Elec 3035 — Control Systems Design 2007/8 — Part II Solutions

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Question 4 a)

In all cases, construct a state-space model of the form $\dot{x} = f(x)$. Equilibrium points are then defined by $f(x) = 0$.

For system i) the equilibrium points are $x_1 = 0, x_2 = 0$ and $x_1 = 5, x_2 = -5$.

For system ii) set $x_1 = x$ and $x_2 = \dot{x}$ to give the state-space model

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1(16 - x_1) - 5x_2 \end{aligned}$$

Equilibrium points at $x_1 = 0, x_2 = 0$ and $x_1 = 16, x_2 = 0$.

For system iii) equilibrium points at $x_1 - \alpha, x_2 = -\alpha$ and $x_1 = -\beta, x_2 = \beta$. [9 marks]

Question 4 b)

Construct the linearised approximation $\dot{z} = Az$ at each equilibrium point where A is Jacobian matrix and evaluate its eigenvalues at this point. If all eigenvalues are stable, i.e. in the open left-half of the complex plane then (local) stability holds, if at least one eigenvalue is in the open right-half of the complex plane then (local) instability holds. If there are eigenvalues on the imaginary axis then the test is inconclusive.

For iii) from part (a) of this question, the Jacobian matrix is

$$A = \begin{bmatrix} \alpha + x_2 & \beta + x_1 \\ -\beta + x_2 & \alpha + x_1 \end{bmatrix}$$

For the equilibrium point at $x_1 = -\alpha, x_2 = -\alpha$

$$A = \begin{bmatrix} 0 & \beta - \alpha \\ -\beta - \alpha & 0 \end{bmatrix}$$

The eigenvalues of this matrix are given by

$$\lambda^2 - (\alpha^2 - \beta^2) = 0$$

Hence if $\alpha > \beta$ the system is unstable and if $\beta > \alpha$ then the test is inconclusive.

For the equilibrium point at $x_1 = -\beta, x_2 = \beta$

$$A = \begin{bmatrix} \alpha - \beta & 0 \\ -0 & \alpha - \beta \end{bmatrix}$$

The eigenvalues of this matrix are given by

$$(\lambda^2 + \gamma)^2 = 0$$

where

$$\gamma = \beta - \alpha$$

Hence if $\beta > \alpha$, the test is inconclusive and if $\alpha < \beta$ the system is unstable. [16 marks]

Question 5 a)

New state variable description

$$\begin{aligned} \dot{z}_1 &= -2z_1 + z_2 \\ \dot{z}_2 &= -2z_1 \cos z_1 + \cos z_1 \sin z_1 + 3u \sin 2z_1 \end{aligned}$$

[5 marks]

Question 5 b)

This system has equilibrium point at $(0, 0)$ and the nonlinearities present can be cancelled by using a control law of the form

$$u = \frac{1}{3\sin 2z_1}(r - \cos z_1 \sin z_1 + 2z_1 \cos z_1)$$

where r is an equivalent input to be designed.

Resulting system is linear and described by the state-space model

$$\begin{aligned} \dot{z}_1 &= -2z_1 + z_2 \\ \dot{z}_2 &= r \end{aligned}$$

The dynamics now are linear and controllable. Hence they can be stabilised by the linear state feedback law

$$r = -k_1 z_1 - k_2 z_2$$

Using

$$u = -4z_2$$

now gives both closed-loop poles at -4 in the open left-half of the complex plane. Finally convert back to the original state variables using

$$\begin{aligned} x_1 &= z_1 \\ x_2 &= \frac{z_2 - \sin z_1}{3} \end{aligned}$$

[15 marks]

Question 5 c)

La Salles invariant principle in application can be applied if it can be shown that $\dot{V}(x) = 0$ only holds for the origin. Hence use the system equations to see if this property holds. If it does then by this principle asymptotic stability holds either locally or globally depending on whether or not $V(x) > 0$ locally or globally and the same for $\dot{V}(x)$. [5 marks]

Question 6 a)

$$u(t) = \begin{cases} \alpha_1 A \sin \omega t, \ 0 \leq t \leq a \\ (\alpha_1 - \alpha_2)\delta + \alpha_2 A \sin \omega t, \ a \leq t \leq \frac{\pi}{2} \end{cases}$$

where $a = \sin^{-1}(\frac{\delta}{A})$.

In the case of saturation, set α_1 equal to the gain of the linear part, set $\alpha_2 = 0$ and δ the value when the system goes into saturation.

In the case of dead-zone, set $\alpha_1 = 0$, and α_2 the gain of the linear part and δ equal to the value when the dead-zone ends. [6 marks]

Question 6 b)

Standard derivation (in the case of a single-valued nonlinearity) of

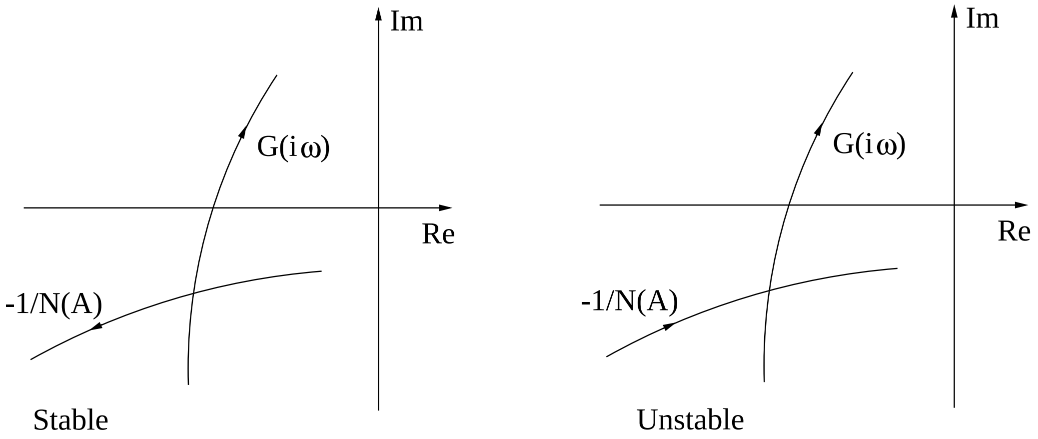
$$G(j\omega) = -\frac{1}{N(A)}$$

starting from the block diagram of a unity negative feedback control scheme (with zero reference signa) with forward path consisting of $G(s)$ in series with the describing function approximation to the nonlinearity.

Also the stability criteria given in Figure 1

Limit cycle is only possible if frequency response $G(j\omega)$ crosses the negative real axis — this is not true for the first case but is for the second.

[9 marks]



(a) (b)

Figure 1:

Question 6 c)

State-space model of system dynamics

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -hx_1 - cx_2^3\end{aligned}$$

Also

$$\dot{V}(x) = hx_1\dot{x}_1 + x_2\dot{x}_2 = -cx_2^4$$

The absence of x_1 terms in $\dot{V}(x)$ means that $\dot{V}(x)$ can only ever be negative semi-definite. To have this property also requires that $c > 0$ — in which case we can predict stability. [10 marks]