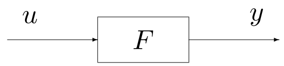
Overview of time-domain analysis

- Linear time-invariant (LTI) filters
- Example: moving average (MA) filter
- Finite impulse response (FIR) filters
- Difference and differential equations representation of LTI filter
- Convolution and causality
- Continuous-time case

Filters

- A filter (or system) F transforms an input signal u into an output signal y



$$y = F(u)$$

- Communication channels can be modelled as filters and therefore analysed
- We need filters to shape communications signals appropriately (synthesis)
- Filters are mathematical objects but they can be realized numerically and simulated
- Filters can also be realized in analog electronics or by mechanical devices, in which case they become physical devices

Linear time-invariant (LTI) filters

- The filter F is linear if

$$F(a_1u_1 + a_2u_2) = a_1F(u_1) + a_2F(u_2), \quad \text{for all inputs } u_1,\ u_2, \text{ and scalars } a_1,\ a_2$$

- Define the backwards time-shift operator σ^τ by

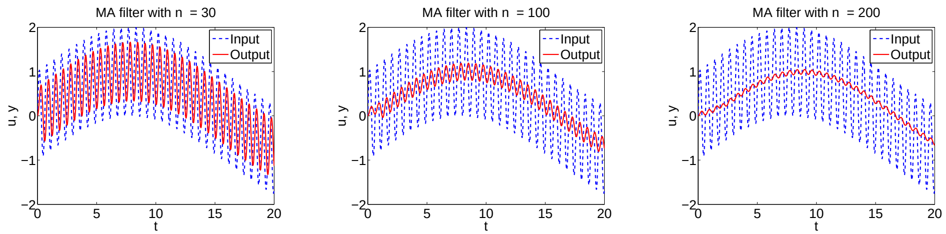
$$\big(\sigma^\tau(u)\big)(t) = u(t+\tau)$$

- The filter F is time-invariant if

$$F(\sigma^\tau u) = \sigma^\tau F(u), \quad \text{for all input } u \text{ and time shifts } \tau$$

Example: moving average (MA) filter

$$y(t) = \frac{1}{m+1}\big(u(t) + u(t-1) + \cdots + u(t-m)\big), \qquad \text{for all } t \qquad \text{(MA)}$$



Exercise: Show that (MA) defines an LTI filter.

Initial conditions

In order to compute the response

$$y = \big(y(0), y(1), \dots\big)$$

of an MA filter (MA) to an input

$$u = \big(u(0), u(1), \dots\big)$$

we need to know m values of the input in the "past"

$$\big(u(-m), \dots, u(-2), u(-1)\big)$$

these are called initial conditions of the MA filter

Finite impulse response (FIR) filter

MA filter is a special case of an FIR filter

$$y(t) = a_0 u(t) + a_1 u(t-1) + \cdots + a_m u(t-m), \qquad \text{for all } t \qquad \text{(FIR)}$$

The response of an FIR filter to a unit pulse input

$$\delta(t) = \begin{cases} 1, & \text{when } t = 0 \\ 0, & \text{otherwise.} \end{cases}$$

under zero initial conditions is

$$\big(a_0, a_1, \dots, a_m, 0, 0, \dots\big)$$

thus the name—finite impulse response.

Difference equation representation of LTI filters

- (FIR) defines $y(t)$ in terms of $u(t)$ and a finite number of past input values.
- This implies that the filter has memory (it "remembers" past values of u).
- Memory is a characteristic property of all dynamical systems.
- More generally, $y(t)$ may depend on $u(t)$ and a finite number of past inputs and outputs

$$\begin{aligned} y(t) + b_1 y(t-1) + \cdots + b_n y(t-n) \\ = a_0 u(t) + a_1 u(t-1) + \cdots + a_m u(t-m), \qquad \text{for all } t \end{aligned}$$

- This is a linear constant coefficients difference equation.

Example

Consider the homogeneous difference equation

$$y(t) = y(t-1) + y(t-2), \qquad \text{for all } t > 1$$

with initial conditions

$$y(0) = y(1) = 1$$

(This equation defines a dynamical system without input.)

Iterating by hand the equation, we find

$$y(2) = 2, \quad y(3) = 3, \quad y(4) = 5, \quad y(5) = 8, \quad y(6) = 13, \quad \dots$$

These numbers are called Fibonacci numbers, see
`http://en.wikipedia.org/wiki/Fibonacci_number`

Another example

Consider the non-homogeneous difference equation

$$y(t) - y(t-1) - y(t-2) = u(t), \qquad \text{for all } t \geq 0\text{, with } y(-2) = y(-1) = 0$$

which defines an LTI filter. (Show this.)
The impulse response of this filter can be computed by hand:

$$y(0) = 1, \quad y(1) = 1, \quad y(2) = 2, \quad y(3) = 3, \quad y(4) = 5, \quad y(5) = 8, \quad y(6) = 13, \quad \ldots \tag{1}$$

Again the Fibonacci numbers.
Note the impulse response is infinite $\rightsquigarrow$ infinite impulse response (IIR) filter.

Solving linear homogeneous difference equations

Given the linear, constant coefficients, homogeneous difference equation

$$y(t) + b_1 y(t-1) + \cdots + b_n y(t-n) = 0, \qquad \text{for all } t \geq 0 \tag{HDE}$$

Form the polynomial equation (called characteristic equation)

$$1 + b_1 z^{-1} + \cdots + b_n z^{-n} = 0 \qquad \iff \qquad z^n + b_1 z^{n-1} + \cdots + b_n = 0$$

Find the roots $z_1, \ldots, z_n$ of this polynomial (this is the hard part).
Any solution of (HDE) is of the form

$$y(t) = c_1 z_1^t + c_2 z_2^t + \cdots + c_3 z_3^t, \qquad \text{for all } t \geq 0$$

The numbers $c_1, \ldots, c_n$ are determined from the initial conditions $y(-1), \ldots, y(-n)$.

Example

Consider again the homogeneous difference equation

$$y(t) = y(t-1) + y(t-2), \qquad \text{for all } t > 1\text{, with } y(0) = y(1) = 1$$

The characteristic equation is

$$z^2 - z - 1 = 0$$

Its roots are

$$z_1 = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad z_2 = \frac{1-\sqrt{5}}{2},$$

so that

$$y(t) = c_1 z_1^t + c_2 z_2^t$$

Example

In order to find c_1 and c_2, we solve the system

$$\begin{aligned} f(0) &= c_1 z_1^0 + c_2 z_2^0 \\ f(1) &= c_1 z_1^1 + c_2 z_2^1 \end{aligned} \qquad \iff \qquad \begin{bmatrix} 1 & 1 \\ z_1 & z_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

From where we find

$$c_1 = \frac{z_2 - 1}{z_2 - z_1}, \qquad c_2 = \frac{1 - z_1}{z_2 - z_1}$$

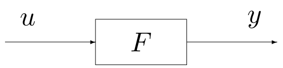
so that

$$f(t) = \frac{z_2 - 1}{z_2 - z_1} z_1^t + \frac{1 - z_1}{z_2 - z_1} z_2^t$$

$\rightsquigarrow$ closed form solution (known as Binet's or Moivre's formula).

Convolution

- Consider a filter F with input u, impulse response h, and output y:



- Represent the input as a sum of shifted delta functions

$$u = \sum_{\tau=-\infty}^{\infty} u(\tau)\sigma^{-\tau}(\delta)$$

- Now using the linearity and time-invariance properties of the filter, we have

$$\begin{aligned} y &= F(u) \\ &= F\left(\sum_{\tau=-\infty}^{\infty} u(\tau)\sigma^{-\tau}(\delta)\right) \\ &= \sum_{\tau=-\infty}^{\infty} u(\tau)\sigma^{-\tau}\big(F(\delta)\big) \\ &= \sum_{\tau=-\infty}^{\infty} u(\tau)\sigma^{-\tau}(h) =: h \star u \end{aligned}$$

- Therefore, the relation between input and output is:

$$y(t) = \sum_{\tau=-\infty}^{\infty} u(\tau)h(t-\tau), \qquad \text{for all } t \tag{2}$$

- Property of convolution
$$y = h \star u = u \star h$$
(show this)

- Special case: Finite Impulse Response (FIR) filter

$$y(t) = \sum_{\tau=0}^{n} h(\tau)u(t-\tau) \tag{3}$$

- Nonzero values of the inputs response in the past, i.e., $h(t) \neq 0$ for some $t < 0$ implies that the response of the filter precedes the action of the input.

- Such systems are called noncausal.

- In order to operate in real-time, the filter must be causal.

Continuous-time case

- Shifts in time become derivatives: linear constant coeff. differential equation

$$\begin{aligned} y(t) &+ b_1 \frac{\mathrm{d}}{\mathrm{d}t} y(t) + \cdots + b_n \frac{\mathrm{d}^n}{\mathrm{d}t^n} y(t) \\ &= a_0 u(t) + a_1 \frac{\mathrm{d}}{\mathrm{d}t} u(t) + \cdots + a_m \frac{\mathrm{d}^m}{\mathrm{d}t^m} u(t), \qquad \text{for all } t > 0 \end{aligned}$$

- The initial conditions are

$$y(0), \quad \frac{\mathrm{d}}{\mathrm{d}t} y(0), \quad \ldots \quad \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} y(0)$$

- Sums over time become integrals: continuous-time convolution

$$y(t) = (h \star u)(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)\, d\tau, \qquad \text{for all } t \tag{4}$$