Special Functions-Based Fixed-Time Estimation and Stabilization for Dynamic Systems

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***Abstract*—Fixed-time stability (FXTS) and fixed-time control (FXTC) of dynamic systems are reconsidered in this article based on special functions from the view of improving the estimate accuracy of settling time (ST) and reducing the chattering caused by the sign function. First, by means of the idea of contradiction and variable transformations, some generic FXTS criteria are established and some upper bounds of ST are directly calculated and expressed by several special functions. It is further proved that these estimates are the most accurate compared with the existing results. Besides, to suppress the chattering caused by the sign function, some saturation functions are constructed to replace the sign function and the FXTS of the new system obtained by replacing is ensured by rigorous theoretical analysis. As applications, the problem of stabilization for chaotic systems in fixed or preassigned time is explored. Especially, an innovative saturation controller is developed to realize preassigned-time stabilization, where the convergence time is prescribed in advance according to actual requirement and the control gains are finite, the existing control methods with time-varying infinite gains are essentially improved. Lastly, three numerical examples are provided to verify the improved estimates of ST and the chattering reduction.**

***Index Terms*—Chattering reduction, fixed-time stability, fixed-time stabilization, preassigned time, special function.**

I. Introduction

FIXED-TIME stability (FXTS) [1] has received extensive concern nowadays and become a hot topic in dynamic analysis of complex systems in view of its potential applications in the space technology [2], power systems [3], hypersonic missiles [4], and marine surface ship [5]. Different from asymptotic stability, it is allowed in FXTS to realize convergence of the system in a finite time, this leads to a faster stability speed and a better anti-interference. Compared to finite-time stability (FNTS) [6]–[8], a uniform upper bound for settling time (ST) is required in FXTS and it must be unrelated to initial values of the addressed model. The unique feature offers major advantages over FNTS because the initial conditions are often unaccessible beforehand for many practical systems.

To our knowledge, FXTS was proposed in [9] and ultimately recognized through Polyakov's work in 2012, in which a sufficient condition and an effective estimate of ST were obtained for FXTS of nonlinear systems [1]. In [10], the problems of FNTS and FXTS of nonautonomous systems were investigated by proposing an implicit Lyapunov technique. In [11] and [12], the problems of FNTS and FXTS were reconsidered from the view of inverse function. In [13], by analyzing the value of a key parameter, a theorem was established to unify FNTS and FXTS. Some criteria were derived in [14] by the Lyapunov method to realize FXTS of continuous autonomous systems. In addition to deterministic systems, nonlinear systems with stochastic factor, uncertain or impulsive effects were also involved in the research of FNTS [15] and FXTS [16]–[18]. For example, a novel systematic backstepping control was developed in [15] to realize FNTS of high-order stochastic models and the effect of noise disturbance was carefully considered. Nonetheless, these results mainly concentrate on how to establish the criteria of FXTS but ignore how to reduce the conservatism and enhance the estimate precision of ST. In [19], a high-precision estimate of ST was obtained based on the theory of optimum values, which is less conservative than the results [1], [20].

On the other hand, fixed-time control (FXTC) has attracted much concern as a type of optimal control in recent decade. In comparison to finite-time control (FNTC) [15], where the convergence time is related to initial values, the superiority of FXTC is that it can stabilize the system within a fixed time even though the initial values are unknown beforehand [21]–[23]. At present, FXTC has been utilized to explore the stock management problem [24], the fixed-time synchronization of dynamic networks [25]–[28], time-varying parameter identification [29], fixed-time consensus of multiagent models [30]–[32] and fixed-time stabilization of generic linear systems [33], [34] and nonlinear models [35]–[37].

Although FXTS and FXTC have been widely discussed, there are still several essential and challenging issues worthy of further deeply study. First, as is known, in the investigation of FXTS, the key is to find an upper bound independent of the initial values for ST. Evidently, it is desirable to obtain some estimates as accurately as possible. In present work [1], [11], [19], [20], to find the upper bound, the common method is to utilize some inequality technique to evaluate an improper

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integral and the estimate of the integral is regarded as the upper bound of ST. Although the method is effective, it may result in some conservative estimates. Evidently, it is undesirable in real applications. Hence, it is interesting to explore some novel methods to reduce the conservatism and improve the accuracy of estimates. On the other hand, the sign function is indispensable in the existing fixed-time controller design [21]–[23], [28], [33], [34], [38], [39]. However, it usually leads to undesired chattering to the system states and the control inputs, which may degrade the performance of the controlled system and disable the controller. In practice, oscillation also causes high wear of moving mechanical parts and damage to the motor [40]. Therefore, a meaningful and challenging problem in FXTC is how to find or construct some new functions to replace the sign function so as to reduce or eliminate the chattering.

Note that the ST in FXTC is dependent of the parameters of system and controller. Whereas, in several real applications, the controlled system is usually required to achieve stability within a predetermined period of time which is preassigned based on the actual requirement and is totally unrelated to initial values and all parameters. In view of the fact, Song *et al.* [41] studied regulation of normal-form nonlinear systems in a prescribed finite time by means of a feedback control design with time-varying and unbounded control gains. Subsequently, the control idea was extended to investigate prescribed-time observer and output feedback for a class of linear systems in [42] and [43] and cluster synchronization in predetermined time for a type of complex coupled systems in [44]. Obviously, this kind of unbounded control gains is undesirable in the actual controller implementation. Naturally, an urgent question is how to develop a new type of control schemes with bounded control gains to realize preassigned-time stabilization of nonlinear models.

Enlightened by the above discussion, the problem of FXTS of dynamic systems is reconsidered in this article based on some special functions and the stabilization of chaotic systems in fixed time and preassigned time is studied by proposing a saturation function. The innovative contents are summarized as follows.

1) First, by use of analytical method, some generic conditions of FXTS are established, and the existing results [1], [11], [12], [14], [19], [45] are essentially improved. Especially, instead of the inequality technique, by using the variable transformation, some upper estimates of ST, denoted by improper integrals with infinite interval, are directly calculated and expressed by several special functions. It is proved that these estimates are the most precise in comparison to the existing work [1], [19].
2) To reduce chattering caused by the sign function, a kind of saturation function is constructed to substitute the sign function in the classic stable scalar system in fixed time. For the modified system, an FXTS condition is derived based on strict theoretical analysis and the ST is estimated by use of cosecant function and incomplete Beta function ratio. The related results obtained in [19] and [20] are greatly improved and developed.
3) As an application, based on the sign function and the developed saturation function, some types of control strategies are developed to discuss the stabilization of chaotic systems in fixed or preassigned time and some criteria are established by matrix inequalities and special functions. Especially, an innovative saturation controller is developed to realize the preassigned-time stabilization, where the convergence time is prescribed in advance according to actual requirement and the control gains are finite, this is totally different from the unbounded gains in [41]–[44]. Moreover, it is revealed in numerical examples that the control scheme based on saturation function is more effective in suppressing chattering.

The structure for the remainder of the paper is provided here. Several necessary definitions and related results are provided in Section II as preliminaries. Based on several special functions, FXTS of dynamic systems is studied in Section III. As some applications in Section IV, FXTS of scalar systems and fixed/preassigned-time stabilization of chaotic system are investigated by proposing some saturation functions. In Section V, three numerical examples are shown to confirm the theoretical results. In Section VI, the summary of the paper and some prospects for future work are given.

Notations: In this article, $\mathcal{M} = \{1, 2, \ldots, m\}$, I_m denotes an m-dimensional unit matrix, here m is a positive integer. R is the set composed of real numbers. $\mathrm{sign}(z)$ denotes the sign function of $z \in R$. R^m represents the m-dimensional Euclidean space, and for any $x = (x_1, x_2, \ldots, x_m)^T$, $y = (y_1, y_2, \ldots, y_m)^T \in R^m$ and $\theta > 0$, $\mathrm{Sign}(x) = (\mathrm{sign}(x_1), \ldots, \mathrm{sign}(x_m))^T$, $[x]^\theta = (|x_1|^\theta, \ldots, |x_m|^\theta)^T$, and $x \circ y = (x_1y_1, \ldots, x_my_m)^T$. $R^{m\times m}$ is the set of real square matrices with m dimension. $\overline{co}$ stands for the convex closure, $\mathcal{L}$ denotes the Lebesgue measure. $R^+ = [0, +\infty)$, $\mathcal{N}(z, \mu)$ represents an open neighborhood with the radius μ and the center z. For any $x \in R$

$$SG(x) = \begin{cases} \{-1\}, & \text{if } x < 0 \\ \{1\}, & \text{if } x > 0 \\ [-1, 1], & \text{if } x = 0. \end{cases}$$

II. Preliminaries

Consider a class of nonlinear dynamic systems

$$\dot{y}(t) = g(y(t)), \quad y(0) = y_0 \tag{1}$$

where $y \in R^m$ is the state variable, $g : R^m \to R^m$ is a function, and $g(0) = 0$. The existence of solution for system (1) is trivial if g is continuous [46]. If function g is discontinuous and locally measurable, the Filippov theory is introduced in the following to define the solution of system (1).

Definition 1 [47]: Function $z(t) : [0, \Gamma) \to R^m$ is called to be a solution of model (1) under the framework of Filippov if it satisfies the following conditions.

1) $z(t)$ is absolutely continuous (AC) on any compact subset of $[0, \Gamma)$.
2) For nearly all $t \in [0, \Gamma)$

$$\dot{z}(t) \in \mathcal{S}[g](z(t)) \tag{2}$$

in which $\mathcal{S}[g] : R^m \to R^m$ represents a set-valued mapping defined as

$$\mathcal{S}[g](z) = \bigcap_{\mu>0} \bigcap_{\mathcal{L}(\Phi)=0} \overline{co}\{g(\mathcal{N}(z,\mu) \setminus \Phi\}.$$

The following definitions and lemmas are provided based on the assumption that function $V : R^m \to R$ is locally Lipschitz.

Definition 2 [47], [48]: $V(z)$ is said to be regular at $z \in R^m$ if $V^0(z, r) = V^C(z, r)$ for all $r \in R^m$, in which $V^0(z, r)$ is the right directional derivative defined as

$$V^0(z, r) = \lim_{s\to 0^+} \frac{V(z + sr) - V(z)}{s}.$$

$V^C(z, r)$ is the Clarke upper generalized derivative given as

$$V^C(z, r) = \overline{\lim}_{k\to z, s\to 0^+} \frac{V(k + sr) - V(k)}{s}.$$

Definition 3 [48], [49]: Function V is C-regular provided that it is positive definite, regular, and radially unbounded.

Definition 4 [50]: Let $z(t)$ be AC on any compact subset of R^+, $V(z)$ is assumed to be C-regular, then $z(t)$ and $V(z(t))$ are differentiable on nearly all R^+ and

$$\frac{dV(z(t))}{dt} = \xi^T \dot{z}(t), \quad \text{for all } \xi \in \partial V(z(t))$$

in which

$$\partial V(z) = co\left\{\lim_{j\to+\infty} \nabla V(z_j) : z_j \to z, z_j \notin S \cup \Delta_V\right\}$$

S is the zero measure set on R^m, Δ_V is the set on R^m composed of all nondifferentiable points of V.

Definition 5 [6]: The zero solution of system (1) is called to be finite-time stable if it is asymptotically stable and there has a $T^*(y_0) > 0$ such that $y(t, y_0) = 0$ for all $t \geq T^*(y_0)$, where $y(t, y_0)$ is an arbitrary solution of model (1). The time $T(y_0) = \inf\{T^*(y_0) \geq 0 : y(t) = 0 \text{ for } t \geq T^*(y_0)\}$ is called the ST.

Definition 6 [1]: The zero solution of system (1) is called to be fixed-time stable if it is finite-time stable and there has a number $\hat{T} > 0$ such that $T(y_0) \leq \hat{T}$ for all $y_0 \in R^m$.

Definition 7 [28], [51]: The incomplete Beta function ratio is defined as the following integral:

$$I(x, p, q) = \frac{1}{B(p, q)} \int_0^x t^{p-1}(1 - t)^{q-1} dt$$

in which $B(p, q)$ is the common Beta function, $p > 0$, $q > 0$.

Lemma 1 [1]: For model (1), the zero solution achieves FXTS provided that there has a function $V(y) : R^m \to R$ to be C-regular and several positive real constants δ, a, θ, k, and b can be found to satisfy $\delta k > 1$, $\theta k < 1$ and

$$\frac{d}{dt}V(y(t)) \leq -\left(aV^\delta(y(t)) + bV^\theta(y(t))\right)^k. \tag{3}$$

Furthermore, the ST $T(y_0)$ is evaluated by

$$T(y_0) \leq T_1 \triangleq \frac{1}{a^k(\delta k - 1)} + \frac{1}{b^k(1 - \theta k)}.$$

In [19], the value of θ is relaxed from $\theta > 0$ to $\theta \geq 0$ and a more accurate estimate is derived for ST. The following is the improved fixed-time result.

Lemma 2 [19]: Suppose that function $V(y) : R^m \to R$ is C-regular, if there have a non-negative number θ and several positive real numbers k, δ, a, and b satisfying $\theta k < 1$, $\delta k > 1$ and

$$\frac{d}{dt}V(y(t)) \leq -\left(aV^\delta(y(t)) + bV^\theta(y(t))\right)^k, \ y(t) \in R^m \setminus \{0\} \tag{4}$$

then the zero solution is fixed-time stable and

$$T(y_0) \leq T_2 \triangleq \frac{1}{b^k}\left(\frac{b}{a}\right)^{\frac{1-\theta k}{\delta-\theta}}\left(\frac{1}{1 - \theta k} + \frac{1}{\delta k - 1}\right).$$

Lemma 3 [28], [52]: The following results are true:

$$B(a, b) = \frac{a + b}{ab} \prod_{n=1}^{\infty} \frac{1 + \frac{a+b}{n}}{\left(1 + \frac{a}{n}\right)\left(1 + \frac{b}{n}\right)}$$

$$B(a, 1 - a) = \frac{\pi}{\sin(\pi a)} = \pi \csc(\pi a).$$

Lemma 4 [53]: Let $\theta_j \geq 0$ for $j \in \mathcal{M}$ and $\delta > 1$, then

$$\sum_{j=1}^m \theta_j^\delta \geq m^{1-\delta}\left(\sum_{j=1}^m \theta_j\right)^\delta.$$

III. Fixed-Time Stability Analysis

In this part, we will propose and obtain several new criteria of FXTS and give some comparisons with the existing results [1], [11], [12], [19], [45].

Theorem 1: On condition that function $V(y) : R^m \to R$ is C-regular and $\psi : R^+ \to R^+$ satisfies $\psi(\mu) > 0$ for $\mu > 0$ and $\int_0^{+\infty}(1/\psi(\mu))d\mu = T_3 < +\infty$, if

$$\frac{d}{dt}V(y(t)) \leq -\psi(V(y(t))), \ y(t) \in R^m \setminus \{0\} \tag{5}$$

then the zero solution of the model (1) achieves FXTS and $T(y_0) \leq T_3$ for any $y_0 \in R^m$.

1) If $\psi(\mu) = c(a\mu^\delta + b)^k$ with positive constants a, b, c, and δ satisfying $\delta\, k > 1$, then $T_3 = (\Delta_1/c)$, where

$$\Delta_1 = \left(\frac{b}{a}\right)^{\frac{1}{\delta}} \frac{1}{\delta b^k} B\left(\frac{1}{\delta}, k - \frac{1}{\delta}\right).$$

2) If $\psi(\mu) = c(a\mu^\delta + b\mu^\theta)^k$ with positive constants a, b, c, δ, and θ satisfying $\delta k > 1$ and $\theta k < 1$, then $T_3 = (\Delta_2/c)$, where

$$\Delta_2 = \left(\frac{b}{a}\right)^{\frac{1-\theta k}{\delta-\theta}} \frac{1}{(\delta - \theta)b^k} B\left(\frac{1 - \theta k}{\delta - \theta}, \frac{\delta k - 1}{\delta - \theta}\right).$$

3) If $\psi(\mu) = c\exp(\zeta\mu)$ with $c > 0$ and $\zeta > 0$, then $T_3 = (1/c\zeta)$.

Proof: Denote

$$\chi(V) = \int_0^V \frac{1}{\psi(\mu)} d\mu.$$

Evidently, $\chi(V) = 0$ is equivalent to $V = 0$ and it is nondecreasing and non-negative. By virtue of (5)

$$\frac{d}{dt}\chi(V(y(t))) \leq -1, \ y(t) \in R^m \setminus \{0\}. \tag{6}$$

First, we prove that there has a $t_1 \in (0, T_3]$ satisfying $y(t_1) = 0$. If not, $y(t) \neq 0$ for all $t \in (0, T_3]$ and according to the inequality (6)

$$\frac{d}{dt}\chi(V(y(t))) \leq -1, \ t \in (0, T_3]$$

which implies that

$$\chi(V(y(T_3)) \leq \chi(V(y_0)) - T_3 \leq 0. \tag{7}$$

If $\chi(V(y_0)) < T_3$, $\chi(V(y(T_3)) < 0$ by (7), which contradicts the non-negativity of χ. If $\chi(V(y_0)) = T_3$, $y(T_3) = 0$ by use of the positive definiteness of V, it is inconsistent with $y(t) \neq 0$ on $t \in (0, T_3]$. All of these indicate that a time instant $t_1 \in (0, T_3]$ can be found such that $y(t_1) = 0$.

In the following, it is revealed that $y(t) = 0$ for any $t \geq t_1$. Otherwise, a $t_2 > t_1$ can be chosen such that $y(t_2) \neq 0$. Let

$$t_3 = \sup\{t \in [t_1, t_2) : y(t) = 0\}. \tag{8}$$

It is evident that $t_1 \leq t_3 < t_2$, $y(t_3) = 0$, and $y(t) \neq 0$ for any $t \in (t_3, t_2]$, which implies that $\chi(V(y(t_3))) = 0$ and

$$\frac{d}{dt}\chi(V(y(t))) \leq -1, \ t \in (t_3, t_2]$$

then

$$\chi(V(y(t_2))) \leq -(t_2 - t_3) < 0 \tag{9}$$

it results in a contradiction with the non-negativity of χ. Hence, the origin achieves FXTS within T_3.

In particular, if $\psi(\mu) = c(a\mu^\delta + b)^k$, introducing the variable transformation

$$v = \frac{b}{a\mu^\delta + b} \tag{10}$$

by the definition of Beta function

$$\begin{aligned} T_3 &= \int_0^{+\infty} \frac{1}{c(a\mu^\delta + b)^k} d\mu \\ &= \frac{1}{c\delta b^k}\left(\frac{b}{a}\right)^{\frac{1}{\delta}} \int_0^1 (1 - v)^{\frac{1}{\delta}-1} v^{k-\frac{1}{\delta}-1} dv \\ &= \frac{1}{c\delta b^k}\left(\frac{b}{a}\right)^{\frac{1}{\delta}} B\left(\frac{1}{\delta}, k - \frac{1}{\delta}\right) \\ &= \frac{\Delta_1}{c}. \end{aligned} \tag{11}$$

If $\psi(\mu) = c(a\mu^\delta + b\mu^\theta)^k$, let

$$W = \frac{V^{1-\theta k}}{1 - \theta k}$$

then for $y(t) \in R^m \setminus \{0\}$

$$\frac{d}{dt}W(y(t)) \leq -c\Big(a(1 - \theta k)^{\frac{\delta-\theta}{1-\theta k}} (W(y(t)))^{\frac{\delta-\theta}{1-\theta k}} + b\Big)^k. \tag{12}$$

Note that $\delta k > 1 > \theta k$, then

$$\frac{(\delta - \theta)k}{1 - \theta k} > 1.$$

According to the case (1), the origin is stable within $T_3 = (\Delta_2/c)$.

If $\psi(\mu) = c\exp(\zeta\mu)$, then

$$T_3 = \frac{1}{c}\int_0^{+\infty} \exp(-\zeta\mu)d\mu = \frac{1}{c\zeta}.$$

The proof is finished. ∎

Remark 1: Actually, FXTS of system (1) has been studied in [1] and [19]. Compared with them, Theorem 1 is more general and less conservative. First, the established condition in Theorem 1 is more generic and flexible since their conditions can be easily derived by case (1) and case (2) in Theorem 1 when $c = 1$. More importantly, different from the inequality technique used in [1] and [19], the upper estimates of ST, described by improper integrals, are directly calculated by means of variable substitution and finally expressed by Beta function. For example, the same upper bound for ST, described by the improper integral $\int_0^{+\infty} [1/((a\mu^\delta + b)^k)]d\mu$, is obtained in [19] and in case (1) with $c = 1$ in this article. To deal with the integral in [19], the following inequality method was first used for any demarcation point r:

$$\begin{aligned} T(y_0) &\leq \int_0^{+\infty} \frac{1}{\left(a\mu^\delta + b\right)^k} d\mu \\ &= \int_0^r \frac{1}{\left(a\mu^\delta + b\right)^k} d\mu + \int_r^{+\infty} \frac{1}{\left(a\mu^\delta + b\right)^k} d\mu \\ &\leq \int_0^r \frac{1}{b^k} d\mu + \int_r^{+\infty} \frac{1}{a^k\mu^{\delta k}} d\mu \\ &= \frac{r}{b^k} + \frac{1}{a^k(\delta k - 1)} r^{1-\delta k} \triangleq \Psi(r) \end{aligned}$$

and further an optimized estimate of ST is obtained by determining the minimum value of $\Psi(r)$ at $r = (b/a)^{(1/\delta)}$, which equals to T_2 with $\theta = 0$. Unlike the inequality method, the improper integral is directly calculated in (11) and then the new estimate T_3 is obtained. In what follows, it is shown that the direct calculation leads to a more accurate estimate for ST compared with the inequality method. In fact, when $c = 1$ for case (2) in Theorem 1, from Lemma 3

$$T_3 < \left(\frac{b}{a}\right)^{\frac{1-\theta k}{\delta-\theta}} \frac{1}{(\delta - \theta)b^k} \frac{k(\delta - \theta)^2}{(1 - \theta k)(\delta k - 1)} = T_2$$

this combines with [19, Th. 2]

$$T_3 < T_2 \leq T_1$$

which demonstrates that the estimates obtained Theorem 1 in this article are the most precise compared with the previous work [1], [19].

Remark 2: In [14], FXTS of continuous systems was investigated and several conditions were provided in Theorem 5, in which $\psi(\mu)$ was required to be continuous and positive definite. Unlike this work, a class of general systems, which can be continuous or discontinuous on the right-hand side of models, is considered in this article and FXTS is ensured by Theorem 1, here $\psi(\mu)$ is relaxed to satisfy the weaker condition that $\psi(\mu) > 0$ for $\mu > 0$.

The following statement is natural from Theorem 1.

Corollary 1: Suppose that function $V(y) : R^m \to R$ is C-regular and there exist several real numbers $0 \leq \theta < 1$, $a > 0$,

$\delta > 1$, $b > 0$ such that the following differential inequality holds:

$$\frac{d}{dt}V(y(t)) \le -aV^{\delta}(y(t)) - bV^{\theta}(y(t)),\ \ y(t) \in R^m \setminus \{0\} \quad (13)$$

then the zero solution of system (1) achieves FXTS and

$$T(y_0) \le T_4 \triangleq \frac{\pi b^{\varepsilon-1}}{a^{\varepsilon}(\delta-\theta)}\csc(\pi\varepsilon)$$

where $\varepsilon = [(1-\theta)/(\delta-\theta)]$.

1) If $\theta = 0$

$$T_4 = \frac{\pi}{\delta b}\left(\frac{b}{a}\right)^{\frac{1}{\delta}}\csc\frac{\pi}{\delta}.$$

2) If $\delta = 1 + (1/2\mu)$ and $\theta = 1 - (1/2\mu)$ with $\mu > 0$

$$T_4 = \frac{\pi\mu}{\sqrt{ab}}.$$

3) If $\delta = (k/n)$, $\theta = (s/r)$, where k, n, s, and r are positive integers satisfying $k > n$ and $r > s$

$$T_4 = \frac{nr\pi}{b(kr-ns)}\left(\frac{b}{a}\right)^{\frac{nr-ns}{kr-ns}}\csc\left(\frac{nr-ns}{kr-ns}\pi\right).$$

Remark 3: It is easy to see that the established condition and the evaluation of ST in [45] are the same as the case (2) in Corollary 1. Hence, our results are more general and flexible.

In Theorem 1, let $\psi(\mu)$ be the following piecewise function:

$$\psi(\mu) = \begin{cases} \psi_1(\mu), & \mu \ge r \\ \psi_2(\mu), & 0 \le \mu < r \end{cases} \quad (14)$$

where $r > 0$ and functions ψ_1 and ψ_2 satisfy the condition.

Assumption 1: For each $i = 1, 2$, $\psi_i : R^+ \to R^+$, $\psi_i(\mu) > 0$ for $\mu > 0$ and

$$\rho_1 = \int_r^{+\infty} \frac{1}{\psi_1(\mu)}d\mu < +\infty,\ \ \rho_2 = \int_0^r \frac{1}{\psi_2(\mu)}d\mu < +\infty.$$

Corollary 2: Assume that function $V(y) : R^m \to R$ is C-regular, if there exist $\psi_1 : R^+ \to R^+$ and $\psi_2 : R^+ \to R^+$ satisfying Assumption 1 such that the following differential inequality holds:

$$\frac{d}{dt}V(y(t)) \le \begin{cases} -\psi_1(V(y(t))), & V(y(t))) \ge r \\ -\psi_2(V(y(t))), & 0 < V(y(t))) < r \end{cases} \quad (15)$$

for some constant $r > 0$, then the zero solution of the model (1) realizes FXTS and $T(y_0) \le T_5 \triangleq \rho_1 + \rho_2$ for any $y_0 \in R^m$.

1) If $\psi_1(\mu) = a\mu^{\delta}$ and $\psi_2(\mu) = b\mu^{\theta}$

$$T_5 = \frac{r^{1-\delta}}{a(\delta-1)} + \frac{r^{1-\theta}}{b(1-\theta)}$$

in which $\delta > 1$, $\theta \in [0, 1)$, $a > 0$, $b > 0$.

2) If $\psi_1(\mu) = a_1\mu^{\delta_1} + b_1\mu^{\theta_1}$ and $\psi_2(\mu) = a_2\mu^{\delta_2} + b_2\mu^{\theta_2}$

$$T_5 = \frac{\pi\csc(\varepsilon_1\pi)}{b_1(\delta_1-\theta_1)}\left(\frac{b_1}{a_1}\right)^{\varepsilon_1} I\left(\frac{b_1}{a_1 r^{\delta_1-\theta_1}+b_1}, 1-\varepsilon_1, \varepsilon_1\right) + \frac{\pi\csc(\varepsilon_2\pi)}{a_2(\delta_2-\theta_2)}\left(\frac{a_2}{b_2}\right)^{\varepsilon_2} I\left(\frac{a_2}{b_2 r^{\theta_2-\delta_2}+a_2}, 1-\varepsilon_2, \varepsilon_2\right) \quad (16)$$

where $a_i > 0$, $b_i > 0$, $\delta_i > 1$, $0 \le \theta_i < 1$ for $i = 1, 2$ and

$$\varepsilon_1 = \frac{1-\theta_1}{\delta_1-\theta_1}, \quad \varepsilon_2 = \frac{\delta_2-1}{\delta_2-\theta_2}.$$

Proof: The result (1) is evident. For case (2), let

$$\omega = \frac{b_1}{a_1\mu^{\delta_1-\theta_1}+b_1}$$

then by Definition 7

$$\begin{aligned}\rho_1 &= \int_r^{+\infty}\frac{1}{a_1\mu^{\delta_1}+b_1\mu^{\theta_1}}d\mu \\ &= \frac{1}{b_1(\delta_1-\theta_1)}\left(\frac{b_1}{a_1}\right)^{\varepsilon_1}\int_0^{\frac{b_1}{a_1 r^{\delta_1-\theta_1}+b_1}} \omega^{-\varepsilon_1}(1-\omega)^{\varepsilon_1-1}d\omega \\ &= \frac{\pi\csc(\varepsilon_1\pi)}{b_1(\delta_1-\theta_1)}\left(\frac{b_1}{a_1}\right)^{\varepsilon_1} I\left(\frac{b_1}{a_1 r^{\delta_1-\theta_1}+b_1}, 1-\varepsilon_1, \varepsilon_1\right).\end{aligned} \quad (17)$$

Similarly, denote

$$v = \frac{a_2}{b_2\mu^{\theta_2-\delta_2}+a_2}$$

then

$$\begin{aligned}\rho_2 &= \int_0^r \frac{1}{a_2\mu^{\delta_2}+b_2\mu^{\theta_2}}d\mu \\ &= \frac{1}{a_2(\delta_2-\theta_2)}\left(\frac{a_2}{b_2}\right)^{\varepsilon_2}\int_0^{\frac{a_2}{b_2 r^{\theta_2-\delta_2}+a_2}} v^{-\varepsilon_2}(1-v)^{\varepsilon_2-1}dv \\ &= \frac{\pi\csc(\varepsilon_2\pi)}{a_2(\delta_2-\theta_2)}\left(\frac{a_2}{b_2}\right)^{\varepsilon_2} I\left(\frac{a_2}{b_2 r^{\theta_2-\delta_2}+a_2}, 1-\varepsilon_2, \varepsilon_2\right).\end{aligned} \quad (18)$$

The proof is finished. ■

Remark 4: For case (1) in Corollary 2, when $\theta = 0$, $\psi_2(\mu) = b > 0$, and $T_5 = [r^{1-\delta}/(a(\delta-1))] + (r/b)$. Similarly, if $\theta_2 = 0$ in case (2), $\psi_2(\mu) = a_2\mu^{\delta_2} + b_2 > 0$ and

$$T_5 = \rho_1 + \frac{\pi\csc\left(\frac{\delta_2-1}{\delta_2}\pi\right)}{a_2\delta_2}\left(\frac{a_2}{b_2}\right)^{\frac{\delta_2-1}{\delta_2}} \times I\left(\frac{a_2}{b_2 r^{-\delta_2}+a_2}, \frac{1}{\delta_2}, 1-\frac{1}{\delta_2}\right).$$

Note that these results cannot be obtained and FXTS of system (1) cannot be ensured when $\psi_2(0) > 0$ according to the previous work given in [11] since the condition $\psi_2(0) = 0$ is required. In addition, when $r = 1$ and $0 < \theta < 1$ in case (1), Corollary 2 is reduced to [12, Lemma 3].

Remark 5: In case (2) in Corollary 2, instead of the common inequality method, some special functions, including incomplete beta function ratio and cosecant function, are introduced to calculate the upper estimate T_5 through variable substitution, which improves the accuracy of the estimate.

The following results can be directly derived from Theorem 1, Corollaries 1 and 2.

Corollary 3: The zero solution is fixed-time stable within $T_p > 0$ for the model (1) provided that there have a function $V(y) : R^m \to R$ to be C-regular and a function $\psi : R^+ \to R^+$, satisfying $\psi(\mu) > 0$ for $\mu > 0$, $\int_0^{+\infty}[1/(\psi(\mu))]d\mu = T_3 < +\infty$, such that the following differential inequality holds:

$$\frac{d}{dt}V(y(t)) \le -\frac{T_3}{T_p}\psi(V(y(t))),\ y(t) \in R^m \setminus \{0\}. \tag{19}$$

Corollary 4: The zero solution for the model (1) achieves FXTS within T_p provided that there have a C-regular function $V(y) : R^m \to R$ and several real numbers $T_p > 0$, $\delta > 1$, $\theta \in [0, 1)$, $a > 0$, $b > 0$ such that the following differential inequality is true:

$$\frac{d}{dt}V(y(t)) \le -\frac{T_4}{T_p}\left(aV^{\delta}(y(t)) + bV^{\theta}(y(t))\right),\ y(t) \in R^m \setminus \{0\}. \tag{20}$$

Corollary 5: Let function $V(y) : R^m \to R$ be C-regular, suppose there have a positive number $T_p > 0$, two functions $\psi_1 : R^+ \to R^+$ and $\psi_2 : R^+ \to R^+$ such that

$$\frac{d}{dt}V(y(t)) \le \begin{cases} -\frac{\rho_1+\rho_2}{T_p}\psi_1(V(y(t))), & V(y(t))) \ge r \\ -\frac{\rho_1+\rho_2}{T_p}\psi_2(V(y(t))), & 0 < V(y(t))) < r \end{cases} \tag{21}$$

here $r > 0$, then the zero solution realizes FXTS within T_p.

Remark 6: Note that the convergence time T_p in Corollaries 3–5 is unrelated to other parameters, this is different from the parameter-dependent counterpart in Theorem 1 and Corollaries 1 and 2. The irrelevance provides a feasible method to investigate the preassigned-time control of dynamic systems, which will be shown in the following section.

IV. Applications

As some applications, FXTS for a kind of typical scalar systems and fixed/preassigned-time stabilization of chaotic systems are investigated in this section. Furthermore, to reduce the chattering, a type of saturation function is constructed to replace the sign function in scalar systems and the FXTS of the replaced systems is strictly proved. The method of saturation is further used to design fixed/preassigned-time controllers.

A. Construction of Saturation Function to Reduce Chattering

First, to compare with the results [19], [20], we consider the following system:

$$\dot{y}(t) = -a\text{sign}(y(t))|y(t)|^{\delta} - b\text{sign}(y(t))|y(t)|^{\theta} \tag{22}$$

in which $y \in R$, $\theta \in [0, 1)$, $\delta > 1$, $a > 0$, $b > 0$.

Corollary 6: The zero solution of system (22) is fixed-time stable and ST satisfies $T(y_0) \le T_4$.

Proof: Define $V(y(t)) = y^2(t)$. If $0 < \theta < 1$, (22) is a continuous system and

$$\frac{d}{dt}V(y(t)) = -2a(V(t))^{\frac{\delta+1}{2}} - 2b(V(t))^{\frac{\theta+1}{2}}.$$

On the other hand, system (22) is discontinuous when $\theta = 0$. According to Filippov theory [47], there has a function $v(t) \in SG(y(t))$ such that

$$\dot{y}(t) = -a\text{sign}(y(t))|y(t)|^{\delta} - bv(t).$$

Then for any $y(t) \in R \setminus \{0\}$

$$\frac{d}{dt}V(y(t)) = -2a(V(t))^{\frac{\delta+1}{2}} - 2b(V(t))^{\frac{1}{2}}.$$

So for any $\theta \in [0, 1)$ and $y(t) \in R \setminus \{0\}$

$$\frac{d}{dt}V(y(t)) = -2a(V(t))^{\frac{\delta+1}{2}} - 2b(V(t))^{\frac{\theta+1}{2}}. \tag{23}$$

Then the origin achieves FXTS within T_4 according to Corollary 1. ■

Remark 7: In [20] and [19], the FXTS of system (22) has been studied with $\delta = (k/n)$, $\theta = (s/r)$, where k, n, s, and r are required to be positive integers satisfying $k > n$ and $r > s$ in [19] and further restricted as odd integers in [20]. Furthermore, similar to Remark 1, it can be shown that the upper bound of ST given in Corollary 6 is the most accurate compared with these previous results [19], [20].

As is well-known, the sign function is a discontinuous switching function, which usually leads to undesired chattering and deteriorates the performance of the controlled system or disable the controller. Although the value of θ is not equal to zero but closely around zero, it is verified in the section of numerical simulations that the chattering is clearly observed. To suppress or reduce the bad phenomenon, a saturation function is constructed to replace the sign function in the following part, and the FXTS of the modified system is discussed.

Consider the following saturation function

$$\text{sat}(x) = \begin{cases} \text{sign}(x), & |x| \ge c \\ h(x), & |x| < c \end{cases} \tag{24}$$

where $c > 0$, $h(x)$ is continuous on the interval $(-c, c)$, $h(0) = 0$, $\lim_{x\to c^-} h(x) = 1$ and $\lim_{x\to -c^+} h(x) = -1$, and $\text{sign}(x)h(x) \ge \lambda|x|^{\beta}$ for some constants $\lambda > 0$ and $0 < \beta < 1$.

By replacing the sign function with the above saturation function in system (22), the following system is obtained:

$$\dot{y}(t) = -a\text{sat}(y(t))|y(t)|^{\delta} - b\text{sat}(y(t))|y(t)|^{\theta} \tag{25}$$

in which $\delta > 1$, $\theta \in [0, 1)$, $a > 0$, $b > 0$.

Corollary 7: The zero solution of system (25) achieves FXTS provided that $0 < \theta + \beta < 1$, and ST is evaluated by

$$T(y_0) \le T_6 \triangleq \frac{\pi\csc(\varepsilon_3\pi)}{b(\delta-\theta)}\left(\frac{b}{a}\right)^{\varepsilon_3} I\left(\frac{b}{ac^{\delta-\theta}+b}, 1-\varepsilon_3, \varepsilon_3\right) + \frac{\pi\csc(\varepsilon_4\pi)}{a\lambda(\delta-\theta)}\left(\frac{a}{b}\right)^{\varepsilon_4} I\left(\frac{a}{bc^{\theta-\delta}+a}, 1-\varepsilon_4, \varepsilon_4\right) \tag{26}$$

where

$$\varepsilon_3 = \frac{1-\theta}{\delta-\theta}, \quad \varepsilon_4 = \frac{\delta+\beta-1}{\delta-\theta}.$$

Proof: Introduce $\hat{V}(y(t)) = |y(t)|$, note that the model (25) belongs to continuous systems, evidently

$$\frac{d}{dt}\hat{V}(y(t)) \le \begin{cases} -a|y(t)|^{\delta} - b|y(t)|^{\theta}, & |y(t)| \ge c \\ -a\lambda|y(t)|^{\delta+\beta} - b\lambda|y(t)|^{\theta+\beta}, & 0 < |y(t)| < c. \end{cases}$$

By virtue of Corollary 2, the zero solution is fixed-time stable within T_6. ■

Remark 8: Actually, the saturation function (24) can be easily constructed. For example

$$\mathrm{sat}(x) = \begin{cases} 1, & x \geq c \\ \left(\frac{x}{c}\right)^{\frac{p}{q}}, & |x| < c \\ -1, & x \leq -c \end{cases} \tag{27}$$

and

$$\mathrm{sat}(x) = \frac{\left|x^{\frac{p}{q}} + d\right| - \left|x^{\frac{p}{q}} - d\right|}{2d} \tag{28}$$

where $q > 0$ and $p > 0$ are odd numbers and satisfy $p < q$, $c > 0$, $d > 0$. Moreover, it is evident that $\lambda = c^{-(p/q)}$ and $\beta = (p/q)$ for the function (27), and $\lambda = (1/d)$, $\beta = (p/q)$, and $c = d^{(q/p)}$ for the function (28).

Remark 9: In several recent results about FXTC such as [54], without any rigorous theoretical proof, the saturation function $\tanh(x)$ was directly used to replace the sign function in numerical simulation to reduce the chattering. Moreover, the rationality of the substitution and the FXTS for the replaced systems are not yet considered. To fill the gap, some rigorous theoretical discussions are provided in this section to show how to construct a suitable saturation function as a substitute of the sign function, and it is further proved that it is still fixed-time stable for the replaced model.

B. Fixed-Time and Preassigned-Time Stabilization of Chaotic Systems

Consider the following controlled chaotic systems

$$\dot{y}(t) = Ly(t) + Pg(y(t)) + u(t) \tag{29}$$

where $y(t) \in R^m$ denotes the state variable, $L, P \in R^{m\times m}$, $u(t)$ is a controller which needs to be designed to achieve stabilization, and $g : R^m \to R^m$ is a function satisfying the following condition.

Assumption 2: $g(0) = 0$ and there exists a positive definite matrix $Q \in R^{m\times m}$ such that

$$(g(u) - g(v))^T(g(u) - g(v)) \leq (u - v)^T Q(u - v)$$

for any $u, v \in R^m$.

First, the controller $u(t)$ is designed in the following form:

$$\begin{aligned} u(t) = &-Ky(t) - \alpha \mathrm{Sign}(y(t)) \circ [y(t)]^{\delta} \\ &- \sigma \mathrm{Sign}(y(t)) \circ [y(t)]^{\theta} \end{aligned} \tag{30}$$

where $K \in R^{m\times m}$, $\alpha > 0$, $\sigma > 0$, and $\delta > 1$, $0 \leq \theta < 1$.

Theorem 2: Under Assumption 2 and the controller (30), the origin of system (29) is fixed-time stabilized if the control gain matrix K satisfies

$$K + K^T \geq L + L^T + \omega PP^T + \omega^{-1}Q \tag{31}$$

where $\omega > 0$. Moreover, the ST is evaluated by

$$T(y_0) \leq T_7 \triangleq \frac{\pi}{\sigma(\delta-\theta)}\left(\frac{m^{\frac{\delta-1}{2}}\sigma}{\alpha}\right)^{\frac{1-\theta}{\delta-\theta}} \csc\left(\frac{1-\theta}{\delta-\theta}\pi\right). \tag{32}$$

Proof: Introduce the Lyapunov candidate function $V(t) = y^T(t)y(t)$. As with similar the derivation of Corollary 6, for $\theta \in [0, 1)$

$$\begin{aligned} \frac{d}{dt}V(t) \leq\ & y^T(t)\big(L + L^T - K - K^T\big)y(t) \\ &+ y^T(t)Pg(y(t)) + g(y(t))^T P^T y(t) \\ &- 2\alpha y^T(t)\big(\mathrm{Sign}(y(t)) \circ [y(t)]^{\delta}\big) \\ &- 2\sigma y^T(t)\big(\mathrm{Sign}(y(t)) \circ [y(t)]^{\theta}\big) \\ \leq\ & y^T(t)\Big(L + L^T - K - K^T + \omega PP^T + \omega^{-1}Q\Big)y(t) \\ &- 2\alpha \sum_{r=1}^{m} |y_r(t)|^{1+\delta} - 2\sigma \sum_{r=1}^{m} |y_r(t)|^{1+\theta} \\ \leq\ & -2m^{\frac{1-\delta}{2}}\alpha(V(t))^{\frac{1+\delta}{2}} - 2\sigma(V(t))^{\frac{1+\theta}{2}}. \end{aligned}$$

By Corollary 1, the origin is fixed-time stabilized. ■

Remark 10: From the above differential inequality and the results of FXTS given in [1] and [19], the following two different estimates can be also obtained:

$$T_7^1 = \frac{m^{\frac{\delta-1}{2}}}{\alpha(\delta-1)} + \frac{1}{\sigma(1-\theta)}$$

$$T_7^2 = \frac{1}{\sigma}\left(\frac{m^{\frac{\delta-1}{2}}\sigma}{\alpha}\right)^{\frac{1-\theta}{\delta-\theta}}\left(\frac{1}{1-\theta} + \frac{1}{\delta-1}\right).$$

Similar to Remark 1, it can be proved that $T_7^1 \geq T_7^2 > T_7$, which implies that the estimate conservatism of ST is essentially decreased in this article compared with existing results [1], [19]. On the other side, from the estimate formulas, although the conservatism can be weaken by adjusting the control parameters, higher control costs may be required to achieve faster convergence rate, which will be illustrated in numerical simulation.

Similarly, to suppress the chattering caused by the sign function, the following controller is designed:

$$\begin{aligned} u(t) = &-Ky(t) - \alpha \mathrm{Sat}(y(t)) \circ [y(t)]^{\delta} \\ &- \sigma \mathrm{Sat}(y(t)) \circ [y(t)]^{\theta} \end{aligned} \tag{33}$$

in which $K \in R^{m\times m}$, $\alpha > 0$, $\sigma > 0$, and $\delta > 1$, $0 \leq \theta < 1$, $\mathrm{Sat}(y) = (\mathrm{sat}(y_1), \ldots, \mathrm{sat}(y_m))^T$ is defined as

$$\mathrm{Sat}(y) = \begin{cases} \mathrm{Sign}(y), & \|y\| \geq c \\ H(y), & \|y\| < c \end{cases} \tag{34}$$

in which $H(y) = (h(y_1), \ldots, h(y_m))^T$, h is defined in (24).

Theorem 3: Under Assumption 2 and the controller (33), the origin of system (29) is fixed-time stabilized if $0 < \theta + \beta < 1$ and the inequality (31) holds. Furthermore

$$\begin{aligned} T(y_0) \leq T_8 \triangleq\ & \frac{\pi \csc(\varepsilon_3\pi)}{\sigma(\delta-\theta)}\left(\frac{\sigma}{\alpha\varepsilon_6}\right)^{\varepsilon_3} \\ &\times I\left(\frac{\sigma}{\alpha\varepsilon_6 c^{\delta-\theta} + \sigma}, 1-\varepsilon_3, \varepsilon_3\right) \\ &+ \frac{\pi \csc(\varepsilon_4\pi)}{\alpha\varepsilon_5\lambda(\delta-\theta)}\left(\frac{\alpha\varepsilon_5}{\sigma}\right)^{\varepsilon_4} \\ &\times I\left(\frac{\alpha\varepsilon_5}{\sigma c^{\theta-\delta} + \alpha\varepsilon_5}, 1-\varepsilon_4, \varepsilon_4\right) \end{aligned} \tag{35}$$

where $\varepsilon_5 = m^{[(1-\delta-\beta)/2]}$ and $\varepsilon_6 = m^{[(1-\delta)/2]}$.

Proof: For the Lyapunov function $V(t) = y^T(t)y(t)$

$$\frac{d}{dt}V(t) \leq -2\alpha y^T(t)\big(\mathrm{Sat}(y(t)) \circ [y(t)]^{\delta}\big) - 2\sigma y^T(t)\big(\mathrm{Sat}(y(t)) \circ [y(t)]^{\theta}\big). \tag{36}$$

When $V(t) \geq c^2$

$$\frac{d}{dt}V(t) \leq -2\alpha \sum_{r=1}^{m} |y_r(t)|^{\delta+1} - 2\sigma \sum_{r=1}^{m} |y_r(t)|^{\theta+1} \leq -2\alpha m^{\frac{1-\delta}{2}} (V(t))^{\frac{\delta+1}{2}} - 2\sigma (V(t))^{\frac{\theta+1}{2}}.$$

For $0 < V(t) < c^2$

$$\frac{d}{dt}V(t) \leq -2\alpha\lambda \sum_{r=1}^{m} |y_r(t)|^{\delta+\beta+1} - 2\sigma\lambda \sum_{r=1}^{m} |y_r(t)|^{\theta+\beta+1} \leq -2\alpha m^{\frac{1-\delta-\beta}{2}} \lambda (V(t))^{\frac{\delta+\beta+1}{2}} - 2\sigma\lambda (V(t))^{\frac{\theta+\beta+1}{2}}.$$

By Corollary 2, the origin is fixed-time stabilized within T_8. ■

Remark 11: If $h(y_i) = (y_i/c)^{(p/q)}$ for each $i \in \mathcal{M}$ in (34), where $q > p > 0$ are odd numbers satisfying $0 < \theta + (p/q) < 1$, it is obvious that $\lambda = c^{-(p/q)}$ and $\beta = (p/q)$ in the estimate (35).

In what follows, we consider the preassigned-time stabilization of system (29). First, the preassigned-time controller is designed as the following form:

$$u(t) = -Ky(t) - \frac{T_7}{T_p}\big(\alpha \mathrm{Sign}(y(t)) \circ [y(t)]^{\delta} + \sigma \mathrm{Sign}(y(t)) \circ [y(t)]^{\theta}\big) \tag{37}$$

in which $K \in R^{m\times m}$, $\alpha > 0$, $\sigma > 0$, and $\delta > 1$, $0 \leq \theta < 1$, $T_p > 0$, T_7 is given in (32).

Theorem 4: Under Assumption 2 and condition (31), the zero solution for system (29) is preassigned-time stabilized within the prespecified time T_p via the controller (37).

The proof of Theorem 4 can be directly obtained from Corollary 4 and the proof of Theorem 2.

Similarly, to reduce the chattering, the following saturation controller is developed:

$$u(t) = -Ky(t) - \frac{T_8}{T_p}\big(\alpha \mathrm{Sat}(y(t)) \circ [y(t)]^{\delta} + \sigma \mathrm{Sat}(y(t)) \circ [y(t)]^{\theta}\big) \tag{38}$$

in which $T_p > 0$ is a preassigned time, $\mathrm{Sat}(y(t))$ and T_8 are defined in (34) and (35), the rest parameters are defined in the controller (33).

From Corollary 5 and the proof of Theorem 3, the following statement is obtained.

Theorem 5: Under Assumption 2 and the controller (38), the zero solution for system (29) is preassigned-time stabilized within the prespecified time T_p if $0 < \theta + \beta < 1$ and the inequality (31) holds.

Remark 12: Different from the dependence of the convergence time T_7 and T_8 on the system and control parameters in Theorems 1 and 2, the stabilized time T_p in Theorems 4 and 5 is independent of other control parameters and can be prespecified according to the actual needs. From this point, the control protocols (37) and (38) are more convenient and more popular in practice.

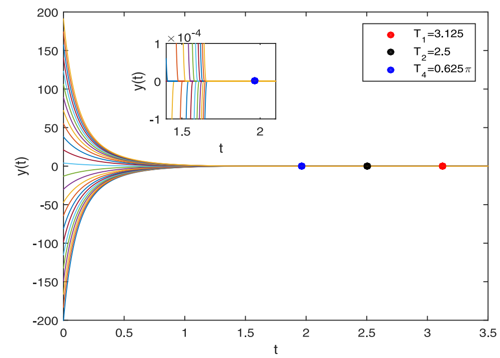

Fig. 1. FXTS of system (39) with $\theta = 0.6$.

Remark 13: In fact, preassigned-time control has been investigated in [41]–[44], in which some time-varying and unbounded gains are required in controller design. Unlike these work, a saturation controller (38) with finite gains is developed in this article to achieve the preassigned-time stabilization.

Remark 14: Actually, many practical systems often suffer from input saturation, input deadzone, unidirectional input constraints. In order to effectively reduce final positioning errors caused by these factors, several control laws based on neural networks or robust integral terms have been developed in [55] and [56]. However, there seems to be little attention about the FXTC of nonlinear systems with unidirectional input constraints or input deadzone. It is meaningful but full of challenge, and will be carefully concerned in our latest work.

V. Numerical Simulations

In this section, three numerical examples are provided to demonstrate the obtained theoretical results.

Example 1: Consider the following scalar system

$$\dot{y}(t) = -\mathrm{sign}(y(t))|y(t)|^{1.4} - 4\mathrm{sign}(y(t))|y(t)|^{\theta}. \tag{39}$$

By Corollary 6, the origin is fixed-time stable. To show this, $\theta = 0.6$ and $\theta = 0$ are both considered in numerical simulations and Figs. 1 and 2 are provided, where the three points, shown in red, black, and blue, represent three different estimates T_1, T_2, and T_4 for ST obtained in [1] and [19] and this article, respectively.

From Figs. 1 and 2, two facts are verified. First, the estimate obtained in this article is less conservative in comparison to the estimates obtained in [1] and [19], which verifies the correctness of the theoretical analysis. In addition, the chattering phenomenon is observed for the case $\theta = 0$, which is shown in Fig. 2.

Example 2: Consider the scalar system

$$\dot{y}(t) = -\mathrm{sat}(y(t))|y(t)|^{1.4} - 4\mathrm{sat}(y(t))|y(t)|^{\theta} \tag{40}$$

in which function $\mathrm{sat}(y)$ is defined in (27) with $p = 3$, $q = 5$, $c = 0.5$, and $\theta = 0$. It follows from Corollary 7 that the origin is fixed-time stable within the time $T_6 = 2.1174$. The

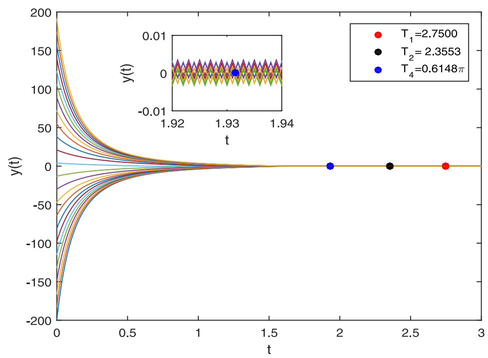

Fig. 2. FXTS of system (39) with $\theta = 0$.

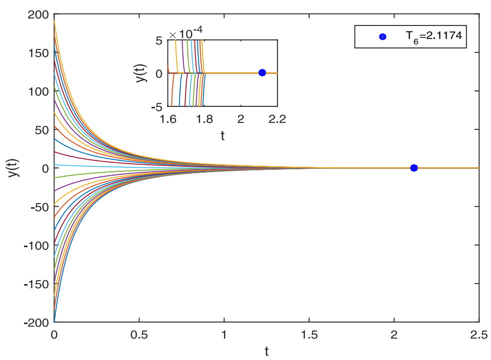

Fig. 3. FXTS of system (40).

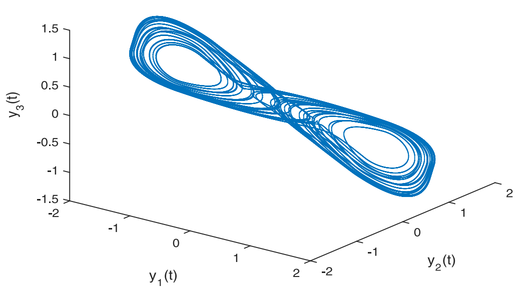

Fig. 4. Chaotic attractor of system (41).

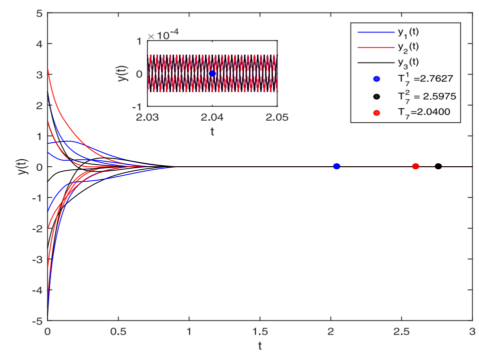

Fig. 5. Fixed-time stabilization of system (41) under controller (30).

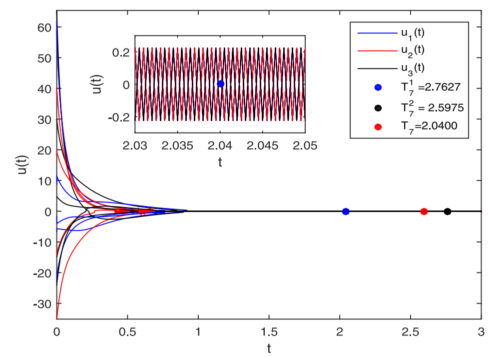

Fig. 6. Evolution of the controller (30).

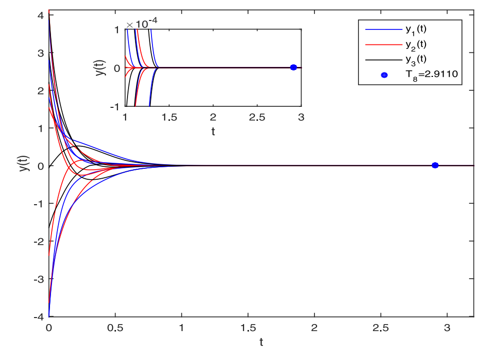

Fig. 7. Fixed-time stabilization of system (41) under the controller (33).

evolutions of solutions for system (40) with different initial values are provided in Fig. 3, from which we can see that the chattering is suppressed when the sign function is replaced by the saturation function.

Example 3: Consider the following controlled three-cell cellular neural network

$$\dot{y}(t) = Ly(t) + Pg(y(t)) + u(t) \tag{41}$$

where $L = -I_3$, $g(y) = (g_1(y_1), g_2(y_2), g_3(y_3))^T$, and $g_i(v) = 0.5(|v+1| - |v-1|)$ and

$$P = \begin{pmatrix} 1.25 & -3.2 & -3.2 \\ -3.2 & 1.1 & -4.4 \\ -3.2 & 4.4 & 1.0 \end{pmatrix}.$$

It has been proved in [57] that neural network (41) exists a chaotic attractor, which is presented in Fig. 4 with initial value $y(0) = (0.1, 0.1, 0.1)^T$.

First, we consider the fixed-time stabilization of system (41) under controller (30). By using LMI toolbox in MATLAB and condition (31), when $\omega = (1/7)$, one can obtain that

$$K = \begin{pmatrix} 5.1998 & 0.1644 & -0.6258 \\ 0.1644 & 5.4147 & 0.3126 \\ -0.6258 & 0.3126 & 5.4653 \end{pmatrix}.$$

Select $\alpha = 2$, $\sigma = 0.6$, $\delta = 1.9$, and$\theta = 0.1$ in controller (30). It follows from Theorem 2 that the origin is fixed-time stabilized within the time $T_7 = 2.04$, which is demonstrated in Fig. 5 and 6, where each initial value $y(0)$ is selected randomly on $[-5, 5]$. From Fig. 5 and 6, the chattering can be clearly found under the controller (30) although the origin is ensured to be fixed-time stabilized.

In addition, two other estimates $T_7^1 = 2.7627$ and $T_7^2 = 2.5975$ can be derived from [1] and [19], which are also shown in Figs. 5 and 6. Evidently, T_7 is the most accurate,

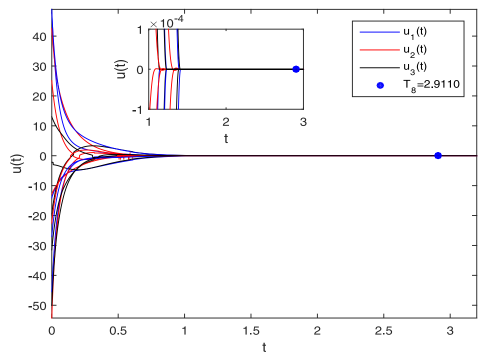

Fig. 8. Evolution of the saturation controller (33).

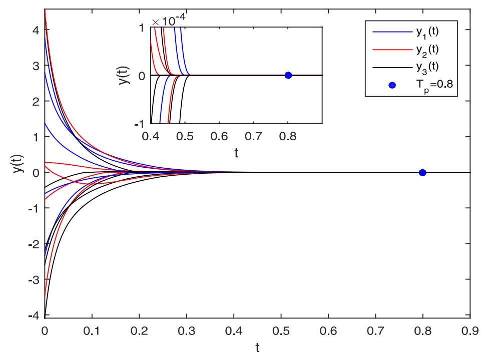

Fig. 9. Preassigned-time stabilization of system (41) under the controller (38).

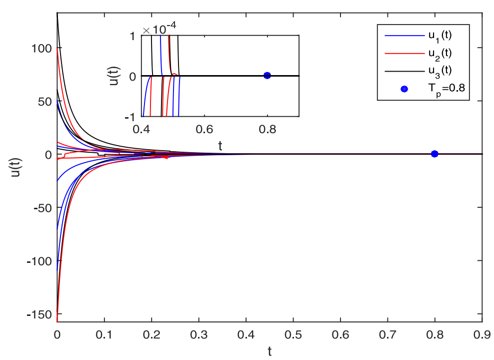

Fig. 10. Evolution of the saturation controller (38).

it is consistent with theoretical analysis. On the other side, under the same parameters $\sigma = 0.6$, $\delta = 1.9$, $\theta = 0.1$, if synchronization is required to be realized within the faster time $T_7 = 2.04$, according to the estimates obtained in [1] and [19], the control gain α must be respectively selected as $\alpha = 9.6819$ and $\alpha = 3.2424$ by simple computation. Evidently, each of them is higher than the gain $\alpha = 2$ given in this article, which implies that higher control costs are required to decrease the estimate conservatism of the results [1], [19].

In the following, we verify the fixed-time stabilization of system (41) based on the controller (33). The matrix of control gains K and the control parameters α, δ, and θ are selected as the same as the controller (30). Choose $p = 3$, $q = 5$, and $c = 0.2$, from Theorem 3, the origin is fixed-time stabilized under the controller (33) within the time $T_8 = 2.9110$, which is verified in Figs. 7 and 8 and it is further revealed that the control scheme (33) based on saturation function is more effective in reducing chattering.

In the following, the preassigned-time stabilization of the chaotic system (41) is verified under the saturation controller (38). The convergence time is prescribed as $T_p = 0.8$ and other parameters are the same as the above selection in fixed-time stabilization simulation. By Theorem 5, the origin is preassigned-time stabilized within the prescribed time $T_p = 0.8$, which is shown in Figs. 9 and 10.

VI. CONCLUSION

The paper explored some essential and challenging problems in the study of FXTS. At first, to reduce estimate conservatism of ST, instead of the single inequality method, several special functions are introduced to calculate the estimates described by improper integrals. The improved estimates are verified to be the most precise compared to the work [1], [19]. In addition, to suppress the chattering phenomenon, some saturation functions are proposed to replace the sign function in the classical fixed-time stable system and the FXTC design. Based on these saturation functions and special functions, some new criteria and estimates about FXTS and stabilization in fixed or preassigned time are derived. Finally, the revised estimates of ST and the chattering reduction are verified by several numerical examples.

Actually, this article dealt with FXTS of nonlinear systems from theoretical viewpoint and the designed controller is a kind of full control schemes which may be inapplicable for practical systems with high dimension. Future work will focus on the investigation on FXTC of practical models, such as human-robot interaction model [58], multiple robots [40], [59], [60], and micro aerial vehicle [61]. In addition, as shown in recent research [62]–[64], stochastic disturbance and delays are unavoidable in practice and have a profound influence on stability and control of systems. So, it is urgent and interesting to explore FXTS and control of stochastic or delayed systems, which will be deeply considered as a future work.

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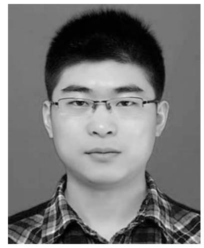

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