$\mathcal{H}_2$-optimal linear parametric design

Ivan Markovsky, Sabine Van Huffel, and Bart De Moor

ESAT-SISTA, K.U.Leuven, Belgium

MTNS, 8 July 2004, Leuven

Motivation

consider an overdetermined system of equations

$$\underbrace{\begin{bmatrix} a_1 & \cdots & a_{n_p} \end{bmatrix}}_{A} p \approx a_0, \qquad \text{where} \quad a_i \in \mathbb{R}^m \text{ and } m \gg n_p$$

interpretation: approximate a_0 by a linear combination of $a_1, \ldots, a_{n_p}$

approximation in the sense $\min_p \|Ap - a_0\|_2$ $\rightsquigarrow$ least squares problem

"dynamic least squares" problem

$$\underbrace{\begin{bmatrix} H_1(z) & \cdots & H_{n_p}(z) \end{bmatrix}}_{H(z)} p \approx H_0(z), \qquad H_i(z) \text{ are transfer functions}$$

Motivation

interpretation: approximate $H_0(z)$ by a linear comb. of $H_1(z), \ldots, H_{n_p}(z)$

approximation in the sense $\min_p \|H(z)p - H_0(z)\|_{\mathcal{H}_2} \rightsquigarrow$ a "nice problem"

$\min_p \|H(z)p - H_0(z)\|_{\mathcal{H}_\infty}$ is also tractable (*i.e.*, convex)

aim: treat more general problems involving the system $H(z,p) := H(z)p$

p is a design parameter
chose p is a design problem $\implies$ parametric design

$H(z,p)$ is a linear function of p $\implies$ linear parametric design

Outline

- (Linearly) parameterized system
- $\mathcal{H}_2$-optimal approximation
- $\mathcal{H}_\infty$-optimal approximation
- Closed loop linear parametric design
- Adaptive parameter estimation
- Conclusion

Parameterized system

$\mathcal{S}$	—	set of systems
$p = \text{col}(p_1, \ldots, p_{n_p})$	—	parameter vector
$\mathbb{R}^{n_p}$	—	parameter space

definition: a parameterized system is a mapping $S : \mathbb{R}^{n_p} \to \mathcal{S}$

in this talk, $\mathcal{S}$ is the set of discrete-time LTI systems

$\Longrightarrow$ $S(p)$ is a discrete-time LTI system for any $p \in \mathbb{R}^{n_p}$

p unknown but bounded $\rightsquigarrow$ parametric disturbance
p to be chosen $\rightsquigarrow$ design parameter

Linearly parameterized system

$S(p)$ is linearly parameterized if S is a linear mapping

$$S(p) = \sum_{i=1}^{n_p} p_i S_i, \qquad S_i \in \mathcal{S} \text{ given}$$

we consider LTI systems and specify $S(p)$ by its transfer function

$$H(z,p) = \sum_{i=1}^{n_p} p_i H_i(z), \qquad H_i(z) \text{ given}$$

$\mathcal{H}_2$-optimal approximation

$H_0(z)$	—	target system
$\{H_i(z)\}_{i=1}^{n_p}$	—	basis systems
$H(z,p) := \sum_{i=1}^{n_p} p_i H_i(z)$	—	approximation
$\tilde{H}(p,z) := -H_0(z) + H(z,p)$	—	error of approximation

$\mathcal{H}_2$-optimal approximation problem: $\min_p \|\tilde{H}(p,z)\|_{\mathcal{H}_2}^2$

Solution of the $\mathcal{H}_2$ approximation problem

$\tilde{H}(z,p)$ is affine in p: $\tilde{H}(z,p) = -H_0(z) + \sum_{i=1}^{n_p} p_i H_i(z)$

state space realization

$H_i(z)$	—	(A_i, B_i, C_i, D_i)	$i = 0, 1, \ldots, n_p$
$\tilde{H}(z,p)$	—	$(\tilde{A}, \tilde{B}, \tilde{C}(p), \tilde{D}(p))$	where

$$\left[\begin{array}{c|c} A & B \\ \hline C(p) & D(p) \end{array}\right] = \left[\begin{array}{cccc|c} A_0 & & & & B_0 \\ & A_1 & & & B_1 \\ & & \ddots & & \vdots \\ & & & A_{n_p} & B_{n_p} \\ \hline -C_0 & p_1 C_1 & \cdots & p_{n_p} C_{n_p} & \sum_{i=1}^{n_p} p_i D_i - D_0 \end{array}\right]$$

Solution of the $\mathcal{H}_2$ approximation problem

$$\|\tilde{H}(z,p)\|^2_{\mathcal{H}_2} = \mathrm{tr}\left(\tilde{C}(p)\tilde{W}_c\tilde{C}^\top(p) + \tilde{D}(p)\tilde{D}^\top(p)\right)$$

where $\tilde{W}_c$ satisfies the discrete-time Lyapunov equation

$$\tilde{A}\tilde{W}_c\tilde{A}^\top - \tilde{W}_c + \tilde{B}\tilde{B}^\top = 0$$

$\|\tilde{H}(z,p)\|^2_{\mathcal{H}_2} = \tilde{p}^\top \tilde{F}\tilde{p}$ quadratic function of p, where

$$\tilde{p} := \mathrm{col}(-1,p) \quad \text{and} \quad \tilde{F}_{ij} := \mathrm{tr}\left(C_i W_{c,ij} C_j^\top\right)$$

$\implies$ $\mathcal{H}_2$-optimal approximation is a convex quadratic problem

$\mathcal{H}_\infty$-optimal approximation: $\min_p \|\tilde{H}(z,p)\|_{\mathcal{H}_\infty}$

$\mathcal{H}_\infty$-norm by bounded-real lemma

$$\|\tilde{H}(z,p)\|^2_{\mathcal{H}_\infty} < \gamma \iff \exists X = X^\top > 0 \quad \text{s.t.}$$

$$\begin{bmatrix} \tilde{A}^\top X\tilde{A} - X & \tilde{A}^\top X\tilde{B} & \tilde{C}^\top(p) \\ \tilde{B}^\top X\tilde{A} & \tilde{B}^\top X\tilde{B} - \gamma I & \tilde{D}^\top(p) \\ \tilde{C}(p) & \tilde{D}(p) & -\gamma I \end{bmatrix} < 0 \qquad \text{(LMI)}$$

$\left(\tilde{A},\tilde{B},\tilde{C}(p),\tilde{D}(p)\right)$ — a realization of $\tilde{H}(z,p)$

$C(\cdot)$ and $D(\cdot)$ are affine functions $\implies$ (LMI) is an LMI in X, p, and γ

$\mathcal{H}_\infty$-optimal approximation is an SDP problem

Closed-loop parametric design

consider a linearly parameterized LTI controller $K(z,p)$, connected in a closed loop to an LTI plant $P(z)$

parametric design: achieve a desired specification for the closed loop system $H(z,p)$ by choosing p (or prove that it is not feasible)

e.g., PID control is a linear combination of three LTI controllers

$\tilde{H}(z,p)$ is affine-fractional in p

$$\tilde{H}(z,p) = \left(I - H_{\mathrm{D}}(z,p)\right)^{-1} H_{\mathrm{N}}(z,p)$$

Closed-loop $\mathcal{H}_2$-optimal parametric design

$$\tilde{H}(z,p) = \Bigg(\underbrace{\begin{bmatrix} A_{\mathrm{N}} & 0 \\ B_{\mathrm{D}}C_{\mathrm{N}}(p) & A_{\mathrm{D}} + B_{\mathrm{D}}C_{\mathrm{D}}(p) \end{bmatrix}}_{\tilde{A}(p)}, \underbrace{\begin{bmatrix} B_{\mathrm{N}} \\ 0 \end{bmatrix}}_{\tilde{B}}, \underbrace{\begin{bmatrix} 0 & C_{\mathrm{D}}(p) \end{bmatrix}}_{\tilde{C}(p)} \Bigg)$$

$$\|\tilde{H}(z,p)\|^2_{\mathcal{H}_2} = \mathrm{tr}\left(\tilde{C}^\top(p)\tilde{W}_c(p)\tilde{C}(p)\right)$$

W_c is a function of p, the Lyapunov equation depends on the parameter

$$\tilde{A}(p)\tilde{W}_c(p)\tilde{A}^\top(p) - \tilde{W}_c(p) + \tilde{B}\tilde{B}^\top = 0$$

closed-loop $\mathcal{H}_2$-optimal parametric design is a non-convex problem

$$\|\tilde{H}(z,p)\|^2_{\mathcal{H}_2} = \tilde{p}^\top \tilde{F}(p)\tilde{p}, \quad \tilde{F}_{ij} := \mathrm{tr}\left(\tilde{C}_i^\top \tilde{W}_c(p)\tilde{C}_j\right)$$

Example: linearly parameterized LQG control

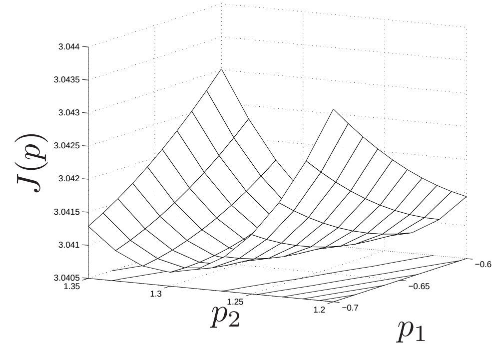
$H_0(z)$ — plant
$K(z,p) := p_1 K_1(z) + p_2 K_2(z)$ — linearly parameterized controller
$J(y,u) := \sum_{t=0}^{\infty} \big(y^\top(t) Q y(t) + u^\top(t) R u(t) \big), \quad Q \geq 0,\ R > 0$

problem: $\min_p J(y,u)$ subject to (y,u) is a trajectory of the close loop system $\tilde{H}(z,p)$

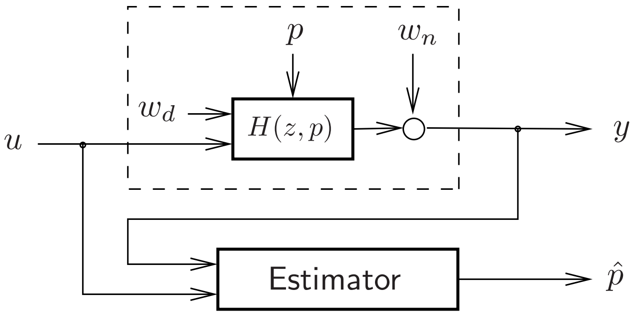
use (local) optimization to find $p_{\min}$

initial guess — $\arg\min_p \|K_0(z) - p_1 K_1(z) - p_2 K_2(z)\|_2^2$,
where $K_0(z)$ is the unstructured LQG controller

Controller	p_1	p_2	$\|\|\tilde{H}(z,p)\|\|_2$
$K_1(z)$	1	0	3.1311
$K_2(z)$	0	1	3.0446
$K(z,p)$	-0.6518	1.2783	3.0407
$K_0(z)$	—	—	2.9958



Adaptive parameter estimation



approximate p from real-time I/O measurements

$$H_i(z) =: \begin{bmatrix} H_{w_d,i}(z) & H_{u,i}(z) \end{bmatrix}$$

Adaptive parameter estimation (cont.)

$$Y := \begin{bmatrix} H_{u,1}(u) & \cdots & H_{u,n_p}(u) \end{bmatrix}$$
$$\tilde{Y} := \begin{bmatrix} H_{w_d,1}(w_d) & \cdots & H_{w_d,n_p}(w_d) \end{bmatrix}$$

Noise free case $\implies$ $Yp = y$

Output noise $\implies$ $Yp = y + w_n$

I/O noise $\implies$ $(Y + \tilde{Y})p = y + w_n$

$\hat{Y} := \begin{bmatrix} \hat{y}_1 & \cdots & \hat{y}_{n_p} \end{bmatrix}$ — filtered outputs by Kalman filters corresponding to the models $H_1, \ldots, H_{n_p}$

select the estimate as $\hat{p} = \arg\min_p \operatorname{tr} \Big(\operatorname{cov}\{\hat{Y}p - y\} \Big)$

Adaptive parameter estimation (cont.)

solution: $\hat{p} = \mathcal{E}\left\{\hat{Y}^T\hat{Y}\right\}^{-1}\mathcal{E}\left\{y^T\hat{Y}\right\} \triangleq F^{-1}h$, where

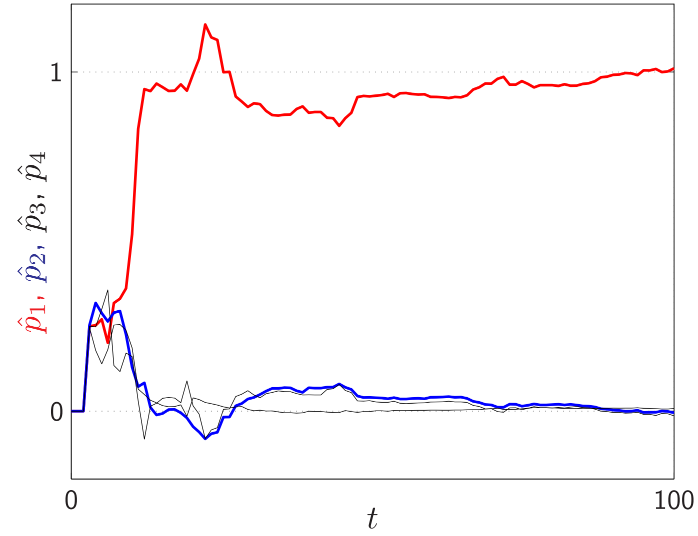
$F := \mathcal{E}\left\{\hat{Y}^T\hat{Y}\right\}$ and $h := \mathcal{E}\left\{y^T\hat{Y}\right\}$ are unknown

$\hat{Y},\ y$ measurable $\implies$ F, h can be estimated in real-time

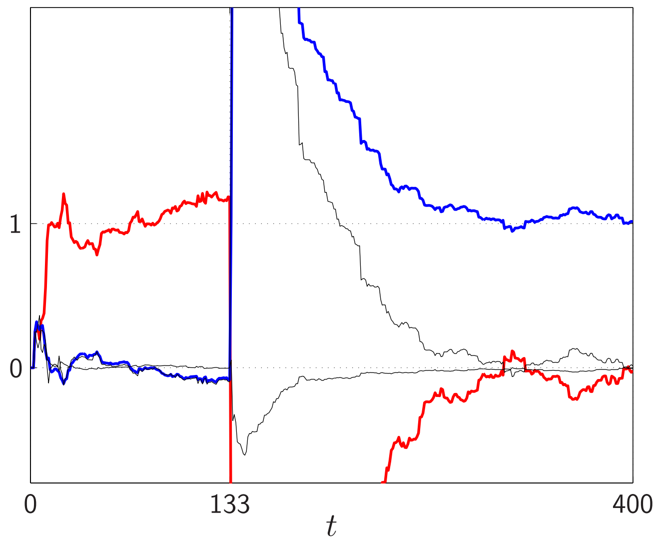
$$\hat{F}(t) = \frac{1}{T}\sum_{\tau=t-T}^{t}\lambda^{t-\tau}\hat{Y}^T(\tau)\hat{Y}(\tau), \quad \hat{h}(t) = \frac{1}{T}\sum_{\tau=t-T}^{t}\lambda^{t-\tau}\hat{y}^T(\tau)\hat{Y}(\tau)$$

T — window length $\qquad$ λ — forgetting factor

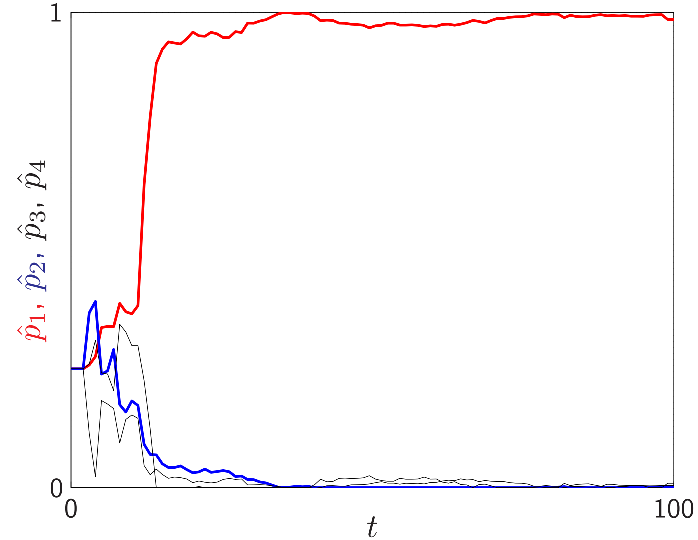
unknown constant parameter $p = (\mathbf{1}, \mathbf{0}, 0, 0)$



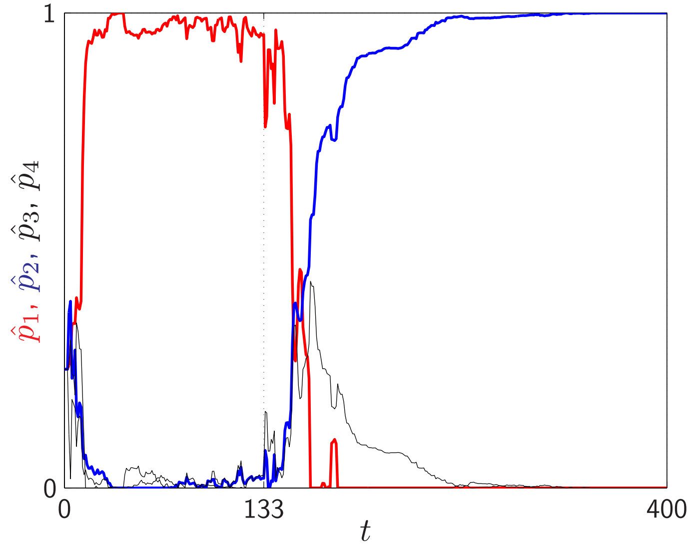
switching from $(\mathbf{1}, \mathbf{0}, 0, 0)$ to $(\mathbf{0}, \mathbf{1}, 0, 0)$ at $t = 133$



the same example as before with constraints $\sum_{i=1}^{n_p} p_i = 1$, and $p \succeq 0$



the same example as before with constraints $\sum_{i=1}^{n_p} p_i = 1$, and $p \succeq 0$



Conclusion

- we considered linearly parameterized systems
- parameterization $\equiv$ structural constraint
- $\mathcal{H}_2$-optimal approximation is convex quadratic
- $\mathcal{H}_\infty$-optimal approximation is SDP
- closed-loop parametric design seems to be NP-hard
- adaptive parameter estimation $\rightsquigarrow$ fault detection