Asian Journal of Control, Vol. 21, No. 4, pp. 1–10, July 2019
Published online in Wiley Online Library (wileyonlinelibrary.com) DOI: 10.1002/asjc.1785

VARIABLE STRUCTURE PREDEFINED-TIME STABILIZATION OF SECOND-ORDER SYSTEMS

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ABSTRACT

A controller that stabilizes second-order vector systems in predefined-time is introduced in this paper. That is, for second-order systems a controller is designed such that the trajectories reach the origin in a time defined in advance. The proposed controller is a variable structure controller that first drive the system trajectories to a linear manifold in predefined time and then drives the system trajectories to a non-smooth manifold with the predefined-time stability property, in predefined time also; this is done in order to avoid the differentiability problem that inherently appears when stabilizing high-order systems in finite time under the block control principle technique. The proposal is applied to the predefined-time exact tracking of fully actuated mechanical systems. As an example, the proposed solution is applied to a two-link planar manipulator, and numerical simulations are conducted to show its performance.

Key Words: Predefined-time stability, variable structure control, second order systems, mechanical systems.

I. INTRODUCTION

Several applications are characterized by their requirement for hard time-response constraints (see for example: [2,16]). In order to deal with those requirements, various developments concerning the concept of *finite-time stability* have been carried out (see for example: [3,8,17–19,23,29,31,32]). Nevertheless, this finite time is usually an unbounded function of the system's initial conditions.

Aiming to eliminate this boundlessness, the notion of *fixed-time stability* has been studied in [1,4,7,12,15,20–22,34,35]. Fixed-time stability represents a significant advantage over finite-time stability due to its valuable feature of convergence time as a function of initial conditions to be globally bounded. That makes fixed-time stability a desired property in, for example, estimation and optimization problems.

Although the fixed-time stability concept represents a significant advantage over finite-time stability, it is often difficult to find a direct relationship between tuning gains and the fixed stabilization time. To overcome this, another class of dynamical system, which exhibits the property of *predefined-time stability*, has been studied [24,25,27]. For these systems, an upper bound (sometimes the least upper bound) of the fixed stabilization time appears explicitly in their tuning gains.

The results [24,25,27] present first-order predefined-time stable dynamical systems. Furthermore, the works [9,26] attempt to extend these results to second-order systems as a nested application of first-order predefined-time stabilizing functions. However, since the predefined-time stabilizing function is non-smooth, these approaches yield singular controllers, which may produce theoretically infinite signals.

Therefore, this paper presents a controller that stabilizes second-order vector systems in predefined-time. The proposed controller is a variable structure controller that first drives the system trajectories to a linear manifold in a predefined-time, and then drives the system trajectories to a non-smooth manifold with a predefined-time stability property (introduced in [9]), also in predefined time. This mechanism allows the controller to avoid the differentiability/singularity problem mentioned above.

Manuscript received September 2, 2016; revised October 16, 2017; accepted December 31, 2017.

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This work was supported by CONACyT, México, under grant 252405. Esteban Jiménez acknowledges CONACyT, México for his MSc scholarship, number 426598.

A previous conference version of this paper was presented in [11].

Furthermore, it is shown that the problem of position tracking in fully actuated mechanical systems is equivalent to the stabilization of second-order systems problem [28]. Hence, the predefined-time exact position tracking in a fully actuated mechanical system is proposed as a direct application of the developed control scheme.

II. MATHEMATICAL BACKGROUND

Consider the system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}; \boldsymbol{\rho}), \qquad \boldsymbol{x}_0 = \boldsymbol{x}(0), \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the system state, $\boldsymbol{\rho} \in \mathbb{R}^b$ represents the parameters of the system and $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$. The initial conditions of this system are $\boldsymbol{x}_0 = \boldsymbol{x}(0)$.

Definition 1 *Global finite-time stability* [3,20]**.** The origin of (1) is *globally finite-time stable* if it is globally asymptotically stable and any solution $\boldsymbol{x}(t, \boldsymbol{x}_0)$ of (1) reaches the equilibrium point at some finite time moment, that is, $\forall t \geq T(\boldsymbol{x}_0) : \boldsymbol{x}(t, \boldsymbol{x}_0) = \boldsymbol{0}$, where $T : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ is called the *settling-time function*.

Definition 2 *Fixed-time stability* [20,21]**.** The origin of (1) is *fixed-time stable* if it is globally finite-time stable and the settling-time function is bounded, that is, $\exists T_{\max} > 0 : \forall \boldsymbol{x}_0 \in \mathbb{R}^n : T(\boldsymbol{x}_0) \leq T_{\max}$.

Remark 1. Note that there are several possible choices for $T_{\max}$; for example, if $T(\boldsymbol{x}_0) \leq T_{\mathrm{m}}$ for a positive number T_m, also $T(\boldsymbol{x}_0) \leq \lambda T_{\mathrm{m}}$ with $\lambda \geq 1$. This motivates the definition of a set that contains all the bounds of the settling-time function.

Definition 3 *Settling-time set and its minimum bound* [24,27]**.** Let the origin be fixed-time-stable for the system (1). The set of all the bounds of the settling-time function is defined as:

$$\mathcal{T} = \left\{ T_{\max} \in \mathbb{R}_+ : T(\boldsymbol{x}_0) \leq T_{\mathrm{m}}, \ \forall \, \boldsymbol{x}_0 \in \mathbb{R}^n \right\}. \tag{2}$$

In addition, the least upper bound of the settling-time function, denoted by T_f, is defined as

$$T_f = \min \mathcal{T} = \sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T(\boldsymbol{x}_0). \tag{3}$$

Remark 2. For several applications it could be desirable for system (1) to stabilize within a time $T_c \in \mathcal{T}$, which can be defined in advance as a function of the system parameters, that is $T_c = T_c(\boldsymbol{\rho})$. The cases where this property is present motivate the definition of predefined-time stability. A strong notion of this class of stability is given when $T_c = T_f$, that is, T_c is the true fixed-time in which the system stabilizes. A weak notion of predefined-time stability is presented when $T_c \geq T_f$, that is, if it is possible to define an upper bound of the settling-time function in terms of the system parameters. This overestimates the true fixed-time in which the system stabilizes.

Definition 4 *Predefined-time stability* [25]**.** For the system parameters $\boldsymbol{\rho}$ and a constant $T_c := T_c(\boldsymbol{\rho}) > 0$, the origin of (1) is said to be the following.

(i) *Globally weakly predefined-time-stable* for system (1) if it is fixed-time-stable and the settling-time function $T : \mathbb{R}^n \to \mathbb{R}$ is such that

$$T(\boldsymbol{x}_0) \leq T_c, \quad \forall \boldsymbol{x}_0 \in \mathbb{R}^n.$$

In this case, T_c is called a *weak predefined time*.

(ii) *Globally strongly predefined-time-stable* for system (1) if it is fixed-time-stable and the settling-time function $T : \mathbb{R}^n \to \mathbb{R}$ is such that

$$\sup_{\boldsymbol{x}_0 \in \mathbb{R}^n} T(\boldsymbol{x}_0) = T_c.$$

In this case, T_c is called the *strong predefined time*.

Definition 5. Let $h \geq 0$. For $\boldsymbol{x} \in \mathbb{R}^n$, define the function

$$\lfloor \boldsymbol{x} \rceil^h = \frac{\boldsymbol{x}}{||\boldsymbol{x}||^{1-h}}, \tag{4}$$

with $||\boldsymbol{x}||$ the euclidean norm of $\boldsymbol{x}$. Since $\lim_{\boldsymbol{x}\to 0} \lfloor \boldsymbol{x} \rceil^h = 0$ for $h > 0$, it is considered that $\lfloor 0 \rceil^h = 0$. Therefore, the function $\lfloor \boldsymbol{x} \rceil^h$ is continuous for $h > 0$ and discontinuous in $x = 0$ for $h = 1$.

Proposition 1. For $h > 0$, the function $\lfloor \boldsymbol{x} \rceil^h$ fulfills:

(i) $\lfloor -\boldsymbol{x} \rceil^h = -\lfloor \boldsymbol{x} \rceil^h$

(ii) $\lfloor \boldsymbol{x} \rceil^0 = \frac{\boldsymbol{x}}{||\boldsymbol{x}||}$, a unit vector.

(iii) $\lfloor \boldsymbol{x} \rceil^1 = \lfloor \boldsymbol{x} \rceil = \boldsymbol{x}$,

(iv) $\frac{d\lfloor \boldsymbol{x} \rceil^h}{dx} = \left[\boldsymbol{I}_n + (h-1)\frac{\boldsymbol{x}\boldsymbol{x}^T}{|\boldsymbol{x}|^2} \right] |\boldsymbol{x}|^{h-1}$ and $\frac{d||\boldsymbol{x}||^h}{dx} = h \left\lfloor \boldsymbol{x}^T \right\rceil^{h-1}$, where $\boldsymbol{I}_n$ is the $n \times n$ identity matrix.

(v) For $h_1, h_2 \in \mathbb{R}$, it follows:

- $||\boldsymbol{x}||^{h_1} \, ||\boldsymbol{x}||^{h_2} = ||\boldsymbol{x}||^{h_1+h_2}$
- $\lfloor \boldsymbol{x} \rceil^{h_1} \, |\boldsymbol{x}|^{h_2} = |\boldsymbol{x}|^{h_1} \, \lfloor \boldsymbol{x} \rceil^{h_2} = \lfloor \boldsymbol{x} \rceil^{h_1+h_2}$
- $\lfloor \boldsymbol{x}^T \rceil^{h_1} \lfloor \boldsymbol{x} \rceil^{h_2} = |\boldsymbol{x}|^{h_1+h_2}$

(vi) For $h_1, h_2 > 0$, then $\left\lfloor \lfloor \boldsymbol{x} \rceil^{h_1} \right\rceil^{h_2} = \lfloor \boldsymbol{x} \rceil^{h_1 h_2}$.

Definition 6 *Predefined-time stabilizing function* [25]**.** For $\boldsymbol{x} \in \mathbb{R}^n$, the *predefined-time stabilizing function* is defined as

$$\boldsymbol{\Phi}_{m,q}(\boldsymbol{x}; T_c) = \frac{1}{mqT_c} \exp\left(||\boldsymbol{x}||^{mq}\right) \lfloor \boldsymbol{x} \rceil^{1-mq}, \tag{5}$$

where $T_c > 0$, $m \geq 1$ and $0 < q \leq \frac{1}{m}$.

Remark 3. From Definition 5, the function $\boldsymbol{\Phi}_{m,q}$ in (5) is continuous for $0 < q < \frac{1}{m}$ and discontinuous for $q = \frac{1}{m}$.

Proposition 2 *Predefined-time stabilizing function derivative* [9]**.** The derivative of the predefined-time stabilizing function (5) is given by

$$\begin{aligned} \frac{\partial \boldsymbol{\Phi}_{m,q}(\boldsymbol{x}; T_c)}{\partial \boldsymbol{x}} = \frac{\exp\left(|\boldsymbol{x}|^{mq}\right)}{mqT_c} \Bigg[& mq\frac{\boldsymbol{x}\boldsymbol{x}^T}{|\boldsymbol{x}|^2} \\ & + \left(\boldsymbol{I}_n - mq\frac{\boldsymbol{x}\boldsymbol{x}^T}{||\boldsymbol{x}||^2}\right)\frac{1}{||\boldsymbol{x}||^{mq}}\Bigg], \end{aligned} \tag{6}$$

for all $\boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$, where $\boldsymbol{I}_n \in \mathbb{R}^{n\times n}$ stands for the $n-$th order identity matrix.

The following two lemmas present dynamical systems with the predefined-time stability property. The predefined-time stabilizing function (5) plays a main role, which justifies its name.

Lemma 1 *A strongly predefined-time stable dynamical system* [25]**.** The origin of the system

$$\dot{\boldsymbol{x}} = -\boldsymbol{\Phi}_{m,q}(\boldsymbol{x}; T_c) \tag{7}$$

with $T_c > 0$, $m \geq 1$ and $0 < q \leq \frac{1}{m}$ is globally strongly predefined-time stable with strong predefined time T_c.

Lemma 2 *A weakly predefined-time stable dynamical system* [25]**.** Let the function $\boldsymbol{\Delta}(t, \boldsymbol{x})$ be considered as a non-vanishing bounded disturbance such that $||\boldsymbol{\Delta}(t, \boldsymbol{x})|| \leq \delta$, with $0 < \delta < \infty$ a known constant. The origin of the system

$$\dot{\boldsymbol{x}} = -k\frac{\boldsymbol{x}}{||\boldsymbol{x}||} - \boldsymbol{\Phi}_{m,q}(\boldsymbol{x}; T_c) + \boldsymbol{\Delta}(t, \boldsymbol{x}) \tag{8}$$

with $k \geq \delta$, $T_c > 0$, $m \geq 1$ and $0 < q \leq \frac{1}{m}$ is globally weakly predefined-time stable with weak predefined time T_c.

In the next section, a variable structure controller to stabilize second-order systems in predefined time is introduced. The Heaviside step function defined below is very useful in the writing of such a controller.

Definition 7. The *Heaviside step function*, denoted by H, is a discontinuous function defined as

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0. \end{cases} \tag{9}$$

III. A PREDEFINED-TIME CONTROLLER FOR SECOND ORDER SYSTEMS

A predefined-time stabilizing controller for a second-order system is presented in this section. This constitutes the main result of this paper, and it will be used in the foregoing to design a predefined-time controller for fully actuated mechanical systems.

Consider the following second-order system subject to a matched perturbation term

$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= \boldsymbol{x}_2 \\ \dot{\boldsymbol{x}}_2 &= \boldsymbol{f}(t, \boldsymbol{x}_1, \boldsymbol{x}_2) + \boldsymbol{B}(t, \boldsymbol{x}_1, \boldsymbol{x}_2)\boldsymbol{u} + \boldsymbol{\Delta}(t, \boldsymbol{x}_1, \boldsymbol{x}_2), \end{aligned} \tag{10}$$

where $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$, $\boldsymbol{f} : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a known nonlinear vector-valued function, $\boldsymbol{B} : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a known nonlinear matrix-valued function assumed to be invertible for all $(t, \boldsymbol{x}_1, \boldsymbol{x}_2) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$, and $\boldsymbol{\Delta} : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is an unknown but globally uniformly bounded function, bounded by $\left|\left|\boldsymbol{\Delta}(t, \boldsymbol{x}_1, \boldsymbol{x}_2)\right|\right| \leq \delta$.

Now, given a time $T_c > 0$, consider the following variable structure controller

$$\begin{aligned} \boldsymbol{u}(t, \boldsymbol{x}_1, \boldsymbol{x}_2) = & \left[1 - H(t - T_{c_0})\right] \boldsymbol{u}_0(t, \boldsymbol{x}_1, \boldsymbol{x}_2) \\ & + H(t - T_{c_0})\boldsymbol{u}_1(t, \boldsymbol{x}_1, \boldsymbol{x}_2), \end{aligned} \tag{11}$$

with the terms $\boldsymbol{u}_0(t, \boldsymbol{x}_1, \boldsymbol{x}_2)$ and $\boldsymbol{u}_1(t, \boldsymbol{x}_1, \boldsymbol{x}_2)$ defined as

$$\begin{cases} \boldsymbol{u}_0(t, \boldsymbol{x}_1, \boldsymbol{x}_2) &= -\boldsymbol{B}^{-1}(t, \boldsymbol{x}_1, \boldsymbol{x}_2)\Big[\boldsymbol{f}(t, \boldsymbol{x}_1, \boldsymbol{x}_2) \\ & \quad + c\boldsymbol{x}_2 + \boldsymbol{\Phi}_{m_0,q_0}(\boldsymbol{\sigma}_0; T_{c_0}) + k\frac{\boldsymbol{\sigma}_0}{||\boldsymbol{\sigma}_0||}\Big] \\ \boldsymbol{\sigma}_0 &= \boldsymbol{x}_2 + c\boldsymbol{x}_1, \end{cases} \tag{12}$$

$$\begin{cases} \boldsymbol{u}_1(t, \boldsymbol{x}_1, \boldsymbol{x}_2) &= -\boldsymbol{B}^{-1}(t, \boldsymbol{x}_1, \boldsymbol{x}_2)\Big[\boldsymbol{f}(t, \boldsymbol{x}_1, \boldsymbol{x}_2) \\ & \quad + \frac{\partial \boldsymbol{\Phi}_{m_1,q_1}(\boldsymbol{x}_1; T_{c_1})}{\partial \boldsymbol{x}_1}\boldsymbol{x}_2 \\ & \quad + \boldsymbol{\Phi}_{m_2,q_2}(\boldsymbol{\sigma}_1; T_{c_2}) + k\frac{\boldsymbol{\sigma}_1}{||\boldsymbol{\sigma}_1||}\Big] \\ \boldsymbol{\sigma}_1 &= \boldsymbol{x}_2 + \boldsymbol{\Phi}_{m_1,q_1}(\boldsymbol{x}_1; T_{c_1}), \end{cases} \tag{13}$$

where the parameters satisfy $c = \frac{\exp(1)}{m_1 q_1 T_{c_1}}$, $k = \delta$, $m_0 \geq 1$, $m_1 \geq 1$, $m_2 \geq 1$, $0 < q_0 < \frac{1}{m_0}$, $0 < q_1 < \frac{1}{2m_1}$, $0 < q_2 < \frac{1}{m_2}$,

$T_{c_0} = \alpha_0 T_c$, $T_{c_1} = \alpha_1 T_c$ and $T_{c_2} = \alpha_2 T_c$, with $\alpha_0 > 0$, $\alpha_1 > 0$, $\alpha_2 > 0$, and $\alpha_0 + \alpha_1 + \alpha_2 = 1$.

Theorem 1. The origin of the closed-loop system (10)-(13) is globally weakly predefined-time stable with a weak predefined time T_c.

The previous controller design is motivated through the stabilization of the scalar double-integrator system in Appendix A. Furthermore, a proof of Theorem 1 is given in Appendix B.

Remark 4. Since the linear manifold $\boldsymbol{\sigma}_0 = 0$ is close to the non-smooth manifold $\boldsymbol{\sigma}_1 = 0$ near the origin (see Appendix 7.1), it is recommended to make $\alpha_0 \gg \alpha_1, \alpha_2$.

Remark 5. In the time interval $0 \leq t \leq T_{c_0}$ the control signal $\boldsymbol{u}_0$ rejects the perturbation $\boldsymbol{\Delta}$ via the unit control term $k\frac{\boldsymbol{\sigma}_0}{||\boldsymbol{\sigma}_0||}$. Then, in the time interval $t \geq T_{c_0}$, the control signal $\boldsymbol{u}_1$ rejects the perturbation $\boldsymbol{\Delta}$ via the unit control term $k\frac{\boldsymbol{\sigma}_1}{||\boldsymbol{\sigma}_1||}$. In this sense, from Lemma 2, the proposed controller is not only robust but insensitive to the considered matched perturbation term $\boldsymbol{\Delta}$ for every time $t \geq 0$.

IV. APPLICATION: PREDEFINED-TIME TRACKING OF FULLY-ACTUATED MECHANICAL SYSTEMS

A fully-actuated mechanical system of n degrees of freedom can be modeled by

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{P}(\dot{\boldsymbol{q}}) + \boldsymbol{\gamma}(\boldsymbol{q}) + \boldsymbol{\theta}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{\tau}, \quad (14)$$

where $\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}} \in \mathbb{R}^n$ are the position, velocity, and acceleration vectors in joint space; $\boldsymbol{M}(\boldsymbol{q}) \in \mathbb{R}^{n\times n}$ is the inertia matrix, $\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \mathbb{R}^{n\times n}$ is the Coriolis and centrifugal effects matrix, $\boldsymbol{P}(\dot{\boldsymbol{q}}) \in \mathbb{R}^n$ is the damping effects vector, usually from viscous and/or Coulomb friction, $\boldsymbol{\gamma}(\boldsymbol{q}) \in \mathbb{R}^n$ is the gravity effects vector and $\boldsymbol{\theta}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ stands for modeling errors and unknown loads.

The main objective is to track the position $\boldsymbol{q}$ to a desired time-dependent position trajectory described by $\boldsymbol{q}_d(t) = [q_{d_1}(t) \quad \cdots \quad q_{d_n}(t)]^T \in \mathbb{R}^n$ after a defined in advance time T_c, by means of the control torque $\boldsymbol{\tau}$. The first and the second time-derivatives of $\boldsymbol{q}_d(t)$, $\dot{\boldsymbol{q}}_d(t)$ (velocity) and $\ddot{\boldsymbol{q}}_d(t)$ (acceleration), are assumed to exist and to be known.

Defining the tracking error variables $\boldsymbol{x}_1 = \boldsymbol{q} - \boldsymbol{q}_d$, $\boldsymbol{x}_2 = \dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_d$ and $\boldsymbol{u} = \boldsymbol{\tau}$, the mechanical model (14) can be rewritten in the state-space form (10), with $\boldsymbol{f}(t, \boldsymbol{x}_1, \boldsymbol{x}_2) = -\boldsymbol{M}^{-1}(\boldsymbol{x}_1 + \boldsymbol{q}_d(t))[\boldsymbol{C}(\boldsymbol{x}_1 + \boldsymbol{q}_d(t), \boldsymbol{x}_2 + \dot{\boldsymbol{q}}_d(t))(\boldsymbol{x}_2 + \dot{\boldsymbol{q}}_d(t)) + \boldsymbol{P}(\boldsymbol{x}_2 + \dot{\boldsymbol{q}}_d(t)) + \boldsymbol{\gamma}(\boldsymbol{x}_1 + \boldsymbol{q}_d(t))] - \ddot{\boldsymbol{q}}_d(t)$, $\boldsymbol{B}(t\boldsymbol{x}_1, \boldsymbol{x}_2) = \boldsymbol{M}^{-1}(\boldsymbol{x}_1 + \boldsymbol{q}_d(t))$ and $\boldsymbol{\Delta}(t, \boldsymbol{x}_1, \boldsymbol{x}_2) = -\boldsymbol{M}^{-1}(\boldsymbol{x}_1 + \boldsymbol{q}_d(t))\boldsymbol{\theta}(t, \boldsymbol{x}_1 + \boldsymbol{q}_d(t), \boldsymbol{x}_2 + \dot{\boldsymbol{q}}_d(t))$.

Thus, the problem of tracking the position $\boldsymbol{q}$ to $\boldsymbol{q}_d$ is equivalent to stabilize the system (10). In this sense, the controller (11) developed in the last section constitutes a predefined-time tracking position controller for fully-actuated mechanical systems modeled by (14).

V. EXAMPLE: TRAJECTORY TRACKING FOR A TWO-LINK MANIPULATOR

5.1 Model description

Consider a planar, two-link manipulator with revolute joints as the one shown in Example 12.1 of [30] (see Fig. 1). The manipulator link lengths are L_1 and L_2, the link masses (concentrated in the end of each link) are M_1 and M_2. The manipulator is operated in the plane, such that the gravity acts along the $z-$axis.

Examining the geometry, it can be seen that the end-effector (the end of the second link, where the mass M_2 is concentrated) position (x_w, y_w) is given by $x_w = L_1 \cos(q_1) + L_2 \cos(q_1 + q_2)$ and $y_w = L_1 \sin(q_1) + L_2 \sin(q_1 + q_2)$, where q_1 and q_2 are the joint positions (angular positions).

Applying the Euler-Lagrange equations, a model according to (14) is obtained, with $m_{11} = L_1^2(M_1 + M_2) + 2(L_2^2 M_2 + L_1 L_1 M_2 \cos q_2) - L_2^2 M_2$, $m_{12} = m_{21} = L_2^2 M_2 + L_1 L_1 M_2 \cos q_2$, $m_{22} = L_2^2 M_2$, $h = L_1 L_2 M_2 \sin q_2$, $c_{11} = -h\dot{q}_2$, $c_{12} = -h(\dot{q}_1 + \dot{q}_2)$, $c_{21} = h\dot{q}_1 \qquad c_{22} = 0$, and

$$\boldsymbol{M}(\boldsymbol{q}) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \quad \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix},$$
$$\boldsymbol{P}(\dot{\boldsymbol{q}}) = [0 \quad 0]^T, \qquad \boldsymbol{\gamma}(\boldsymbol{q}) = [0 \quad 0]^T.$$

For this example, the end-effector of the manipulator is required to follow a circular trajectory of radius r_d and center in the origin.

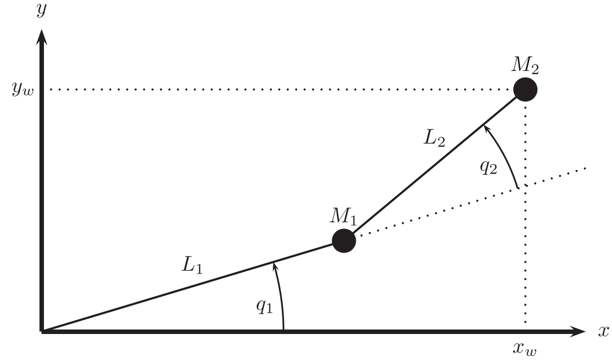

Fig. 1. Two-link manipulator.

The two-link manipulator parameters are $M_1 = M_2 = 0.2\,\mathrm{kg}$ and $L_1 = L_2 = 0.2\,\mathrm{m}$.

5.2 Simulation results

The simulations were conducted using the Euler integration method, with a fundamental step size of 1×10^{-4} s. The initial conditions for the two-link manipulator were selected as: $\boldsymbol{x}_1(0) = \begin{bmatrix} -\frac{3\pi}{4} & -\frac{\pi}{4} \end{bmatrix}^T$ and $\boldsymbol{x}_2(0) = [0 \quad 0]^T$. In addition, the controller gains were adjusted to: $k = 0$, $T_{c_0} = T_{c_1} = 1$, $T_{c_2} = 0.1$, $m_0 = m_1 = m_2 = 1$, $q_0 = q_2 = \frac{1}{2}$ and $q_1 = 0.3$.

The desired circular trajectory in the joint coordinates is described by the equations

$$\boldsymbol{q}_d(t) = \begin{bmatrix} q_{d_1}(t) \\ q_{d_2}(t) \end{bmatrix} = \begin{bmatrix} \frac{\pi}{2}t - \pi \\ -\frac{\pi}{2} \end{bmatrix},$$

and it corresponds to a circumference of radius 0.2828 m.

The following figures show the behavior of the proposed controller.

Note that $\boldsymbol{\sigma}_1(t) = 0$ for $t \geq 1.1\ \mathrm{s} = T_{c_0} + T_{c_2}$ (Fig. 2). Once the error variables slide over the manifold $\boldsymbol{\sigma}_1 = 0$, this motion is governed by the reduced order system

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{x}_2 = -\boldsymbol{\Phi}_{m_1,q_1}(\boldsymbol{x}_1; T_{c_1}).$$

This implies that the error variables are exactly zero for $t > T_{c_0} + T_{c_1} + T_{c_2} = 2.1$ s. In fact, from Fig. 3, it can be seen that $\boldsymbol{x}_1(t) = \boldsymbol{x}_2(t) = 0$ for $t \geq 1.5\ \mathrm{s} < T_{c_0} + T_{c_1} + T_{c_2} = 2.1$ s. Fig. 4 shows the control signal (torque) versus time, where the switching effect can be seen at $t = 2.1$ s =

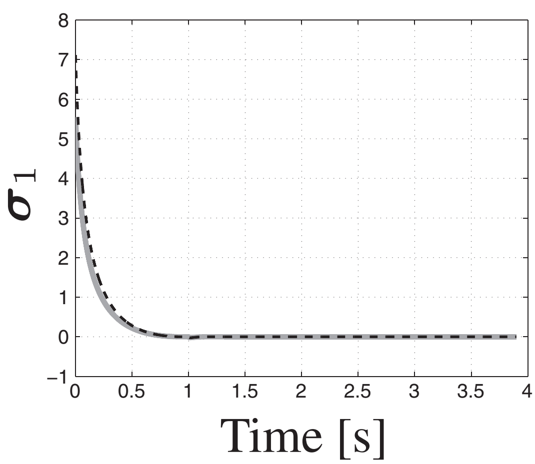

Fig. 2. Variable $\boldsymbol{\sigma}_1$. First component (gray and solid) and second component (black and dashed). Note that $\boldsymbol{\sigma}_1(t) = 0$ for $t > T_{c_2} = 0.5$ *s*.

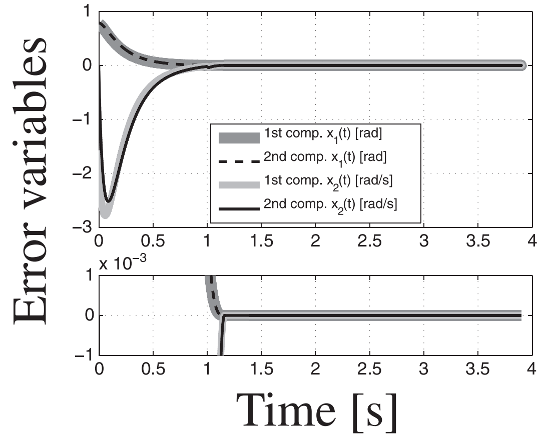

Fig. 3. Error variables. First component of $\boldsymbol{x}_1$ (dark gray and thick), second component of $\boldsymbol{x}_1$ (black and dashed), first component of $\boldsymbol{x}_2$ (light gray and solid) and second component of $\boldsymbol{x}_2$ (black and solid).

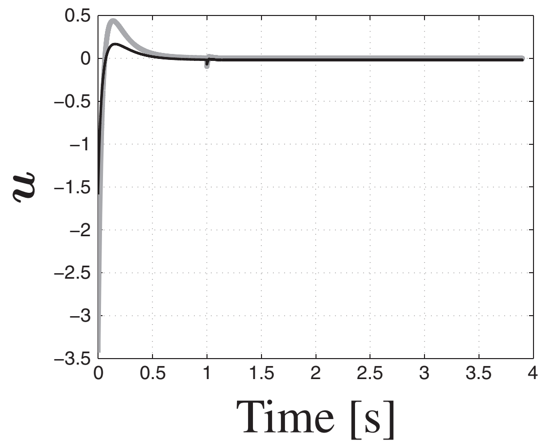

Fig. 4. Control signal. First component (gray and solid) and second component (black and solid).

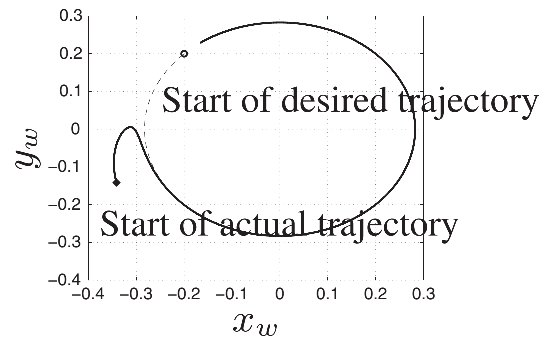

Fig. 5. Reference tracking in rectangular coordinates.

T_{c_0}. Finally, from Fig. 5, it can be seen that the reference tracking is in rectangular coordinates.

Remark 6. Note that the tracking error variables, $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, become zero at the same time because of the relation $\boldsymbol{x}_2 = -\boldsymbol{\Phi}_{m_1,q_1}(\boldsymbol{x}_1; T_{c_1})$. This can be observed in Fig. 3.

VI. CONCLUSION

The predefined-time stabilization of second-order vector systems using a variable structure controller was studied in this paper. The proposed control scheme was designed such that the system trajectories are firstly driven to a linear manifold and then to a non-smooth manifold with the predefined-time stability property, to avoid the differentiability problem, which inherently appears when stabilizing high-order systems in finite time under the block control principle technique. The developed controller was applied to the predefined-time exact tracking of fully actuated mechanical systems. As an example, the proposed solution was applied to a two-link planar manipulator, and numerical simulations were conducted to show its performance.

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VII. APPENDIX A

7.1 Some remarks on the stabilization of second order systems

Consider the scalar double-integrator system as a base study case

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u, \end{aligned} \tag{15}$$

where $x_1, x_2, u \in \mathbb{R}$.

The objective is to stabilize the origin $(x_1, x_2) = (0, 0)$ of the system (15) in some desired way (asymptotically, in finite time, in fixed time, or in predefined time), by means of the control signal u. Based on the block control principle [5,6], a good candidate for desired compensated dynamics is

$$\dot{x}_1 + \phi_1(x_1) = 0,$$

where the function $\phi_1(\cdot)$ is to be designed to stabilize the origin $x_1 = 0$ (asymptotically, in finite time, in fixed time, or in predefined time) of the above reduced order system.

To achieve the above desired compensated dynamics, introduce a new variable σ as

$$\sigma = x_2 + \phi_1(x_1). \tag{16}$$

Using (15) and (16), the dynamics of the variable σ is

$$\begin{aligned}\dot{\sigma} &= u + \frac{d\phi_1(x_1)}{dx_1}x_2 \\ &= u + \frac{d\phi_1(x_1)}{dx_1}(\sigma - \phi_1(x_1)).\end{aligned}$$

From the above, one may propose the following controller u

$$\begin{aligned}u &= -\frac{d\phi_1(x_1)}{dx_1}x_2 - \phi_2(\sigma) \\ &= \frac{d\phi_1(x_1)}{dx_1}(\phi_1(x_1) - \sigma) - \phi_2(\sigma),\end{aligned} \tag{17}$$

where, similarly to $\phi_1(\cdot)$, the function $\phi_2(\cdot)$ is to be designed to stabilize the origin $\sigma = 0$ (asymptotically, in finite time, in fixed time, or in predefined time).

Remark 7. Note that the existence of the controller (17) is conditioned by the differentiability of the function $\phi_1(\cdot)$. Hence, for the case of finite-time stabilization (as well as its stronger forms of fixed-time and predefined-time stabilization) the above approach is inadmissible, since the finite-time stability property can only be induced using non Lipschitz continuous functions.

Remark 8. A similar approach is followed in [13] for finite-time stabilization, but the term $\frac{d\phi_1(x_1)}{dx_1}x_2$ is not included in the controller. Yet not canceling this term, the finite-time stability property of the closed-loop is preserved.

Remark 9. The works [14,33] circumvent the differentiability issue for both finite-time and fixed-time under the concept of singularity free terminal sliding mode.

Now, in the rest of this appendix the case of predefined-time stabilization under the block control approach will be deeply analyzed.

This is, the case $\phi_1(\cdot) = \Phi_{m_1,q_1}(\cdot; T_{c_1})$ and $\phi_2 = \Phi_{m_2,q_2}(\cdot; T_{c_2})$ with $m_1 \geq 1$, $0 < q_1 < \frac{1}{2m_1}$, $T_{c_1} > 0$, $m_2 \geq 1$, $0 < q_2 < \frac{1}{m_2}$ and $T_{c_2} > 0$, is considered.

The explicit form of the controller is

$$\begin{aligned}u &= -\frac{d\Phi_{m_1,q_1}(x_1; T_{c_1})}{dx_1}x_2 - \Phi_{m_2,q_2}(\sigma; T_{c_2}) \\ &= \frac{d\Phi_{m_1,q_1}(x_1; T_{c_1})}{dx_1}(\Phi_{m_1,q_1}(x_1; T_{c_1}) - \sigma) \\ &\qquad - \Phi_{m_2,q_2}(\sigma; T_{c_2}).\end{aligned} \tag{18}$$

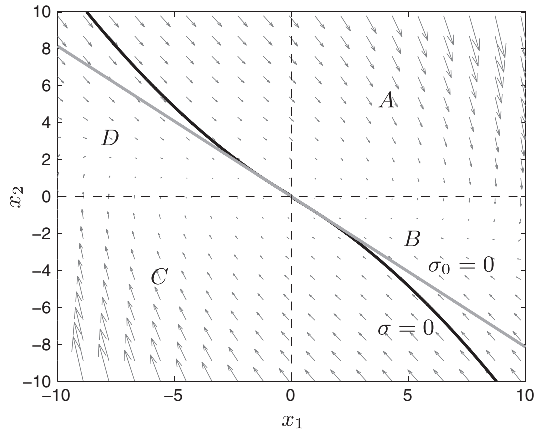

Fig. A1. Phase portrait of the closed-loop system (15)- (17) (gray arrows), manifold $\sigma = 0$ (black line) and manifold $\sigma_0 = 0$ (gray line).

Some aspects of the controller (18) related to Remark 7 should be noted.

(i) The continuity of, the term $\frac{d\Phi_{m_1,q_1}(x_1;T_{c_1})}{dx_1}\Phi_{m_1,q_1}(x_1; T_{c_1})$ is assured by the choice of q_1. Even so, the term $\frac{d\Phi_{m_1,q_1}(x_1;T_{c_1})}{dx_1}\sigma$, which produces theoretically infinite signals whenever the system solutions cross the axis $x_1 = 0$ unless $\sigma = 0$ (see (6)), is also present in the controller.

(ii) In fact, the stability analysis in [9,10] assumes implicitly that the system solutions do not cross the axis $x_1 = 0$ before $\sigma = 0$. However, this assumption does not hold in general. For instance, consider the cases $x_1(0) = 0$ and $x_2(0) \neq 0$, or $|x_1(0)| \approx 0$ and $x_1(0)x_2(0) \ll 0$.

Although the controller (18) is not global, it will be helpful to state a sufficient condition for it to work. With this aim, consider the phase portrait of the closed-loop system in Fig. A1.

The regions labeled as A, B, C, D can be described as:

- $A = \{x \in \mathbb{R}^2 : x_1 > 0, x_2 \geq 0\}$
- $B = \{x \in \mathbb{R}^2 : x_1 > 0, x_2 < 0, x_1\sigma \geq 0\}$
- $C = \{x \in \mathbb{R}^2 : x_1 < 0, x_2 \leq 0\}$
- $D = \{x \in \mathbb{R}^2 : x_1 < 0, x_2 > 0, x_1\sigma \geq 0\}$

On region A, $\dot{x}_1 = x_2 \geq 0$ and $\dot{x}_2 = -\frac{d\Phi_{m_1,q_1}(x_1;T_{c_1})}{dx_1}x_2 - \Phi_{m_2,q_2}(\sigma; T_{c_2}) \leq -\Phi_{m_2,q_2}(\sigma; T_{c_2}) < -\Phi_{m_2,q_2}(x_2; T_{c_2})$. Then, every solution starting on A

enters B (without crossing the line $x_1 = 0$) in at most T_{c_2} time units.

On region B, it is clearly impossible to cross the line $x_1 = 0$ without crossing the manifold $\sigma = 0$. Moreover $\dot{\sigma} = -\Phi_{m_2,q_2}(\sigma; T_{c_2})$, hence, every solution starting on B will reach the manifold $\sigma = 0$ in predefined-time T_{c_2} (without crossing the line $x_1 = 0$), and will stay on it thereafter.

In fact, every solution starting on $A \cup B$ will reach the manifold $\sigma = 0$ in predefined-time T_{c_2} (without crossing the line $x_1 = 0$), and will stay on it thereafter. By symmetry, the same happens in the region $C \cup D$, which means that the controller (17) will work for every initial condition on

$$A \cup B \cup C \cup D = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_1\sigma \geq 0\right\}. \tag{19}$$

The above analysis is summarized in the following lemma.

Lemma 3. For the system (15) closed-loop with the controller (18), if the initial conditions of system satisfy $x_1(0)\sigma(0) \geq 0$ and $x_1(0) \neq 0$, then $x_1(t) = 0$ and $x_2(t) = 0$ for $t > T_{c_1} + T_{c_2}$.

The above result can be used to construct a variable structure global predefined-time stabilizing controller for system (15), exploiting the predefined-time feature.

To this end, a smooth manifold on the region $\left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_1\sigma \geq 0\right\}$ will be constructed. We will consider smooth manifolds of the form

$$\sigma_0 = x_2 + cx_1 = 0, \qquad c > 0, \tag{20}$$

that is, linear manifolds. Note that for this linear manifold to be in the region (19), it must be that

$$c \leq \frac{1}{m_1 q_1 T_{c_1}} \frac{\exp\left(|x_1|^{m_1 q_1}\right)}{|x_1|^{m_1 q_1}}.$$

To find such a c, let us minimize the right side of the above inequality.

Definition 8. Let $m \geq 1$, $0 < q < \frac{1}{m}$ and $T_c > 0$. The function $f_{m,q,T_c} : \mathbb{R}_+ \to \mathbb{R}_+$ is defined as

$$f_{m,q,T_c}(s) = \frac{1}{mqT_c} \frac{\exp(s^{mq})}{s^{mq}}. \tag{21}$$

Lemma 4. Let $m \geq 1$, $0 < q < \frac{1}{m}$ and $T_c > 0$. Then,

$$\min_{s \in \mathbb{R}_+} f_{m,q,T_c}(s) = f_{m,q,T_c}(1).$$

Proof. Note that

$$\frac{df_{m,q,T_c}(s)}{ds} = \frac{\exp(s^{mq})}{T_c s^{mq+1}} \left[s^{mq} - 1\right].$$

It can be easily seen then that

$$\frac{df_{m,q,T_c}(s)}{ds} \begin{cases} < 0 \text{ if } s < 1 \\ = 0 \text{ if } s = 1 \\ > 0 \text{ if } s > 1, \end{cases}$$

which implies that $\min_{s \in \mathbb{R}_+} f_{m,q,T_c}(s) = f_{m,q,T_c}(1)$.

From Definition 8 and Lemma 4, a suitable value for the parameter c is

$$c = f_{m_1,q_1,T_{c_1}}(1) = \frac{\exp(1)}{m_1 q_1 T_{c_1}}.$$

With this selection, not only the linear manifold $\sigma_0 = 0$ (20) lies in the region (19), but is also close to the non-smooth manifold $\sigma = 0$ (16) near the origin (see Fig. A1).

Having constructed this linear manifold, a variable structure predefined-time controller will be used. In the first stage, the controller will drive the system trajectories to the linear manifold (which is in the region $\left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_1\sigma \geq 0\right\}$). In the second stage, the controller (17) will be used.

This is the main idea of the predefined-time stable second-order variable structure system described in Section III.

VIII. APPENDIX B

8.1 Proof of theorem 1

Proof. Note that system (10) can be written component-wise as

$$\begin{aligned} \dot{x}_{1,i} &= x_{2,i} \\ \dot{x}_{2,i} &= f_i(\boldsymbol{x}_1, \boldsymbol{x}_2) + \boldsymbol{b}_i^T(\boldsymbol{x}_1, \boldsymbol{x}_2)\boldsymbol{u} + \Delta_i, \end{aligned}$$

for $i = 1, \ldots, n$, where $\boldsymbol{x}_1 = [x_{1,1} \ \ldots \ x_{1,n}]^T$, $\boldsymbol{x}_2 = [x_{2,1} \ \ldots \ x_{2,n}]^T$, $\boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2) = [f_1(\boldsymbol{x}_1, \boldsymbol{x}_2) \ \ldots \ f_n(\boldsymbol{x}_1, \boldsymbol{x}_2)]^T$, $\boldsymbol{B}^T(\boldsymbol{x}_1, \boldsymbol{x}_2) = [\boldsymbol{b}_1(\boldsymbol{x}_1, \boldsymbol{x}_2) \ \ldots \ \boldsymbol{b}_n(\boldsymbol{x}_1, \boldsymbol{x}_2)]$ and $\boldsymbol{\Delta} = [\Delta_1 \ \ldots \ \Delta_n]^T$. Furthermore, the component-wise expressions of the variables $\boldsymbol{\sigma}_0$ and $\boldsymbol{\sigma}_1$ are

$$\begin{aligned} \sigma_{0,i} &= x_{2,i} + cx_{1,i} \\ \sigma_{1,i} &= x_{2,i} + f_{m_1,q_1,T_{c_1}}(\|\boldsymbol{x}_1\|)x_{1,i}. \end{aligned}$$

A similar analysis to that of Lemma 3, yields that a sufficient condition for the controller (13) to work is $x_{1,i}(0)\sigma_{1,i}(0) \geq 0$ and $x_{1,i} \neq 0$. Then, the selection of c is justified by Lemma 4.

For $0 \leq t \leq T_{c_0}$, the derivative of $\boldsymbol{\sigma}_0$ (12) is

$$\begin{aligned}\dot{\boldsymbol{\sigma}}_0 &= \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2) + \boldsymbol{B}(\boldsymbol{x}_1, \boldsymbol{x}_2)\boldsymbol{u} + \boldsymbol{\Delta} + c\boldsymbol{x}_2 \\ &= -k\frac{\boldsymbol{\sigma}_0}{||\boldsymbol{\sigma}_0||} - \boldsymbol{\Phi}_{m_0,q_0}(\boldsymbol{\sigma}_0; T_{c_0}) + \boldsymbol{\Delta}.\end{aligned}$$

Thus, applying Lemma 2, $\boldsymbol{\sigma}_0 = 0$ is weakly predefined-time stable with weak predefined time T_{c_0}. This is, $\boldsymbol{\sigma}_0(t) = 0$ for $t \geq T_{c_0}$.

Now, for $t > T_{c_0}$, the derivative of $\boldsymbol{\sigma}_1$ (13) is

$$\begin{aligned}\dot{\boldsymbol{\sigma}}_1 &= \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2) + \boldsymbol{B}(\boldsymbol{x}_1, \boldsymbol{x}_2)\boldsymbol{u} + \boldsymbol{\Delta} + \frac{\partial \boldsymbol{\Phi}_{m_1,q_1}(\boldsymbol{x}_1; T_{c_1})}{\partial \boldsymbol{x}_1}\boldsymbol{x}_2 \\ &= -k\frac{\boldsymbol{\sigma}_1}{||\boldsymbol{\sigma}_1||} - \boldsymbol{\Phi}_{m_2,q_2}(\boldsymbol{\sigma}_1; T_{c_2}) + \boldsymbol{\Delta}.\end{aligned}$$

Hence, applying Lemma 2, $\boldsymbol{\sigma}_1 = 0$ is weakly predefined-time stable with weak predefined time T_{c_2}. This is, $\boldsymbol{\sigma}_1(t) = 0$ for $t \geq T_{c_0} + T_{c_2}$.

Finally, for $t > T_{c_0} + T_{c_2}$, since $\boldsymbol{\sigma}_1 = 0$,

$$\dot{\boldsymbol{x}}_1 = -\boldsymbol{\Phi}_{m_1,q_1}(\boldsymbol{x}_1; T_{c_1}),$$

and applying Lemma 1, $\boldsymbol{x}_1(t) = 0$ for $t > T_{c_0} + T_{c_1} + T_{c_2} = T_c$. Also note that $\boldsymbol{x}_2(t) = 0$ for $t > T_c$. Then, the origin of the system (10) closed-loop with (11) is weakly predefined-time stable with weak predefined time T_c.

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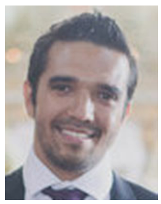

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