线性时不变系统（LTI）和卷积
(章节预备知识)

Contents

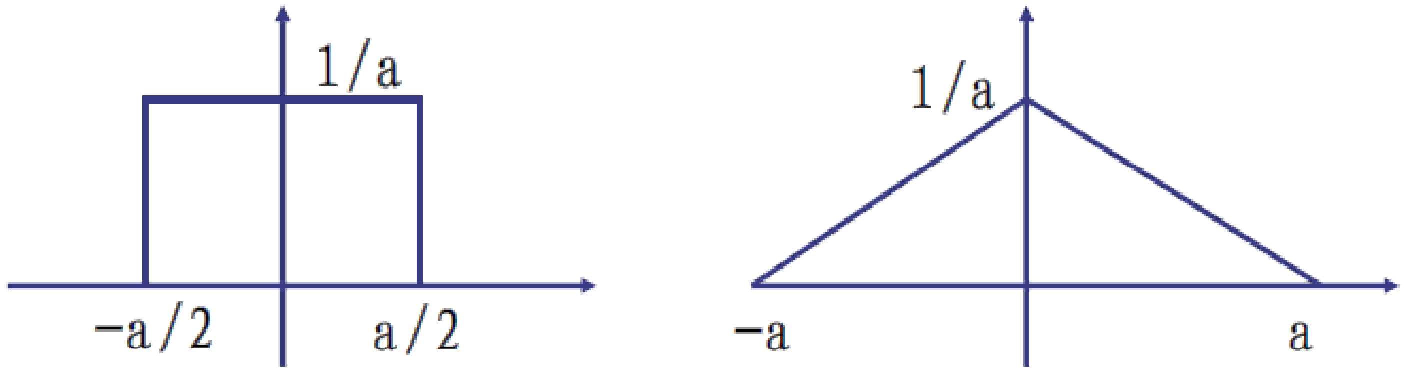
- **Discrete-Time LTI Systems**
  - The representation of discrete-time signals in terms of impulse （以脉冲信号表示离散时间信号）
  - The discrete-time unit impulse response and the convolution sum representation of LTI systems
    （以脉冲响应卷积和表示离散信号通过LTI系统的响应）
- **Continuous-Time LTI Systems**
  - The representation of continuous-time signals in terms of impulse
  - The continuous-Time unit impulse response and the convolution integral representation

脉冲函数的定义：也叫δ函数

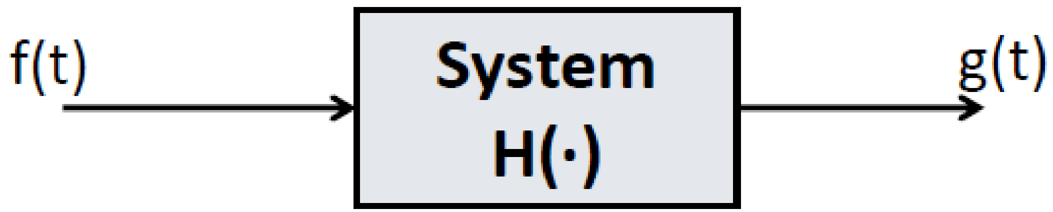
$$\delta(t-t_0)=0 \quad t\neq t_0$$

$$\int_{-\infty}^{\infty}\delta(t-t_0)dt=1$$

脉冲函数的极限定义：脉冲函数可以看成是一系列函数的极限，这些函数的振幅逐渐增大，持续时间逐渐减少，而保持面积不变。



线性系统（Linear System）



假设 $H(f_1(t)) = g_1(t) \qquad H(f_2(t)) = g_2(t)$

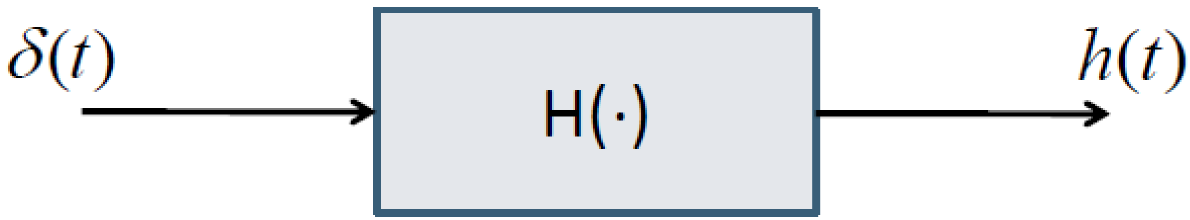
H(·)为线性系统:

$$H(\alpha \cdot f_1(t) + \beta \cdot f_2(t)) = \alpha \cdot g_1(t) + \beta \cdot g_2(t)$$

时不变系统（**Time-invariant system**）：

$$H(f(t-\tau)) = g(t-\tau)$$

单位冲激响应（Unit Impulse Response）



1. Discrete-Time LTI Systems: The Convolution Sum

- The representation of discrete-time signals in terms of impulse
  - According to the sampling property of impulse signal, discrete-time signals can be represented in terms of

$$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$$

1. Discrete-Time LTI Systems: The Convolution Sum

- The representation of discrete-time signals in terms of impulse

  weight，或x(t)在k的抽样

  - According to the sampling property of impulse signal, discrete-time signals can be represented in terms of

$$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$$

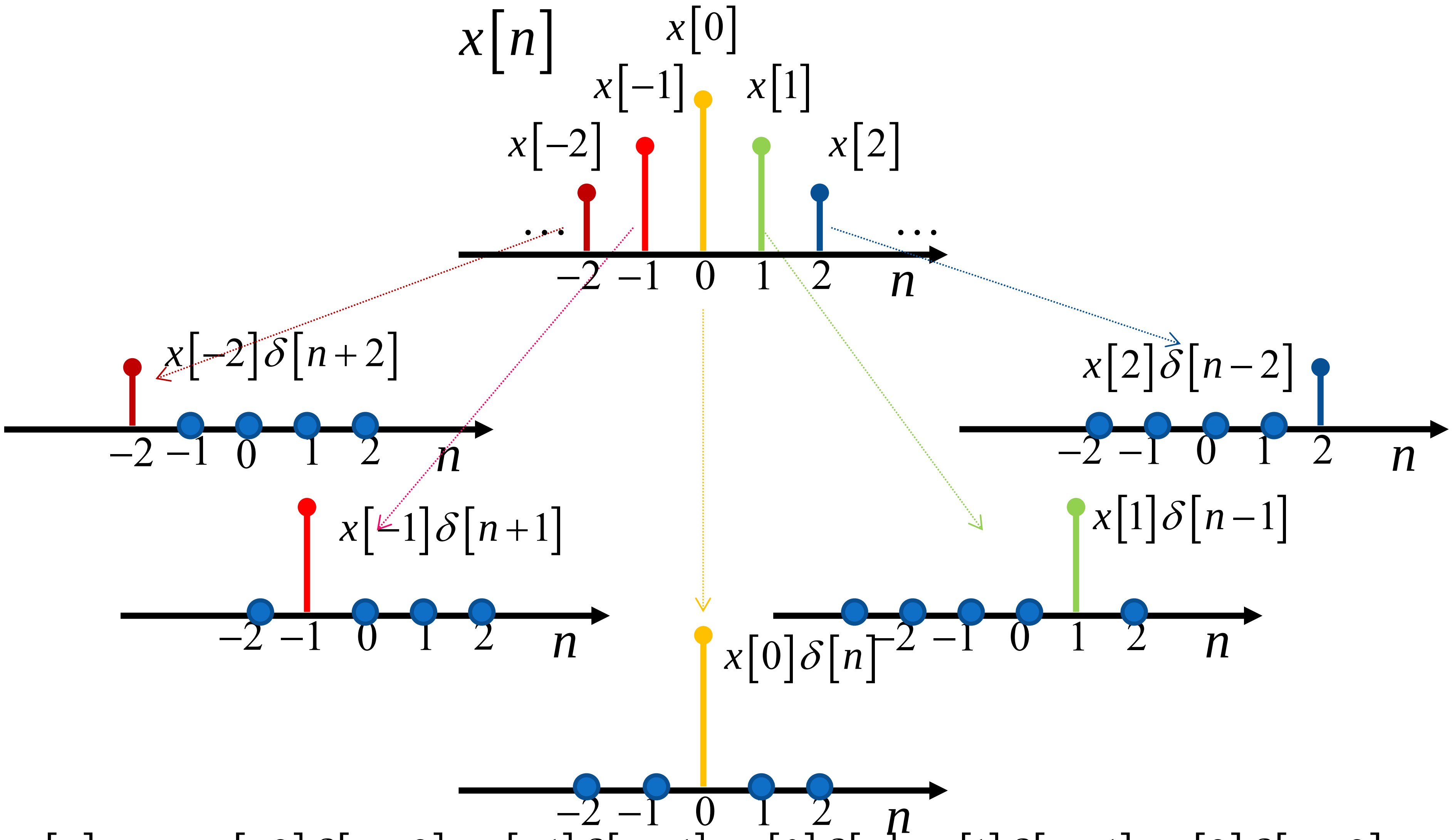
location

  - Regard a discrete-time signal as a sequence of individual impulses.

Do you remember the property of unit impulse ?

$$x[n]\delta[n-k]=x[k]\delta[n-k]$$

Any discrete-time signal can be represented as the sum of several weighted impulse.



$$x[n] = \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \dots$$

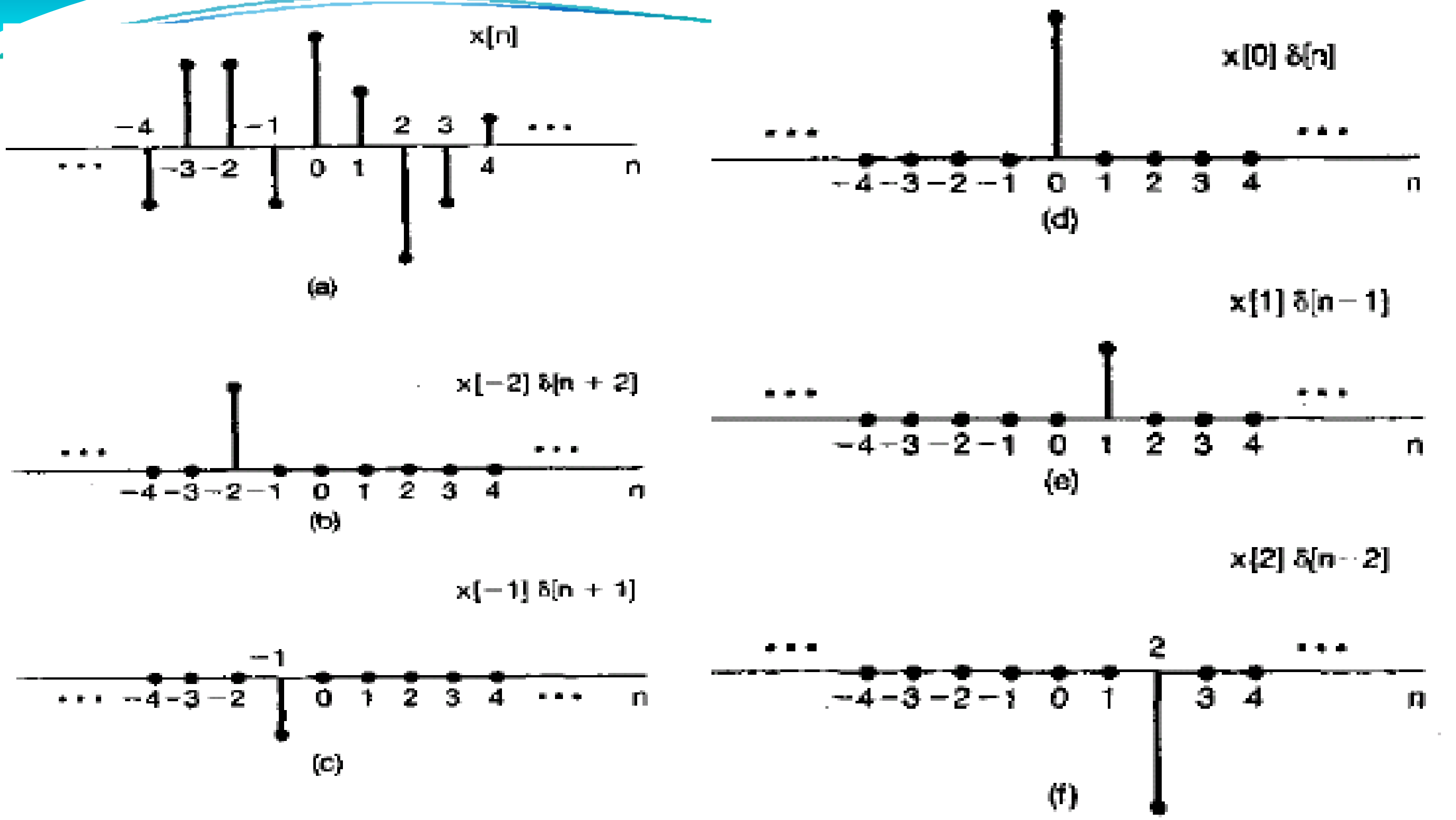
$$x[n]=\cdots+x[-2]\delta[n+2]+x[-1]\delta[n+1]+x[0]\delta[n]+x[1]\delta[n-1]+x[2]\delta[n-2]+\ldots$$

$$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$$

$$\Downarrow$$

$$x[n]*\delta[n]\overset{def}{=}\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$$

卷积和定义



$$x[n] = \cdots + x[-3]\delta[n+3] + x[-2]\delta[n+2] + x[-1]\delta[n+1]$$
$$+ x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3] + \cdots$$

The Discrete-Time unit Impulse Response and the Convolution-sum Representation of LTI Systems

- The ***shifting property*** （***shift v.筛选***）
  - it represents x[n] as ***a superposition*** （***叠加***）***of scaled versions*** of a very simple of elementary functions, shifted unit impulses.
- The response of a linear system to x[n]
  - ***the superposition of the scaled responses*** of the system to each of these shifted impulses.

- From the property of time invariance
  - the responses of a time-invariant system to the time-shifted unit impulses are simply ***time-shifted versions*** of one another.
- The convolution-sum（卷积和） representation for a LTI discrete-time system results from putting the above two basic facts together.

Denote $h_k[n]$

the response of the linear system to the shifted unit impulse $\delta[n-k]$

如果 $h_k[n]=h_0[n]$，则为线性时不变系统，否则为线性时变系统

Consider an input:

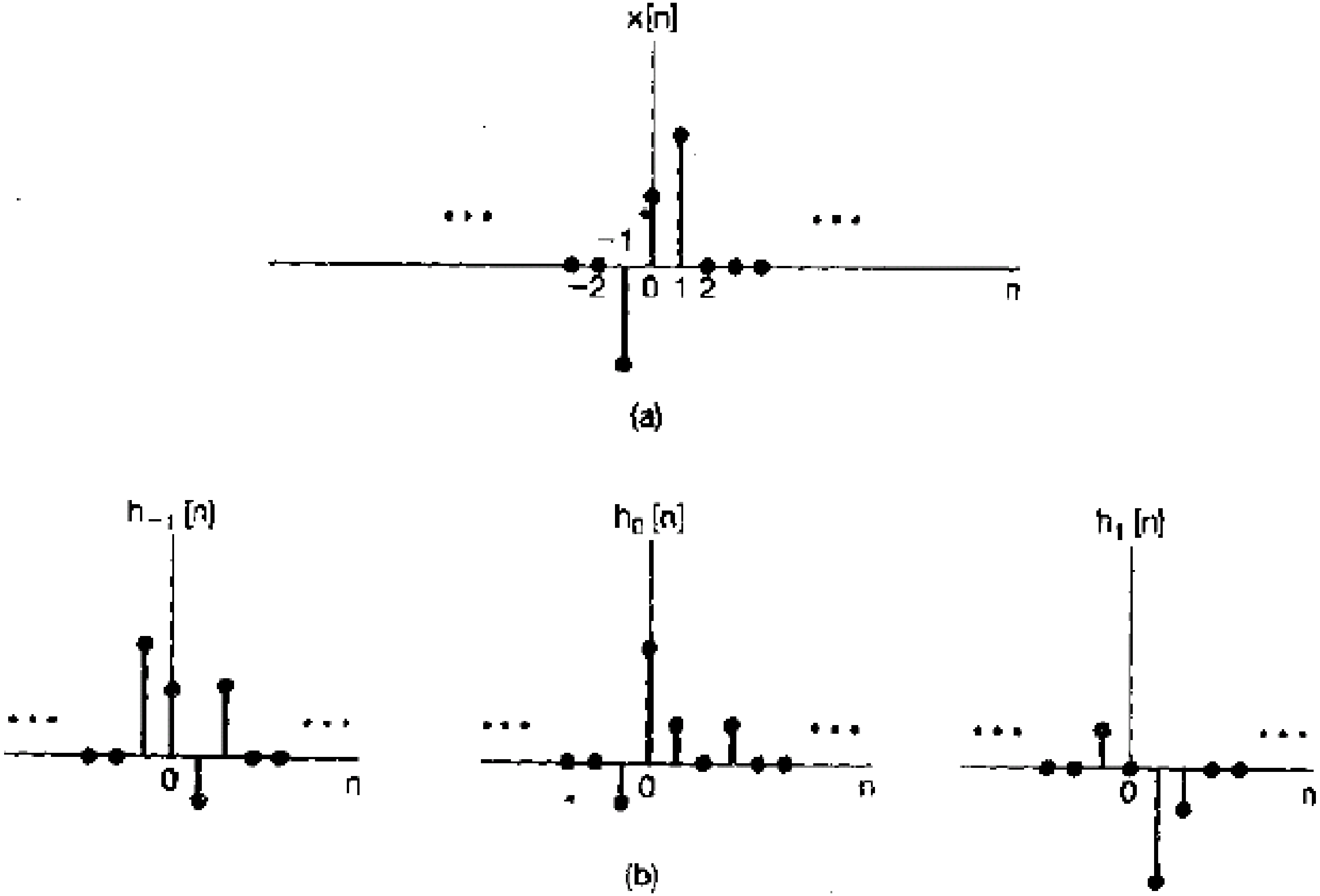
$$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$$

线性系统

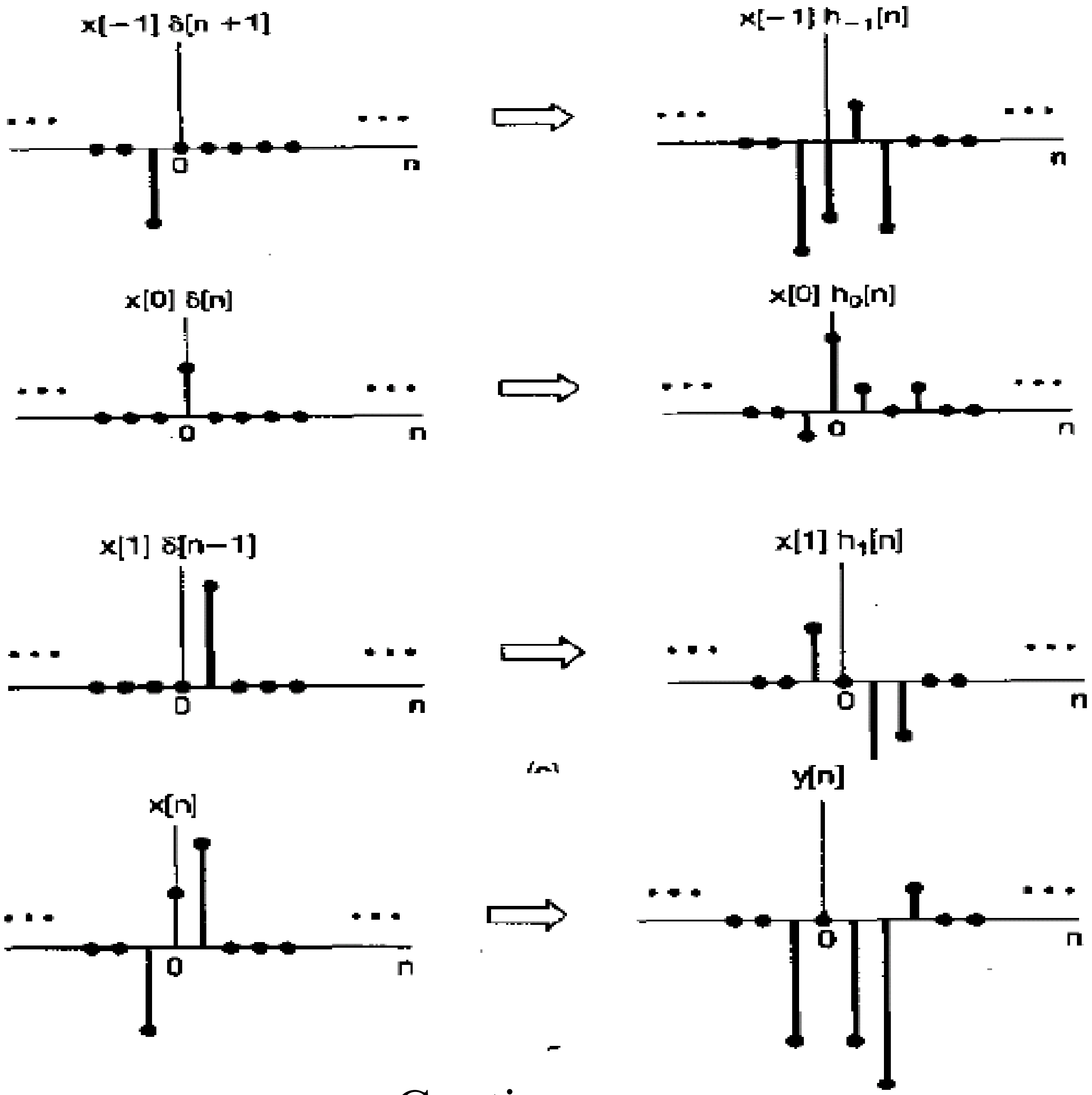
The response of the above system to this input can be expressed as

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h_k[n]$$

It is correct for either time-variant or time-invariant system



Explanation for $y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n]$



Continue

If we restrict our consideration onto TIME-INVARIANT systems

$$h_k[n] = h_0[n-k]$$

DO YOU REMEMBER THE TIME INVARIANCE PROPERTY INTRODUCED IN CHAPTER ONE?

$$e(t) \longrightarrow r(t)$$

$$e(t-t_0) \longrightarrow r(t-t_0)$$

What would happen if the system is a Linear Time Invariant one?

Denote the input to an LTI system as

$$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$$

Define the unit impulse response

$$h[n]=h_0[n]$$

The response of this LTI system to the input can be expressed as

$$\boxed{y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]}$$

时不变

THIS IS REFERRED TO AS THE ***CONVOLUTION SUM*** OR SUPERPOSITION SUM. （卷积和）

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Always be represented symbolically as

$$y[n] = x[n] * h[n]$$

Now, let's call back the definition of LTI system …

Time Invariant

$$x_1[n] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow y_1[n]$$

$$x_1[n-n_0] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow y_1[n-n_0]$$

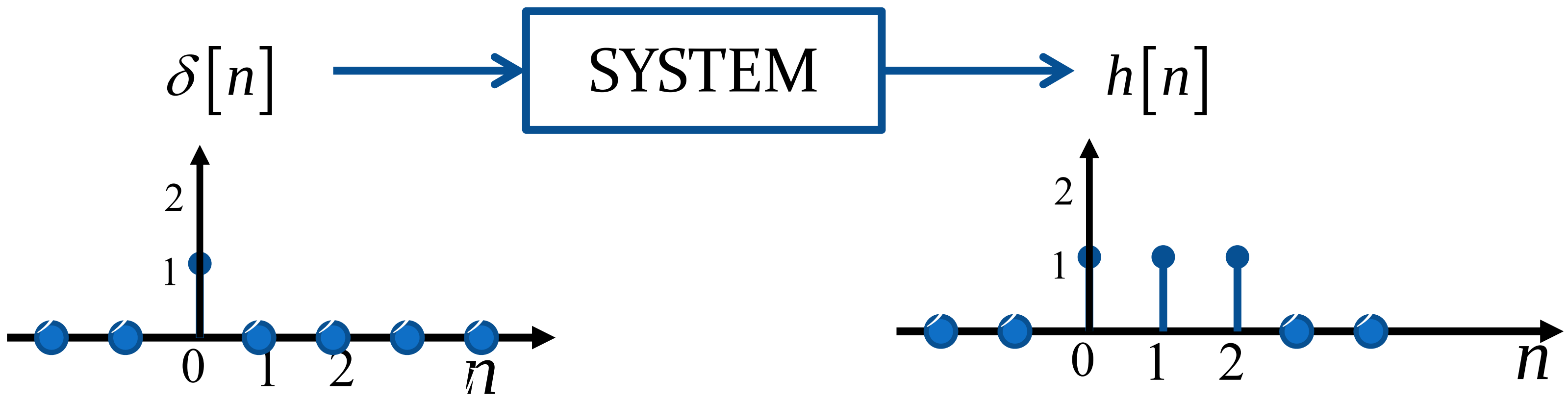
Linear

$$x_1[n] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow y_1[n]$$

$$x_2[n] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow y_2[n]$$

$$ax_1[n]+bx_2[n] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow ay_1[n]+by_2[n]$$

Unit Impulse Response



Unit impulse response

$$\delta[n] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow h[n]$$

Unit impulse response

$$\delta[n] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow h[n]$$

Time invariant

$$\delta[n-k] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow h[n-k]$$

Unit impulse response

$$\delta[n] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow h[n]$$

Time invariant

$$\delta[n-k] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow h[n-k]$$

Homogeneity

$$x[k]\delta[n-k] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow x[k]h[n-k]$$

Unit impulse response

$$\delta[n] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow h[n]$$

Time invariant

$$\delta[n-k] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow h[n-k]$$

Homogeneity

$$x[k]\delta[n-k] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow x[k]h[n-k]$$

Additivity

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Unit impulse response

$$\delta[n] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow h[n]$$

Time invariant

$$\delta[n-k] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow h[n-k]$$

Homogeneity

$$x[k]\delta[n-k] \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow x[k]h[n-k]$$

Additivity

$$\underbrace{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]}_{x[n]} \longrightarrow \boxed{\text{SYSTEM}} \longrightarrow \underbrace{\sum_{k=-\infty}^{\infty} x[k]h[n-k]}_{y[n]}$$

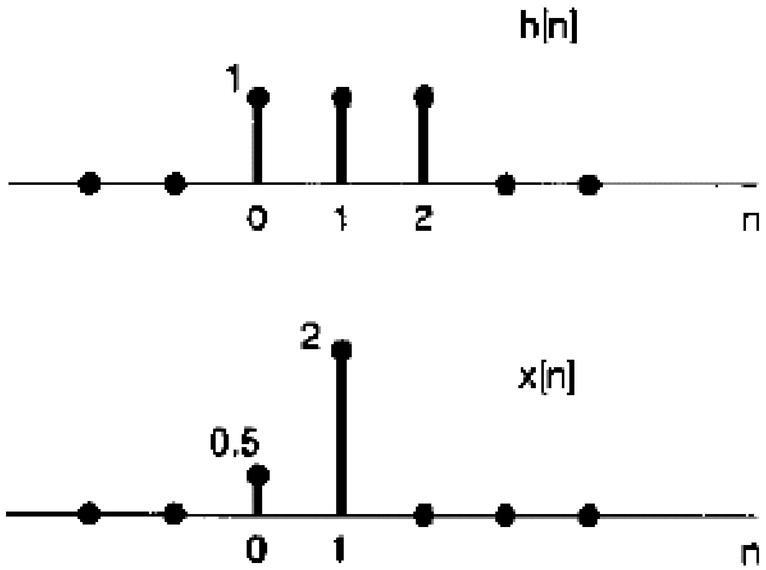
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Remember, it is correct for the case when the system is an LTI one.

WHY?

Example

Consider an LTI system with impulse response h[n] and input x[n], as illustrated in below. What will the response of this system be with input x[n]?



What does h[n] mean?

$$\delta[n] \rightarrow h[n]$$

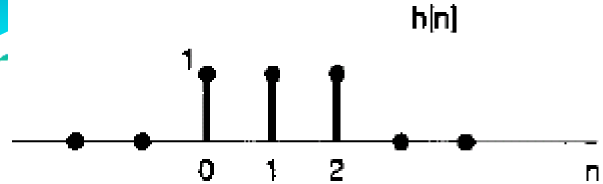
$$\delta[n-1] \rightarrow h[n-1]$$

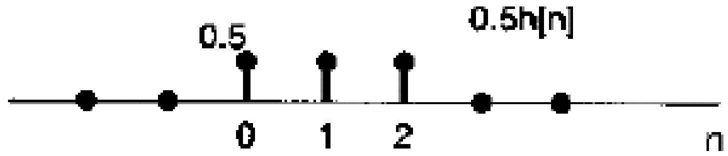
$$\vdots$$

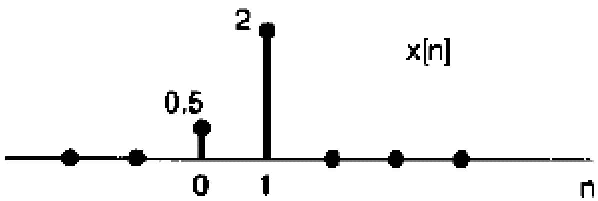
What does x[n] tell us?

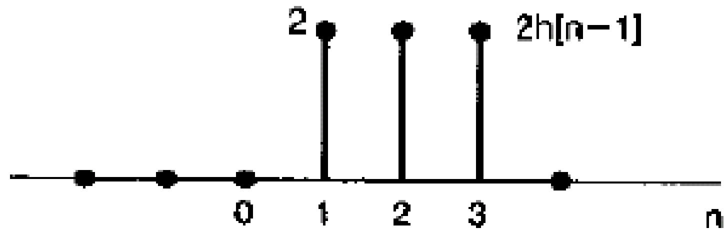
$$x[n] = x[0]\delta[n-0] + x[1]\delta[n-1]$$

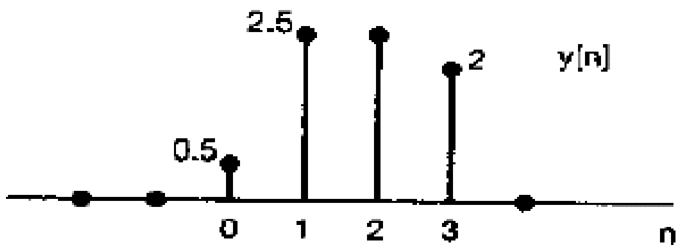
$$= 0.5\delta[n-0] + 2\delta[n-1]$$









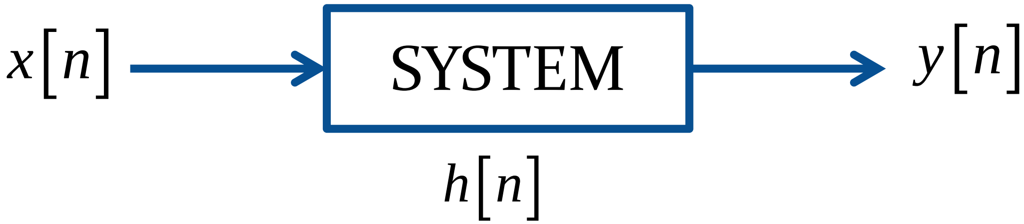


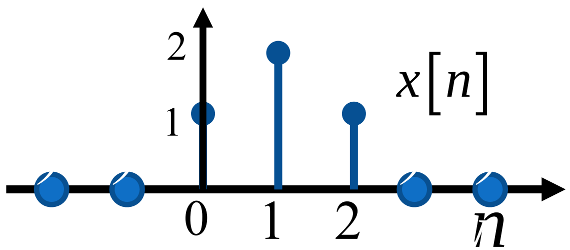
$$x[n] = x[0]\delta[n-0] + x[1]\delta[n-1]$$
$$= 0.5\delta[n-0] + 2\delta[n-1]$$

$$y[n] = x[0]h[n-0] + x[1]h[n-1]$$
$$= 0.5h[n] + 2h[n-1]$$

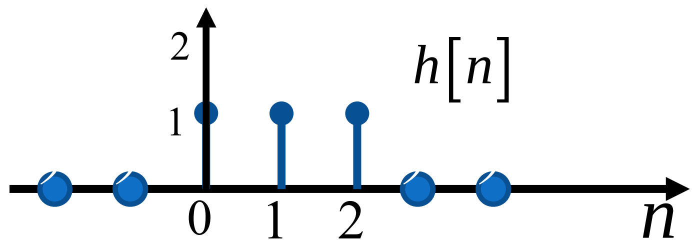
Example: Calculation of convolution

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

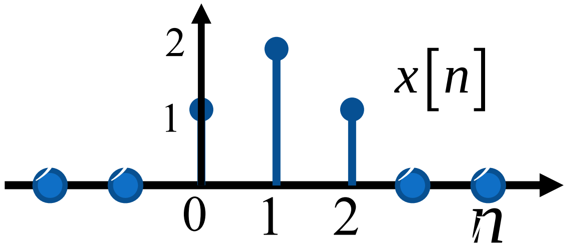




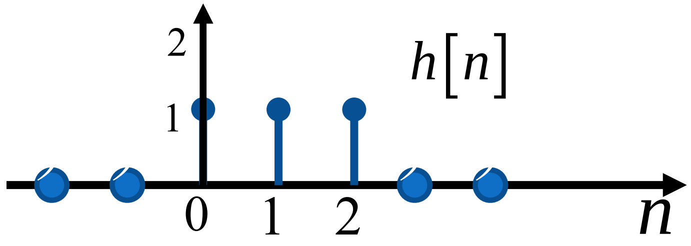
$*$



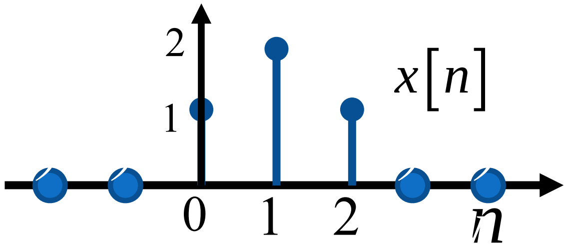
$$y[n]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$$



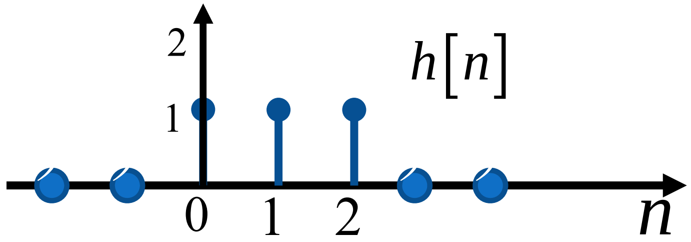
$*$



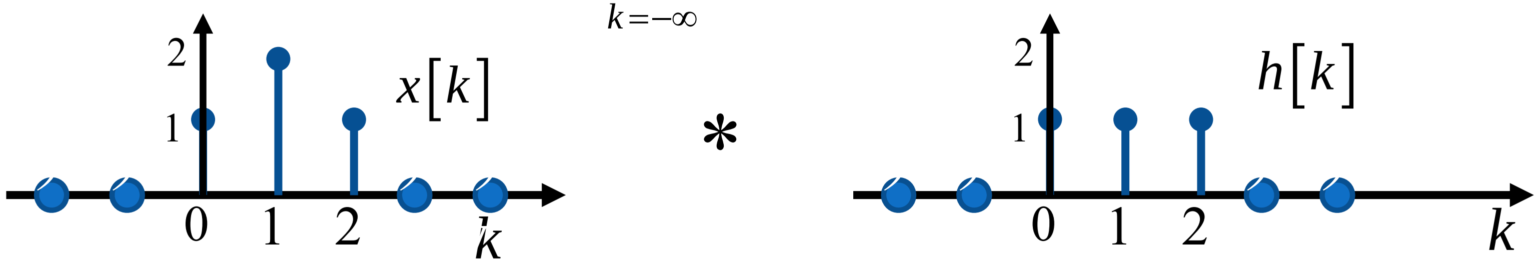
$$y[0]=\sum_{k=-\infty}^{\infty} x[k]h[0-k]$$



$*$

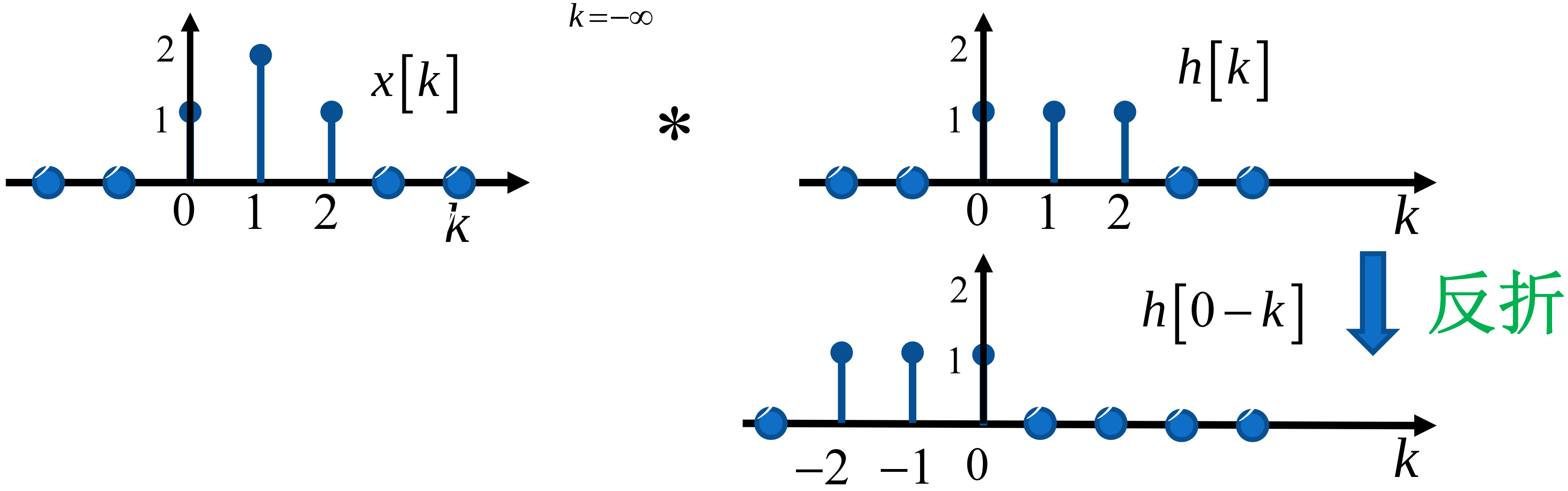


$$y[0]=\sum_{k=-\infty}^{\infty} x[k]h[0-k]$$

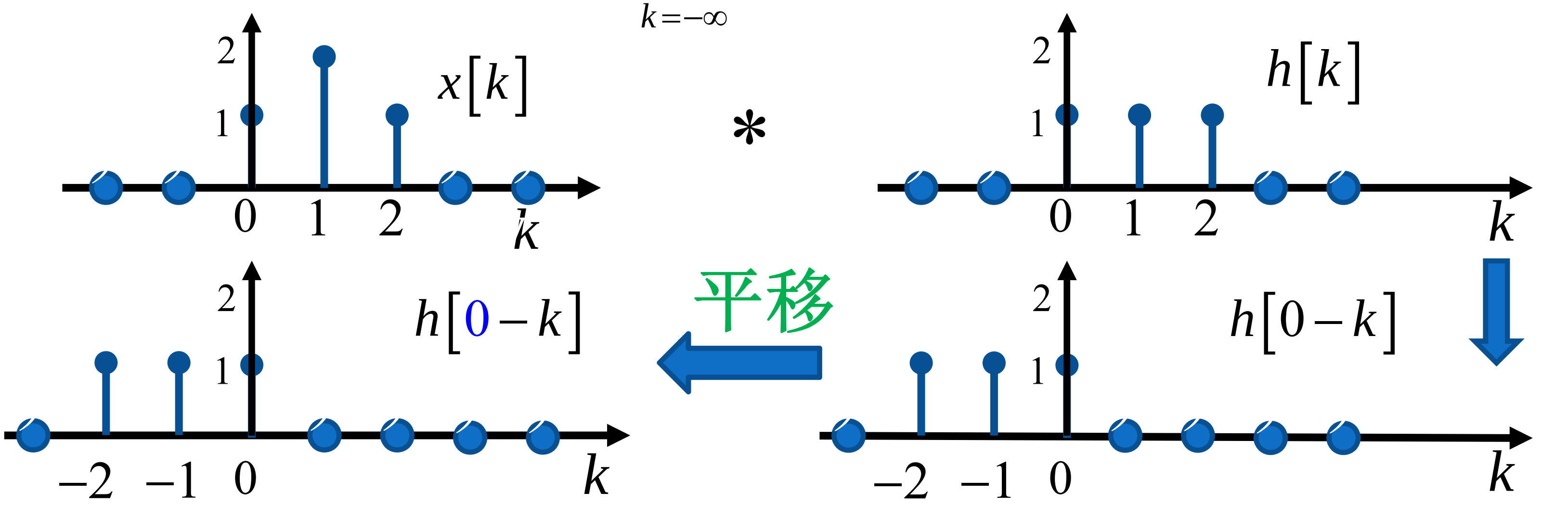


变量代换

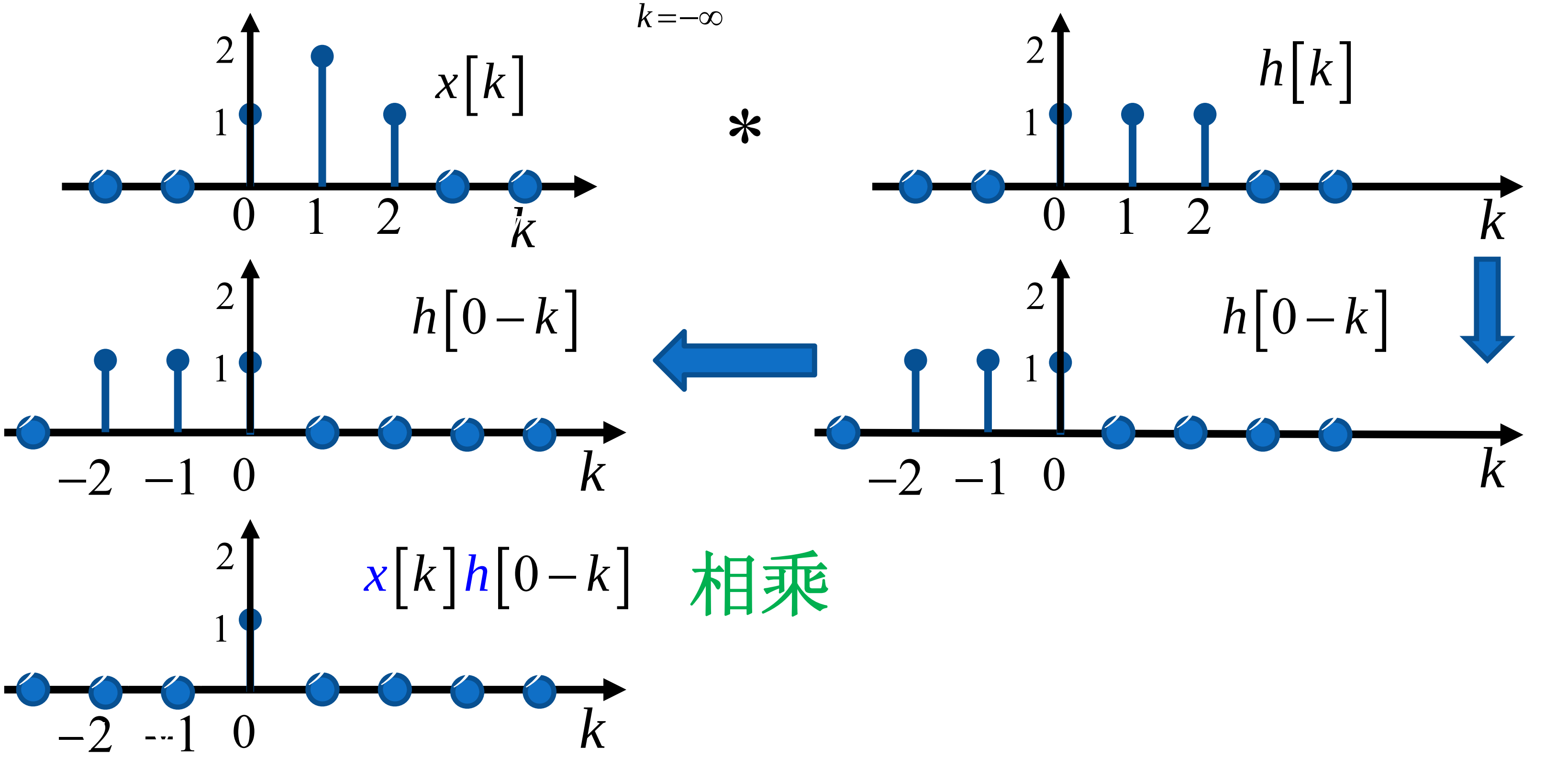
$$y[0]=\sum_{k=-\infty}^{\infty}x[k]h[0-k]$$



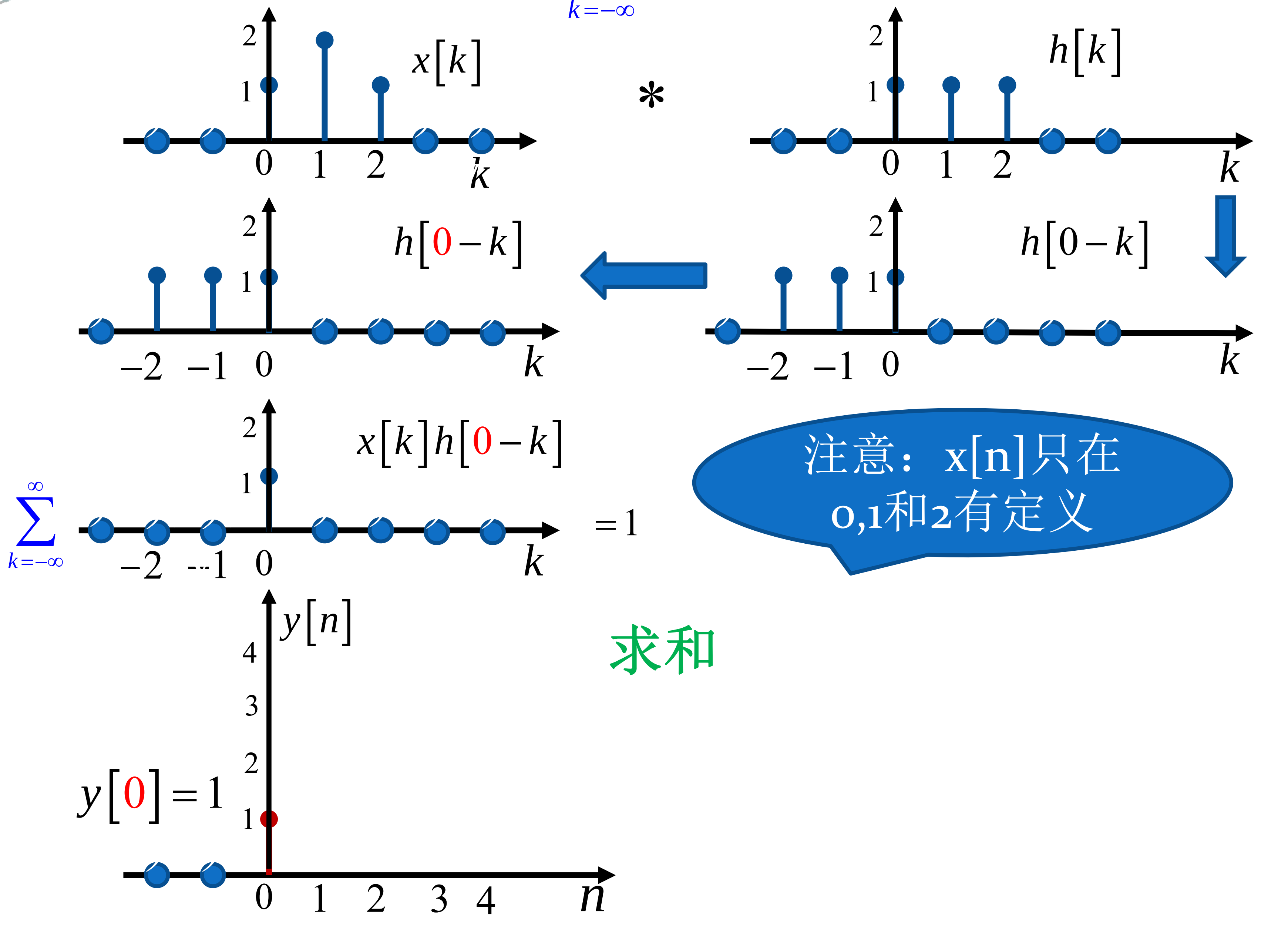
$$y[0]=\sum_{k=-\infty}^{\infty}x[k]h[0-k]$$



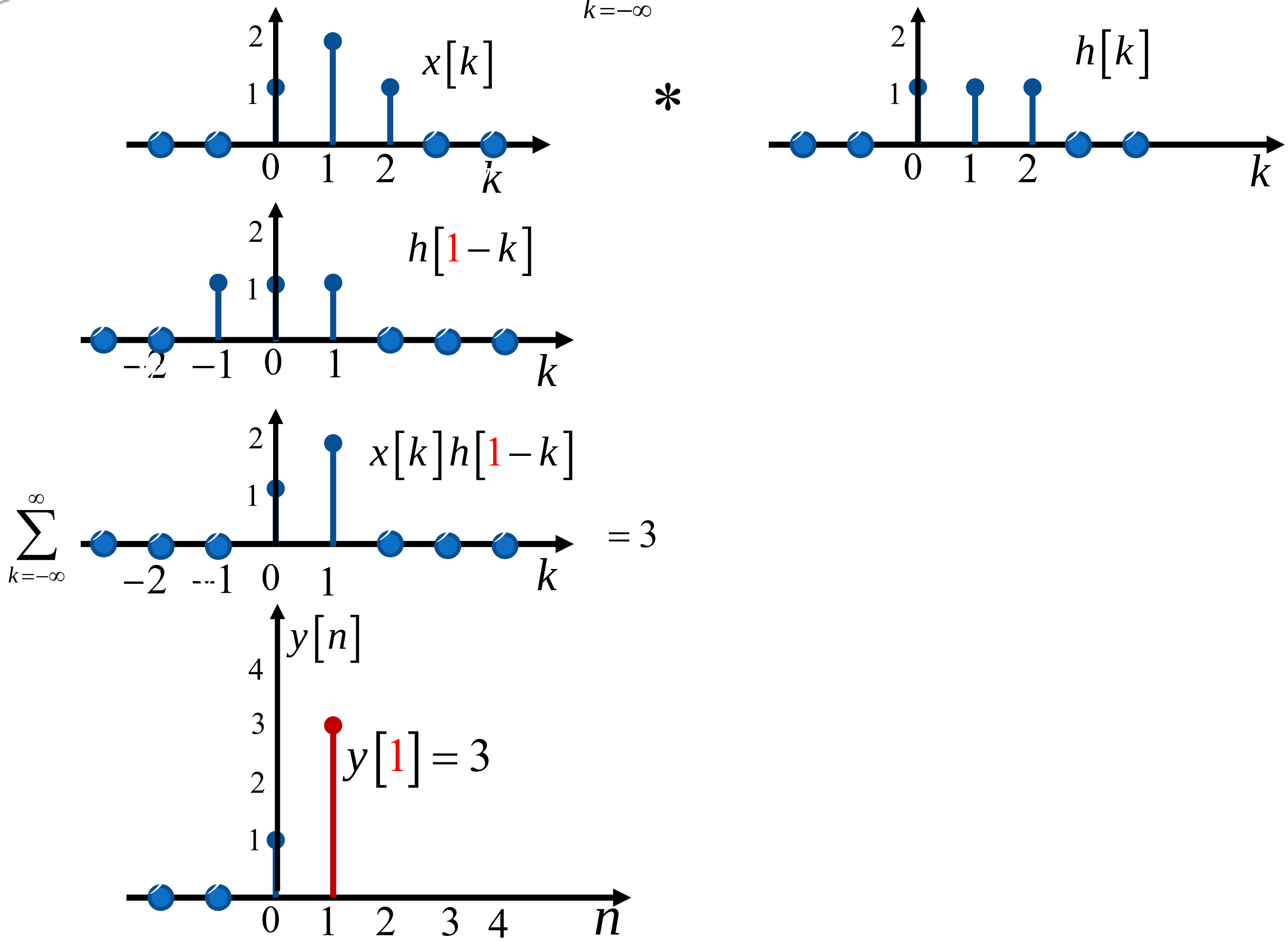
$$y[0]=\sum_{k=-\infty}^{\infty} x[k]h[0-k]$$



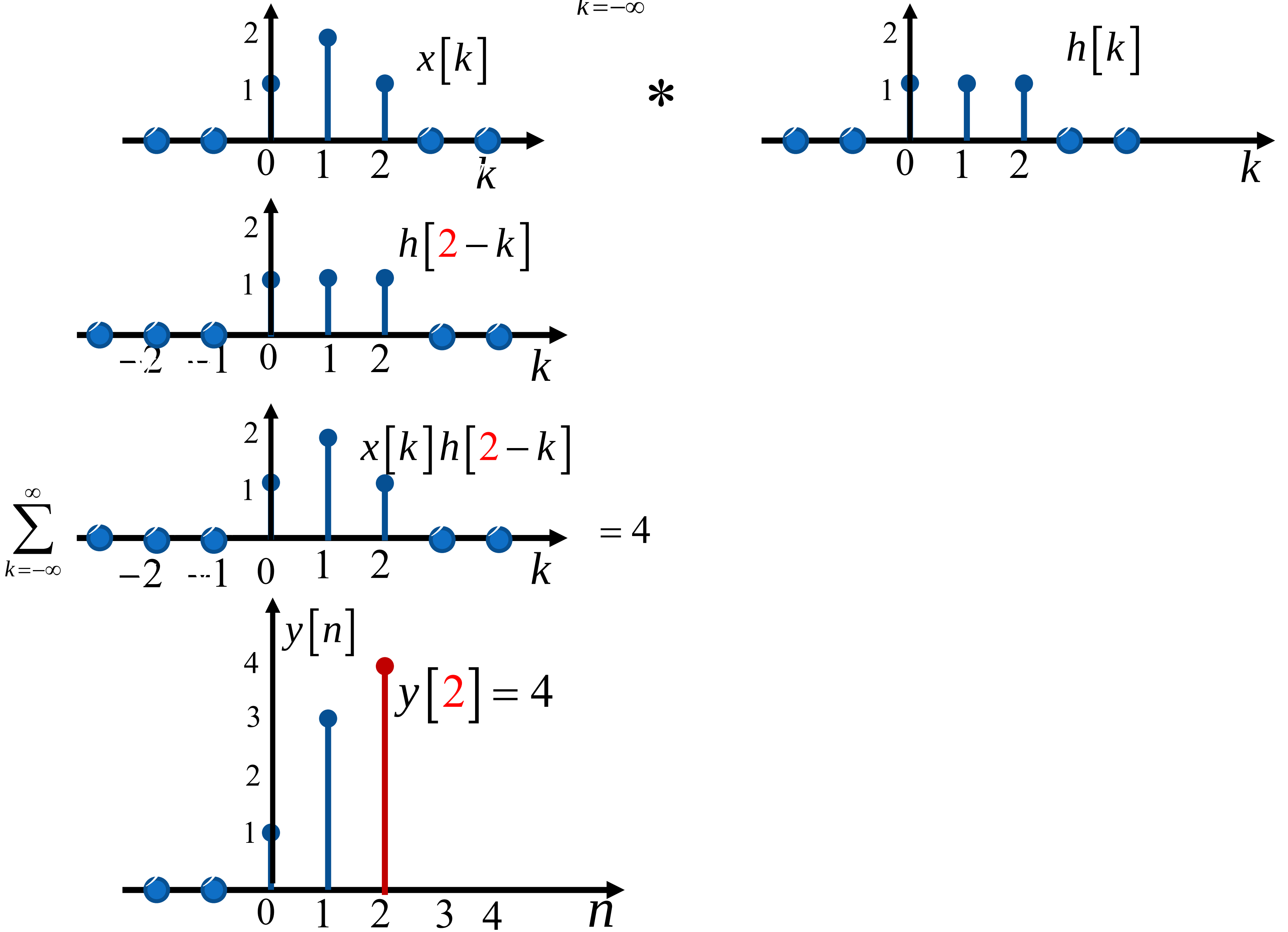
$$y[0]=\sum_{k=-\infty}^{\infty} x[k]h[0-k]$$



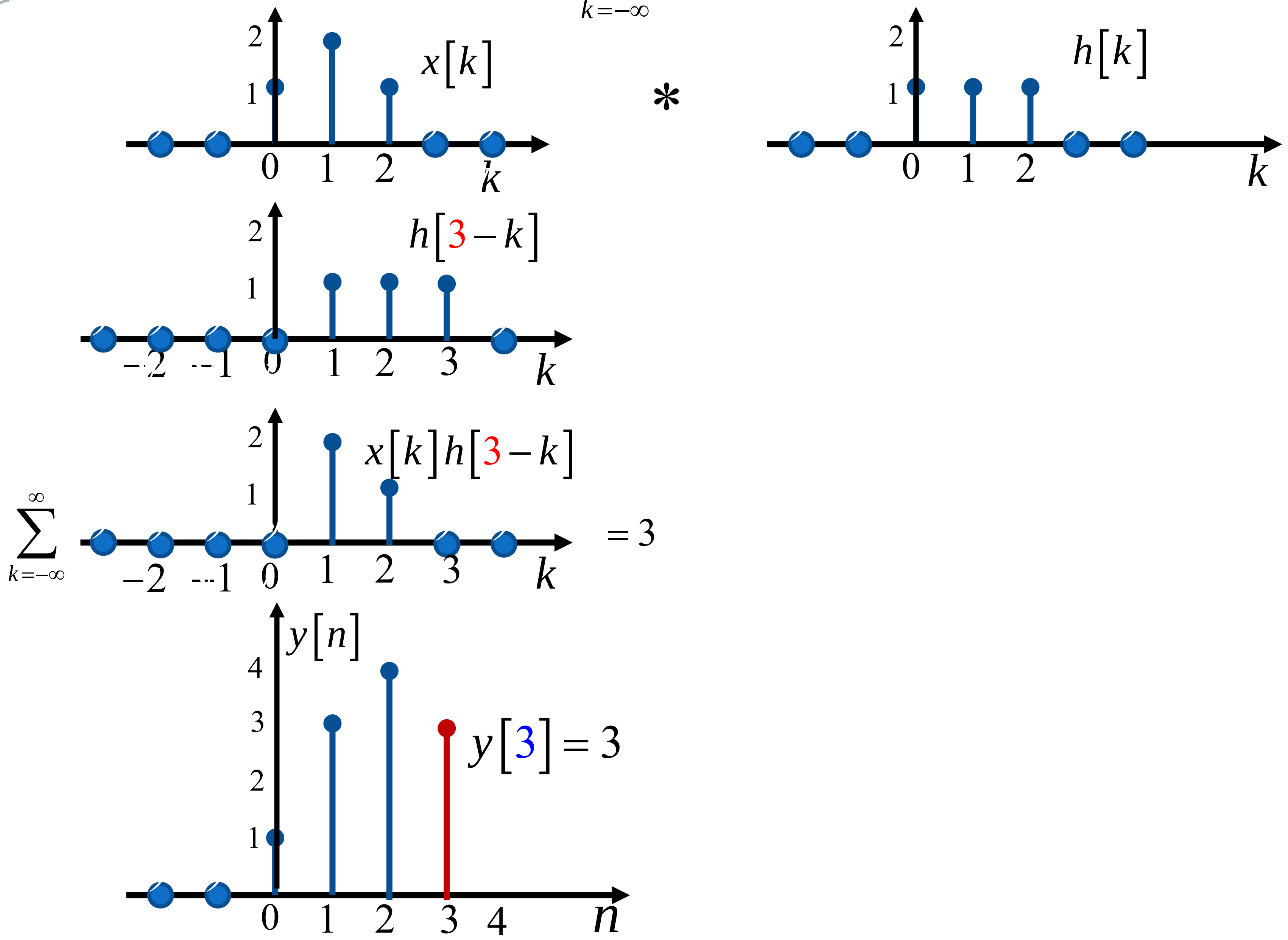
$$y[1]=\sum_{k=-\infty}^{\infty} x[k]h[1-k]$$



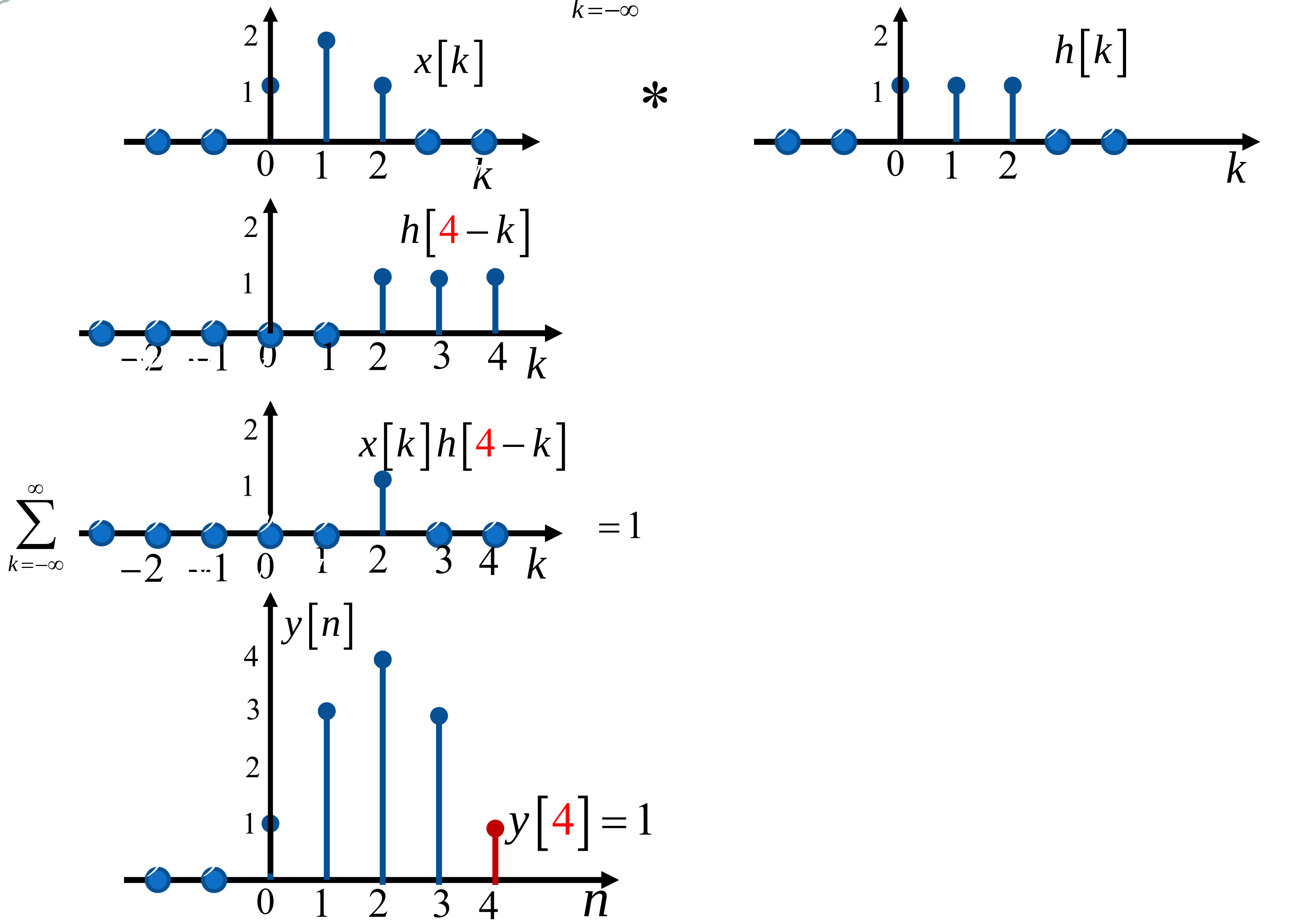
$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k]$$



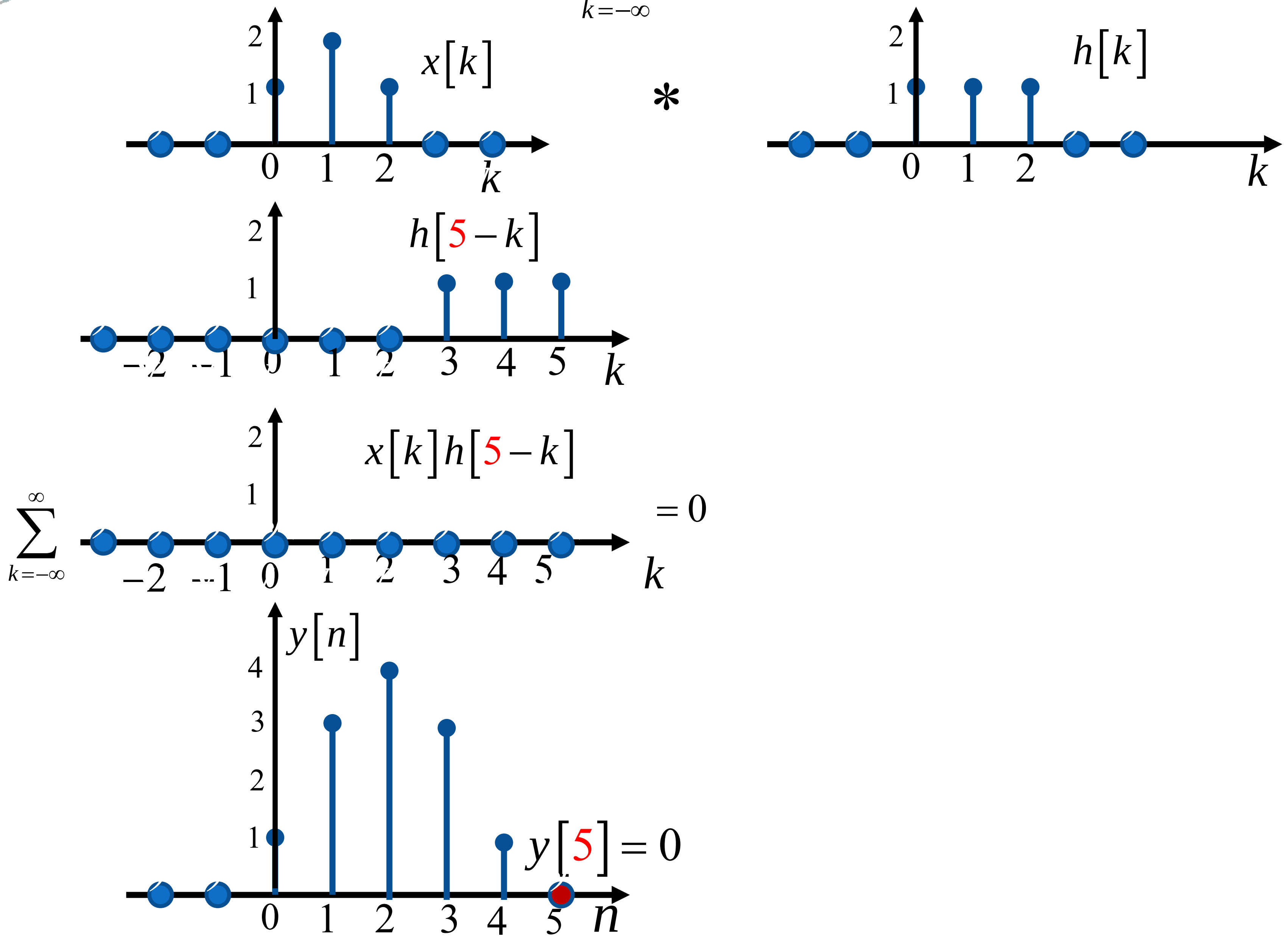
$$y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k]$$



$$y[4] = \sum_{k=-\infty}^{\infty} x[k]h[4-k]$$



$$y[5] = \sum_{k=-\infty}^{\infty} x[k]h[5-k]$$

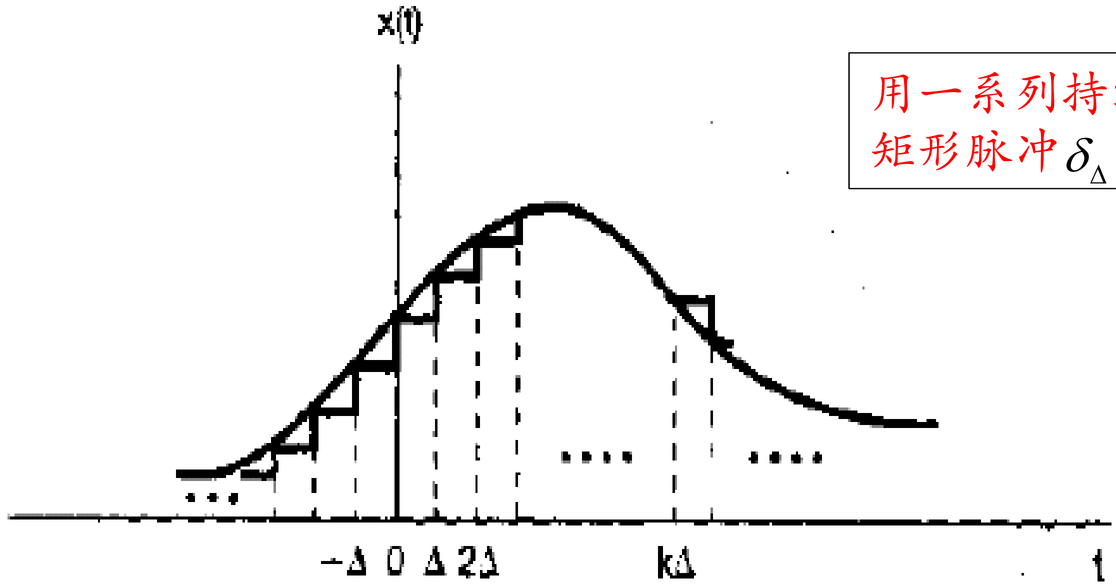


2. Continuous-Time LTI Systems: The Convolution Integral

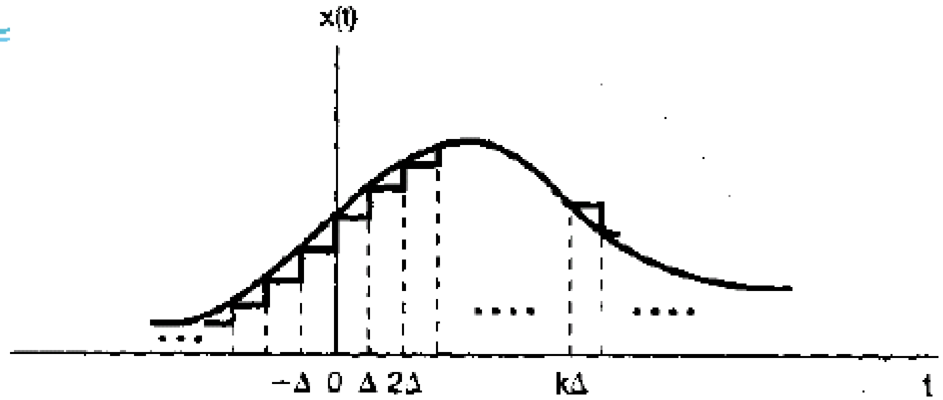
- Consider the unit impulse as the idealization of a pulse which is so short that its duration is inconsequential for any real, physical system.
- A representation for arbitrary continuous-time signals in terms of these idealized pulse with vanishingly small duration.

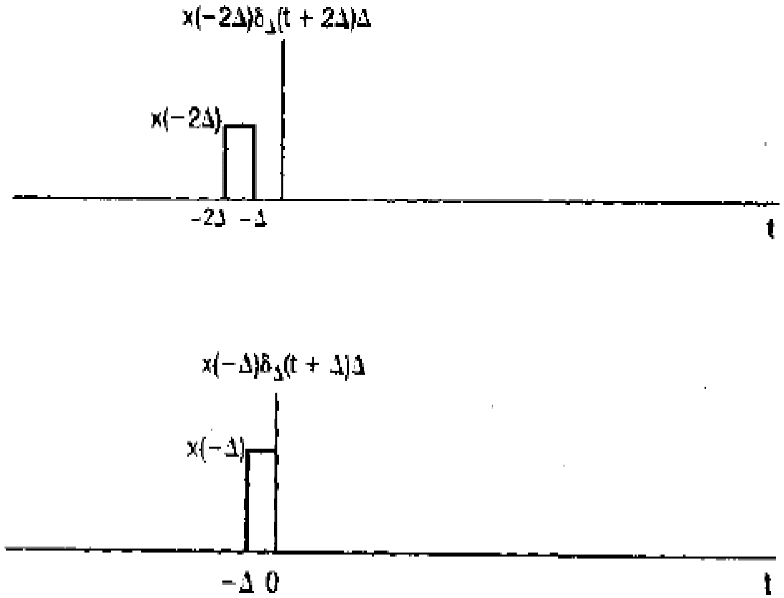
- How do you accurately represent an arbitrary signal by a series 'simple' signals?
  - ***Simple***???
  - 'Small' duration → accurate
  - Easy to deal with → easy

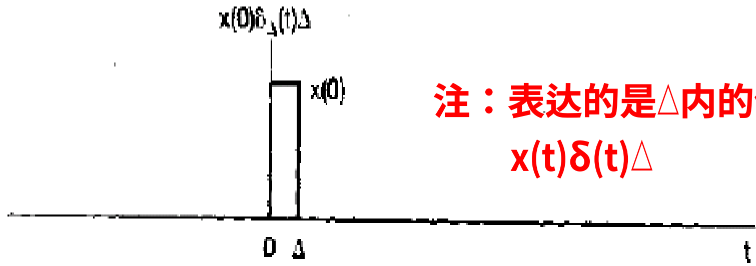
The Representation of Continuous-Time Signals in Terms of Impulse



用一系列持续时间为 Δ 的矩形脉冲 δ_Δ 抽样 $x(t)$

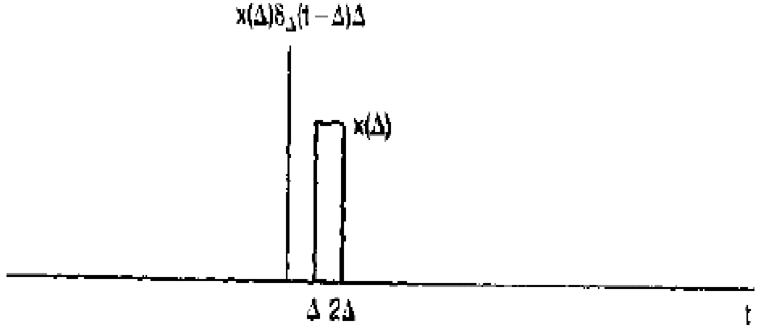






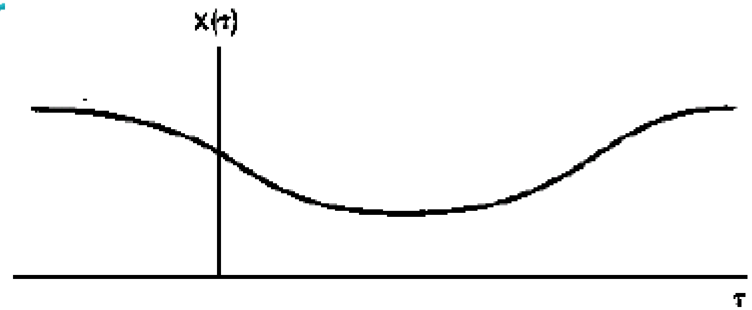
注：表达的是△内的信号，故有：

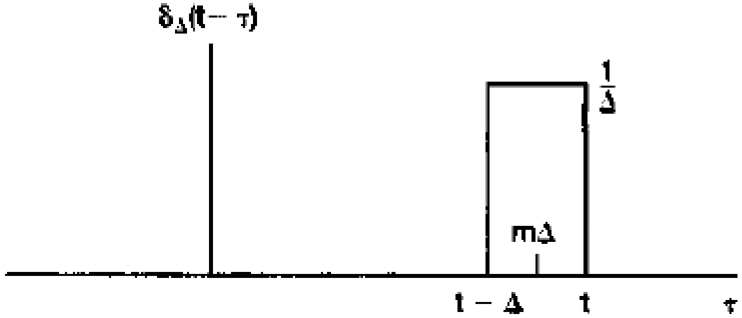
x(t)δ(t)△

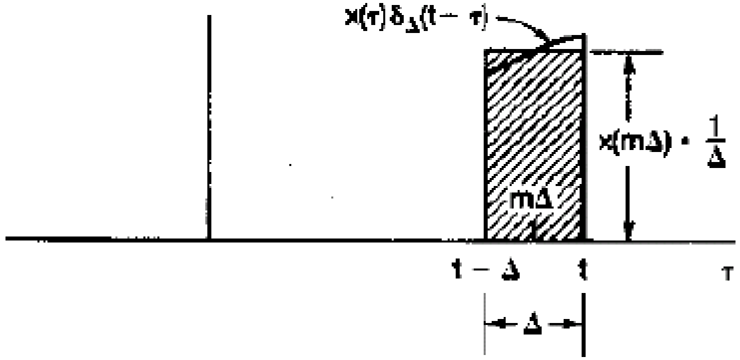


Represent x(t) as a series of weighted rectangular pulse

加权矩形脉冲







$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t - k\Delta)\Delta$$

What does it mean?

Define

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & 0 \le t < \Delta \\ 0 & others \end{cases}$$

$$\Delta \cdot \delta_\Delta(t) = \begin{cases} 1 & 0 \le t < \Delta \\ 0 & others \end{cases}$$

Then

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

The approximation becomes better and better as Δ approaches 0, and the limit equals x(t).

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

From another perspective …

$$x(t)=\lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty}x(k\Delta)\left[u\left(t-k\Delta\right)-u\left(t-k\Delta-\Delta\right)\right]$$

$$=\lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty}x(k\Delta)\frac{\left[u\left(t-k\Delta\right)-u\left(t-k\Delta-\Delta\right)\right]}{\Delta}\Delta$$

$$=\lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty}x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta$$

2个阶跃函数相减形成一个宽度为 Δ 的矩形脉冲

As Δ approach 0

$$k\Delta \to \tau$$

$$x(k\Delta) \to x(\tau)$$

$$\delta_\Delta(t-k\Delta) \to \delta(t-\tau)$$

$$\Delta \to d\tau$$

$$\sum \to \int$$

So, $x(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$

It can be expressed as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$ ***Shifting Property***

It is easy to represent ***unit step function*** in terms of integral of impulse function.

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)d\tau = \int_{0}^{\infty} \delta(t-\tau)d\tau$$

As for ***an arbitrary signal*** x(t)

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = \int_{-\infty}^{\infty} x(t)\delta(t-\tau)d\tau$$

$$= x(t)\int_{-\infty}^{\infty} \delta(t-\tau)d\tau = x(t)$$

The Continuous-Time Unit Impulse Response and the Convolution Integral Representation of LTI System

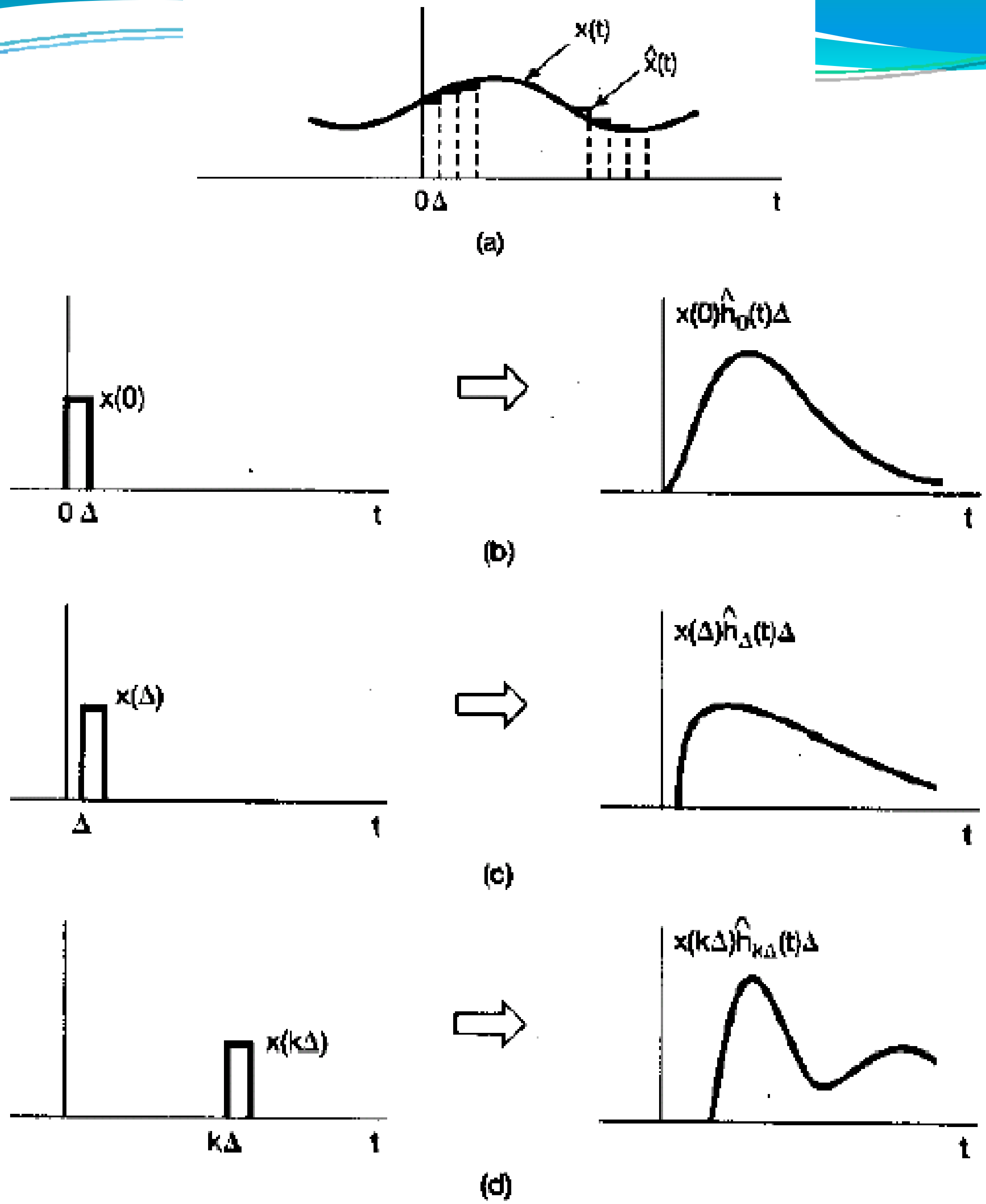
- An arbitrary continuous-time signal can be viewed as the superposition of scaled and shifted pulses.
- The response of a linear system to this signal will be the superposition of the response to the scaled and shifted versions of pulses.

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta)\Delta$$

Define $\hat{h}_{k\Delta}(t)$

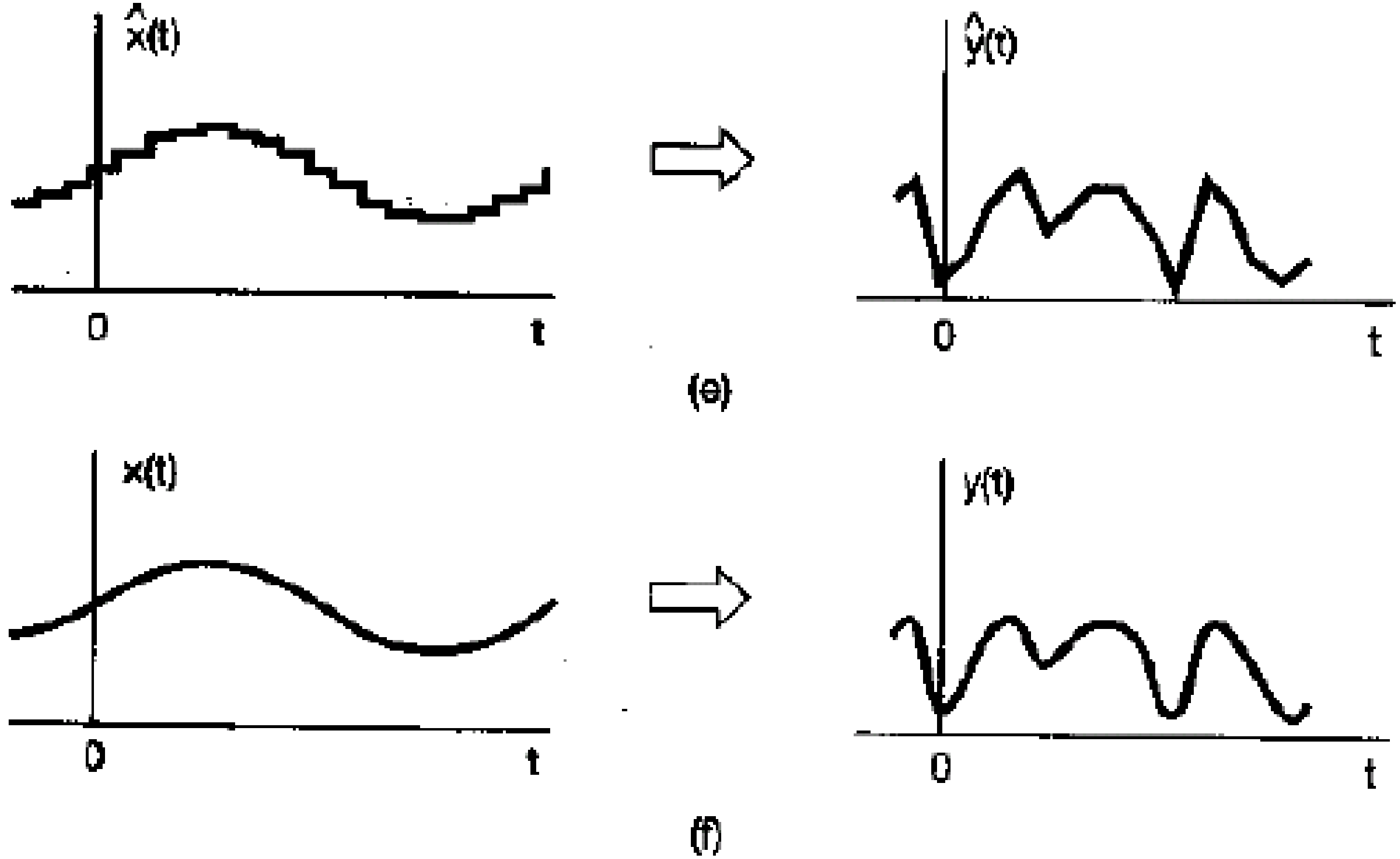
as the response of an linear system to the input $\delta_\Delta(t - k\Delta)$

Then, we can see $\hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta$



线性系统，

不一定时不变


x̂(t)
0
t
ŷ(t)
0
t
(e)
x(t)
0
t
y(t)
0
t
(f)

As Δapproaches 0,

$$y(t)=\lim_{\Delta\to 0}\hat{y}(t)=\lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty}x(k\Delta)\hat{h}_{k\Delta}(t)\Delta$$

$$y(t)=\int_{-\infty}^{+\infty}x(\tau)h_{\tau}(t)d\tau$$

What is the time invariant property???

Consider the time invariant property of LTI system

Define $$h(t)=h_0(t),\ h_{\tau}(t)=h(t-\tau)$$

Then $$y(t)=\int_{-\infty}^{+\infty}x(\tau)h(t-\tau)d\tau$$

线性时不变

THE CONVOLUTION INTEGRAL （卷积积分）
OR THE SUPERPOSITION ITEGRAL

$$y(t)=\int_{-\infty}^{+\infty}x(\tau)h(t-\tau)d\tau$$

Always be represented symbolically as

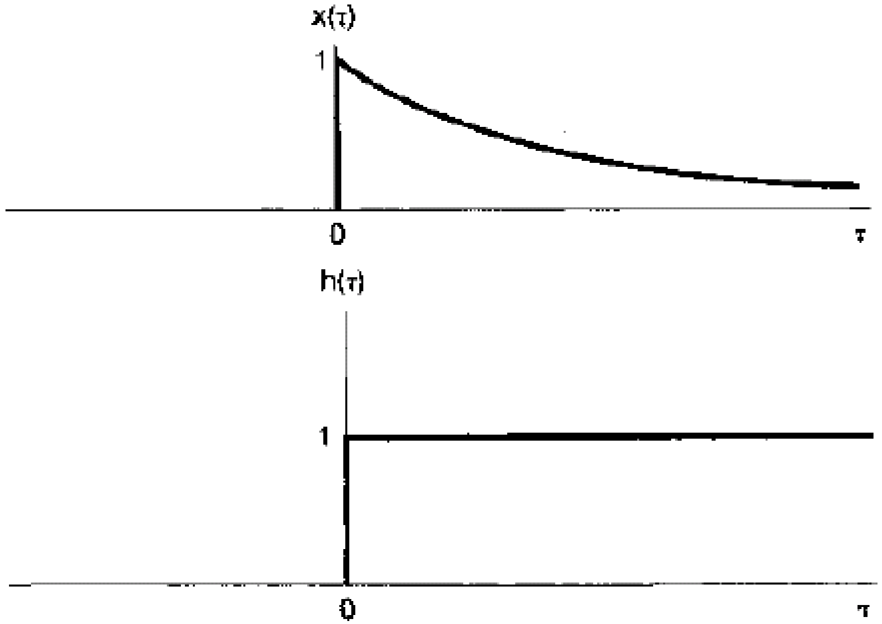
$$y(t)=x(t)*h(t)$$

Example

Let $x(t)$ be the input to an LTI system with unit impulse response $h(t)$, where $x(t) = e^{-at}u(t), a > 0$

$$h(t) = u(t)$$

What is the response of this system in such condition?



Since $x(t) = e^{-at}u(t), a > 0$

$$h(t) = u(t)$$

We can see $x(\tau)h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 < \tau < t \\ 0, & otherwise \end{cases}$

Then, for $\boxed{t>0}$

$\tau \le t$

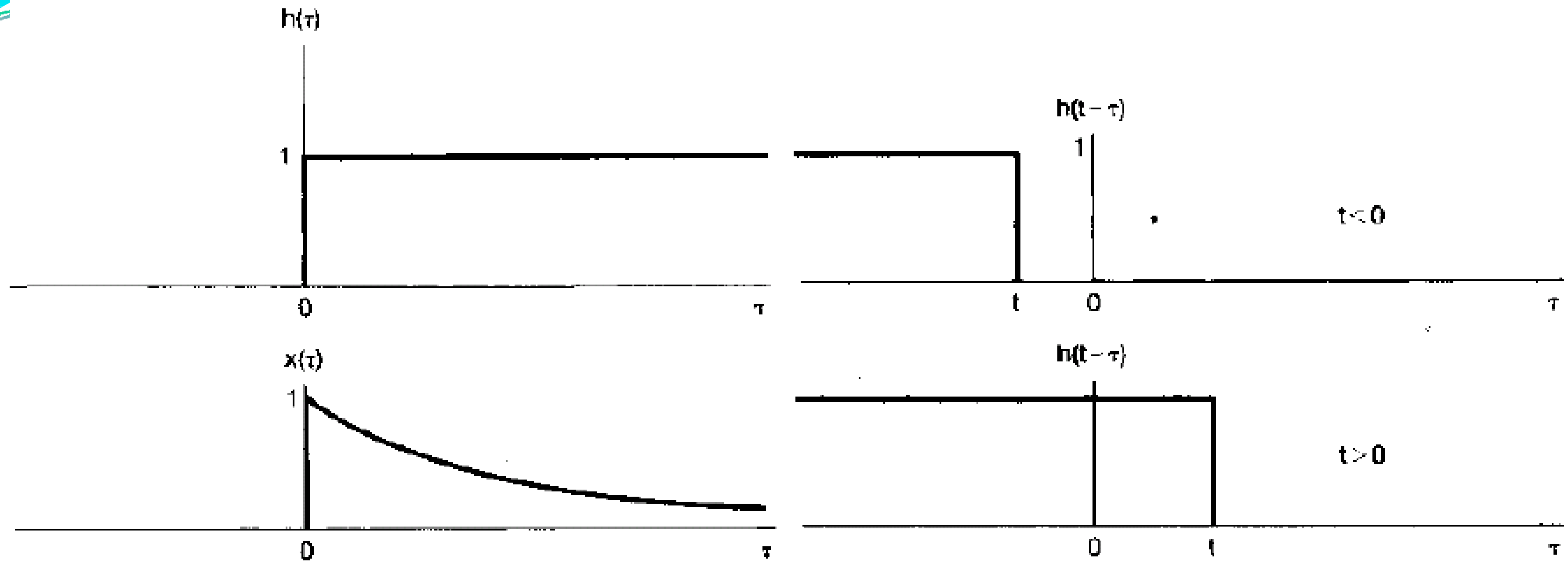
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t-\tau)d\tau$$

$$= \int_{0}^{t} e^{-a\tau}d\tau = -\frac{1}{a}e^{-a\tau}\Big|_{0}^{t} = \frac{1}{a}(1-e^{-at})$$

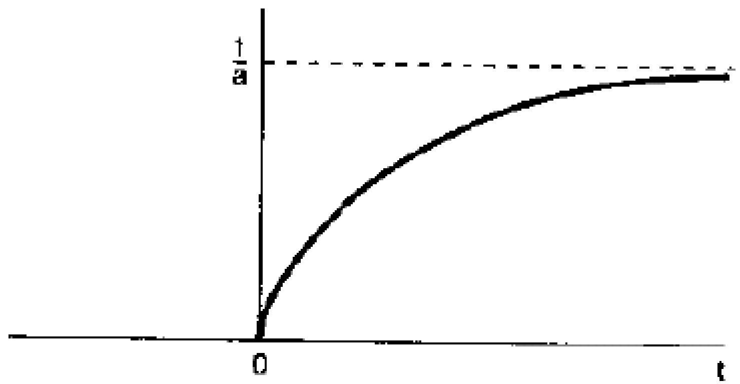
Then, for all t,

$$y(t) = \frac{1}{a}(1-e^{-at})u(t)$$

Graphical Interpretation



Calculation of the convolution integral



Response of the system

How to Calculate Convolution Integral and Convolution Sum?

Step One: Replace the independent variable ‘t’ with ‘τ’

Step Two: Reverse one signal

Step Three: Shifting **with t**

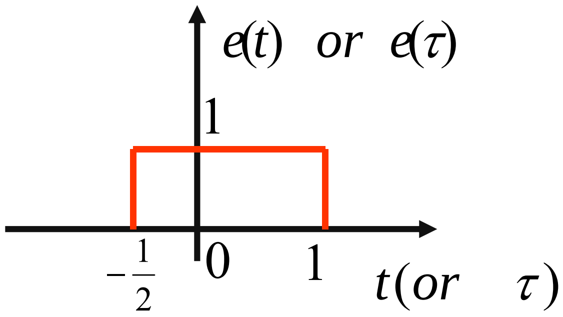
Step Four: Multiply

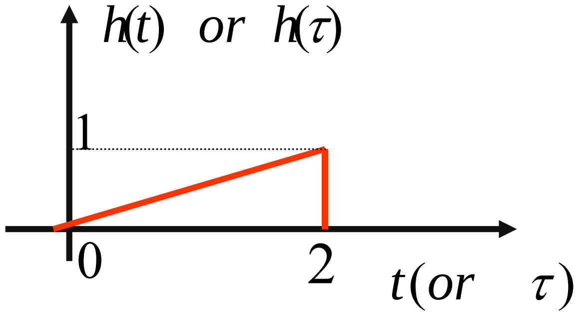
Step Five: Integral or Add

• Example

Determine and sketch $e(t) * h(t)$

(1)

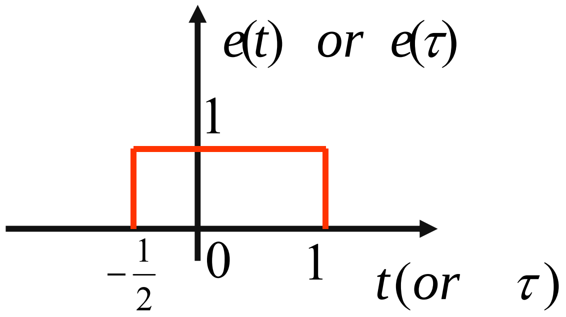


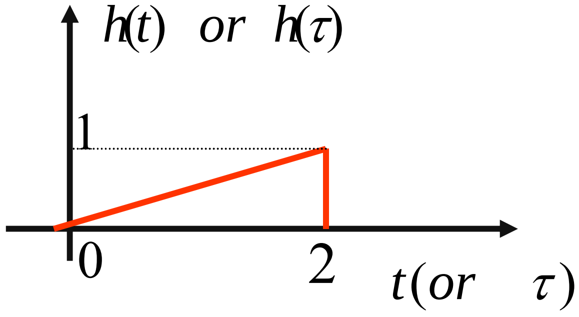


Example

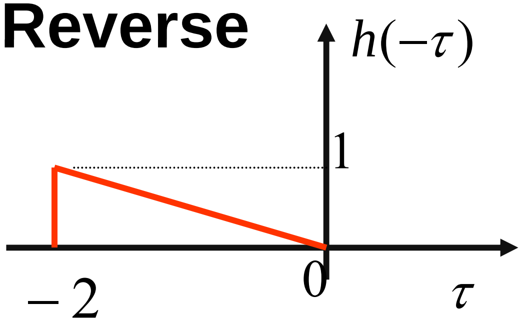
Determine and sketch $e(t) * h(t)$

(1)



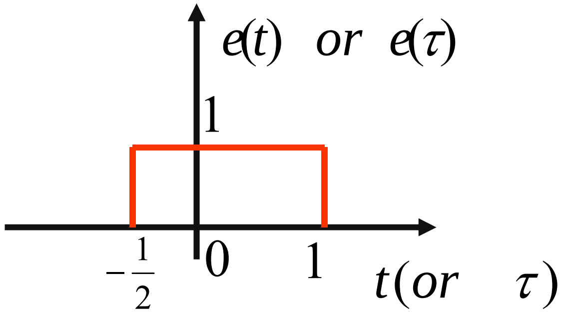


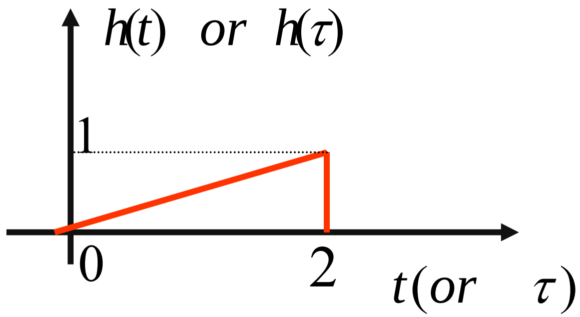
(2) Reverse



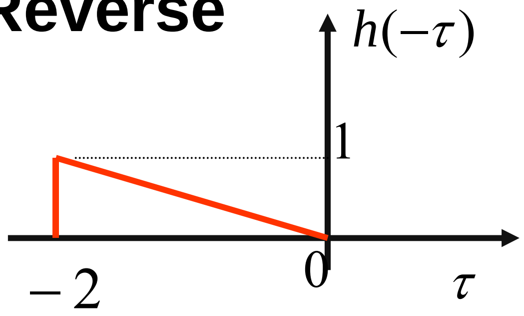
Example Determine and sketch $e(t) * h(t)$

(1)

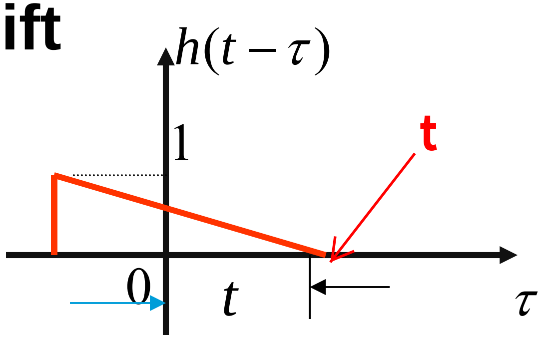




(2) Reverse

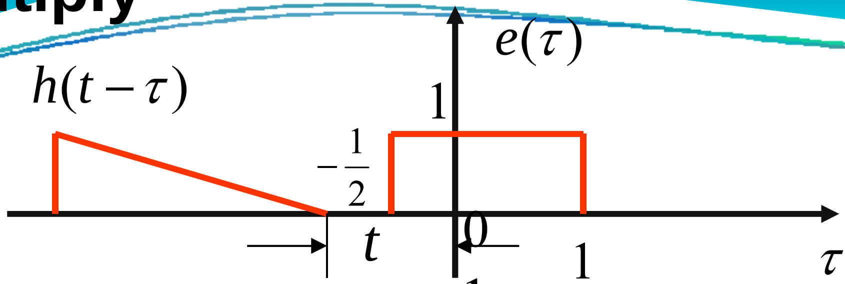


(3) Shift

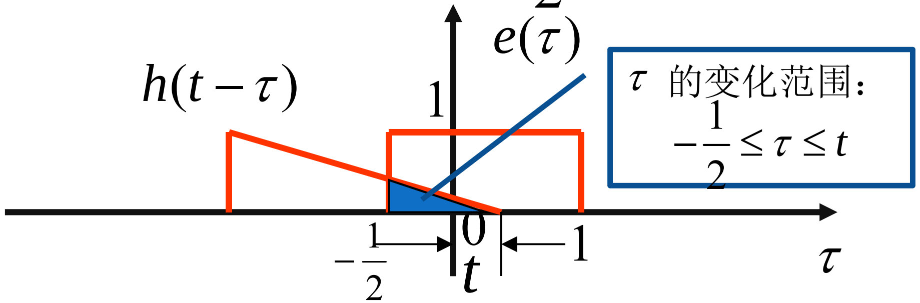


t∈[-∞,∞]，但要考虑2个函数交集

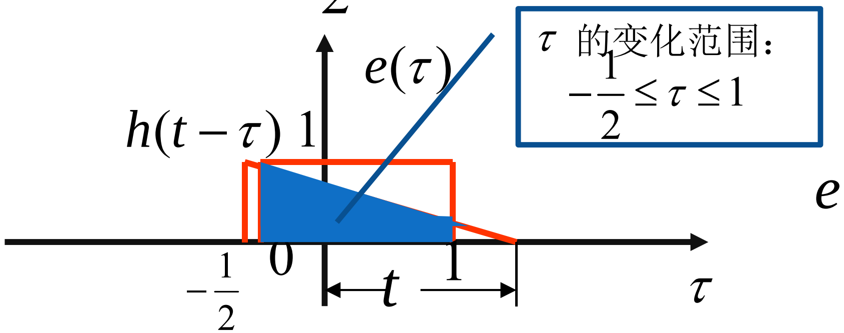
(4)Multiply



(a) $-\infty < t \le -\frac{1}{2}$



(b) $-\frac{1}{2} \le t \le 1$



(c) $1 \le t \le \frac{3}{2}$

(a) $-\infty < t \le -\frac{1}{2}$

$$e(t) * h(t) = 0$$

(b) $-\frac{1}{2} \le t \le 1$

$$e(t) * h(t) = \int_{-\frac{1}{2}}^{t} 1 \times \frac{1}{2}(t-\tau)\,d\tau$$

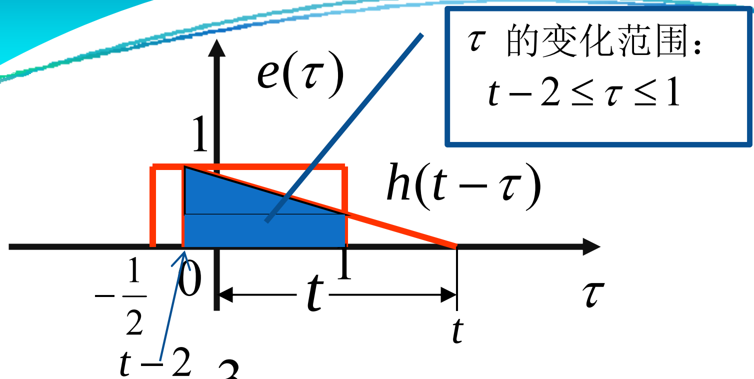
$$= \frac{t^2}{4} + \frac{t}{4} + \frac{1}{16}$$

(c) $1 \le t \le \frac{3}{2}$

$$e(t) * h(t) = \int_{-\frac{1}{2}}^{1} 1 \times \frac{1}{2}(t-\tau)\,d\tau$$

$$= \frac{3}{4}t - \frac{3}{16}$$

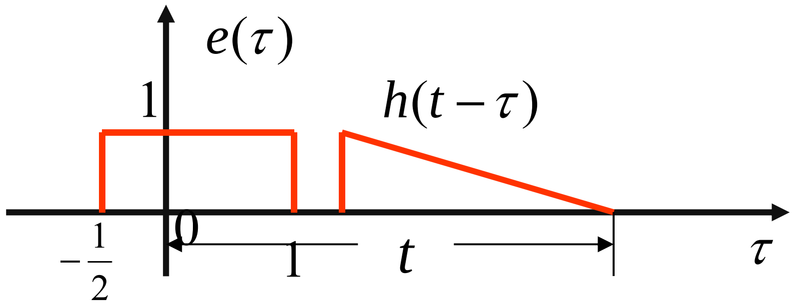
$(d)\quad \frac{3}{2} \le t \le 3$



$(d)\quad \frac{3}{2} \le t \le 3$

$$e(t) * h(t) = \int_{t-2}^{1} 1 \times \frac{1}{2}(t-\tau)d\tau$$

$$= -\frac{t^2}{4} + \frac{t}{2} + \frac{3}{4}$$

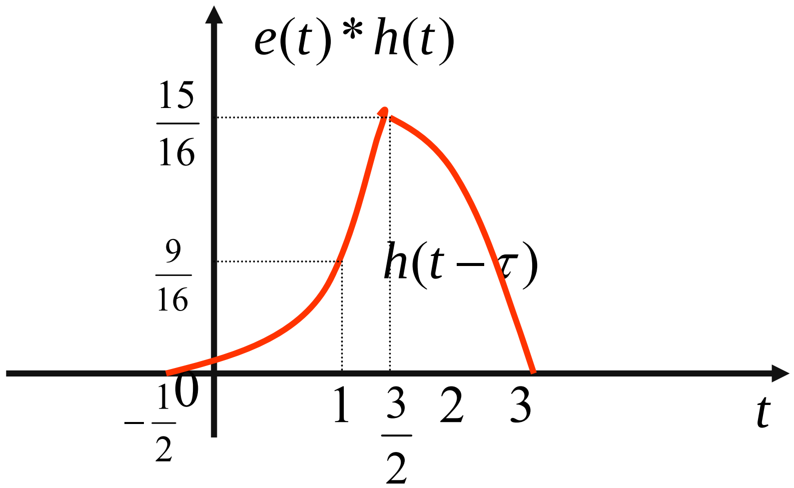


$(e)\quad 3 < t \le \infty$

$(e)\quad 3 < t \le \infty$

$$e(t) * h(t) = 0$$

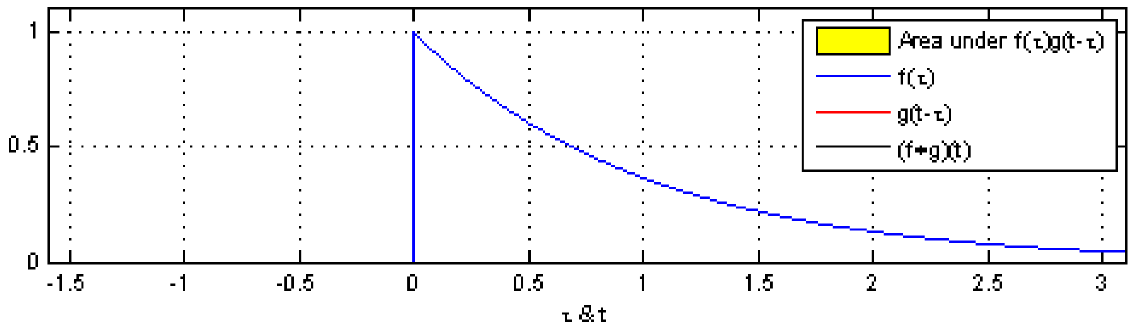
（5）Integral



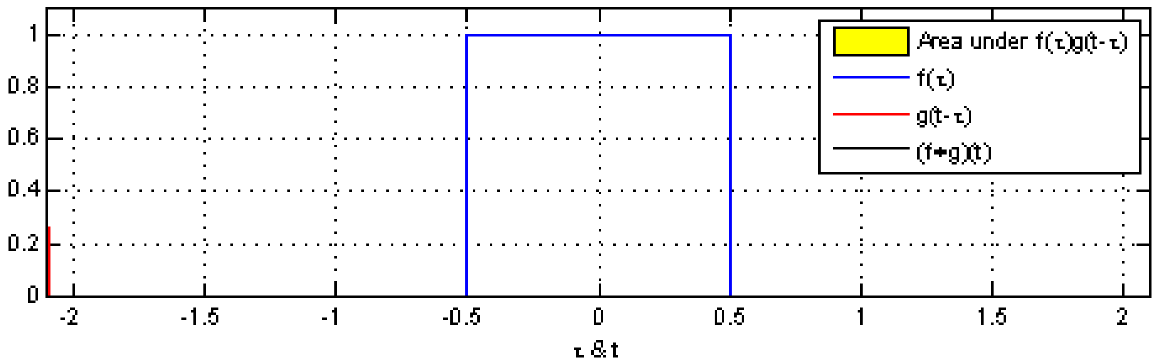
$$e(t) * h(t)$$

Example

Convolution of a square pulse with the impulse response of an RC circuit to obtain the output signal waveform



Convolution of two square pulses

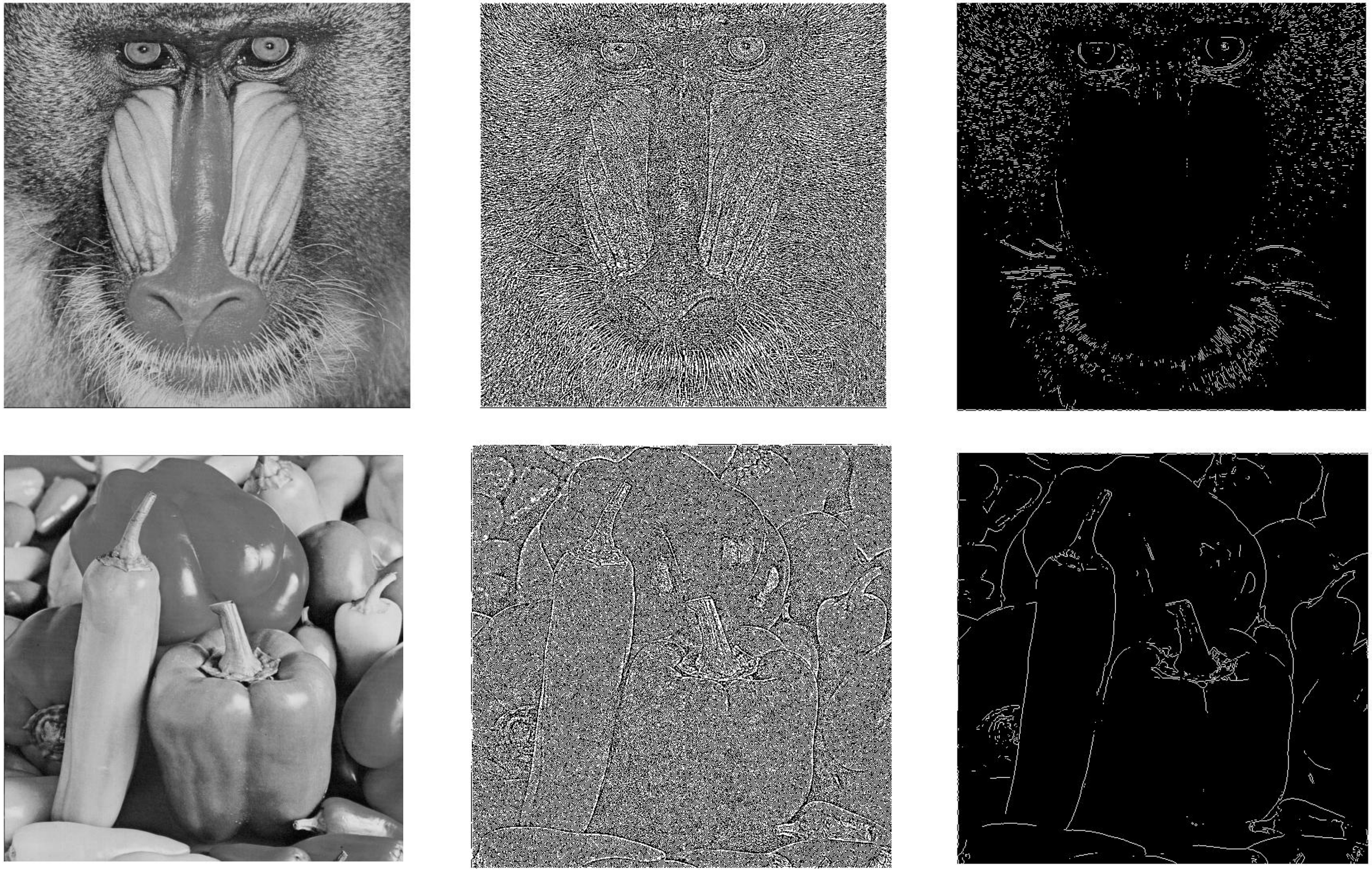


Applications of convolution

- Electrical engineering
  - Digital signal processing
  - Image processing
- Statistics
  - Moving average model…
- Probability theory
  - E.g. Pdf of X+Y
- …

```
% Example of the application of convolution in image processing
Im_in = imread('E:\standard picture\gray_image\baboon.bmp','bmp');
figure;
imshow(Im_in);
 LD = [0  -1 0;
     -1  4  -1;
     0  -1  0];
 Im_out = conv2(double(Im_in), LD, 'same');
figure;
imshow(Im_out);
Im_out2 = edge(Im_in, 'sobel');
figure;
imshow(Im_out2);
```

Matlab



Original After convolution Edge

Some useful properties of Convolution Integral

(1) $f_1(t) * f_2(t) = f_2(t) * f_1(t)$

(2) $f_1(t) * [f_2(t) + f_3(t)] = f_1(t) * f_2(t) + f_1(t) * f_3(t)$

(3) $f_1(t) * f_2(t) * f_3(t) = f_1(t) * f_3(t) * f_2(t)$

(4) $$\frac{d[f_1(t) * f_2(t)]}{dt} = f_1(t) * \frac{df_2(t)}{dt} = \frac{df_1(t)}{dt} * f_2(t)$$

(5) $$\int_{-\infty}^{t} [f_1(\tau) * f_2(\tau)] d\tau = f_1(t) * \int_{-\infty}^{t} f_2(\tau) d\tau = \int_{-\infty}^{t} f_1(\tau) d\tau * f_2(t)$$

(6) $\quad if \quad f(t) = f_1(t) * f_2(t)$

$\quad then \quad f^{(i)}(t) = f_1^{(j)}(t) * f_2^{(i-j)}(t)$

(7) $\quad f(t) * \delta(t) = f(t)$

$\quad f(t) * \delta(t - t_0) = f(t - t_0)$

(8) $\quad f(t) * u(t) = f^{(-1)}(t) = \int_{-\infty}^{t} f(\tau) d\tau$

Example

Suppose that $f_1(t)=(1+t)[u(t)-u(t-1)],$

$f_2(t)=u(t-1)$

Determine $f_1(t)*f_2(t)$

利用前页卷积性质（6）

$f_1^{(-1)}(t)=\int_{-\infty}^{t} f_1(\tau)d\tau$

$f_2^{(1)}(t)=\dfrac{df_2(t)}{dt}$，注意: $(-1)+1=0$

$$f_1(t)*f_2(t)=\int_{-\infty}^{t} f_1(\tau)d\tau*\frac{df_2(t)}{dt}$$

构造成f(t)*δ(t)的形式

$$=\int_{-\infty}^{t}(1+\tau)\left[u(\tau)-u(\tau-1)\right]d\tau*\delta(t-1)$$

$$=\left[\int_{-\infty}^{t}(1+\tau)u(\tau)d\tau-\int_{-\infty}^{t}(1+\tau)u(\tau-1)d\tau\right]*\delta(t-1)$$

$$=\left[\int_{0}^{t}(1+\tau)d\tau u(t)-\int_{1}^{t}(1+\tau)d\tau u(t-1)\right]*\delta(t-1)$$

$$f_1(t) * f_2(t) = \left[\int_0^t (1+\tau) d\tau u\left(t\right) - \int_1^t (1+\tau) d\tau u\left(t-1\right) \right] * \delta(t-1)$$

$$= \left[\tau + \frac{\tau^2}{2} \Bigg|_0^t u\left(t\right) - \tau + \frac{\tau^2}{2} \Bigg|_1^t u\left(t-1\right) \right] * \delta(t-1)$$

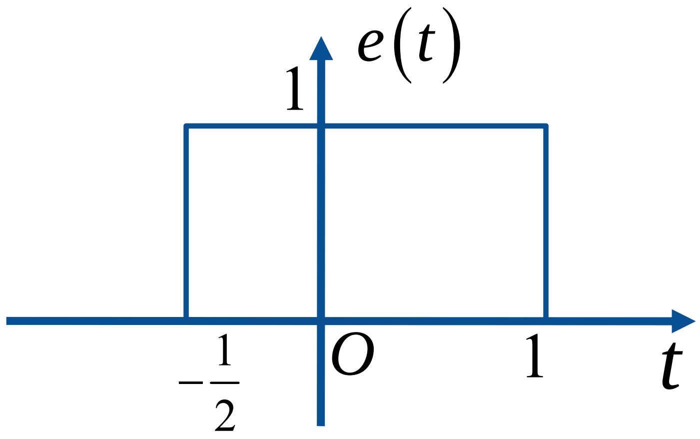
$$= \left[\left(t + \frac{t^2}{2} \right) u(t) - \left(t - 1 + \frac{t^2 - 1}{2} \right) u(t-1) \right] * \delta(t-1)$$

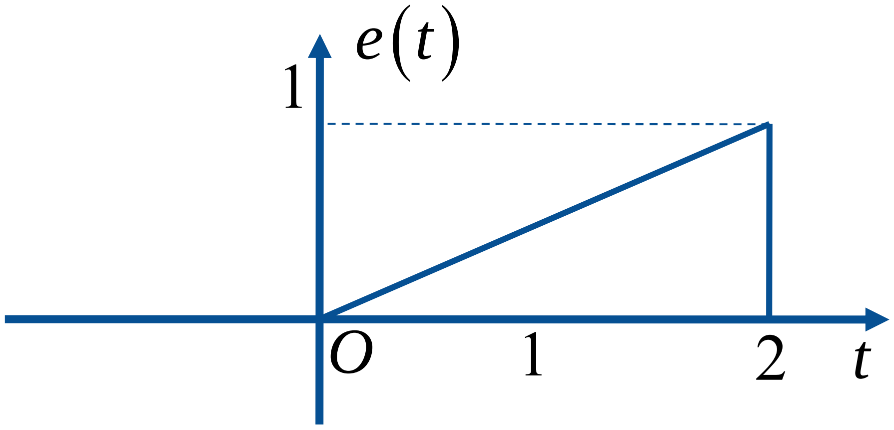
$$f_1(t) * f_2(t) = \left[t - 1 + \frac{(t-1)^2}{2}\right]u(t-1)$$

$$- \left[t - 2 + \frac{(t-1)^2 - 1}{2}\right]u(t-2)$$

$$= \frac{t^2 - 1}{2}u(t-1) - \frac{t^2 - 4}{2}u(t-2)$$

Exercise. Determine the convolution of e(t) and h(t).





$$e(t) = u\left(t + \frac{1}{2}\right) - u(t-1) \qquad h(t) = \frac{1}{2}t[u(t) - u(t-2)]$$

$$e(t)*h(t)=\frac{de(t)}{dt}*\int_{-\infty}^{t}h(\tau)d\tau$$

性质6: 0 = 1 + （-1）

$$\because \frac{de(t)}{dt}=\delta\left(t+\frac{1}{2}\right)-\delta(t-1)$$

$$\int_{-\infty}^{t}h(\tau)d\tau=\frac{1}{2}\int_{-\infty}^{t}\tau[u(\tau)-u(\tau-2)]d\tau$$

$$=\frac{1}{2}\left(\int_{0}^{t}\tau d\tau u(t)-\int_{2}^{t}\tau d\tau u(t-2)\right)$$

$$=\frac{1}{4}t^{2}[u(t)-u(t-2)]+u(t-2)$$

$$\therefore e(t)*h(t)=\left[\delta\left(t+\frac{1}{2}\right)-\delta(t-1)\right]*$$

$$\left\{\frac{1}{4}t^2\left[u(t)-u(t-2)\right]+u(t-2)\right\}$$

$$=\delta\left(t+\frac{1}{2}\right)*\left\{\frac{1}{4}t^2\left[u(t)-u(t-2)\right]\right\}$$

$$+\delta\left(t+\frac{1}{2}\right)*u(t-2)$$

$$-\delta(t-1)*\left\{\frac{1}{4}t^2\left[u(t)-u(t-2)\right]\right\}$$

$$-\delta(t-1)*u(t-2)$$

利用性质（7）：函数和脉冲函数的卷积

$$e(t)*h(t)=\left\{\frac{1}{4}t^2[u(t)-u(t-2)]\right\}*\delta\left(t+\frac{1}{2}\right)$$

$$+u(t-2)*\delta\left(t+\frac{1}{2}\right)$$

$$-\left\{\frac{1}{4}t^2[u(t)-u(t-2)]\right\}*\delta(t-1)$$

$$-u(t-2)*\delta(t-1)$$

注意:

$$f(t)*\delta(t-t_0)=f(t-t_0)$$

$$e(t)*h(t)=\frac{1}{4}\left(t+\frac{1}{2}\right)^2\left[u\left(t+\frac{1}{2}\right)-u\left(t-\frac{3}{2}\right)\right]+u\left(t-\frac{3}{2}\right)$$

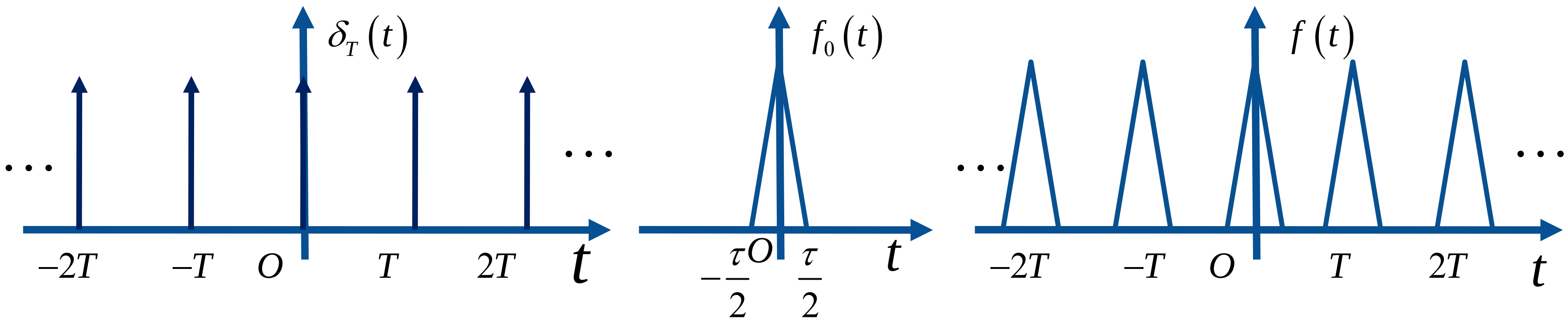
$$-\frac{1}{4}(t-1)^2[u(t-1)-u(t-3)]-u(t-3)$$

$$\therefore r_{zs}(t) = e(t) * h(t) = \begin{cases} \frac{1}{4}\left(t + \frac{1}{2}\right)^2 & -\frac{1}{2} \le t < 1 \\ \frac{3}{4}\left(t - \frac{1}{4}\right) & 1 \le t < \frac{3}{2} \\ 1 - \frac{1}{4}(t-1)^2 & \frac{3}{2} \le t < 3 \end{cases}$$

Example

Let $\delta_T(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT)$,

$f_0(t)$ is shown in the figure below. Compute $\delta_T(t) * f_0(t)$。



$$\delta_T(t) * f_0(t) = f_0(t) * \delta_T(t)$$

$$= f_0(t) * [\sum_{m=-\infty}^{\infty} \delta(t - mT)] = \sum_{m=-\infty}^{\infty} [f_0(t) * \delta(t - mT)]$$

$$= \sum_{m=-\infty}^{\infty} f_0(t - mT)$$

以B的中点5为中心顺时针旋转180度，滑动B逐点相乘和相加

9 8 7

6 5 4

3 2 1

```
A = [1 2 3
  4 5 6
  7 8 9];
B = A;
C = conv2(A,B,'same');
% C=
26  56    54
 84   165   144
 134   236   186
```

2-D Convolution

$$C(n_1,n_2)=\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}a(k_1,k_2)b(n_1-k_1,n_2-k_2)$$

Note that matrix indices in MATLAB always start at 1 rather than 0.

Therefore, matrix elements A(1,1), B(1,1), and C(1,1) correspond to mathematical quantities a (0,0), b (0,0), and c (0,0).

$$C(1,1)=1\times5+2\times4+4\times2+5\times1=26$$

B旋转后的中心点5 - 对齐A（1,1）

$$C(1,2)=1\times6+2\times5+3\times4+4\times3+2\times5+6\times1=56$$

$$C(1,3)=2\times6+3\times5+5\times3+6\times2=54$$

$$\vdots$$

Summary

- Represent an arbitrary signal by means of unit impulse signal
- Convolution
  - Properties of convolution
  - Properties of unit impulse signal