Unfalsified Linear Time-Invariant Behaviors of Bounded Complexity ⋆

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Abstract: We modify the notion of the most powerful unfalsified model (MPUM) for a given finite time series in order to allow the identification of models with inputs by assuming that the number of inputs is a priori known. First, necessary and sufficient conditions are established for the existence and uniqueness of the MPUM in this setting. Then, in case of non-uniqueness of unfalsified models, it is shown that the set of unfalsified models admits an affine structure.

Keywords: Behavioral approach, exact system identification, most powerful unfalsified model.

1. INTRODUCTION

The notion of the most powerful unfalsified model (MPUM), introduced by Willems (1986) in the late 1980s, led to a growing interest in exact model identification by researchers from the systems and control community, see for example (Van Overschee and De Moor, 1996, Chapter 2). It is defined for an exact (noise-free) infinite time series as the least complex linear time-invariant (LTI) model explaining the given time series. The complexity of a model is defined by the pair *(number of inputs, system order)*, where a model with a higher number of inputs is always considered a more complex model. There are algorithms to compute the MPUM and it has been proven that it exists and is also unique (Willems 1986). Under certain assumptions called *identifiability conditions*, it can be shown that the MPUM coincides with the data generating system (Willems et al. 2005).

In real-life applications, however, time series are finite. Further, the concept of the MPUM is fundamental in studying the *data-driven simulation and control* problems (Markovsky and Rapisarda 2008; Hou and Wang 2013; Maupong and Rapisarda 2017). Therefore, it is desirable to define the MPUM for a given finite time series. However, in the finite case, there are two crucial issues: i) the MPUM may not exist or be non-unique, and ii) due to the above definition of model complexity, the MPUM if exists is always a finite dimensional autonomous system. This work is devoted to resolve these issues. Specifically, we aim to modify the definition of the MPUM in order to allow the identification of systems with inputs and derive necessary and sufficient conditions for its existence and uniqueness. Our main result is that every unfalsified model can be expressed as a sum of the MPUM and an autonomous model of bounded order.

2. LINEAR TIME-INVARIANT BEHAVIORS

A dynamical system is defined by the triplet $(\mathbb{T}, \mathbb{W}, \mathscr{B})$, where $\mathbb{T} \subseteq \mathbb{R}$ is the time axis, $\mathbb{W} \subseteq \mathbb{R}^q$ is the signal space, and $\mathscr{B} \subseteq \mathbb{W}^{\mathbb{T}}$ is the behavior with $\mathbb{W}^{\mathbb{T}}$ the set of all functions $w : \mathbb{T} \to \mathbb{W}$. This work is devoted to discrete-time behaviors, so that $\mathbb{T} \subseteq \mathbb{N}$. By $\mathscr{L}^q$, we denote the set of LTI behaviors with q variables that are closed in the topology of point-wise convergence. The sum of two behaviors $\mathscr{B}_i \in \mathscr{L}^q, i = 1,2$ is a behavior defined as

$$\mathscr{B}_1 + \mathscr{B}_2 := \{w : w = w_1 + w_2, w_1 \in \mathscr{B}_1, w_2 \in \mathscr{B}_2\}.$$

Each $\mathscr{B} \in \mathscr{L}^q$ admits a *kernel representation* $\mathscr{B} = \{w : R(\sigma)w = 0\}$, where $R \in \mathbb{R}^{g \times q}[\xi]$ is a polynomial matrix and σ is the *backward shift* operator defined as $(\sigma w)(t) := w(t+1)$. The minimal degree of R in a kernel representation of the system is invariant of the representation. It is called the *lag* of the system and is denoted by $\mathbf{l}(\mathscr{B})$. The restriction of the behavior $\mathscr{B}$ to the interval $[1, L]$ is defined as

$$\mathscr{B}|_L := \{\, w \in (\mathbb{R}^q)^L \mid \text{there is } v \in \mathscr{B}, \text{ such that } w(t) = v(t) \text{ for all } 1 \le t \le L \,\}.$$

The order and the number of inputs of $\mathscr{B} \in \mathscr{L}^q$ are denoted by, respectively, $\mathbf{n}(\mathscr{B})$ and $\mathbf{m}(\mathscr{B})$. The ordered pair $\big(\mathbf{m}(\mathscr{B}), \mathbf{n}(\mathscr{B})\big) =: \mathbf{c}(\mathscr{B})$ define the complexity of $\mathscr{B}$ in lexicographic manner. By $\mathscr{L}^{q,n}_m$, we denote the set of LTI systems with q variables and complexity bounded by (m, n).

The Hankel matrix with $L \in \mathbb{N}$ block-rows for a time series

$$w_d := \big(w_d(1), w_d(2), \dots, w_d(T)\big) \in (\mathbb{R}^q)^T$$

is defined as follows:

$$\mathscr{H}_L(w_d) := \begin{bmatrix} w_d(1) & w_d(2) & \cdots & w_d(T-L+1) \\ w_d(2) & w_d(3) & \cdots & w_d(T-L+2) \\ \vdots & \vdots & \ddots & \vdots \\ w_d(L) & w_d(L+1) & \cdots & w_d(T) \end{bmatrix}.$$

3. MODIFICATION OF THE MPUM

The MPUM for $w_d \in (\mathbb{R}^q)^{\mathbb{N}}$ in the model class $\mathscr{L}^q$ is defined as follows (Willems, 1986, Definition 4):

$$\mathrm{MPUM}(w_d) := \arg\min_{\mathscr{B} \in \mathscr{L}^q, w_d \in \mathscr{B}} \mathbf{c}(\mathscr{B}). \qquad (1)$$

The MPUM for a finite time series, however, coincides with the finite dimensional behavior corresponding to an autonomous model (Willems 2006). Here, we resolve this issue by assuming

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that the number of inputs is a priori known. Thus, we define the MPUM for $w_d \in (\mathbb{R}^q)^T$ as the minimization of the order:

$$\mathrm{MPUM}_m(w_d) := \arg\min_{\mathscr{B}\in\mathscr{L}^q_m, w_d\in\mathscr{B}|_T} \mathbf{n}(\mathscr{B}). \tag{2}$$

As discussed in the introduction, an unfalsified model in a model class of bounded complexity may not exist or be nonunique. To this end, we define the set of unfalsified models:

$$\Sigma_{m,n}(w_d) = \{\mathscr{B} : w_d \in \mathscr{B}|_T, \mathscr{B} \in \mathscr{L}^{q,n}_m\}. \tag{3}$$

4. MAIN RESULTS

Theorem 1. Let $w_d \in (\mathbb{R}^q)^T$ be a given time series that is generated by $\mathscr{B} \in \mathscr{L}^{q,n}_m$. Then

$$\mathrm{MPUM}_m(w_d) = \mathscr{B} \qquad \text{and} \qquad \mathrm{colspan}\big(\mathscr{H}_L(w_d)\big) = \mathscr{B}|_L$$

if and only if the following two conditions hold:

(i) $w_d \in \mathscr{B}|_T$, and

(ii) $\mathrm{rank}\,\mathscr{H}_L(w_d) = \mathbf{n}(\mathscr{B}) + \mathbf{m}(\mathscr{B})L$, where $L = \left\lceil \frac{T+1}{q+1} \right\rceil$.

Further, $\mathbf{n}\big(\mathrm{MPUM}_m(w_d)\big) = \mathrm{rank}\,\mathscr{H}_L(w_d) - \mathbf{m}(\mathscr{B})L$.

Proof. Let (i) and (ii) hold. Clearly, $\mathrm{colspan}\big(\mathscr{H}_L(w_d)\big) \subseteq \mathscr{B}|_L$. Note that

$$\begin{aligned}\dim\Big(\mathrm{colspan}\big(\mathscr{H}_L(w_d)\big)\Big) &= \mathrm{rank}\,\mathscr{H}_L(w_d)\\ &= \mathbf{n}(\mathscr{B}) + \mathbf{m}(\mathscr{B})L = \dim(\mathscr{B}|_L).\end{aligned}$$

Hence, $\mathrm{colspan}\big(\mathscr{H}_L(w_d)\big) = \mathscr{B}|_L$. Next, we need to prove that $\mathrm{MPUM}_m(w_d) = \mathscr{B}$. Since $\mathscr{B} \in \mathscr{L}^{q,n}_m$ and $w_d \in \mathscr{B}|_T$. Thus, $\mathscr{B}$ is unfalsified. Now, it remains to show that $\mathscr{B}$ is the most powerful. Let there be another exact model $\bar{\mathscr{B}} \in \mathscr{L}^{q,n}_m$ such that $\mathbf{n}(\bar{\mathscr{B}}) < \mathbf{n}(\mathscr{B})$. Then $\mathrm{rank}\,\mathscr{H}_L(w_d) < \mathbf{n}(\mathscr{B}) + \mathbf{m}(\mathscr{B})L$, which is a contradiction. Hence, $\mathrm{MPUM}_m(w_d) = \mathscr{B}$.

Conversely, $\mathrm{MPUM}_m(w_d) = \mathscr{B}$ implies $w_d \in \mathscr{B}|_T$. Next, $\mathrm{colspan}\big(\mathscr{H}_L(w_d)\big) = \mathscr{B}|_L$ implies $\mathrm{rank}\,\mathscr{H}_L(w_d) = \dim(\mathscr{B}|_L) = \mathbf{n}(\mathscr{B}) + \mathbf{m}(\mathscr{B})L$. Now, it is obvious that $\mathbf{n}\big(\mathrm{MPUM}_m(w_d)\big) = \mathrm{rank}\,\mathscr{H}_L(w_d) - \mathbf{m}(\mathscr{B})L$.

We have the following theorem that provides a characterization of the set of unfalsified models.

Theorem 2. The set of unfalsified models of bounded complexity for a given time series $w_d \in (\mathbb{R}^q)^T$ is given as

$$\begin{aligned}\Sigma_{m,n}(w_d) = \mathrm{MPUM}_m(w_d) + \mathscr{L}^{q,n-k}_0,&\\ \text{where } k = \mathbf{n}\big(\mathrm{MPUM}_m(w_d)\big).&\end{aligned} \tag{4}$$

Proof. Clearly,

$$\mathrm{MPUM}_m(w_d) + \mathscr{L}^{q,n-k}_0 \subseteq \Sigma_{m,n}(w_d).$$

For reverse inclusion, let $\mathscr{B} \in \Sigma_{m,n}(w_d)$ and let

$$\mathbf{n}\big(\mathrm{MPUM}_m(w_d)\big) = k \leq \mathbf{n}(\mathscr{B}) \leq n.$$

Then, $\mathrm{MPUM}_m(w_d) \subseteq \mathscr{B}$. Thus,

$$\mathscr{B} = \mathrm{MPUM}_m(w_d) + \mathscr{B}', \quad \text{where } \mathscr{B}' \in \mathscr{L}^{q,n-k}_0,$$

and hence (4) holds.

Thus, the set of models explaining the data w_d is described as a sum of the MPUM of order k and any autonomous model of order bounded by $n-k$. We can distinguish three cases:

(I) if $n < \mathbf{n}\big(\mathrm{MPUM}_m(w_d)\big)$, there is no exact model, *i.e.*, $\Sigma_{m,n}(w_d) = \emptyset$,

(II) if $n = \mathbf{n}\big(\mathrm{MPUM}_m(w_d)\big)$, there is a unique exact model, *i.e.*, $\Sigma_{m,n}(w_d) = \mathrm{MPUM}_m(w_d)$, and

(III) if $n > \mathbf{n}\big(\mathrm{MPUM}_m(w_d)\big)$, there are infinitely many exact models (4).

5. EXAMPLES

Example 1. (SISO system). Consider a single-input single-output system $\mathscr{B} \in \mathscr{L}^{2,k}_1$ defined by the input/output representation:

$$P(\sigma)y = Q(\sigma)u,$$

where P, Q, with $\det P \neq 0$, are scalar polynomials. Equivalently,

$$\underbrace{\begin{bmatrix}-Q(\sigma) & P(\sigma)\end{bmatrix}}_{R(\sigma)} \begin{bmatrix}u\\ y\end{bmatrix} = 0,$$

is a kernel representation of $\mathscr{B}$. Let $w_d := \left[\begin{smallmatrix}u_d\\ y_d\end{smallmatrix}\right] \in (\mathbb{R}^2)^T$ be a trajectory of $\mathscr{B}$ that satisfies conditions (i) and (ii) of Theorem 1. Then, we have that $\mathscr{H}_{k+1}(w_d)$ has one-dimensional left kernel and $R\mathscr{H}_{k+1}(w_d) = 0$. Hence, the Hankel matrix with $n+1$ block-rows $\mathscr{H}_{n+1}(w_d)$ has $n-k+1$ dimensional left kernel. A basis for the left kernel of $\mathscr{H}_{n+1}(w_d)$ is given by the rows of the generalized Sylvester matrix

$$\mathscr{S}(R) := \begin{bmatrix} R_0 & R_1 & \cdots & R_k & & & \\ & R_0 & R_1 & \cdots & R_k & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & R_0 & R_1 & \cdots & R_k \end{bmatrix}.$$

Any $\hat{R}$ in the left kernel of $\mathscr{H}_{n+1}(w_d)$ is given by $\hat{R} = z\mathscr{S}(R)$, where $z \in \mathbb{R}^{n-k+1}$. Since $\hat{R}$ correspond to an exact model for w_d in $\mathscr{L}^{2,n}_1$, we have again special structure of the set of exact models due to the special structure of the left kernel of $\mathscr{H}_{n+1}(w_d)$. Note that, with $\hat{R} =: \begin{bmatrix}-\hat{Q} & \hat{P}\end{bmatrix}$, $\hat{P}$ and $\hat{Q}$ are such that $\hat{P} = zP$ and $\hat{Q} = zQ$. This corresponds to Theorem 2 (*cf.* (4)): the k "fixed" poles and zeros are the ones of the MPUM and the $n-k$ "spurious" ones are common for the $\hat{P}$ and $\hat{Q}$.

6. CONCLUSIONS

The problem of exact system identification in the behavioral framework has been investigated. We have modified the notion of the MPUM for a finite time series and have shown that every unfalsified model must include the MPUM.

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