ELEC 3035: Tutorial on state space and polynomial representations

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1. *Mass–damper-springer mechanical system* The mechanical system shown on Figure 1 is described by the following differential equation

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}y + d\frac{\mathrm{d}}{\mathrm{d}t}y + ky = u, \qquad \text{(M-D-S eqn)}$$

where m is the mass of the body, k is the elasticity constant of the spring, d is the damping of the damper, u is the external force applied on the body, and y is the body displacement from its equilibrium position. Find an input/state/output representation $\mathscr{B}_{\mathrm{i/s/o}}(A,B,C,D)$ of the system

$$\mathscr{B} = \{\,\mathrm{col}(u,y) \mid \text{(M-D-S eqn) holds}\,\}$$

defined by (M-D-S eqn). In other words, find matrices A, B, C, and D, such that

$$\mathscr{B} = \{\,\mathrm{col}(u,y) \mid \text{exists } x \text{ such that } \frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu,\ y = Cx + Du\,\}.$$

Hint: Take as a state vector position and velocity, *i.e.*, $x = \mathrm{col}(y, \frac{\mathrm{d}}{\mathrm{d}t}y)$.

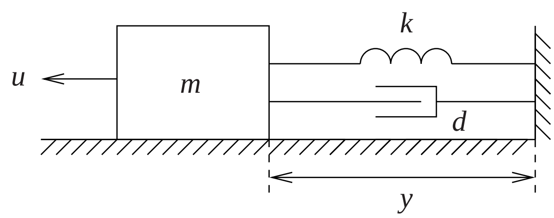

Figure 1: Mass-damper-springer mechanical system.

Solution: Let

$$x = \begin{bmatrix} y \\ \frac{\mathrm{d}}{\mathrm{d}t}y \end{bmatrix}. \qquad (1)$$

We can rewrite (M-D-S eqn) as

$$m\frac{\mathrm{d}}{\mathrm{d}t}x_2 + dx_2 + kx_1 = u \quad \Longleftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}x_2 = -\frac{d}{m}x_2 - \frac{k}{m}x_1 + \frac{1}{m}u.$$

The latter equation together with the trivial equation $\frac{\mathrm{d}}{\mathrm{d}t}x_1 = x_2$ (see (1)) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}x = \underbrace{\begin{bmatrix} 0 & 1 \\ -k/m & -d/m \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}_{B} u,$$

which has the form of a state equation of an input/state/output representation. The corresponding output equation is $y = x_1$ (see (1)) or

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} x + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} u.$$

With the definition of A, B, C, and D given above we have that

$$\text{(M-D-S eqn) holds} \quad \Longleftrightarrow \quad (1),\ x = Ax + Bu,\ \text{and } y = Cx + Du \text{ hold},$$

so that $\mathscr{B}_{\mathrm{i/s/o}}(A,B,C,D) = \mathscr{B}$ as required. □

2. *Moving average filter* In class we defined the forward shift σ, $(\sigma w)(t) = w(t+1)$. By tradition, in signal processing is used the backward shift σ^{-1}, $(\sigma^{-1}w)(t) = w(t-1)$. The difference equation

$$\sigma^n y = q_n \sigma^n u + q_{n-1}\sigma^{n-1}u + \cdots + q_1\sigma^1 u + q_0\sigma^0 u$$

can be written, using σ^{-1}, as

$$y = q_n u + q_{n-1}\sigma^{-1}u + \cdots + q_1\sigma^{-n+1}u + q_0\sigma^{-n}u. \tag{MA}$$

The LTI dynamical system defined by (MA) is called a moving average filter. Taking as a state

$$x = \operatorname{col}\left(\sigma^{-1}u, \ldots, \sigma^{-n}u\right)$$

write the moving average filter in a state space form.

Solution: Direct verification shows that

$$x = \begin{bmatrix} \sigma^{-1}u \\ \vdots \\ \sigma^{-n}u \end{bmatrix} \iff \sigma x = \underbrace{\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B} u$$

and

$$\text{(MA)} \iff y = \underbrace{\begin{bmatrix} q_{n-1} & q_{n-2} & \cdots & q_1 & q_0 \end{bmatrix}}_{C} x + \underbrace{\begin{bmatrix} q_n \end{bmatrix}}_{D} u.$$

□

3. *Autoregressive filter* The difference equation

$$y = p_{n-1}\sigma^{-1}y + \cdots + \sigma^{-n}p_0 y(t) + u(t) \tag{AR}$$

describes an LTI dynamical system called autoregressive filter. Taking as a state

$$x = \operatorname{col}\left(\sigma^{-1}y, \ldots, \sigma^{-n}y\right)$$

write the autoregressive filter in a state space form.

Solution: Direct verification shows that

$$x = \begin{bmatrix} \sigma^{-1}y \\ \vdots \\ \sigma^{-n}y \end{bmatrix} \text{ and (AR)} \iff \sigma x = \underbrace{\begin{bmatrix} p_{n-1} & p_{n-2} & p_{n-3} & \cdots & p_0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{B} u$$

and

$$\text{(AR)} \iff y = \underbrace{\begin{bmatrix} p_{n-1} & p_{n-2} & \cdots & p_1 & p_0 \end{bmatrix}}_{C} x + \underbrace{\begin{bmatrix} 1 \end{bmatrix}}_{D} u.$$

□

4. *Moving average-autoregressive filter* The difference equation

$$p_n y(t) + p_{n-1} y(t-1) + \cdots + p_0 y(t-n) = q_0 u(t-n) + q_1 u(t-n+1) + \cdots + q_n u(t)$$

describes an LTI dynamical system called moving average-autoregressive filter. Taking as a state

$$x(t) = \operatorname{col}\big(u(t-1)), \ldots, u(t-n), y(t-1)), \ldots, y(t-n)\big)$$

write the moving average-autoregressive filter in a state space form.

Solution: Combine the solutions of the moving average and autoregressive filters to obtain

$$\sigma x = \underbrace{\left[\begin{array}{cccc|cccc} 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\ 1 & \ddots & & \vdots & \vdots & & & \vdots \\ & \ddots & \ddots & \vdots & \vdots & & & \vdots \\ 0 & & 1 & 0 & 0 & \cdots & \cdots & 0 \\ \hline q_{n-1} & \cdots & q_1 & q_0 & p_{n-1} & \cdots & p_1 & p_0 \\ 0 & \cdots & \cdots & 0 & 1 & \ddots & & \vdots \\ \vdots & & & \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & & 1 & 0 \end{array}\right]}_{A} x + \underbrace{\left[\begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \\ \hline q_n \\ 0 \\ \vdots \\ 0 \end{array}\right]}_{B} u$$

$$y = \underbrace{\left[\begin{array}{cccc|cccc} q_{n-1} & \cdots & q_1 & q_0 & p_{n-1} & \cdots & p_1 & p_0 \end{array}\right]}_{C} x + \underbrace{\left[q_n\right]}_{D} u$$

$\square$

5. *Convolution* Recall the definition of the convolution operation $\star$

$$y(t) = (h \star u)(t) := \sum_{\tau=0}^{\infty} h(\tau) u(t-\tau), \qquad \text{for } t > 0, \quad \text{where } u(t) = 0, \text{ for all } t \leq 0. \tag{CONV}$$

Equation (CONV) defines an LTI dynamical system

$$\mathscr{B} = \{\operatorname{col}(u,y) \mid \text{(CONV) holds}\}.$$

Define the input and output vectors

$$Y = \operatorname{col}\big(y(1), \ldots, y(T)\big), \qquad U = \operatorname{col}\big(u(1), \ldots, u(T)\big) \tag{U,Y}$$

composed of the first T samples. The signals u and y being a trajectory of the LTI systems $\mathscr{B}$ means that there is a linear function that maps U to Y. Find a matrix representation of this function. In other words find a matrix $\mathscr{T}$, such that $Y = \mathscr{T} U$. The matrix $\mathscr{T}$ is called a Toeplitz matrix.

Solution: From (CONV), we have

$$\begin{aligned} y(1) &= h(0)u(1) \\ y(2) &= h(0)u(2) + h(1)u(1) \\ &\vdots \\ y(T) &= h(0)u(T) + \cdots + h(T-1)u(1) \end{aligned}$$

Written in a matrix form this system of equations is

$$\underbrace{\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(T) \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} h(0) & 0 & \cdots & 0 \\ h(1) & h(0) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ h(T-1) & \cdots & h(1) & h(0) \end{bmatrix}}_{\mathscr{T}} \underbrace{\begin{bmatrix} u(1) \\ u(2) \\ \vdots \\ u(T) \end{bmatrix}}_{U}.$$

Note that all elements of $\mathscr{T}$ above the main diagonal are zero (such matrices are called lower-triangular) and has the same elements along the diagonals (such matrices are called Toeplitz structured). The lower-triangular property corresponds to causality (current output does not depend on the future inputs) and the Toeplitz structured corresponds to time-invariance. $\square$

6. *Matrix-vector representation of an LTI system* The response y of a discrete-time LTI system $\mathscr{B}_{\mathrm{i/s/o}}(A,B,C,D)$ is a linear function of the initial state $x(1)$ and the input u. This means that (with the definition of U and Y, given in (U,Y)) there is a linear function that maps U and $x(1)$ to Y. Find a matrix representation of this function. In other words find matrices $\mathscr{O}$ and $\mathscr{T}$, such that $Y = \mathscr{O}x(1) + \mathscr{T}U$. The matrix $\mathscr{O}$ is called an observability matrix of the state space representation.

   Solution: Substitute $x(1)$ and $u(1)$ in the state and output equations to obtain $x(2)$ and $y(1)$

$$x(2) = Ax(1) + Bu(1)$$
$$y(1) = Cx(1) + Du(1)$$

   Now substitute $x(1)$ and $u(1)$ in the state and output equations to obtain $x(3)$ and $y(2)$

$$x(3) = Ax(2) + Bu(2) = A^2x(1) + ABu(1) + Bu(2)$$
$$y(1) = Cx(1) + Du(1) = CAx(0) + CBu(0) + Du(1)$$

   After T steps we obtain for the output

$$y(T) = CA^{T-1}x(1) + CA^{T-2}Bu(0) + \cdots + CBu(T-1) + Du(T).$$

   Written in a matrix form the system of equations for $y(1), y(2), \ldots, y(T)$ is

$$\underbrace{\begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(T) \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{T-1} \end{bmatrix}}_{\mathscr{O}} x(1) + \underbrace{\begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ CA^{T-2}B & \cdots & CB & D \end{bmatrix}}_{\mathscr{T}} \underbrace{\begin{bmatrix} u(1) \\ u(2) \\ \vdots \\ u(T) \end{bmatrix}}_{U}. \tag{2}$$

   The term $\mathscr{O}x(1)$ is the response due to the initial conditions and the term $\mathscr{T}U$ is the response due to the input. Because of linearity the response of the system due to nonzero initial conditions $x(1)$ and input u is the sum of $\mathscr{O}x(1)$ and $\mathscr{T}U$. Note that $\mathscr{O}x(1)$ is the response of the autonomous system $\mathscr{B}(A,C)$ to $x(1)$ and $\mathscr{T}U$ describes a convolution system. $\square$

7. *Impulse response* The discrete-time impulse signal δ is defined as

$$\delta(t) = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{otherwise.} \end{cases}$$

   Find the responses y_i of a discrete-time LTI system $\mathscr{B}_{\mathrm{i/s/o}}(A,B,C,D)$ to inputs $e_i\delta$ under zero initial conditions. (e_i is the ith unit vector). The signal $H := \begin{bmatrix} y_1 & \cdots & y_{\mathrm{col\,dim}(D)} \end{bmatrix}$ is called *impulse response* of the system.

   Solution: Let $D_i := De_i$ and $B_i := Be_i$. From (2), taking $x(0) = 0$ (zero initial conditions) and $u = u_i = e_i\delta$, we have

$$y(t) = y_i(t) = \begin{cases} D_i & \text{if } t = 0 \\ CA^{t-1}B_i & \text{if } t > 0 \end{cases}$$

   so that

$$H(t) = \begin{cases} D & \text{if } t = 0 \\ CA^{t-1}B & \text{if } t > 0. \end{cases}$$

   The formula gives the impulse response of a system in terms of the state space parameters $\square$