Exam questions for "Signal theory: Part 1"

Work alone. You can use printed materials but no electronic devices. Time allowed: 60 minutes.

1. (2 points) The signal $y(t) = e^t + e^{2t} + e^{3t}$ is a response of an autonomous linear time-invariant system. What can you say about the order $\mathtt{n}$ of the system?

   (a) $\mathtt{n} = 1$ (b) $\mathtt{n} = 2$ (c) $\mathtt{n} = 3$ (d) $\mathtt{n} \geq 3$

   Answer: (d)

2. (3 points) Find *all* least-squares approximate solutions of the linear system of equations

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} x = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}.$$

   Answer: $\left\{ \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}$

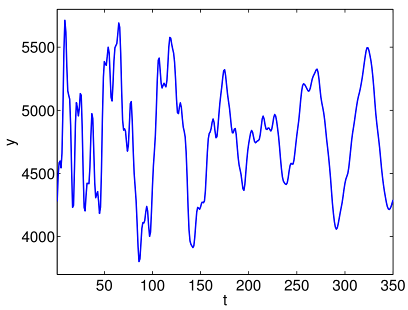
3. (5 points) An invariant space of a state space system is a subset $\mathscr{I}$ of its state space, such that if the initial state $x(0)$ is in $\mathscr{I}$, then the state $x(t)$ remains in $\mathscr{I}$ for all $t > 0$.

   3.1. (3 points) Explain how to find all one dimensional invariant spaces of a system defined by $x(t+1) = Ax(t)$.
   Answer: Find all simple (multiplicity one) eigenvectors $\{v_1, \dots, v_k\}$ of A. There are k one dimensional invariant spaces of the system, given by $\mathscr{V}_i = \mathrm{span}(v_i)$, $i = 1, \dots, k$.

   3.2. (2 point) Apply the solution on the system with $A := \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]$.

   Answer: $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

4. (5 points) A colleague of yours shows you the following signal



   and says:

   > "I think the data generating system is not linear time-invariant, because the response of such a system is a sum of terms that are exponentially decaying, exponentially growing, or periodic while the behavior of the given signal is more complicated: it is obviously not periodic and it is neither exponentially decaying nor exponentially growing."

   Do you agree? If so, how would you make the argument rigorous? If not, what is wrong with the argument and how would you prove that it is wrong?

You should give a specific answer (correct or wrong) and justify it theoretically. In addition, you can assume that you have the observed signal numerically and a computer available to process the data. Can you refine your answer by being more specific about the data generating system? What computations would you do?

Answer: The argument (but not necessarily the conclusion) is *wrong*. 1) The argument is based on the silent assumption that there is no (unobserved) input. 2) Even if there is no input, the class of signals obtained as a sum of (complex) exponentials is rich an includes the signal in the figure. (Think of the Fourier expansion. Almost all signals have a Fourier representation.)

If the signal is available numerically and we can do computations, we can decompose the signal (exactly or approximately) in a sum of a minimal number of exponentials. We've discussed this problem in the course and derived two methods for solving it. Such a representation is a proof that the signal is a sum of exponentials, *i.e.*, it is a response of a linear time-invariant system.

5. (5 points) We discussed in the course two identification methods for autonomous linear time-invariant systems—one based on a state space representation and one based on a polynomial representation. These methods use as data a trajectory $y = \big(y(1), \dots, y(T)\big)$ and produce as an output the model parameters.

   5.1. (3 points) Extend the methods to use *two* given trajectories

   $$y^{(1)} = \big(y^{(1)}(1), \dots, y^{(1)}(T_1)\big) \quad \text{and} \quad y^{(2)} = \big(y^{(2)}(1), \dots, y^{(2)}(T_2)\big)$$

   of the system instead of one trajectory. Discuss both the exact data case and the noisy data case.

   Answer: *Polynomial method:* solve exactly or approximately in the least-squares sense the system of equations

   $$\begin{bmatrix} p_0 & p_1 & \cdots & p_{\mathtt{n}-1} & -1 \end{bmatrix} \begin{bmatrix} \mathscr{H}_{\mathtt{n}+1}(y^{(1)}) & \mathscr{H}_{\mathtt{n}+1}(y^{(2)}) \end{bmatrix} = 0,$$

   where $\mathscr{H}_{\mathtt{n}+1}(y)$ is the Hankel matrix constructed from the signal y.

   State space method: solve exactly or approximately in the least-squares sense the system of equations

   $$\begin{bmatrix} y^{(1)} & y^{(2)} \end{bmatrix} = \mathscr{O} \begin{bmatrix} x_{\text{ini}}^{(1)} & x_{\text{ini}}^{(2)} \end{bmatrix},$$

   where $\mathscr{O}$ is the (extended) observability matrix of the system and $x_{\text{ini}}^{(i)}$ are the initial conditions that generate the trajectories $y^{(1)}$ and $y^{(2)}$, respectively. In class. we assumed that $\mathscr{O}$ is given (state estimation problem). If $\mathscr{O}$ is not given, the problem is a realization one and is related again the Hankel matrix $\begin{bmatrix} \mathscr{H}_{\mathtt{n}+1}(y^{(1)}) & \mathscr{H}_{\mathtt{n}+1}(y^{(2)}) \end{bmatrix}$. We obtain $\mathscr{O}$ from the rank revealing factorization

   $$\begin{bmatrix} \mathscr{H}_{\mathtt{n}+1}(y^{(1)}) & \mathscr{H}_{\mathtt{n}+1}(y^{(2)}) \end{bmatrix} = \mathscr{O}\mathscr{C}.$$

   5.2. (2 points) Apply your method on the data $y^{(1)} = (0,1,1)$ and $y^{(2)} = (1,0,2)$, assuming that the data is exact and the data generating system is of second order.

   Answer: The system of linear equations is

   $$\begin{bmatrix} p_0 & p_1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} = 0$$

   or

   $$\begin{bmatrix} p_0 & p_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix} \qquad \Longrightarrow \qquad \begin{bmatrix} p_0 & p_1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix}.$$