UNIVERSITY OF SOUTHAMPTON ELEC3035W1

EXAMINATION 2008/09

Control Systems Design

Duration: 120 mins

Answer THREE questions with at least one from each part.

University approved calculators MAY be used.

An approximate marking scheme is indicated.

Number of pages: 7

Section A

Question 1

This problem involves three static linear systems

Plant:	$\mathscr{P} = \{\, (u,y) \mid y = Pu \,\}$,	where	$P \in \mathbb{R}^{\mathtt{p} \times \mathtt{m}}$,
Controller:	$\mathscr{C} = \{\, (r,u) \mid u = Cr \,\}$,	where	$C \in \mathbb{R}^{\mathtt{m} \times \mathtt{r}}$,
Reference model:	$\mathscr{M} = \{\, (r,y) \mid y = Mr \,\}$,	where	$M \in \mathbb{R}^{\mathtt{p} \times \mathtt{r}}$.

The controller and the plant are interconnected in the feedforward configuration, shown in Figure 1.

$$r \longrightarrow \boxed{C} \xrightarrow{u} \boxed{P} \longrightarrow y$$

Figure 1: Interconnection of the plant and the controller.

Given a plant and a reference model, the aim is to find a controller that makes the controlled system as close to the reference model M as possible.

a) Define the controlled system $\mathscr{P}_{\mathscr{C}}$, corresponding to the feedforward interconnection of Figure 1. Find an input/output representation of $\mathscr{P}_{\mathscr{C}}$. Is $\mathscr{P}_{\mathscr{C}}$ a static linear system? Justify your answer. [5 marks]

b) Under what conditions on $\mathscr{P}$, for any choice of $\mathscr{M}$, there is a controller $\mathscr{C}$, such that $\mathscr{P}_{\mathscr{C}} = \mathscr{M}$. Under what conditions on $\mathscr{P}$, the controller is moreover unique? When the controller is unique, show how one can obtain its input/output representation C from the given P and M matrices. When the controller is not unique, display one particular solution. [10 marks]

c) Consider the case when there is no controller $\mathscr{C}$ that achieves the reference model $\mathscr{M}$ exactly. Define a problem of finding a controller that is as close to achieving the specification as possible. In what sense a controlled system $\mathscr{P}_{\mathscr{C}}$ with a controller solving this problem is "close" to the reference model $\mathscr{M}$? Is your controller design problem equivalent to a standard approximation problem. If so, give a solution. [10 marks]

Question 2

Consider the system defined by the state space representation

$$x(t+1) = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{b} u(t).$$

a) Show that this system is state uncontrollable. Sketch the controllable subspace. [5 marks]

b) Consider an initial state $x(0) = 0$ and a target state $x_{\text{target}} = \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$. Show that the target state is not reachable from the initial state in two time steps. Determine a state x^* that is as close (in the 2-norm distance) to x_{target} as possible and is reachable from $x(0)$ in two time steps. [10 marks]

c) Find all inputs $u = \big(u(0), u(1)\big)$ that transfers the system from the initial state $x(0) = 0$ to a final state $x(2) = x^*$ and select the minimum 2-norm input. [10 marks]

Question 3

Consider the system defined by the state space representation

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \qquad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t) + e(t), \end{aligned} \tag{1}$$

where e is measurement noise. This problem is concerned with the estimation of the initial state x_0 from given input/output observations.

a) Suppose that the observations of the input are

$$\big(u(1), u(2), \ldots, u(T)\big) \tag{2}$$

and the observations of the output are

$$\big(y(t_1), y(t_2), \ldots, y(t_K)\big), \quad \text{where } t_1 < t_2 < \cdots < t_K \text{ are integers in the interval } [1, T]. \tag{3}$$

Give verbal interpretation of the data (2–3). Is your interpretation practically relevant? If so, give (engineering) examples where it may occur. [10 marks]

b) Assuming that the system is observable and its order is smaller than the number of output samples K, find the least squares estimate of the initial state x_0 from the observations (2–3). [10 marks]

c) Assuming that $t_K < T$, predict the output at $t_K + 1, \ldots, T$ from the observations (2–3). Is your predicted output optimal in some sense? If so, in what sense? [5 marks]

Section B

Question 4

a) A nonlinear system is described by the following differential equations

$$\begin{aligned} ml^2\ddot{z} + mgl\sin z &= -b\dot{z} + c(y - \dot{z}) - u \\ J\dot{y} &= -c(y - \dot{z}) + u \end{aligned}$$

where b, c, J, l, g and m are positive real constants and u is the input. Use the state variables $x_1 = z$, $x_2 = \dot{z}$, $x_3 = y$ to obtain the following state-space model for this system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{b}{ml^2}x_2 + \frac{c}{ml^2}(x_3 - x_2) - \frac{1}{ml^2}u \\ \dot{x}_3 &= -\frac{c}{J}(x_3 - x_2) + \frac{1}{J}u \end{aligned}$$

Also calculate its equilibrium points. [9 marks]

b) State the theorem associated with Lyapunov's first method of stability analysis. In the case of the system of part (a) above show that the Jacobian matrix can be written in the form

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -f\cos x_1 & -g & h \\ 0 & i & j \end{bmatrix}$$

and give the formulas for the constants $f, g, h, i,$ and j. [9 marks]

c) Apply Lyapunov's first method to the equilibrium points of the system of parts (a) and (b) above. You may make use of the fact that if all roots of a real polynomial the form

$$s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3$$

are to have strictly negative real parts then $\alpha_i > 0$, $i = 1, 2, 3$. [7 marks]

Question 5

a) For the nonlinear system described by

$$\begin{aligned}\dot{x}_1 &= -2x_1e^{x_1} + 3x_2 + x_1^5 + u_1\\ \dot{x}_2 &= -x_1 + u_2\end{aligned}$$

show that the feedback linearisation controller

$$\begin{aligned}u_1 &= 2x_1e^{x_1} - 5x_1 - 3x_2 - x_1^5\\ u_2 &= -x_2 + x_1\end{aligned}$$

gives the closed loop stable linear system

$$\begin{aligned}\dot{x}_1 &= -5x_1\\ \dot{x}_2 &= -x_2\end{aligned}$$

It can be shown that

$$\begin{aligned}u_1 &= -x_1^5\\ u_2 &= -x_2\end{aligned}$$

is an alternative stabilising control law. Explain why this control law requires less control effort than the first one. [8 marks]

b) Consider a nonlinear system described by

$$\begin{aligned}\dot{x}_1 &= -3f(x)x_1 + x_2\\ \dot{x}_2 &= -f(x)x_2 - x_1^3\end{aligned}$$

for which $(0,0)$ is an equilibrium point. Use the Lyapunov function

$$V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$$

to investigate the stability of this point in the cases when

i) $f(x) > 0$ for all $\sqrt{x_1^2 + x_2^2} \leq 3$

ii) $f(x) > 0$ for all x, and

iii) the first state equation is replaced by $\dot{x}_1 = 3f(x)x_1 + x_2$.

[12 marks]

c) In the analysis of a nonlinear system with Lyapunov function $V(x)$, it is only possible to establish that

$$\dot{V}(x) \leq 0$$

Explain how LaSalles principle can be applied here. Also what is the difference between local and global properties in this form of analysis? [5 marks]

Question 6

a) A unity negative feedback control scheme consists of a forward path of a nonlinear element in series with a linear transfer-function $G(s)$. It is known that the nonlinearity can be approximated as shown in Figure 2.

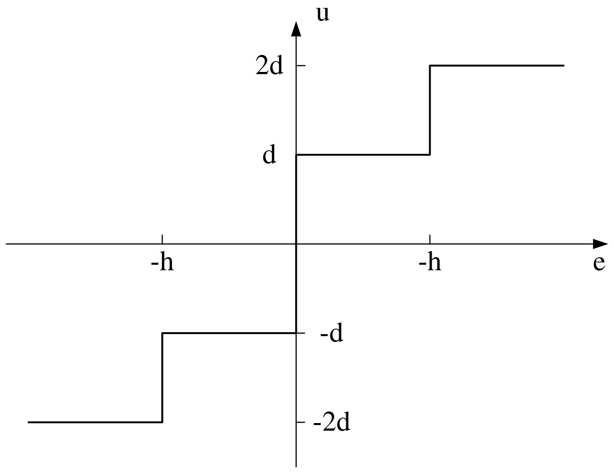

Figure 2:

Show that the describing function for this nonlinearity is given by

$$N(A) = \begin{cases} \frac{4d}{\pi A}(1+\frac{\sqrt{A^2-h^2}}{A}),\ A > h \\ \frac{4d}{\pi A},\ A \leq h \end{cases}$$

You may use the following formula where all symbols have their normal meanings.

$$N(A) = \frac{1}{\pi A}\int_0^{2\pi} f(A\sin\theta)\sin\theta d\theta$$

[8 Marks]

b) In the feedback control scheme of part (a) above

$$G(s) = \frac{k}{(s+1)^3},\ k > 0$$

Show that a limit cycle of amplitude $A = \frac{5h}{4}$ is possible and find the corresponding values of the limit cycle frequency and transfer-function gain k.

[10 Marks]

c) Determine the stability characteristics of the limit cycle of part (b) above and give clearly labelled sketches to illustrate your method. Describe also the main limitations of the describing function method. [7 Marks]

END OF PAPER