Algorithms for exact identification

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An exact identification problem

Problem P1 (Exact identification)

Given two vector time series

$$u_{\mathrm{d}} = \big(u_{\mathrm{d}}(1), \dots, u_{\mathrm{d}}(T)\big) \in (\mathbb{R}^{\mathtt{m}})^T \qquad \text{“inputs”}$$
$$y_{\mathrm{d}} = \big(y_{\mathrm{d}}(1), \dots, y_{\mathrm{d}}(T)\big) \in (\mathbb{R}^{\mathtt{p}})^T \qquad \text{“outputs”}$$

find an LTI system Σ of order $\mathtt{n}$, such that $(u_{\mathrm{d}}, y_{\mathrm{d}})$ is a trajectory of Σ.

What does it mean “is a trajectory of”?

What does it mean “is a trajectory of”?

let σ be the shift operator $\sigma x(t) = x(t+1)$ and let Σ be defined by a state space representation

$$\Sigma: \quad \sigma x = Ax + Bu, \quad y = Cx + Du \qquad \text{(I/S/O)}$$

$(u_{\mathrm{d}}, y_{\mathrm{d}})$ is a trajectory of Σ if there exists $x_{\text{ini}} \in \mathbb{R}^{\mathtt{n}}$, such that

$$y_{\mathrm{d}} = \underbrace{\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{T-1} \end{bmatrix}}_{\mathscr{O}_T(A,C)} x_{\text{ini}} + \begin{bmatrix} D & & & & \\ CB & D & & & \\ CAB & CB & D & & \\ \vdots & \ddots & \ddots & \ddots & \\ CA^{T-1}B & \cdots & CAB & CB & D \end{bmatrix} u_{\mathrm{d}}$$

i.e., y_{d} is the response of Σ to input u_{d} and some initial condition x_{ini}

Comments

- P1 is an exact fitting problem, a most basic system id. problem
- easily generalizable to a set of N time series
- the realization problem (impulse response $\mapsto (A,B,C,D)$) is a special case of P1 for a set of $\mathtt{m}$ time series
- while $\mathtt{m}$ is given, finding $\mathtt{n}$ is part of the problem

  in fact, $\mathtt{n} \geq \mathtt{p}T \quad \rightsquigarrow \quad$ trivial solution
- we are interested is a solution of a minimal order $\mathtt{n}$

An exact identification problem (revised)

Problem P1' (Exact identification)

Given two vector time series

$$u_{\mathrm{d}} = \big(u_{\mathrm{d}}(1), \dots, u_{\mathrm{d}}(T)\big) \in (\mathbb{R}^{\mathtt{m}})^T \qquad \text{“inputs”}$$
$$y_{\mathrm{d}} = \big(y_{\mathrm{d}}(1), \dots, y_{\mathrm{d}}(T)\big) \in (\mathbb{R}^{\mathtt{p}})^T \qquad \text{“outputs”}$$

find the smallest $\mathtt{n} \in \mathbb{N}$ and an LTI system Σ of order $\mathtt{n}$, with $\mathtt{m}$ inputs and $\mathtt{p}$ outputs, such that $(u_{\mathrm{d}}, y_{\mathrm{d}})$ is a trajectory of Σ.

Behavior and representation of a system

the behavior of an LTI system Σ is the set $\mathscr{B}$ of all trajectories $w := (u, y)$ that Σ can possibly generate

$\mathscr{B}|_{[1,t]}$ — restriction of the behavior to the interval $[1,t]$

a representation of Σ is an equation whose solution set is equal to $\mathscr{B}$

e.g., (I/S/O), *e.g.*, the difference eqn repr.

$R_0 w(t) + R_1 w(t+1) + \cdots + R_l w(t+l) = 0, \quad$ for all t, $\quad$ where $R_i \in \mathbb{R}^{g \times (\mathtt{m}+\mathtt{p})}$

also called kernel representation because

$$\mathscr{B} = \ker\big(R(\sigma)\big), \qquad \text{where} \quad R(\xi) := \sum_{i=0}^{l} R_i \xi^i$$

Set of LTI systems with a fixed complexity

$\mathscr{L}_{\mathtt{m},\mathtt{l}}^{\mathtt{w},\mathtt{n}}$ — set of all LTI systems with

- $\mathtt{w}$ (external) variables,
- at most $\mathtt{m}$ inputs,
- state dimension at most $\mathtt{n}$, and
- lag (= observability index = order of $R(\xi)$) at most $\mathtt{l}$

for $t \geq \mathtt{n}$, $\quad \dim(\mathscr{B}|_{[1,t]}) \leq t\mathtt{m} + \mathtt{n} \leq t\mathtt{m} + \mathtt{p}\mathtt{l} \qquad (\mathtt{p}(\mathtt{l}-1) \leq \mathtt{n} \leq \mathtt{p}\mathtt{l})$

$\Longrightarrow \quad (\mathtt{m}, \mathtt{n})$ and $(\mathtt{m}, \mathtt{l})$ bound the complexity of the system

An exact identification problem (revised)

Problem P2 (Exact identification)

Given a vector time series

$$w_{\mathrm{d}} = \big(w_{\mathrm{d}}(1), \dots, w_{\mathrm{d}}(T)\big) \in (\mathbb{R}^{\mathtt{w}})^T$$

find the smallest $\mathtt{m} \in \mathbb{N}$, $\mathtt{l} \in \mathbb{N}$ and $\mathscr{B} \in \mathscr{L}_{\mathtt{m},\mathtt{l}}^{\mathtt{w}}$, such that $w_{\mathrm{d}} \in \mathscr{B}$.

comments:

- no separation between inputs and outputs
- the complexity is bounded by $(\mathtt{m}, \mathtt{l})$

Most powerful unfalsified model

The most powerful unfalsified model in the model class $\mathscr{L}^{\mathtt{w}}_{\mathtt{m},\mathtt{l}}$ of a time series $w_{\mathrm{d}} \in (\mathbb{R}^{\mathtt{w}})^T$ is the system $\mathscr{B}_{\mathsf{mpum}}$ that is

1. in the model class, *i.e.*, $\mathscr{B}_{\mathsf{mpum}} \in \mathscr{L}^{\mathtt{w}}_{\mathtt{m},\mathtt{l}}$,
2. unfalsified, *i.e.*, $w_{\mathrm{d}} \in \mathscr{B}_{\mathsf{mpum}}|_{[1,T]}$, and
3. most powerful among all LTI unfalsified systems, *i.e.*,

$$\mathscr{B} \in \mathscr{L}^{\mathtt{w}}_{\mathtt{m},\mathtt{l}} \text{ and } w_{\mathrm{d}} \in \mathscr{B}|_{[1,T]} \quad \Longrightarrow \quad \mathscr{B}_{\mathsf{mpum}}|_{[1,T]} \subseteq \mathscr{B}|_{[1,T]}.$$

the MPUM need not exist, but if it does, then it is unique

Identifiability question

P2 is the problem of computing the MPUM

the following related question is of interest:

Suppose that $w_{\mathrm{d}} \in \mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ and upper bounds $\mathtt{n}_{\max}$ and $\mathtt{l}_{\max}$ of the order $\mathtt{n}$ and the lag $\mathtt{l}$ of $\mathscr{B}$ are given.

Under what conditions is $\mathscr{B}_{\mathsf{mpum}}(w_{\mathrm{d}})$ equal to the system $\mathscr{B}$ that generated w_{d}?

the answer is given by the following lemma

Fundamental Lemma

Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w},\mathtt{n}}_{\mathtt{m}}$ be controllable and let $w_{\mathrm{d}} := (u_{\mathrm{d}}, y_{\mathrm{d}}) \in \mathscr{B}|_{[1,T]}$. Then, if u_{d} is persistently exciting of order $L+\mathtt{n}$,

$$\text{image}\left(\begin{bmatrix} w_{\mathrm{d}}(1) & w_{\mathrm{d}}(2) & w_{\mathrm{d}}(3) & \cdots & w_{\mathrm{d}}(T-L+1) \\ w_{\mathrm{d}}(2) & w_{\mathrm{d}}(3) & w_{\mathrm{d}}(4) & \cdots & w_{\mathrm{d}}(T-L+2) \\ w_{\mathrm{d}}(3) & w_{\mathrm{d}}(4) & w_{\mathrm{d}}(5) & \cdots & w_{\mathrm{d}}(T-L+3) \\ \vdots & \vdots & \vdots & & \vdots \\ w_{\mathrm{d}}(L) & w_{\mathrm{d}}(L+1) & w_{\mathrm{d}}(L+2) & \cdots & w_{\mathrm{d}}(T) \end{bmatrix}\right) = \mathscr{B}|_{[1,L]}$$

$\Longrightarrow$ under the conditions of the FL, any L samples long response y of $\mathscr{B}$ can be obtained as $y = \mathscr{H}_L(y_{\mathrm{d}})g$, for certain g $\rightsquigarrow$ algorithms

$\Longrightarrow$ with $L = \mathtt{l}_{\max}+1$, the FL gives conditions for identifiability

Persistency of excitation

the sequence $u_{\mathrm{d}} = \big(u_{\mathrm{d}}(1), \ldots, u_{\mathrm{d}}(T)\big)$ is persistently exciting of order $L+\mathtt{n}$ if the Hankel matrix

$$\mathscr{H}_{L+\mathtt{n}}(u_{\mathrm{d}}) := \begin{bmatrix} u_{\mathrm{d}}(1) & u_{\mathrm{d}}(2) & u_{\mathrm{d}}(3) & \cdots & u_{\mathrm{d}}(T-L-\mathtt{n}+1) \\ u_{\mathrm{d}}(2) & u_{\mathrm{d}}(3) & u_{\mathrm{d}}(4) & \cdots & u_{\mathrm{d}}(T-L-\mathtt{n}+2) \\ u_{\mathrm{d}}(3) & u_{\mathrm{d}}(4) & u_{\mathrm{d}}(5) & \cdots & u_{\mathrm{d}}(T-L-\mathtt{n}+3) \\ \vdots & \vdots & \vdots & & \vdots \\ u_{\mathrm{d}}(L+\mathtt{n}) & u_{\mathrm{d}}(L+\mathtt{n}+1) & u_{\mathrm{d}}(L+\mathtt{n}+2) & \cdots & u_{\mathrm{d}}(T) \end{bmatrix}$$

is of full row rank

Algorithms for exact identification

($w_{\rm d} \mapsto$ representation of the MPUM)

Overview of available algorithms

1. $w_{\rm d} \mapsto R(\xi)$
2. $w_{\rm d} \mapsto$ impulse response H
3. $w_{\rm d} \mapsto (A,B,C,D)$ (possibly balanced)

3.a. $w_{\rm d} \mapsto R(\xi) \mapsto (A,B,C,D)$ or $w_{\rm d} \mapsto H \mapsto (A,B,C,D)$

3.b. $w_{\rm d} \mapsto \mathscr{O}_{\mathtt{l}_{\max}+1}(A,C) \mapsto (A,B,C,D)$

3.c. $w_{\rm d} \mapsto \big(x_{\rm d}(1),\ldots,x_{\rm d}(\mathtt{n}_{\max}+\mathtt{m}+1)\big) \mapsto (A,B,C,D)$

$$\boxed{w_{\rm d} \mapsto R(\xi)}$$

under the assumptions of the FL, $\text{image}\big(\mathscr{H}_{\mathtt{l}_{\max}+1}(w_{\rm d})\big) = \mathscr{B}|_{[1,\mathtt{l}_{\max}+1]}$

$\implies$ a basis for left $\ker\big(\mathscr{H}_{\mathtt{l}_{\max}+1}(w_{\rm d})\big)$ defines a kernel repr. of $\mathscr{B}$

let

$$\begin{bmatrix}\tilde R_0 & \tilde R_1 & \cdots & \tilde R_{\mathtt{l}_{\max}}\end{bmatrix} \mathscr{H}_{\mathtt{l}_{\max}+1}(w_{\rm d}) = 0, \quad \text{where } \tilde R_i \in \mathbb{R}^{g\times \mathtt{w}}$$

and define $\tilde R(\xi) = \sum_{i=0}^{\mathtt{l}_{\max}} \xi^i \tilde R_i$

then $\mathscr{B} = \ker\big(\tilde R(\sigma)\big)$ is, in general, a "nonminimal" kernel representation

$$\boxed{w_{\rm d} \mapsto R(\xi)}$$

$\tilde R$ can be made "minimal" by standard polynomial linear algebra alg.

find a unimodular polynomial matrix U, such that

$$U\tilde R = \begin{bmatrix} R \\ 0 \end{bmatrix} \quad \text{and } R \text{ is full row rank}$$

then $\ker\big(R(\sigma)\big) = 0$ is minimal kernel representation

refinements:

- efficient recursive computation (exploiting the Hankel structure)
- as a byproduct an input/output partition of the variables
- a shortest lag kernel representation (*i.e.*, R row proper)

$w_\mathrm{d} \mapsto H$

under the conditions of FL, there is G, such that $H = \mathscr{H}_t(y_\mathrm{d})G$

the problem reduces to the one of finding a particular G. Define

$$\begin{bmatrix} \mathscr{H}_{\mathtt{l}_{\max}+t}(u_\mathrm{d}) \\ \mathscr{H}_{\mathtt{l}_{\max}+t}(y_\mathrm{d}) \end{bmatrix} =: \begin{bmatrix} U_\mathrm{p} \\ U_\mathrm{f} \\ Y_\mathrm{p} \\ Y_\mathrm{f} \end{bmatrix} \qquad \begin{aligned} \text{row dim}(U_\mathrm{p}) &= \text{row dim}(Y_\mathrm{p}) = \mathtt{l}_{\max} \\ \text{row dim}(U_\mathrm{f}) &= \text{row dim}(Y_\mathrm{f}) = t \end{aligned}$$

let u_d be p.e. of order $t + \mathtt{l}_{\max} + \mathtt{n}_{\max}$. Then there is G, such that

$$\begin{bmatrix} U_\mathrm{p} \\ Y_\mathrm{p} \\ U_\mathrm{f} \end{bmatrix} G = \begin{bmatrix} 0 \\ 0 \\ \begin{bmatrix} I_\mathtt{m} \\ 0 \end{bmatrix} \end{bmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{zero ini. conditions} \\ \leftarrow \text{impulse input} \end{matrix} \tag{1}$$

$$Y_\mathrm{f} \quad G = H$$

$w_\mathrm{d} \mapsto H$

block algorithm for computation of $\big(H(0), \dots, H(t-1)\big)$:

1. Input: u_d, y_d, $\mathtt{l}_{\max}$, and t.
2. Solve the system of equations (1). Let $\bar{G}$ be the computed solution.
3. Compute $H = Y_\mathrm{f}\bar{G}$.
4. Output: the first t samples of the impulse response H.

refinements:

- solve (1) efficiently by exploiting the Hankel structure
- do the computations iteratively for pieces of $H \rightsquigarrow$ iterative algorithm
- automatically choose t, for a sufficient decay of H

$w_\mathrm{d} \mapsto (A,B,C,D)$

- $w_\mathrm{d} \mapsto H(0:2\mathtt{l}_{\max})$ or $R(\xi) \xrightarrow{\text{realization}} (A,B,C,D)$
- $w_\mathrm{d} \mapsto \mathscr{O}_{\mathtt{l}_{\max}+1}(A,C) \xrightarrow{(2)} (A,B,C,D)$
- $w_\mathrm{d} \mapsto \big(x_\mathrm{d}(1), \dots, x_\mathrm{d}(\mathtt{n}_{\max}+\mathtt{m}+1)\big) \xrightarrow{(3)} (A,B,C,D)$

(2) and (3) are easy:

$$\mathscr{O}_{\mathtt{l}_{\max}+1}(A,C) \mapsto (A,C) \quad \text{and} \quad (u_\mathrm{d}, y_\mathrm{d}, A, C) \mapsto (B, C, x_\mathrm{ini}) \tag{2}$$

$$\begin{bmatrix} x_\mathrm{d}(2) & \cdots & x_\mathrm{d}(\mathtt{n}_{\max}+\mathtt{m}+1) \\ y_\mathrm{d}(1) & \cdots & y_\mathrm{d}(\mathtt{n}_{\max}+\mathtt{m}) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_\mathrm{d}(1) & \cdots & x_\mathrm{d}(\mathtt{n}_{\max}+\mathtt{m}) \\ u_\mathrm{d}(1) & \cdots & u_\mathrm{d}(\mathtt{n}_{\max}+\mathtt{m}) \end{bmatrix} \tag{3}$$

$\mathscr{O}_{\mathtt{l}_{\max}+1}(A,C) \mapsto (A,B,C,D)$

C is the first block entry of $\mathscr{O}_{\mathtt{l}_{\max}+1}(A,C)$ and A is given by

$$\big(\sigma^* \mathscr{O}_{\mathtt{l}_{\max}+1}(A,C)\big)A = \big(\sigma \mathscr{O}_{\mathtt{l}_{\max}+1}(A,C)\big) \quad \text{shift equation}$$

(σ^* removes the last block entry and σ removes the first block entry)

once C and A are known, the system of equations

$$y_\mathrm{d}(t) = CA^t x_\mathrm{d}(1) + \sum_{\tau=1}^{t-1} CA^{t-1-\tau} B u_\mathrm{d}(\tau) + D\delta(t+1), \quad \text{for } t = 1, \dots, \mathtt{l}_{\max}+1$$

is linear in D, B, $x_\mathrm{d}(1)$ and can be solved explicitly (*e.g.*, by using Kronecker products)

$$\boxed{w_\mathrm{d} \mapsto \mathscr{O}_{\mathtt{l}_\mathrm{max}+1}(A,C)}$$

the columns of $\mathscr{O}_{\mathtt{l}_\mathrm{max}+1}(A,C)$ are $\mathtt{n}$ linearly indep. free responses of Σ

under the conditions of FL, such responses can be computed from data

$$\begin{bmatrix} \mathscr{H}_t(u_\mathrm{d}) \\ \mathscr{H}_t(y_\mathrm{d}) \end{bmatrix} G = \begin{bmatrix} 0 \\ Y_0 \end{bmatrix} \quad \begin{matrix} \leftarrow & \text{zero inputs} \\ \leftarrow & \text{free responses} \end{matrix}$$

in order to obtain lin. indep. free responses, G should be maximal rank

once we have a maximal rank matrix of free responses Y_0

$$Y_0 = \mathscr{O}_{\mathtt{l}_\mathrm{max}+1}(A,C) \underbrace{\begin{bmatrix} x_{\mathrm{ini},1} & \cdots & x_{\mathrm{ini},j} \end{bmatrix}}_{X_\mathrm{ini}} \qquad \text{rank revealing factorization}$$

$\rightsquigarrow$ $\mathscr{O}_{\mathtt{l}_\mathrm{max}+1}(A,C)$ and X_ini, the factorization fixes the state space basis

$$\boxed{w_\mathrm{d} \mapsto \big(x_\mathrm{d}(1), \dots, x_\mathrm{d}(\mathtt{n}_\mathrm{max}+\mathtt{m}+1)\big)}$$

if the free responses are sequential, *i.e.*, if Y_0 is block-Hankel, then X_ini is a state sequence of Σ

computation of sequential free responses is achieved as follows

$$\begin{bmatrix} U_\mathrm{p} \\ Y_\mathrm{p} \\ U_\mathrm{f} \end{bmatrix} G = \begin{bmatrix} U_\mathrm{p} \\ Y_\mathrm{p} \\ 0 \end{bmatrix} \begin{matrix} \Big\} \text{ sequential ini. conditions} \\ \leftarrow \text{ zero inputs} \end{matrix} \tag{4}$$

$$Y_\mathrm{f} \quad G = \quad Y_0$$

note: now we use the splitting of the data into “past” and “future”

$$Y_0 = \mathscr{O}_{\mathtt{l}_\mathrm{max}+1}(A,C) \begin{bmatrix} x_\mathrm{d}(1) & \cdots & x_\mathrm{d}(\mathtt{n}_\mathrm{max}+\mathtt{m}+1) \end{bmatrix} \qquad \text{rank revealing factorization}$$

Relation to other exact identification algorithms

MOESP type algorithms

project orthogonally the rows of $\mathscr{H}_{\mathtt{n}_\mathrm{max}}(y_\mathrm{d})$ on $\Big(\text{row span}\big(\mathscr{H}_{\mathtt{n}_\mathrm{max}}(u_\mathrm{d})\big)\Big)^\perp$

$$Y_0 := \mathscr{H}_{\mathtt{n}_\mathrm{max}}(y_\mathrm{d}) \Pi_{u_\mathrm{d}}^\perp$$

where

$$\Pi_{u_\mathrm{d}}^\perp := \Big(I - \mathscr{H}_{\mathtt{n}_\mathrm{max}}^\top(u_\mathrm{d}) \big(\mathscr{H}_{\mathtt{n}_\mathrm{max}}(u_\mathrm{d}) \mathscr{H}_{\mathtt{n}_\mathrm{max}}^\top(u_\mathrm{d})\big)^{-1} \mathscr{H}_{\mathtt{n}_\mathrm{max}}(u_\mathrm{d})\Big)$$

observe that $\Pi_{u_\mathrm{d}}^\perp$ is maximal rank and

$$\begin{bmatrix} \mathscr{H}_{\mathtt{n}_\mathrm{max}}(u_\mathrm{d}) \\ \mathscr{H}_{\mathtt{n}_\mathrm{max}}(y_\mathrm{d}) \end{bmatrix} \Pi_{u_\mathrm{d}}^\perp = \begin{bmatrix} 0 \\ Y_0 \end{bmatrix}$$

$\implies$ the orthogonal projection computes free responses

Comments

- $T - \mathtt{n}_{\max} + 1$ free responses are computed via the orth. proj. while $\mathtt{n}_{\max}$ such responses suffice for the purpose of exact identification
- the orth. proj. is a geometric operation, whose system theoretic meaning is not revealed
- the condition for $\operatorname{rank}(Y_0) = \mathtt{n}$, given in the MOESP literature,

$$\operatorname{rank}\left(\begin{bmatrix} X_{\text{ini}} \\ \mathscr{H}_{\mathtt{n}_{\max}}(u_{\text{d}}) \end{bmatrix}\right) = \mathtt{n} + \mathtt{n}_{\max}\mathtt{m}$$

  is not verifiable from the data $(u_{\text{d}}, y_{\text{d}})$

  $\Longrightarrow$ can not be checked whether the computation gives $\mathscr{O}(A,C)$

N4SID-type algorithms

consider the splitting of the data into "past" and "future"

$$\mathscr{H}_{2\mathtt{n}_{\max}}(u_{\text{d}}) =: \begin{bmatrix} U_{\text{p}} \\ U_{\text{f}} \end{bmatrix}, \qquad \mathscr{H}_{2\mathtt{n}_{\max}}(y_{\text{d}}) =: \begin{bmatrix} Y_{\text{p}} \\ Y_{\text{f}} \end{bmatrix}$$

with $\text{row dim}(U_{\text{p}}) = \text{row dim}(U_{\text{f}}) = \text{row dim}(Y_{\text{p}}) = \text{row dim}(Y_{\text{f}}) = \mathtt{n}_{\max}$ and let

$$W_{\text{p}} := \begin{bmatrix} U_{\text{p}} \\ Y_{\text{p}} \end{bmatrix}$$

the key step of the N4SID algorithms is the oblique projection of the rows of Y_{f} along $\text{row span}(U_{\text{f}})$ onto $\text{row span}(W_{\text{p}})$

$$Y_0 := Y_{\text{f}} /_{U_{\text{f}}} W_{\text{p}} := Y_{\text{f}} \underbrace{\begin{bmatrix} W_{\text{p}}^\top & U_{\text{f}}^\top \end{bmatrix} \begin{bmatrix} W_{\text{p}} W_{\text{p}}^\top & W_{\text{p}} U_{\text{f}}^\top \\ U_{\text{f}} W_{\text{p}}^\top & U_{\text{f}} U_{\text{f}}^\top \end{bmatrix}^+ \begin{bmatrix} W_{\text{p}} \\ 0 \end{bmatrix}}_{\Pi_{\text{obl}}}$$

N4SID-type algorithms

observe that

$$\begin{bmatrix} W_{\text{p}} \\ U_{\text{f}} \\ Y_{\text{f}} \end{bmatrix} \Pi_{\text{obl}} = \begin{bmatrix} W_{\text{p}} \\ 0 \\ Y_0 \end{bmatrix}$$

(in fact Π_{obl} is the least-norm, least-squares solution)

$\Longrightarrow$ the oblique projection computes sequential free responses

Comments

- $T - 2\mathtt{n}_{\max} + 1$ sequential free responses are computed via the oblique projection while $\mathtt{n}_{\max} + \mathtt{m} + 1$ such responses suffice for exact ident.
- the oblique proj. is a geometric operation, whose system theoretic meaning is not revealed
- the conditions for $\operatorname{rank}(Y_0) = \mathtt{n}$, given in the N4SID literature,

  1. u_{d} persistently exciting of order $2\mathtt{n}_{\max}$ and
  2. $\text{row span}(X_{\text{ini}}) \cap \text{row span}(U_{\text{f}}) = \{0\}$

  are not verifiable from the data $(u_{\text{d}}, y_{\text{d}})$

Conclusions

- the ARMAX problem is invariably considered in the sys. id. literature

  we discussed the less usual exact deterministic identification problem

- exact identification is interesting and nontrivial; it has two parts:

  1. check if w_d completely specifies $\mathscr{B}$ $\rightsquigarrow$ FL
  2. find a desired representation of $\mathscr{B}$ from w_d

  $$w_\mathrm{d} \mapsto R(\xi) \qquad w_\mathrm{d} \mapsto H \qquad w_\mathrm{d} \mapsto (A,B,C,D) \qquad w_\mathrm{d} \mapsto \text{balanced}$$

- of course we want to find approximate (stochastic) model

Conclusions

The Fundamental Lemma:

Let $\mathscr{B} \in \mathscr{L}^{\mathtt{w},\mathtt{n}}_{\mathtt{m}}$ be controllable and let $w_\mathrm{d} := (u_\mathrm{d}, y_\mathrm{d}) \in \mathscr{B}|_{[1,T]}$. Then, if u_d is persistently exciting of order $L+\mathtt{n}$,

$$\text{image}\left(\begin{bmatrix} w_\mathrm{d}(1) & w_\mathrm{d}(2) & w_\mathrm{d}(3) & \cdots & w_\mathrm{d}(T-L+1) \\ w_\mathrm{d}(2) & w_\mathrm{d}(3) & w_\mathrm{d}(4) & \cdots & w_\mathrm{d}(T-L+2) \\ w_\mathrm{d}(3) & w_\mathrm{d}(4) & w_\mathrm{d}(5) & \cdots & w_\mathrm{d}(T-L+3) \\ \vdots & \vdots & \vdots & & \vdots \\ w_\mathrm{d}(L) & w_\mathrm{d}(L+1) & w_\mathrm{d}(L+2) & \cdots & w_\mathrm{d}(T) \end{bmatrix}\right) = \mathscr{B}|_{[1,L]}$$

is a convenient tool for computing responses from data.

Conclusions

- we gave system theoretic interpretation of the orth. and oblique proj.

- MOESP and N4SID alg. are computationally inefficient; more than what is necessary for exact ident. is computed $\rightsquigarrow$ cheaper algorithms

- the FL gives sharp conditions for identifiability, verifiable from the data

  $\rightsquigarrow$ our alg. might be applicable in cases when the classical alg. are not

- we clarified the role of the splitting: the “past” assigns the initial conditions and in the “future” a desired response is computed

Thank you!

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