# Lecture 1: Classical vs behavioral paradigms for data modeling

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# Exercise 1: Constant approximation

- the approximate modeling problem:

$$\begin{aligned} &\text{minimize} \quad \text{over } \widehat{D} \quad \|D - \widehat{D}\|_{\mathrm{F}} \\ &\text{subject to} \quad \{\widehat{d}_1, \dots, \widehat{d}_N\} \subset \text{line through } 0 \end{aligned} \qquad (*)$$

  is equivalent to low-rank approximation

- the solution is given by the EYM theorem (SVD of $D$)

- solve $(*)$ with additional constraint

$$\widehat{d}_i = \widehat{d}, \quad \text{for some } \widehat{d} \in \mathbb{R}^q \qquad (**)$$

- $(**) \iff \widehat{D} = \widehat{d}\begin{bmatrix}1 & \cdots & 1\end{bmatrix}$ (image representation)
- then, $(*)$ is equivalent to

$$\text{minimize} \quad \text{over } \widehat{d} \quad \|D - \widehat{d}\begin{bmatrix}1 & \cdots & 1\end{bmatrix}\|_{\mathrm{F}}$$

- linear least-squares $\rightsquigarrow$ the solution is

$$\widehat{d} = \frac{1}{N} D \begin{bmatrix}1 & \cdots & 1\end{bmatrix}^\top = \texttt{mean(D,2)}$$

- HW: compare with $\widehat{R}_{\mathrm{EV}}$ in the motivating example of Johan's and Rik's lectures

# Exercise 2: Line fitting

- we considered the problem of fitting

$$\mathscr{D} = \{ d_1, \dots, d_N \} \subset \mathbb{R}^2$$

  to a line passing through the origin

- the solution is rank-1 approximation of the matrix

$$D = \begin{bmatrix} d_1 & \cdots & d_N \end{bmatrix}$$

- consider now the problem of fitting $\mathscr{D}$ to any line in $\mathbb{R}^2$

- is it also equivalent to low-rank approximation?

the points $d_i = (a_i, b_i)$, $i = 1, \ldots, N$ lie on a line

$$\Updownarrow$$

there is $(R_1, R_2, R_3) \neq 0$, such that
$R_1 a_i + R_2 b_i + R_3 = 0$, for $i = 1, \ldots, N$

$$\Updownarrow$$

there is $(R_1, R_2, R_3) \neq 0$, such that

$$\begin{bmatrix} R_1 & R_2 & R_3 \end{bmatrix} \begin{bmatrix} a_1 & \cdots & a_N \\ b_1 & \cdots & b_N \\ 1 & \cdots & 1 \end{bmatrix} = 0$$

$$\Updownarrow$$

$$\operatorname{rank}\left( \begin{bmatrix} a_1 & \cdots & a_N \\ b_1 & \cdots & b_N \\ 1 & \cdots & 1 \end{bmatrix} \right) \leq 2$$

- $\{\, d \mid Rd = 0 \,\}$ — linear static model
- $\{\, d \mid R \left[\begin{smallmatrix} d \\ 1 \end{smallmatrix}\right] = 0 \,\}$ — affine static model
- in exact modeling

$$\text{affine fitting} \quad \Updownarrow \quad \text{data centering + linear modeling}$$

- HW: is the same true in approximate modeling?

# Exercise 3: Conic section fitting

- conic section model

$$\mathscr{B}(S,u,v) = \{\, d \in \mathbb{R}^2 \mid d^\top S d + u^\top d + v = 0 \,\}$$

where $S = S^\top$, $u$, $v$ are model parameters

- express the exact fitting condition $\mathscr{D} \subset \mathscr{B}(S,u,v)$ as a rank constraint on a matrix $D(\mathscr{D})$

- find a conic section fitting the points

$$d_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad d_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad d_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad d_4 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

the points $d_i = (a_i, b_i)$, $i = 1, \ldots, N$ lie on a conic section

$$\Updownarrow$$

$\exists\ S = S^\top$, $u$, $v$, at least one of them nonzero, such that
$d_i^\top S d_i + u^\top d_i + v = 0$, for $i = 1, \ldots, N$

$$\Updownarrow$$

there is $(s_{11}, s_{12}, s_{22}, u_1, u_2, v) \neq 0$, such that

$$\begin{bmatrix} s_{11} & 2s_{12} & u_1 & s_{22} & u_2 & v \end{bmatrix} \begin{bmatrix} a_1^2 & \cdots & a_N^2 \\ a_1 b_1 & \cdots & a_N b_N \\ a_1 & \cdots & a_N \\ b_1^2 & \cdots & b_N^2 \\ b_1 & \cdots & b_N \\ 1 & \cdots & 1 \end{bmatrix} = 0$$

the points $d_i = (a_i, b_i)$, $i = 1, \dots, N$ lie on a conic section

$$\Updownarrow$$

$$\operatorname{rank}\left(\begin{bmatrix} a_1^2 & \cdots & a_N^2 \\ a_1 b_1 & \cdots & a_N b_N \\ a_1 & \cdots & a_N \\ b_1^2 & \cdots & b_N^2 \\ b_1 & \cdots & b_N \\ 1 & \cdots & 1 \end{bmatrix}\right) \leq 5$$

```
f = @(a, b) [a .^ 2; a .* b; a; b .^ 2; b;
             ones(size(a))];
```

- finding exact models

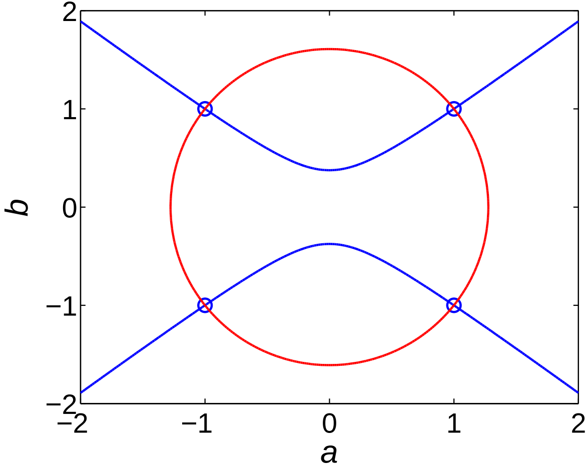
```
R = null(f(d(1, :), d(2, :))')';
```

- plotting a model

```
function H = plot_model(th, f, ax, c)
H = ezplot(@(a, b) th * f(a, b), ax);
for h = H', set(h, 'color', c, 'linewidth',
```

- show results

```
plot(d(1, :), d(2, :), 'o', 'markersize', 12
ax = 2 * axis;
for i = 1:size(R, 1)
  hold on, plot_model(R(i, :), f, ax, c(i));
end
```




- HW: parameterize all solutions

# Exercise 4: Subspace clustering

- this problem is a special case of the Generalized PCA

- union of two lines model

$$\mathscr{B}(R^1, R^2) = \{\, d \in \mathbb{R}^2 \mid (R^1 d)(R^2 d) = 0 \,\}$$

  where $R^1, R^2 \in \mathbb{R}^{1\times 2}$, $R^1, R^2 \neq 0$ are model parameters

- express the exact fitting condition $\mathscr{D} \subset \mathscr{B}(R^1, R^2)$ as a rank constraint on a matrix $D(\mathscr{D})$

- find a union of two lines model fitting the points

$$d_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad d_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad d_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad d_4 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

the points $d_i \in \mathbb{R}^2$, $i = 1, \dots, N$ lie on a union of two lines

$$\Updownarrow$$

there are $R^1 \neq 0$ and $R^2 \neq 0$, $v$, such that
$(R^1 d_i)(R^2 d_i) = 0$, for $i = 1, \dots, N$

$$\Updownarrow$$

there are $\begin{bmatrix} R^1_1 & R^1_2 \end{bmatrix} \neq 0$ and $\begin{bmatrix} R^2_1 & R^2_2 \end{bmatrix} \neq 0$, such that

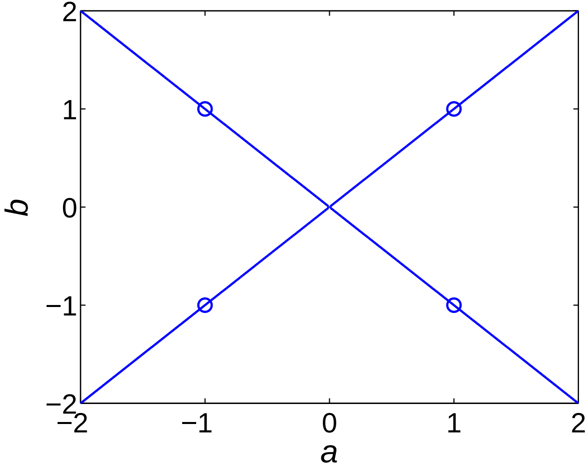
$$\begin{bmatrix} R^1_1 R^2_1 & R^1_1 R^2_2 + R^1_2 R^2_1 & R^1_2 R^2_2 \end{bmatrix} \begin{bmatrix} a_1^2 & \cdots & a_N^2 \\ a_1 b_1 & \cdots & a_N b_N \\ b_1^2 & \cdots & b_N^2 \end{bmatrix} = 0$$

- if $d_i \in \mathbb{R}^2$, $i = 1, \ldots, N$ lie on a union of two lines, then

$$\operatorname{rank}\left(\begin{bmatrix} a_1^2 & \cdots & a_N^2 \\ a_1 b_1 & \cdots & a_N b_N \\ b_1^2 & \cdots & b_N^2 \end{bmatrix}\right) \leq 2$$

- in this case, the rank condition is only necessary
- additional constraint

$$\ker(D) = \operatorname{image}(\begin{bmatrix} 1 & \alpha + \beta & \alpha\beta \end{bmatrix}) \quad \text{for some } \alpha, \beta$$




- HW: how to “extract” $R^1$ and $R^2$ from $\ker(D)$

# Exercise 5: LTI autonomous system fitting

- linear time-invariant autonomous system

$$\mathscr{B}|_T(R) = \{\, \big(w(1),\dots,w(T)\big) \mid R_0 w(t) + R_1 w(t+1) + \cdots + R_\ell w(t+\ell) = 0, \text{ for } t = 1,\dots,T-\ell \,\}$$

- express the exact fitting condition $w \in \mathscr{B}(R)$ as a rank constraint on a matrix constructed from $w$

- find the smallest $\ell$, for which $\exists\, R \in \mathbb{R}^{1\times(\ell+1)}$, such that

$$w_\mathrm{d} := (1,2,4,7,13,24,44,81) \in \mathscr{B}_8(R)$$

$$
\begin{aligned}
w \in \mathscr{B}|_T(R) \quad &\iff \quad R\,\mathscr{H}_{\ell+1}(w) = 0 \\
&\implies \quad \operatorname{rank}\big(\mathscr{H}_{\ell+1}(w)\big) \le \ell
\end{aligned}
$$

where

$$
\mathscr{H}_{\ell+1}(w) := \begin{bmatrix} w(1) & w(2) & \cdots & w(T-\ell) \\ w(2) & w(3) & \cdots & w(T-\ell+1) \\ \vdots & \vdots & & \vdots \\ w(\ell+1) & w(\ell+2) & \cdots & w(T) \end{bmatrix}
$$

- for $\ell = 1, 2, \ldots$, if $\operatorname{rank}\big(\mathscr{H}_{\ell+1}(w)\big) = \ell$, stop
- $w_{\mathrm{d}} \in \mathscr{B}|_8\big(\begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix}\big)$

# Exercise 6: Polynomial common divisor

- the polynomials

$$p(z) = p_0 + p_1 z + \cdots + p_{\ell_p} z^{\ell_p}$$
$$q(z) = q_0 + q_1 z + \cdots + q_{\ell_q} z^{\ell_q}$$

  have a common divisor

$$c(z) = c_0 + c_1 z + \cdots + c_{\ell_c} z^{\ell_c}$$

  iff $p = ca$ and $q = cb$ for some polynomials $a$ and $b$

- express the common divisor condition as a rank constraint on a matrix constructed from $p$, $q$

$$
c(z) = a(z)b(z) \iff \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{\ell_c} \end{bmatrix} = \begin{bmatrix} a_0 & & & \\ a_1 & a_0 & & \\ \vdots & a_1 & \ddots & \\ a_{\ell_a} & \vdots & \ddots & a_0 \\ & a_{\ell_a} & & a_1 \\ & & \ddots & \vdots \\ & & & a_{\ell_a} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{\ell_b} \end{bmatrix}
$$

$$
\iff: \; c = S_{\ell_b}(a)b \iff c = S_{\ell_a}(b)a
$$

$$\begin{array}{c} p \in \mathbb{R}[z] \text{ and } q \in \mathbb{R}[z] \\ \text{have common divisor} \\ c \in \mathbb{R}[z],\ \deg(c) = \ell_c \end{array} \iff \begin{array}{l} \exists\, a \in \mathbb{R}[z],\ \deg(a) = \ell_p - \ell_c \\ \exists\, b \in \mathbb{R}[z],\ \deg(b) = \ell_q - \ell_c \\ \text{such that } {\color{red} p = ca \text{ and } q = cb} \end{array}$$

$$\iff \quad qa - pb = 0$$

$$\iff \quad \begin{bmatrix} S_{\ell_a}(q) & S_{\ell_b}(p) \end{bmatrix} \begin{bmatrix} a \\ -b \end{bmatrix} = 0$$

$$\iff \quad \begin{bmatrix} S_{\ell_a}(q) & S_{\ell_b}(p) \end{bmatrix} \quad \begin{array}{c} \text{is rank} \\ \text{deficient} \end{array}$$

- HW: how this result can be used to find the GCD $c$?