FROM TIME SERIES to BALANCED REPRESENTATION

Part II: Algorithms

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Outline

- A new algorithm for balanced subspace identification
- Comparison with Van Overschee–De Moor algorithm
- Comparison with Moonen–Ramos algorithm
- Simulations
- Conclusions and discussion

A new algorithm for balanced subspace identification

The problem and an outline of the basic algorithm

problem: **given:** $u, y : [1, T] \to \mathbb{R}^{\mathtt{m}} \times \mathbb{R}^{\mathtt{p}}$

satisfying the conditions of the fundamental lemma

determine: an associated balanced state model

basic algorithm (with finite matrices):

1. find **sequential zero input responses** Y_0, row dim$(Y_0) = \Delta\mathtt{p}$
2. find the **impulse response** $H : [0, 2\Delta - 1] \to \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$
3. compute the **SVD** of the Hankel matrix of Markov parameters $\mathfrak{H}$

$$\mathfrak{H} = U\Sigma V^\top, \qquad \text{where} \qquad \mathfrak{H} \in \mathbb{R}^{\Delta\mathtt{p}\times\Delta\mathtt{m}}$$

4. find a **balanced state sequence** $X := \sqrt{\Sigma^{-1}}U^\top Y_0$
5. find a **balanced realization** A, B, C, D (by LS)

Notation

part I distinguishes $\begin{array}{ll} w = (u,y) & \text{— general trajectory} \\ \tilde w = (\tilde u, \tilde y) & \text{— particular one} \end{array}$

later on: $w = (u,y)$ is a particular (measured) trajectory

$\mathcal{H}_L(\cdot)$ is a block-Hankel matrix with L block-rows

e.g., with $u : [1,T] \to \mathbb{R}^{\mathtt{m}}$

$$\mathcal{H}_t(u) = \begin{bmatrix} u(1) & u(2) & u(3) & \cdots & u(T-L+1) \\ u(2) & u(3) & u(4) & \cdots & u(T-L+2) \\ u(3) & u(4) & u(5) & \cdots & u(T-L+3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u(L) & u(L+1) & u(L+2) & \cdots & u(T) \end{bmatrix}$$

Notation

$\mathcal{T}_L(\cdot)$ is a lower triangular $L \times L$ block-Toeplitz matrix

e.g., with $H(0) := D,\ H(t) := CA^{t-1}B,\ t = 1, \ldots, L-1$

$$\mathcal{T}_L(H) = \begin{bmatrix} H(0) & 0 & 0 & \cdots & 0 \\ H(1) & H(0) & 0 & \ddots & \vdots \\ \vdots & H(1) & H(0) & \ddots & 0 \\ H(L-2) & \vdots & \ddots & \ddots & 0 \\ H(L-1) & H(L-2) & \cdots & H(1) & H(0) \end{bmatrix}$$

Impulse response from data

let $\quad H := \begin{bmatrix} H^\top(0) & H^\top(1) & \cdots & H^\top(2\Delta - 1) \end{bmatrix}^\top$

given $w = (u,y)$, $\quad$ find $\quad H$

col span$\big(\mathcal{H}_{2\Delta}(w)\big) = \mathfrak{B}|_{[1,2\Delta]} \implies \exists\, G$ s.t. $H = \mathcal{H}_{2\Delta}(y)G$

how to do that?

let $\mathtt{n}_{\max}$ be an estimate of an upper bound on the system order $\mathtt{n}$

$$\mathcal{H}_{\mathtt{n}_{\max}+2\Delta}(u) := \begin{bmatrix} U_{\mathsf{p}} \\ U_{\mathsf{f}} \end{bmatrix} \begin{matrix} \}\mathtt{n}_{\max}\mathtt{m} \\ \}2\Delta\mathtt{m} \end{matrix}, \quad \mathcal{H}_{\mathtt{n}_{\max}+2\Delta}(y) := \begin{bmatrix} Y_{\mathsf{p}} \\ Y_{\mathsf{f}} \end{bmatrix} \begin{matrix} \}\mathtt{n}_{\max}\mathtt{p} \\ \}2\Delta\mathtt{p} \end{matrix}$$

Impulse response from data (cont.)

with G a solution of the system

$$\begin{bmatrix} U_{\mathsf{p}} \\ U_{\mathsf{f}} \\ Y_{\mathsf{p}} \end{bmatrix} G = \begin{bmatrix} 0_{\mathtt{n}_{\max}\times\mathtt{m}} \\ \begin{bmatrix} I_{\mathtt{m}} \\ 0_{(2\Delta-\mathtt{m})\times\mathtt{m}} \end{bmatrix} \\ 0_{\mathtt{n}_{\max}\times\mathtt{m}} \end{bmatrix} \begin{matrix} \longrightarrow \text{zero initial conditions} \\ \longrightarrow \text{impulse inputs} \\ \longrightarrow \text{zero initial conditions} \end{matrix}$$

$$\boxed{Y_{\mathsf{f}}G = H}$$

note: a solution G exists whenever u is persistently exciting of order at least $2(\Delta + \mathtt{n}_{\max})$

More samples of the impulse response

H computed above is with length Δ at most $\frac{1}{2\mathtt{m}}T - \mathtt{n}_{\max}$

moreover for efficiency and accuracy we want to keep Δ small

it is possible, however, to find an arbitrary long H

we will compute iteratively blocks of $L < \frac{1}{2\mathtt{m}}T - \mathtt{n}_{\max}$ consecutive samples of the impulse response

there are conflicting criteria in the choice of L, we want:

small L for efficiency and statistical accuracy (under noise) but large L for numerical stability

More samples of the impulse response (cont.)

let $F_{\mathsf{u}}^{(1)} := \begin{bmatrix} 0_{\mathtt{n}_{\max}\times\mathtt{m}} \\ \begin{bmatrix} I_{\mathtt{m}} \\ 0_{(L\mathtt{m}-\mathtt{m})\times\mathtt{m}} \end{bmatrix} \end{bmatrix}$ and $F_{\mathsf{y}}^{(1)} := \begin{bmatrix} 0_{\mathtt{n}_{\max}\times\mathtt{m}} \\ * \end{bmatrix}$

for $k = 1, 2, \ldots$ solve the system

$$\begin{bmatrix} U_{\mathsf{p}} \\ U_{\mathsf{f}} \\ Y_{\mathsf{p}} \end{bmatrix} G^{(k)} = \begin{bmatrix} F_{\mathsf{u}}^{(k)} \\ F_{\mathsf{y,p}}^{(k)} \end{bmatrix} \quad \text{where} \quad F_{\mathsf{y}}^{(k)} =: \begin{bmatrix} F_{\mathsf{y,p}}^{(k)} \\ F_{\mathsf{y,f}}^{(k)} \end{bmatrix}$$

define $H^{(k)} := Y_{\mathsf{f}} G^{(k)}$, $F_{\mathsf{y,f}}^{(k)} := H^{(k)}$, and shift $F_{\mathsf{u}}, F_{\mathsf{y}}$

$$F_{\mathsf{u}}^{(k+1)} := \begin{bmatrix} \sigma^L F_{\mathsf{u}}^{(k)} \\ 0_{L\mathtt{m}\times\mathtt{m}} \end{bmatrix}, \quad F_{\mathsf{y}}^{(k+1)} := \begin{bmatrix} \sigma^L F_{\mathsf{y}}^{(k)} \\ * \end{bmatrix}$$

More samples of the impulse response (cont.)

σM is the matrix obtained from M by deleting its 1st block-row

the result $H := \begin{bmatrix} H^{(1)} \\ H^{(2)} \\ \cdots \end{bmatrix}$ of the algorithm is the impulse response

monitor $||H^{(k)}||$ and stop when it is small enough

note: gives an automatic way to determine the "depth" constant Δ

this is our method for computing the impulse response

Sequential zero input responses

let $y_0 : [0, 1, \ldots, \Delta] \longrightarrow \mathbb{R}^{\mathtt{p}}$ be a zero input response (due to an initial condition $x(1)$)

given $w = (u, y)$, find a zero input response y_0

with a computed impulse response H of length Δ

$$y_0 = y(1{:}\Delta) - \mathcal{T}_\Delta(H) u(1{:}\Delta)$$

in particular

$$\boxed{Y_0 = \mathcal{H}_\Delta(y) - \mathcal{T}_\Delta(H)\mathcal{H}_\Delta(u)}$$

is a matrix of sequential zero input responses

Sequential zero input responses (cont.)

another approach: with g a solution of the system

$$\begin{bmatrix} U_{\mathsf{p}} \\ U_{\mathsf{f}} \\ Y_{\mathsf{p}} \end{bmatrix} g = \begin{bmatrix} * \\ 0 \\ * \end{bmatrix} \begin{array}{l} \longrightarrow \text{set initial conditions} \\ \longrightarrow \text{zero input} \\ \longrightarrow \text{set initial conditions} \end{array}$$

$$Y_{\mathsf{f}}\, g = y_0$$

in particular with G a solution of the system $\begin{bmatrix} U_{\mathsf{p}} \\ U_{\mathsf{f}} \\ Y_{\mathsf{p}} \end{bmatrix} G = \begin{bmatrix} U_{\mathsf{p}} \\ 0 \\ Y_{\mathsf{p}} \end{bmatrix}$

$\boxed{Y_{\mathsf{f}} G = Y_0}$ is a matrix of sequential zero input responses

i.e., the oblique projection in the classical subspace algorithms

More samples of the free response

let $F_{\mathsf{u}}^{(1)} := \begin{bmatrix} U_{\mathsf{p}} \\ 0 \end{bmatrix}$ and $F_{\mathsf{y}}^{(1)} := \begin{bmatrix} Y_{\mathsf{p}} \\ * \end{bmatrix}$

for $k = 1, 2, \ldots$ solve the system

$$\begin{bmatrix} U_{\mathsf{p}} \\ U_{\mathsf{f}} \\ Y_{\mathsf{p}} \end{bmatrix} G^{(k)} = \begin{bmatrix} F_{\mathsf{u}}^{(k)} \\ F_{\mathsf{y,p}}^{(k)} \end{bmatrix} \quad \text{where} \quad F_{\mathsf{y}}^{(k)} =: \begin{bmatrix} F_{\mathsf{y,p}}^{(k)} \\ F_{\mathsf{y,f}}^{(k)} \end{bmatrix}$$

define $Y_0^{(k)} := Y_{\mathsf{f}} G^{(k)}$, $F_{\mathsf{y,f}}^{(k)} := Y_0^{(k)}$, and shift F_{u}, F_{y}

$$F_{\mathsf{u}}^{(k+1)} := \begin{bmatrix} \sigma^L F_{\mathsf{u}}^{(k)} \\ 0 \end{bmatrix}, \quad F_{\mathsf{y}}^{(k+1)} := \begin{bmatrix} \sigma^L F_{\mathsf{y}}^{(k)} \\ * \end{bmatrix}$$

Balanced state sequence

Hankel matrix of the Markov parameters: $\mathfrak{H} = \mathcal{H}_{\Delta}(\sigma H)$

$$\mathfrak{H} = U\Sigma V^{\top} = \underbrace{U\sqrt{\Sigma}}_{\Gamma_{\mathsf{bal}}}\underbrace{\sqrt{\Sigma}V^{\top}}_{\Delta_{\mathsf{bal}}}$$

$$\Gamma_{\mathsf{bal}} = \begin{bmatrix} C_{\mathsf{bal}} \\ C_{\mathsf{bal}} A_{\mathsf{bal}} \\ \vdots \\ C_{\mathsf{bal}} A_{\mathsf{bal}}^{\Delta-1} \end{bmatrix}, \quad \Delta_{\mathsf{bal}} = \begin{bmatrix} B_{\mathsf{bal}} & A_{\mathsf{bal}} B_{\mathsf{bal}} & \cdots & A_{\mathsf{bal}}^{\Delta-1} B_{\mathsf{bal}} \end{bmatrix}$$

matrix of sequential zero input responses: Y_0

$$Y_0 = \Gamma X = \Gamma_{\mathsf{bal}} X_{\mathsf{bal}} \implies \boxed{X_{\mathsf{bal}} = \sqrt{\Sigma^{-1}}U^{\top} Y_0}$$

Balanced model estimation by LS

$$X_{\mathsf{bal}} = \begin{bmatrix} x(\mathrm{n}_{\max}+1) & x(\mathrm{n}_{\max}+2) & \cdots & x(\mathrm{n}_{\max}+T+1-L) \end{bmatrix}$$

$$\begin{bmatrix} x(\mathrm{n}_{\max}+2) & x(\mathrm{n}_{\max}+3) & \cdots & x(\mathrm{n}_{\max}+T+1-L) \\ y(\mathrm{n}_{\max}+1) & y(\mathrm{n}_{\max}+2) & \cdots & y(\mathrm{n}_{\max}+T-L) \end{bmatrix} =$$

$$\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \begin{bmatrix} x(\mathrm{n}_{\max}+1) & x(\mathrm{n}_{\max}+2) & \cdots & x(\mathrm{n}_{\max}+T-L) \\ u(\mathrm{n}_{\max}+1) & u(\mathrm{n}_{\max}+2) & \cdots & u(\mathrm{n}_{\max}+T-L) \end{bmatrix}$$

A new algorithm

input: $u(1), \dots, u(T), \quad y(1), \dots, y(T)$
an upper bound $\mathrm{n_{max}}$ for the system order

1. zero input responses: $Y_0 = Y_\mathrm{f} G$, where $\begin{bmatrix} U_\mathrm{p} \\ U_\mathrm{f} \\ Y_\mathrm{p} \end{bmatrix} G = \begin{bmatrix} U_\mathrm{p} \\ 0 \\ Y_\mathrm{p} \end{bmatrix}$
2. impulse response: $H = Y_\mathrm{f} G$, where $\begin{bmatrix} U_\mathrm{p} \\ U_\mathrm{f} \\ Y_\mathrm{p} \end{bmatrix} G = \begin{bmatrix} 0 \\ \begin{bmatrix} I \\ 0 \end{bmatrix} \\ 0 \end{bmatrix}$
3. SVD: $\mathfrak{H} = \mathcal{H}_\Delta(\sigma H) = U \Sigma V^\top$
4. balanced state sequence: $X = \sqrt{\Sigma^{-1}} U^\top Y_0$
5. balanced model: solve the LS problem (LS)

output: $\hat{A}, \hat{B}, \hat{C}, \hat{D}$

Comparison with the algorithm Van Overschee–De Moor

Algorithm Van Overschee–De Moor

input: $u_0, \dots, u_T \quad y_0, \dots, y_T$ and $i, \quad i \geq \mathrm{n_{max}}$

$\begin{bmatrix} U_\mathrm{p} \\ U_\mathrm{f} \end{bmatrix} := \mathcal{H}_{2i}(u), \quad \begin{bmatrix} Y_\mathrm{p} \\ Y_\mathrm{f} \end{bmatrix} := \mathcal{H}_{2i}(y)$ row dim$(U_\mathrm{p}) = i\,\mathrm{m}$, row dim$(U_\mathrm{f}) = i\,\mathrm{m}$

1. oblique projection: $Y_0 := Y_\mathrm{f} / _{U_\mathrm{f}} \begin{bmatrix} U_\mathrm{p} \\ Y_\mathrm{p} \end{bmatrix}$
2. weight matrix: $W = U_\mathrm{p}^\top (U_\mathrm{p} U_\mathrm{p}^\top)^{-1} J$
3. SVD: $Y_0 W = U \Sigma V^\top$
4. balanced state sequence: $X_\mathrm{f} = \sqrt{\Sigma^{-1}} U^\top Y_0$
5. balanced model: solve the LS problem (LS)

output: $\hat{A}, \hat{B}, \hat{C}, \hat{D}$

Comments

- the oblique proj. $Y_\mathrm{f} / _{U_\mathrm{f}} \begin{bmatrix} U_\mathrm{p} \\ Y_\mathrm{p} \end{bmatrix}$ contains sequential zero input responses
- $Y_0 W$ contains impulse responses $+$ initial condition responses
- $Y_0 W$ is only approximately a Hankel matrix of Markov param.
- for large i the initial conditions responses die out and the impulse responses dominate
- due to the Hankel structure most elements are recomputed many times
- in approximate case the matrix $Y_0 W$ is not Hankel

Comparison

- both VO–DM and the new algorithm match the basic outline
- steps 4 (balanced state seq.) and 5 (LS) are the same
- different are the methods for computing the impulse response and the zero input response
- algorithm VO–DM computes the Hankel matrix itself
- the new algorithm computes the impulse response (and constructs the Hankel matrix from the response)

The oblique projection

the oblique projection $A/_B C$ is closely related to the solution of the system $\left[\begin{smallmatrix} C \\ B \end{smallmatrix}\right] G = \left[\begin{smallmatrix} C \\ 0 \end{smallmatrix}\right]$ that we use

$A/_B C$ — project A obliquely onto C along B

$$A/_B C := A \begin{bmatrix} C^\top & B^\top \end{bmatrix} \begin{bmatrix} CC^\top & CB^\top \\ BC^\top & BB^\top \end{bmatrix}^+ \begin{bmatrix} C \\ 0 \end{bmatrix} \qquad \text{(OBL)}$$

$Y_\mathrm{f}/_{U_\mathrm{f}} \left[\begin{smallmatrix} U_\mathrm{p} \\ Y_\mathrm{p} \end{smallmatrix}\right]$ is the standard way of computing $Y_0 = \Gamma X$

let G be the least-norm, least-squares solution of the system

$$\begin{bmatrix} C \\ B \end{bmatrix} G = \begin{bmatrix} C \\ 0 \end{bmatrix} \qquad \text{then} \qquad A/_B C = AG$$

Comparison with Moonen–Ramos algorithm

Algorithm Moonen–Ramos

$$\begin{bmatrix} U_\mathrm{p} \\ U_\mathrm{f} \end{bmatrix} := \mathcal{H}_{2i}(u) \qquad \begin{bmatrix} Y_\mathrm{p} \\ Y_\mathrm{f} \end{bmatrix} := \mathcal{H}_{2i}(y)$$

row dim$(U_\mathrm{p}) = i\,\mathtt{m}$ row dim$(Y_\mathrm{p}) = i\,\mathtt{p}$
row dim$(U_\mathrm{f}) = i\,\mathtt{m}$ row dim$(Y_\mathrm{f}) = i\,\mathtt{p}$

let the rows of $\begin{bmatrix} T_1 & T_2 & T_3 & T_4 \end{bmatrix}$ form a basis for the left kernel of $\left[\begin{smallmatrix} U_\mathrm{p} \\ Y_\mathrm{p} \\ U_\mathrm{f} \\ Y_\mathrm{f} \end{smallmatrix}\right]$

$$\begin{bmatrix} T_1 & T_2 & T_3 & T_4 \end{bmatrix} \begin{bmatrix} U_\mathrm{p} \\ Y_\mathrm{p} \\ U_\mathrm{f} \\ Y_\mathrm{f} \end{bmatrix} = 0$$

Algorithm Moonen–Ramos

input: $u_0, \ldots, u_T, \quad y_0, \ldots, y_T$ **and** $i, \quad i \geq \mathrm{n}_{\max}$

$$\begin{bmatrix} U_\mathrm{p} \\ U_\mathrm{f} \end{bmatrix} := \mathcal{H}_{2i}(u), \quad \begin{bmatrix} Y_\mathrm{p} \\ Y_\mathrm{f} \end{bmatrix} := \mathcal{H}_{2i}(y) \qquad \text{row dim}(U_\mathrm{p}) = i\,\mathrm{m}, \quad \text{row dim}(U_\mathrm{f}) = i\,\mathrm{m}$$

0. **annihilators:** $[T_1 \ T_2 \ T_3 \ T_4]$
1. **zero input responses:** $Y_0 = T_4^+ [T_1 \ T_2] \begin{bmatrix} U_\mathrm{p} \\ Y_\mathrm{p} \end{bmatrix}$
2. **Hankel matrix:** $\mathfrak{H} = T_4^+ (T_2 T_4^+ T_3 - T_1)$
3. **SVD:** $\mathfrak{H} = U \Sigma V^\top$
4. **balanced state sequence:** $X_\mathrm{f} = \sqrt{\Sigma^{-1}} U^\top Y_0$
5. **balanced model:** solve the LS problem (LS)

output: $\hat{A}, \hat{B}, \hat{C}, \hat{D}$

Comments

- the main computation is to find the annihilators $[T_1 \cdots T_4]$ efficient implementation should **exploit the Hankel structure**
- we have a **"dual" algorithm**, to the one presented, that recursively computes the left kernel of the data matrix
- $[T_1 \ T_2] \begin{bmatrix} U_\mathrm{p} \\ Y_\mathrm{p} \end{bmatrix}$ is a **state sequence** (shift-and-cut operator)
- $T_4^+ [T_1 \ T_2] \begin{bmatrix} U_\mathrm{p} \\ Y_\mathrm{p} \end{bmatrix}$ is a matrix of **zero input responses**
- $T_4^+ (T_2 T_4^+ T_3 - T_1)$ is the **Hankel matrix**

Comparison

- Moonen–Ramos algorithm also **fits into the basic outline**
- steps 4 (balanced state seq.) and 5 (LS) are the same
- the impulse and a free responses are computed via the annihilators $\begin{bmatrix} T_1 & T_2 & T_3 & T_4 \end{bmatrix}$
- again most elements are **recomputed** many times

  therefore under noise $T_4^+ (T_2 T_4^+ T_3 - T_1)$ is **not Hankel**

Simulations

Simulation setup

aim: to show correctness and advantages of the new algorithm

but the algorithms are not optimized in efficiency

example used in all experiments:

third order random stable SISO system

$T = 100$, $\tilde{u}$ is unity variance white noise

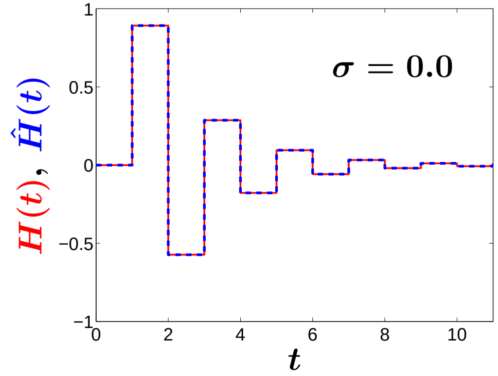
$\tilde{w}$ **is corrupted** by white noise with standard deviation σ

in all simulations: $\mathtt{n}_{\max} = \mathtt{n}$ and $L = \mathtt{n}$

Impulse response estimation

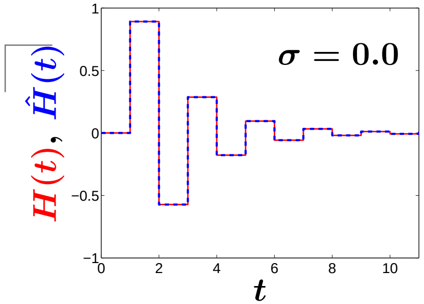
solid red — exact impulse response H

dashed blue — impulse response computed from data $\hat{H}$

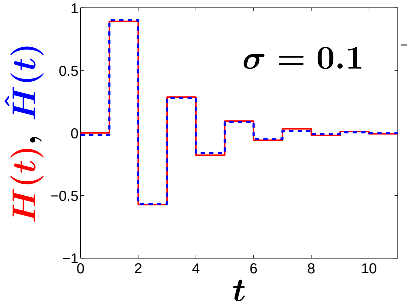


$||H - \hat{H}||_F = 10^{-15}$ $\Longrightarrow$ up to the numerical precision **exact match**

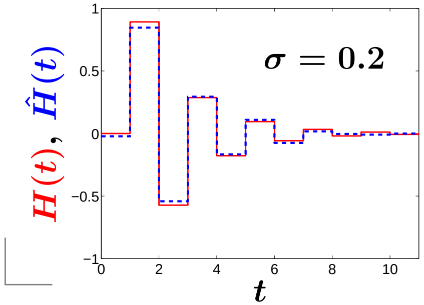
$||H - \hat{H}|| = 10^{-15}$



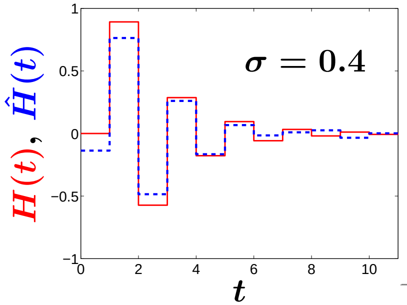
$||H - \hat{H}|| = 0.02$



$||H - \hat{H}|| = 0.05$



$||H - \hat{H}|| = 0.21$



Free response estimation

$Y_0 = \Gamma X$ — exact sequence of free responses ($\Delta = 10$)

$\hat{Y}_0$ — estimated sequence of free responses

error of estimation: $e = ||Y_0 - \hat{Y}_0||_F$

σ	0.0	0.1	0.2	0.4
new algorithm	10^{-14}	1.33	2.84	4.48
oblique proj.	10^{-11}	2.02	4.03	5.44

the oblique projection is computed by (OBL)

note: the new algorithm uses **more overdetermined system** of equations and **does not square the data**

Closeness to balancing

the algorithms return a finite time balanced model

we illustrate the effect of the depth parameter Δ on the balancing

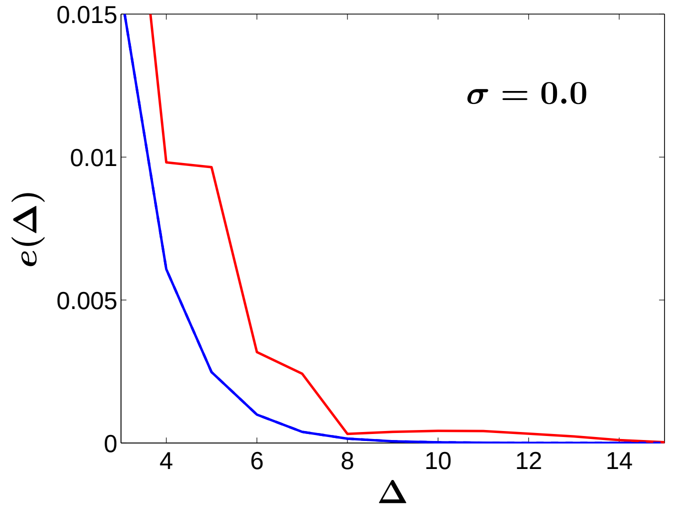
closeness to exact balancing

$\mathcal{C}/\mathcal{O}$ — contr./obsrv. Gramian of the exact balanced model
$\hat{\mathcal{C}}/\hat{\mathcal{O}}$ — contr./obsrv. Gramian of the identified model

$$e^2 := \frac{||\mathcal{C} - \hat{\mathcal{C}}||_F^2 + ||\mathcal{O} - \hat{\mathcal{O}}||_F^2}{||\mathcal{C}||_F^2 + ||\mathcal{O}||_F^2}$$

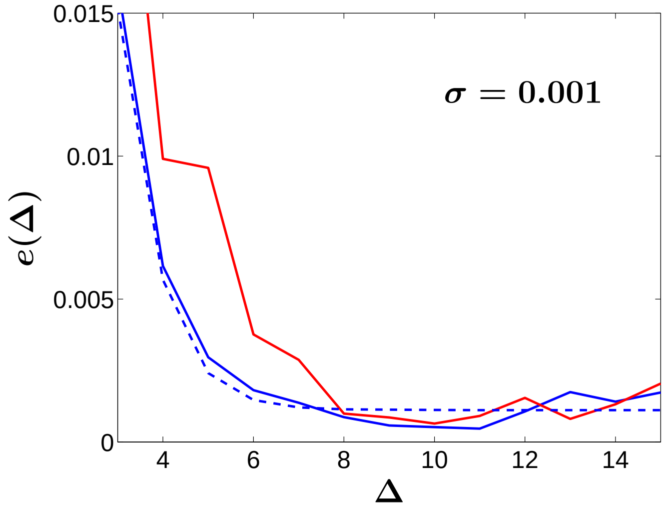
Closeness to balancing (cont.)

red — VO–DM blue — M–R new $\equiv$ M–R



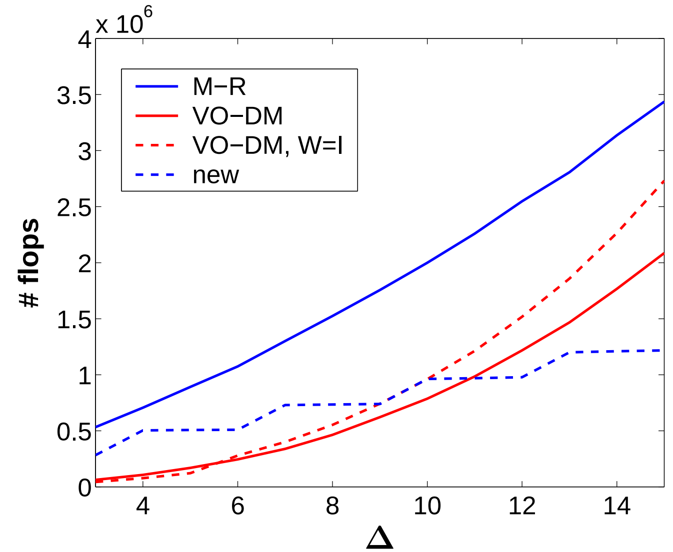
Closeness to balancing (cont.)

red — VO–DM blue — M–R dashed blue — new



Flops count

solid red — VO–DM
dashed red — VO–DM, $W = I$
solid blue — M–R
dashed blue — new



Conclusions and discussion

Conclusions

- impulse response and sequence of zero input responses are the **main tools for balanced model identification**
- classically they are computed via the **oblique projection**
- we gave **system theoretic interpretation** of the oblique proj.
- arbitrary **long responses** can be computed **from finite data set**
- computation of impulse response instead of Hankel matrix of Markov parameters can **improve efficiency and accuracy**
- next goal: **optimize efficiency and implement in C/FORTRAN**