UNIVERSITY OF SOUTHAMPTON ELEC3035W1

EXAMINATION 2007/08

Control Systems Design

Duration: 120 mins

Answer THREE questions with at least one from each part.

University approved calculators MAY be used.

An approximate marking scheme is indicated.

Number of pages: 7

Part I

Question 1

a) An nth order scalar autonomous linear time-invariant system can be represented by an nth order linear constant-coefficients difference equation

$$p_0 y(t) + p_1 y(t+1) + \cdots + y(t+n) = 0, \qquad \text{for all } t \in \mathbb{Z}, \tag{1}$$

where $p_0, p_1, \ldots, p_{n-1}$ are parameters of the model. Explain how to find the smallest n and corresponding parameters $p_0, p_1, \ldots, p_{n-1}$, such that the system defined by (1) is an exact model for a given sequence

$$y_\mathrm{d} = \big(y_\mathrm{d}(1), \ldots, y_\mathrm{d}(T)\big),$$

i.e., find the minimal n, for which there exist $p_0, p_1, \ldots, p_{n-1}$ satisfying the system of equations

$$p_0 y_\mathrm{d}(t) + p_1 y_\mathrm{d}(t+1) + \cdots + y_\mathrm{d}(t+n) = 0, \qquad \text{for } t = 1, \ldots, T-n. \tag{2}$$

[15 marks]

b) Apply the method of a) on the sequence of the first nine Fibonacci numbers

$$y_\mathrm{d} = (0, 1, 1, 2, 3, 5, 8, 13, 21).$$

[5 marks]

c) Suppose that the system order n is fixed and there is no exact model for y_d of order n. How would you modify the method of step a) for obtaining an approximate model? In what sense does your approximate model approximate y_d? [5 marks]

Question 2

Consider an nth order controllable discrete-time state space dynamical system defined by the difference equation

$$x(t+1) = Ax(t) + Bu(t).$$

We will call the squared 2-norm of the input sequence

$$U_T := \text{col}\big(u(0), u(1), \dots, u(T-1)\big),$$

i.e., $\|U_T\|_2^2$, the "energy" of the input $\big(u(0), u(1), \dots, u(T-1)\big)$.

a) Find the reachable set in $T \geq n$ seconds with bounded 2-norm input

$$\mathscr{R}_{T,\delta} := \{\, x(T) \mid x(t+1) = Ax(t) + Bu(t),\ x(0) = 0,\ \|U_T\|_2^2 \leq \delta \,\}.$$

You answer should be given in terms of the system parameters A, B, the time limit $T \geq n$, and the norm bound $\delta \geq 0$. [15 marks]

b) Find the set of reachable states with bounded 2-norm input without time limit, i.e.,

$$\mathscr{R}_{\delta} := \lim_{T \to \infty} \mathscr{R}_{T,\delta}.$$

[5 marks]

c) What are the sets $\mathscr{R}_{T,\delta}$ and $\mathscr{R}_{\delta}$ in the cases of first and second order systems? Compute $\mathscr{R}_{\delta}$ for the system $x(t+1) = 0.5x(t) + u(t)$ and $\delta = 1$. [5 marks]

Question 3

Design a dead-beat controller for the system defined by the difference equation

$$P(\sigma)y = Q(\sigma)u,$$

where $(\sigma y)(t) := y(t+1)$ is the shift operator,

$$P(z) := (z-1)^2 \qquad \text{and} \qquad Q(z) := z+1$$

using polynomial and state space methods.

a) *Design by polynomial methods*

i) Define the desired closed-loop characteristic polynomial and the Diophantine equation corresponding to the controller input/output representation

$$R(\sigma)u = -S(\sigma)y.$$

ii) Find the solution of the Diophantine equation of minimal degree.

iii) Give the controller and the resulting closed-loop system.

[10 marks]

b) *Design by state space methods*

i) Derive a state space representation of the plant.

ii) Calculate the state feedback gain.

iii) Give the state-feedback controller and the resulting closed-loop system.

[10 marks]

c) *Compare the results* Comment on the similarities and differences between the approaches used and results obtained in a) and b). What is missing and what would make the two approaches equivalent? [5 marks]

Part II

Question 4

a) Determine all equilibrium points for each of the following three systems

i)

$$\begin{aligned}\dot{x}_1 &= (x_2+5)x_1 \\ \dot{x}_2 &= (x_1-5)x_2\end{aligned}$$

ii)

$$\ddot{x}+5\dot{x}+x(16-x) = 0$$

iii)

$$\begin{aligned}\dot{x}_1 &= (x_2+\alpha)(x_1+\beta) \\ \dot{x}_2 &= (x_1+\alpha)(x_2-\beta)\end{aligned}$$

where α and β are constants and $\alpha \neq \beta$.

[9 marks]

b) State the theorem associated with Lyapunov's first method for stability analysis. Apply this method to the system listed under iii) in part a). [16 marks]

Question 5

a) Consider the nonlinear system described by

$$\begin{aligned}\dot{x}_1 &= -2x_1 + 3x_2 + \sin x_1 \\ \dot{x}_2 &= -x_2 \sin x_1 + u \sin 2x_1\end{aligned}$$

Obtain an equivalent description for this system in terms of the following state variables

$$\begin{aligned}z_1 &= x_1 \\ z_2 &= 3x_2 + \sin x_1\end{aligned}$$

[5 marks]

b) Consider the problem of designing the control input for a single-input nonlinear system of the form

$$\dot{x} = f(x, u)$$

Illustrate the main steps of the input-state approach by designing u for the system in part (a) of this question to leave all poles of the closed-loop linear system at -4 in the complex plane [15 marks]

c) In the analysis of a nonlinear system with Lyapunov function $V(x)$, it is only possible to establish that

$$\dot{V}(x) \leq 0$$

Explain how LaSalles principle can be applied here. Also what is the difference between local and global properties in this form of analysis? [5 marks]

Question 6

a) The describing function of the nonlinearity shown in **Figure 1** is given by

$$N(A) = \begin{cases} \alpha_1, \ 0 \le A \le \delta \\ \frac{2(\alpha_1-\alpha_2)}{\pi}(\beta + \frac{\delta}{A}\cos\beta) + \alpha_2, \ A > \delta \end{cases}$$

where $\sin\beta = \frac{\delta}{A}$.

Write down the basic description of this nonlinearity which is the starting point for obtaining this describing function. Explain how the $N(A)$ here can be directly used to obtain the describing function of the saturation and dead-zone nonlinearities. [6 Marks]

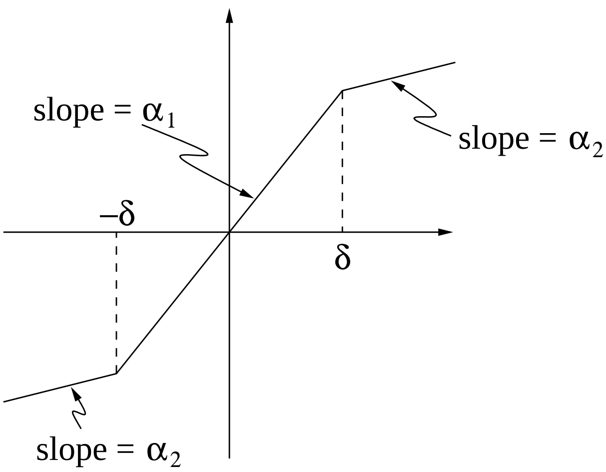

Figure 1:

b) Detail, with the aid of clearly labelled diagrams, how the describing function can be used to predict the existence or otherwise of limit cycles in a unity negative feedback control scheme whose forward path is given by a nonlinearity which can be approximated by a describing function in series with a transfer function $G(s)$. For the case when the nonlinearity is that of part (a), for which of the following choices of $G(s)$ could a limit cycle be predicted?

$$G(s) = \frac{1}{s(1+7s)^2}, \quad \text{or} \quad G(s) = \frac{1}{s(1+7s)}$$

[9 Marks]

c) Apply Lyapunov's second method to investigate the stability of the origin of the system

$$\ddot{x} + c\dot{x}^3 + hx = 0$$

using the candidate Lyapunov function

$$V = \frac{1}{2}(hx^2 + \dot{x}^2)$$

[10 marks]

END OF PAPER