}} \notin \ker(P(z))$. An ad hock way of correcting y_{d} in the specific example is to set $y_{\mathrm{d}}(7) = 65$. □

5. $\mathscr{B}(A,C) \overset{?}{=} \ker(P(z))$ (2 points)
Check if the system defined by (SS) is the same system as the one defined by (KER).

Solution: We need to verify that the eigenvalues $z_1 = 1$ and $z_2 = 2$ of A are the same as the roots of $P(z)$. Indeed,

$$(z-1)(z-2) = 2 - 3z + z^2,$$

so that the two representations define the same system. □

6. $\mathscr{B}(A_1,C_1) + \mathscr{B}(A_2,C_2)$ (2 points)
Let $\mathscr{B}$ be the system obtained by adding the outputs of two autonomous LTI systems $\mathscr{B}_1$ and $\mathscr{B}_2$ of orders n_1 and n_2. Is $\mathscr{B}$ linear time-invariant? What is its order?

Solution: $\mathscr{B}$ is linear because the sum of two subspaces is a subspace. $\mathscr{B}$ is also time-invariant because, for any $y \in \mathscr{B}$, it follows that $y = y_1 + y_2$, where $y_1 \in \mathscr{B}_1$ and $y_2 \in \mathscr{B}_2$, but $\sigma^\tau y_1 \in \mathscr{B}_1$ and $\sigma^\tau y_2 \in \mathscr{B}_2$ for all τ, so that

$$\sigma^\tau y = \sigma^\tau y_1 + \sigma^\tau y_2 \in \mathscr{B}.$$

The order of $\mathscr{B}$ is

$$n = n_1 + n_2 - \text{"\# of common poles of } \mathscr{B}_1 \text{ and } \mathscr{B}_2\text{"}.$$

□

7. *Fast method for computing* A^{100} (4 points)
How many scalar multiplications requires the direct computation of A^{100} as $\underbrace{A\cdots A}_{100}$ for a 2×2 matrix A?

Suggest a faster method. Using the method, find a good approximation of $\begin{bmatrix} -1/4 & 1/4 \\ -3/2 & 1 \end{bmatrix}^{100}$.

Solution: The matrix product AB, for $A,B \in \mathbb{R}^{2\times 2}$, requires 8 multiplications, so that A^{100} requires 99×8 multiplications. A fast method is obtained by the eigenvalue decomposition $A = V\Lambda V^{-1}$ of A, because

$$A^{100} = V\Lambda^{100}V^{-1} = V \begin{bmatrix} \lambda_1^{100} & \\ & \lambda_2^{100} \end{bmatrix} V^{-1},$$

requires only $2\times 99 + 12$ multiplications once the eigenvalue decomposition is computed. In the example the eigenvalues of A are $1/2$ and $1/4$. We have $(1/2)^{100} < 10^{-30}$ and $(1/4)^{100} < 10^{-60}$, so that $A^{100} \approx 0$. □

8. A thermometer reading 21°C, which has been inside a house for a long time, is taken outside. After one minute the thermometer reads 15°C; after two minutes it reads 11°C. What is the outside temperature? (According to Newton's law of cooling, an object of higher temperature than its environment cools at a rate that is proportional to the difference in temperature.)

   Solution: Let $y(t)$ be the reading of the thermometer at time t and let $\bar{u}$ be the environmental temperature. From Newton's law of cooling, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}y = a\big(\bar{u}s - y\big)$$

   for some unknown constant $a \in \mathbb{R}$, $a > 0$, which describes the cooling process. Integrating the differential equation, we obtain an explicit formula for y in terms of the constant a, the environmental temperature $\bar{u}$, and the initial condition $y(0)$

$$y(t) = e^{-at}y(0) + (1 - e^{-at})\bar{u}, \qquad \text{for } t \geq 0 \tag{1}$$

   The problem is to find $\bar{u}$ from (1) given that $y(0) = 21$, $y(1) = 15$, and $y(2) = 11$. Substituting the data in (1), we obtain a nonlinear system of two equations in the unknowns $\bar{u}$ and $f := e^{-a}$

$$\left\{ \begin{array}{l} y(1) = fy(0) + (1 - f)\bar{u} \\ y(2) = f^2 y(0) + (1 - f^2)\bar{u} \end{array} \right. \tag{2}$$

   We may stop here and declare that the solution can be computed by a method for solving numerically a general nonlinear system of equations.

   System (2), however, can be solved without using "nonlinear" methods. Define Δy to be the temperature increment from one measurement to the next, *i.e.*, $\Delta y(t) := y(t) - y(t-1)$, for all t. The increments satisfy the homogeneous differential equation $\frac{\mathrm{d}}{\mathrm{d}t}\Delta y(t) = a\Delta y(t)$, so that

$$\Delta y(t+1) = e^{-a}\Delta y(t) \qquad \text{for } t = 0, 1, \ldots \tag{3}$$

   From the given data we evaluate

$$\Delta y(0) = y(1) - y(0) = 15 - 21 = -6, \qquad \Delta y(1) = y(2) - y(1) = 11 - 15 = -4.$$

   Substituting in (3), we find the constant $f = e^{-a} = 2/3$. With f known, the problem of solving (2) in $\bar{u}$ is linear, and the solution is found to be $\bar{u} = 3$°C. $\square$