ELEC 3035, Lecture 9:
Polynomial approach to pole placement

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1. Review of the state space approach
2. Polynomial approach and the Diophantine equation
3. Example

Review of the state space pole placement approach $(A,B,C,D) \mapsto (A_c,B_c,C_c,D_c)$

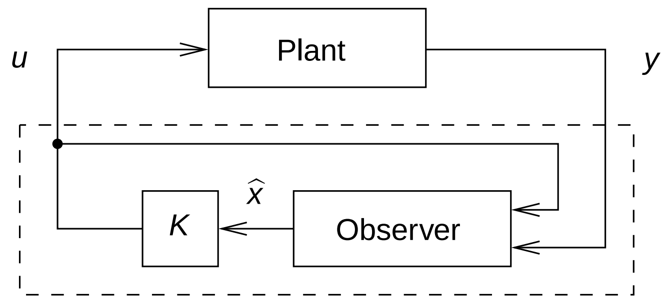
1. State feedback pole placement: compute the controller gain K
2. Pole placement observer design: compute the observer gain L
3. Dynamic output feedback controller:

$$\begin{aligned} \sigma\widehat{x} &= A_c\widehat{x} + B_c y \\ u &= C_c\widehat{x} + D_c y \end{aligned} \qquad \text{where}$$

$$A_c = A + LC + BK + LDK\,, \quad B_c = -L, \quad C_c = K\,, \quad D_c = 0$$

Computational tool: transition to controller/observer canonical forms

Involves computing controllability/observe. matrices and their inverses.



Output feedback controller

Main results for output feedback pole placement of $\mathscr{B}_{\mathrm{i/s/o}}(A,B,C,D)$:

The poles of the closed loop system can be assigned arbitrarily by dynamic output feedback if A,B is controllable and A,C is observable.

Suppose that the plant is given by an I/O representation $\mathscr{B}_{\mathrm{i/o}}(P,Q)$. Then it is natural to ask for an I/O repr. $\mathscr{B}_{\mathrm{i/o}}(P_c,Q_c)$ of the controller.

An approach to obtain a pole placement controller $\mathscr{B}_{\mathrm{i/o}}(P_c,Q_c)$ is:

- Derive a state space repr. $\mathscr{B}_{\mathrm{i/s/o}}(A,B,C,D)$ of $\mathscr{B}_{\mathrm{i/o}}(P,Q)$

$$(P,Q) \mapsto (A,B,C,D)$$

- Apply the state space output feedback pole placement approach

$$(A,B,C,D) \mapsto (A_c,B_c,C_c,D_c)$$

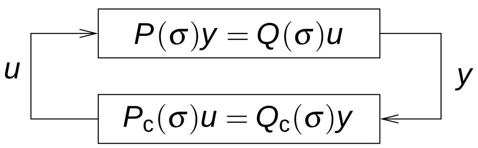
- Derive an I/O repr. $\mathscr{B}_{\mathrm{i/o}}(P_c,Q_c)$ of $\mathscr{B}_{\mathrm{i/s/o}}(A_c,B_c,C_c,D_c)$

$$(A_c,B_c,C_c,D_c) \mapsto (P_c,Q_c)$$

We will study direct approach $(P,Q) \mapsto (P_c,Q_c)$ as an alternative to

$$(P,Q) \mapsto (A,B,C,D) \mapsto (A_c,B_c,C_c,D_c) \mapsto (P_c,Q_c)$$

Polynomial approach to pole placement



Plant: $P(\sigma)y = Q(\sigma)u$

Controller: $P_c(\sigma)u = Q_c(\sigma)y$

The closed-loop system is autonomous. In the SISO case

Closed-loop system: $\big(p_c(\sigma)p(\sigma) - q_c(\sigma)q(\sigma)\big)y = 0$

and the closed-loop characteristic polynomial is

$$p_{cl}(z) := p_c(z)p(z) - q_c(z)q(z)$$

Diophantine equation

For SISO pole placement we need to solve the polynomial equation

$$p_c(z)p(z) - q_c(z)q(z) = p_{des}(z) \qquad \text{(D)}$$

in p_c, q_c with $\text{degree}(p_c) \geq \text{degree}(q_c)$ (for causality of the controller).

Notes:

- p_{des} is the desired char. polynomial of the closed-loop system
- $\underbrace{\text{degree}(p_{des})}_{\text{CL sys's order } n_{cl}} = \underbrace{\text{degree}(p)}_{\text{plant order } n} + \underbrace{\text{degree}(p_c)}_{\text{controller order } n_c}$
- In state space, p_{des} includes plant and observer's desired poles.

The equation (D) is called Diophantine equation (also Bezout eqn).

polynomial $\times$ polynomial $\iff$ Toeplitz matrix $\times$ vector

$$c(z) = a(z)b(z) \quad \iff \quad \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{\ell_c} \end{bmatrix} = \begin{bmatrix} a_0 & & & \\ a_1 & a_0 & & \\ \vdots & a_1 & \ddots & \\ a_{\ell_a} & \vdots & \ddots & a_0 \\ & a_{\ell_a} & & a_1 \\ & & \ddots & \vdots \\ & & & a_{\ell_a} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{\ell_b} \end{bmatrix}$$

$$\iff: \quad c = S_{\ell_b}(a)b \quad \iff \quad c = S_{\ell_a}(b)a$$

polynomial $c(z) \in \mathbb{R}[z]$, $\deg(c) = \ell_c$ $\longleftrightarrow$ vector $c \in \mathbb{R}^{\ell_c+1}$

polynomial operations $\longleftrightarrow$ structured matrix operations

Diophantine equation

With $n_c := \text{degree}(p_c)$ and $m_c := \text{degree}(q_c)$ given,

$$p_c(z)p(z) - q_c(z)q(z) = p_{des}(z)$$

can be written as

$$\begin{bmatrix} S_{n_c}(p) & S_{m_c}(q) \end{bmatrix} \begin{bmatrix} p_c \\ q_c \end{bmatrix} = p_{des} \qquad \text{(D')}$$

where

$$p_c = \text{col}(p_{c,0}, p_{c,1}, \ldots, p_{c,n_c}) \quad , \quad q_c = \text{col}(q_{c,0}, q_{c,1}, \ldots, q_{c,m_c}),$$

$$p_{des} = \text{col}(p_{des,0}, p_{des,1}, \ldots, p_{des,n_{cl}})$$

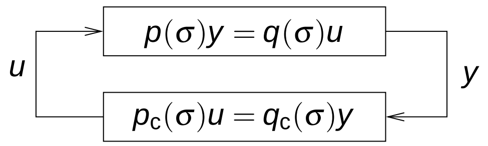
$\implies$ solving (D) (with n_c, m_c given) is a standard linear algebra problem

Main result in polynomial approach for pole placement

Controller $\mathscr{B}_{\mathrm{i/o}}(p_{\mathrm{c}}, q_{\mathrm{c}})$ solving the pole placement problem exists if and only if the Diophantine equation (D) has solution.

Fact: (D) has solution if and only if the GCD of p and q divides p_{des}.

Corollary: The poles of the closed-loop system



can be assigned to arbitrary locations iff $\mathscr{B}_{\mathrm{i/o}}(p, q)$ is controllable.

(Recall that $\mathscr{B}_{\mathrm{i/o}}(p, q)$ is controllable iff p, q have no common factor.)

Main computational problem in the polynomial approach is solving (D).

Example

Consider the plant defined by the difference equation

$$(2 - 3\sigma + \sigma^2)y = (-3 + 2\sigma)u$$

Our goal is to design a feedback deadbeat controller for this plant.

We need to solve the Diophantine equation

$$(2 - 3z + z^2)p_{\mathrm{c}}(z) - (-3 + 2z)q_{\mathrm{c}}(z) = z^{2+n_{\mathrm{c}}} \qquad (*)$$

however, we do not know the controller order n_{c}.

We know that a controller of order $n_{\mathrm{c}} = 2$ exists, however, it turns out that there is a controller of lower order.

Try $n_{\mathrm{c}} = 0$ (static controller). Then $(*)$ can be written (see (D')) as

$$\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} p_{\mathrm{c},0} + \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} q_{\mathrm{c},0} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This system has no solution, so a controller of order 0 does not exist.

Try $n_{\mathrm{c}} = 1$. Then $(*)$ can be written (see (D')) as

$$\begin{bmatrix} 2 & 0 & 3 & 0 \\ -3 & 2 & -2 & 3 \\ 1 & -3 & 0 & -2 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{\mathrm{c},0} \\ p_{\mathrm{c},1} \\ q_{\mathrm{c},0} \\ q_{\mathrm{c},0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

This system has solution, so there is a controller of order 1.

$$\begin{bmatrix} p_{\mathrm{c},0} \\ p_{\mathrm{c},1} \\ q_{\mathrm{c},0} \\ q_{\mathrm{c},0} \end{bmatrix} = \begin{bmatrix} 2 & 0 & -3 & 0 \\ -3 & 2 & 2 & -3 \\ 1 & -3 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -15 \\ 1 \\ 10 \\ -9 \end{bmatrix}$$

Therefore the controller is given by the difference equation

$$(-15 + \sigma)u = (10 - 9\sigma)y$$

and the closed loop system is $\sigma^3 y = 0$.

Comments:

- In general, the minimal order of the controller is $n_{\mathrm{c}} = n - 1$ (count the # of equations and unknowns in (D'))
- Reduced order observer design in the state space approach would give us the same result
- State controllability + observability vs I/O controllability