

Complex dynamics in the Leslie–Gower type of the food chain system with multiple delays



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ABSTRACT

In this paper, we present a Leslie–Gower type of food chain system composed of three species, which are resource, consumer, and predator, respectively. The digestion time delays corresponding to consumer–eat–resource and predator–eat–consumer are introduced for more realistic consideration. It is called the resource digestion delay (RDD) and consumer digestion delay (CDD) for simplicity. Analyzing the corresponding characteristic equation, the stabilities of the boundary and interior equilibrium points are studied. The food chain system exhibits the species coexistence for the small values of digestion delays. Large RDD/CDD may destabilize the species coexistence and induce the system dynamic into recurrent bloom or system collapse. Further, the present of multiple delays can control species population into the stable coexistence. To investigate the effect of time delays on the recurrent bloom of species population, the Hopf bifurcation and periodic solution are investigated in detail in terms of the central manifold reduction and normal form method. Finally, numerical simulations are performed to display some complex dynamics, which include multiple periodic solution and chaos motion for the different values of system parameters. The system dynamic behavior evolves into the chaos motion by employing the period-doubling bifurcation.

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1. Introduction

In the real ecological system, a number of species populations can compose a complex network through the predator–prey and parasite–host interactions [1,2]. The dynamic relationship between predator and prey has long been and will continue to be one of the dominant themes in both ecology and mathematical biology. A ubiquitous and important problem to take into account is the stable coexistence of all species populations [3,4].

The Leslie–Gower formulation [5,6] characterized by a logistic-type predator growth equation plays an important role in the modeling of species population dynamics, where the environmental carrying capacity of predator is proportional to the available prey quantity. The growth rate of predator population depends on its ratio to prey ones. The importance of the Leslie–Gower model is highlighted by Collings [7], who stated “ratio-dependent Leslie model provides a way to avoid the biological control paradox wherein classical prey-dependent exploitation models generally do not allow for a pest (prey) equilibrium density that is both low and stable”. Many researchers have extensively studied the population growth of prey–predator system using the Leslie–Gower formulation. Aziz-Alaoui et al. presented a set of the prey–predator models based on the modified Leslie–Gower scheme. The complex dynamic behaviors have been investigated, such as the local

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and global stability [8], boundedness and chaos behaviors [9], and stability coexistence [10]. Furthermore, some sufficient conditions for the global stability of the positive equilibrium point were obtained by constructing the Lyapunov function [11,12]. The Leslie–Gower model system incorporated with seasonally varying parameters [13] and impulsive term [14] were also studied, respectively. Chen et al. taken into consideration of the diffusion effect [15], stage structure [16], prey refuge [17], and feedback control [18] in the Leslie–Gower prey–predator system. Two and three limit cycles behaviors were exhibited in the Leslie–Gower model system with Allee effect [19,20].

Furthermore, in the most of ecosystems, both natural and manmade, population of one species does not respond instantaneously to the interactions of other species. Kuang [21] mentioned that animals must take some times to digest their foods before further activities and responses. Any species population dynamical system without delays is an approximation at best. To incorporate this idea in the modeling approach, the ecological systems with time delay have been developed extensively [22,23]. The ordinary differential equation (ODE), which is the heart of mathematical biology, has been replaced by the delay differential equation (DDE). In general, DDEs exhibit much more complex dynamics than ODEs since dynamical system with delay has an infinite-dimensional phase space. Recently, the phenomena of bifurcation and chaos have been widely investigated in various ecological systems [24,25]. These results indicate that delay is an important parameter. It destabilizes the equilibrium points of system models and causes the recurrent bloom of species population.

In this paper, we consider a three-species food chain system with multiple delays. The dynamical behavior, such as the stability of equilibrium point, bifurcation and chaos will be presented by using the theory analyses and numerical simulations. The rest of the paper is structured as follows. In the following section, we present a brief sketch of the model construction, which may indicate the biological relevance. In Section 3, the stability of equilibrium point is analyzed by considering the corresponding system eigenvalue. The food chain system presents the species coexistence. Time delays may destabilize the species coexistence and evolve the system dynamic into recurrent bloom or system collapse. However, the present of multiple delays can control the species population into the stable coexistence, which is illustrated in Section 4. To investigate the recurrent bloom of species population (existence of the periodic solution), the Hopf bifurcation direction and stability of periodic solution are given in detail by using the central manifold reduction and normal form method in Section 5. In Section 6, the Poincare section is introduced to investigate the evolution of the dynamical behaviors. The route of system entering chaos is the periodic-1, periodic-2, and periodic-4 solution through the period-doubling bifurcation. The concluding remarks are given in Section 7.

2. The model

In this section, the dynamical interaction of a three-species food chain system is presented, where the population density of resource species is denoted by X , consumer and predator are Y and Z , respectively. The entire community is assumed to arise from the interacting species where Z preys on and only on Y and Y preys on X . That is to say, we consider that the predator subsistence exclusively depends on the prey population [1,22,24,26]. The growth rates of consumer and predator are described by Leslie–Gower law. The carrying capacity of predator depends on the available amount of the corresponding prey.

The model is governed by the following system of the nonlinear delay differential equation.

$$\begin{cases} \frac{dX(T)}{dT} = r_1 X(T) \left(1 - \frac{X(T)}{K}\right) - \alpha_1 Y(T), \\ \frac{dY(T)}{dT} = r_2 Y(T) \left(1 - \frac{Y(T)}{c_1 X(T-\theta_1)}\right) - \alpha_2 Z(T), \\ \frac{dZ(T)}{dT} = r_3 Z(T) \left(1 - \frac{Z(T)}{c_2 Y(T-\theta_2)}\right), \end{cases} \quad (2.1)$$

where $r_i > 0$, ($i = 1, \dots, 3$) represent the intrinsic growth rate of species X , Y , Z , respectively; the positive parameter K is the environment carrying capacity of resource X ; $c_1 > 0$ and $c_2 > 0$ are the measure of food quality measure that prey provides for conversion into the births of consumer and predator; the positive parameters α_1 and α_2 are the predation rate of consumer and predator; two delays ($\theta_1, \theta_2 > 0$) are introduced for digestion periods corresponding to consumer-eat-resource and predator-eat-consumer, which are called the resource digestion delay (RDD) and consumer digestion delay (CDD), respectively.

For simplicity, one non-dimensionalizes the system (2.1) with the following scaling:

$$x = \frac{X}{K}, \quad y = \frac{\alpha_1 Y}{r_1 K}, \quad z = \frac{\alpha_1 \alpha_2 Z}{r_1^2 K}, \quad t = r_1 T.$$

Then system (2.1) takes the form

$$\begin{cases} \frac{dx(t)}{dt} = x(t)(1 - x(t)) - y(t), \\ \frac{dy(t)}{dt} = y(t) \left(a_1 - \frac{a_2 y(t)}{x(t-\tau_1)}\right) - z(t), \\ \frac{dz(t)}{dt} = z(t) \left(a_3 - \frac{a_4 z(t)}{y(t-\tau_2)}\right), \end{cases} \quad (2.2)$$

where

$$a_1 = \frac{r_2}{r_1}, \quad a_2 = \frac{r_2}{\alpha_1 c_1}, \quad a_3 = \frac{r_3}{r_1}, \quad a_4 = \frac{r_3}{\alpha_2 c_2}, \quad \tau_1 = r_1 \theta_1, \quad \tau_2 = r_1 \theta_2.$$

The initial conditions for system (2.2) take the form

$$x(\theta) = \varphi_1(\theta) \geq 0, \quad y(\theta) = \varphi_2(\theta) \geq 0, \quad z(\theta) = \varphi_3(\theta) \geq 0, \quad \theta \in [-\tau, 0], \quad \varphi_i(0) > 0, \quad \varphi_i(\theta) \in C([-\tau, 0], \mathbb{R}_{+0}), \\ i = 1, 2, 3,$$

where $\tau = \max(\tau_1, \tau_2)$.

3. Stable coexistence of the species population

In this section, we investigate the stable coexistence of species population in the food chain system (2.2) by analyzing the existence and stability of the system equilibrium points. That is to say, the food chain system is considered to present the stable coexistence of species population when their dynamical behaviors evolve into the stable equilibrium points. Otherwise, if the dynamic behavior of equilibrium point is unstable, the system behavior will not evolve into the corresponding equilibrium points. It implies that the species populations of system will present the recurrent bloom or system collapse.

First, let us begin to determine the equilibrium point in the closed first octant $\mathbb{R}^{3+} = \{(x, y, z) | x \geq 0, y \geq 0, z \geq 0\}$. It is easy to verify that system (2.2) has the boundary equilibrium point $E^0 : (x^0, y^0, 0)$, $x^0 > 0, y^0 > 0$ when the condition $a_2 > a_1$ is satisfied, where $x^0 = (a_2 - a_1)/a_2$ and $y^0 = a_1 x^0/a_2$. On the other hand, the interior equilibrium point $E^c : (x^c, y^c, z^c)$ of system (2.2) exists if and only if conditions $a_3 < a_1 a_4 < a_3 + a_2 a_4$ are satisfied, where $x^c > 0, y^c > 0, z^c > 0$ are given by the following equations.

$$\begin{cases} x^c = (a_3 - a_1 a_4)/a_2 a_4 + 1, \\ y^c = (a_1 a_4 - a_3)x^c/a_2 a_4, \\ z^c = (a_3/a_4)y^c. \end{cases} \quad (3.1)$$

Now, we, respectively determine the stability of equilibrium points $E^0 : (x^0, y^0, 0)$ and $E^c : (x^c, y^c, z^c)$, which are labeled as $\bar{E} : (\bar{x}, \bar{y}, \bar{z})$ for simplicity. Linearization of system (2.2) in the neighborhood of the equilibrium point $\bar{E} : (\bar{x}, \bar{y}, \bar{z})$ produces the following linearized system

$$\begin{cases} \frac{dx(t)}{dt} = (1 - 2\bar{x})x(t) - y(t), \\ \frac{dy(t)}{dt} = \frac{a_1 \bar{x} - 2a_2 \bar{y}}{\bar{x}} y(t) - z(t) + \frac{a_2 \bar{y}^2}{\bar{x}^2} x(t - \tau_1), \\ \frac{dz(t)}{dt} = \frac{a_3 \bar{y} - 2a_4 \bar{z}}{\bar{y}} z(t) + \frac{a_4 \bar{z}^2}{\bar{y}^2} y(t - \tau_2). \end{cases} \quad (3.2)$$

The corresponding characteristic equation is

$$\Delta(\bar{E}, \tau_1, \tau_2) = \begin{vmatrix} -1 + \lambda + 2\bar{x} & 1 & 0 \\ (-a_2 \bar{y}^2/\bar{x}^2)e^{-\lambda \tau_1} & (a_1 \bar{x} - 2a_2 \bar{y})/\bar{x} + \lambda & 1 \\ 0 & (a_4 \bar{z}^2/\bar{y}^2)e^{-\lambda \tau_2} & (a_3 \bar{y} - 2a_4 \bar{z})/\bar{y} + \lambda \end{vmatrix} = 0. \quad (3.3)$$

Then, the stability of system (2.2) in the neighborhood of the equilibrium point can be determined, respectively. Based on the dynamical theory, we have

Theorem 1. The boundary equilibrium point $E^0 : (x^0, y^0, 0)$ is an unstable node whatever the system parameter values are.

Proof. Submitting the coordinate value of the boundary equilibrium point $E^0 : (x^0, y^0, 0)$ into Eq. (3.3), one has the corresponding characteristic equation as follows

$$\Delta(E^0, \tau_1, \tau_2) = (\lambda - a_3)(a_2 \lambda(1 + \lambda) + a_1(a_2 - 2\lambda + a_2 \lambda) + a_1^2(1 - 2e^{\lambda \tau_1})e^{-\lambda \tau_1}) = 0. \quad (3.4)$$

It is easy to check that Eq. (3.4) has the positive root $\lambda = a_3 > 0$. That is to say, the characteristic equation corresponded to equilibrium point $E^0 : (x^0, y^0, 0)$ exhibits the eigenvalue with the positive real part. The boundary equilibrium point is an unstable node. \square

Remark. Theorem 1 implies that the boundary equilibrium point is an impossible steady state for system (2.2). The predator species will not evolve into the population extinction when resource and predator exist.

Next, we consider the stability of the interior equilibrium point $E^c : (x^c, y^c, z^c)$ in system (2.2). Then, we have the following theorem.

Theorem 2. The interior equilibrium point $E^c : (x^c, y^c, z^c)$ is locally asymptotically stable for the non-delayed system (2.2) when the parameter value is satisfied with the conditions $m_2 > 0$ and $m_1 m_2 - m_0 m_3 > 0$, where

$$\begin{aligned} m_3 &= a_2 a_4^2, \quad m_2 = a_4(2a_3 + a_2 a_3(a_4 - 2) + a_1 a_4(a_2 - 2) + a_2 a_4), \\ m_1 &= a_1(a_2 - a_1)a_4^2 - a_3^2(3 + a_2 a_4 - 2a_4) + a_3 a_4(4a_1 - 2a_2 + a_2 a_4 - 2a_1 a_4 + a_1 a_2 a_4), \\ m_0 &= a_3(a_1 a_4 - a_3)(a_3 + a_2 a_4 - a_1 a_4). \end{aligned}$$

Proof. Submitting $E^c : (x^c, y^c, z^c)$ into Eq. (3.3), one obtains

$$\begin{aligned} \Delta(E^c, \tau_1, \tau_2) &= (\lambda + a_3)(a_1 a_4 - 2a_3 + \lambda a_4)(2a_3 + a_2 a_4 - 2a_1 a_4 + a_2 a_4 \lambda) + (a_3 - a_1 a_4)^2(\lambda + a_3)e^{-\lambda \tau_1} + a_3^2 e^{-\lambda \tau_2}(2a_3 \\ &\quad + a_2 a_4 - 2a_1 a_4 + a_2 a_4 \lambda) \\ &= 0. \end{aligned} \quad (3.5)$$

If $\tau_1 = \tau_2 = 0$, the original system (2.2) is an ODE model. The characteristic equation of its linearized system is given by

$$\Delta(E^c, 0, 0) = m_3 \lambda^3 + m_2 \lambda^2 + m_1 \lambda + m_0, \quad (3.6)$$

It follows from the Routh-Hurwitz criterion that the necessary and sufficient conditions for all roots of Eq. (3.6) having negative real parts are given by

$$m_2 > 0, \quad m_1 m_2 - m_0 m_3 > 0. \quad (3.7)$$

$E^c : (x^c, y^c, z^c)$ is a stable equilibrium point. This completes the proof. \square

To obtain the effect of digestion delay on the stability of equilibrium point, let us consider the case $\tau_1 > 0$ and $\tau_2 = 0$ under the conditions $m_1 m_2 - m_0 m_3 > 0$, $m_2 > 0$ for simplify, then the characteristic equation (3.5) is

$$\Delta(E^c, \tau_1, 0) = (2a_3 + a_2 a_4 - 2a_1 a_4 + a_2 a_4 \lambda)(a_3^2 + (\lambda + a_3)(a_1 a_4 - 2a_3 + \lambda a_4)) + (a_3 - a_1 a_4)^2(\lambda + a_3)e^{-\lambda \tau_1} = 0. \quad (3.8)$$

The interior equilibrium point $E^c : (x^c, y^c, z^c)$ may lose its stability when τ_1 varies. Notice that $\Delta(E^c, \tau_1, \tau_2)|_{\lambda=0} = a_3(a_1 a_4 - a_3)(a_3 + a_2 a_4 - a_1 a_4) > 0$ if $a_3 < a_1 a_4 < a_3 + a_2 a_4$ holds. It implies that $\lambda = 0$ is not a root of the characteristic equation (3.5) when the interior equilibrium point $E^c : (x^c, y^c, z^c)$ exists. The critical condition of the $E^c : (x^c, y^c, z^c)$ stability is determined by the pair of pure conjugate imaginary roots. Then, we have the following theorem.

Theorem 3. The interior equilibrium point $E^c : (x^c, y^c, z^c)$ is locally asymptotically stable for the arbitrary delay τ_1 if the polynomial $G_1(v, \tau_1) = 0$ has no positive root, which is called the delay-independent stability. Further, if the polynomial $G_1(v, \tau_1) = 0$ has only one positive and simple root v_c , there exists exactly a critical delay $\tau_1^c > 0$ such that $E^c : (x^c, y^c, z^c)$ is a stable interior equilibrium point when $\tau_1 \in (0, \tau_1^c)$ holds, where $G_1(v, \tau_1) = n_1 v^8 + n_2 v^6 + n_3 v^4 + n_4 v^2 + n_5$, and

$$\begin{aligned} n_1 &= a_2^2 a_4^4, \\ n_2 &= a_4^2(-4(-a_2 + a_1(2 + a_2^2))a_3 a_4 + (-4a_1 a_2 + a_2^2 + a_1^2(4 + a_2^2))a_4^2 + 2a_3^2(2 + a_2^2(2 + (-1 + a_4)a_4))), \\ n_3 &= -(a_3 - a_1 a_4)^4 + 2a_2 a_3^2 a_4^2(a_3 - a_1 a_4)(2a_3 + (-2a_1 + a_2)a_4) - 2a_3^2 a_4(-2(-1 + a_2)a_3 \\ &\quad + (a_1(-2 + a_2) + a_2)a_4)(a_3(2 + (-2 + a_2)a_4) + a_4(-2a_1 + a_2 - (a_1(-2 + a_2) + a_2)a_4)) \\ &\quad + (2(-3a_1 + a_2)a_3 a_4 + a_1(-2a_1 + a_2)a_4^2 + a_3^2(4 + a_2(-1 + a_4)a_4))^2, \\ n_4 &= a_3^2(-2(a_3 - a_1 a_4)^4 + a_3^2(a_3(2 + (-2 + a_2)a_4) + a_4(-2a_1 + a_2 - (a_1(-2 + a_2) + a_2)a_4))^2 \\ &\quad + 2(a_3 - a_1 a_4)(2a_3 + (-2a_1 + a_2)a_4)(2(-3a_1 + a_2)a_3 a_4 + a_1(2a_1 - a_2)a_4^2 + a_3^2(4 + a_2(-1 + a_4)a_4))), \\ n_5 &= a_3^4(a_3 - a_1 a_4)^2(3a_3 + (-3a_1 + a_2)a_4)(a_3 + (-a_1 + a_2)a_4). \end{aligned}$$

Proof. Supposing $\lambda = \pm iv$ ($v > 0$) is a pair of purely imaginary roots of Eq. (3.8), one has

$$\begin{aligned} \Delta(E^c, \tau_1, 0)|_{\lambda=iv} &= (2a_3 + a_2 a_4 - 2a_1 a_4 + a_2 a_4 iv)(a_3^2 + (iv + a_3)(a_1 a_4 - 2a_3 + iv a_4)) \\ &\quad + (a_3 - a_1 a_4)^2(iv + a_3)(\cos(v\tau_1) + i \sin(v\tau_1)) = 0. \end{aligned} \quad (3.9)$$

Separating Eq. (3.9) into the real and imaginary parts yields

$$\begin{cases} a_3(a_1 a_4 - a_3)(2a_3 + a_2 a_4 - 2a_1 a_4) - (2 - 2a_2 + a_2 a_4)a_3 a_4 v^2 + (a_2 - 2a_1 + a_1 a_2)a_4^2 v^2 + (a_3 - a_1 a_4)^2(a_3 \cos v\tau_1 + v \sin v\tau_1) = 0, \\ (a_2 a_4 - 2a_2 + 6a_1 + a_1 a_2 a_4 - 2a_1 a_4)a_3 a_4 v - a_3^2(4 - 2a_4 + a_2 a_4)v + (a_1 a_2 - 2a_1^2 - a_2 v^2)a_4^2 v - (a_3 - a_1 a_4)^2(-v \cos v\tau_1 + a_3 \sin v\tau_1) = 0. \end{cases} \quad (3.10)$$

Solving $\cos v\tau_1$ and $\sin v\tau_1$ from Eq. (3.10) and employing $\cos^2 v\tau_1 + \sin^2 v\tau_1 = 1$, we have

$$G_1(v, \tau_1) = n_1 v^8 + n_2 v^6 + n_3 v^4 + n_4 v^2 + n_5, \quad (3.11)$$

Based on the dynamical theory, the equilibrium point is locally asymptotically stable, i.e. species population exhibits the stability coexistence if and only if each of eigenvalues has the negative real parts. This completes the proof. \square

Remark. Under the condition of system parameters $m_2 > 0$ and $m_1 m_2 - m_0 m_3 > 0$, the species population can evolve into the stable coexistence corresponding to the steady state $E^c : (x^c, y^c, z^c)$ for arbitrary value of the resource digestion delay when the polynomial $G_1(v, \tau_1) = 0$ has no positive root. However, with the resource digestion delay τ_1 increasing, the species population may evolve into the recurrent bloom or system collapse. The time delay impacts the stable coexistence of the species population.

Example 1. We choose the parameter values of system (2.2) as $a_1 = 1.2$, $a_2 = 2.5$, $a_3 = 1.5$, $a_4 = 1.8$ and $\tau_2 = 0$, where the system parameters are chosen as $a_i > 1$, $i = 1, \dots, 4$. It implies that the intrinsic growth rate of resource is less than the consumer and predator. At this time, system has the interior equilibrium point $x^c = 0.853333$, $y^c = 0.125156$, $z^c = 0.104296$. Furthermore, the conditions $m_2 > 0$ and $m_1 m_2 - m_0 m_3 > 0$ hold. It follows from Eq. (3.11) that $G_1(v, \tau_1) = 0$ has no positive root (see Fig. 1(a)). The corresponding eigenvalue has the negative real parts for τ_1 varying, as shown in Fig. 1(b). The interior equilibrium point $E^c : (x^c, y^c, z^c)$ of system (2.2) is locally asymptotically stable for the arbitrary delay τ_1 . Figs. 1(c) and (d) illustrate the time histories when the resource digestion delay τ_1 is fixed as 0.5 and 12, respectively.

However, fixed the system parameter $a_1 = 1.2$, $a_2 = 2.5$, $a_3 = 1.5$, $a_4 = 4.8$ and $\tau_2 = 0$, the interior equilibrium point of system (2.2) is $x^c = 0.645$, $y^c = 0.228975$, $z^c = 0.0715547$. Following Eq. (3.11), the equation $G_1(v, \tau_1) = 0$ has one positive root $v_c = 0.1988$, as shown in Fig. 2(a). It follows from theorem 3 that there exists the critical delay $\tau_1^c = 11.8916 > 0$ (Fig. 2(b)). The interior equilibrium point $E^c : (x^c, y^c, z^c)$ is locally asymptotically stable when the resource digestion delay $\tau_1 \in (0, \tau_1^c)$ holds. The time history for the fixed delay $\tau_1 = 1$ is shown in Fig. 2(c). However, when delay τ_1 increases and passes through the critical value $\tau_1^c = 11.8916$, the interior equilibrium point $E^c : (x^c, y^c, z^c)$ of system (2.2) lose its stability. The system enters into the recurrent bloom of species population as shown in Fig. 2(d).

4. Controlled species coexistence by the multiple delays

In order to investigate the combined effect of multiple delays on the stability of the interior equilibrium point $E^c : (x^c, y^c, z^c)$ in system (2.2), we regard τ_2 as the varying parameter for the fixed delay τ_1 . Based on the dynamical theory, one has the following theorem.

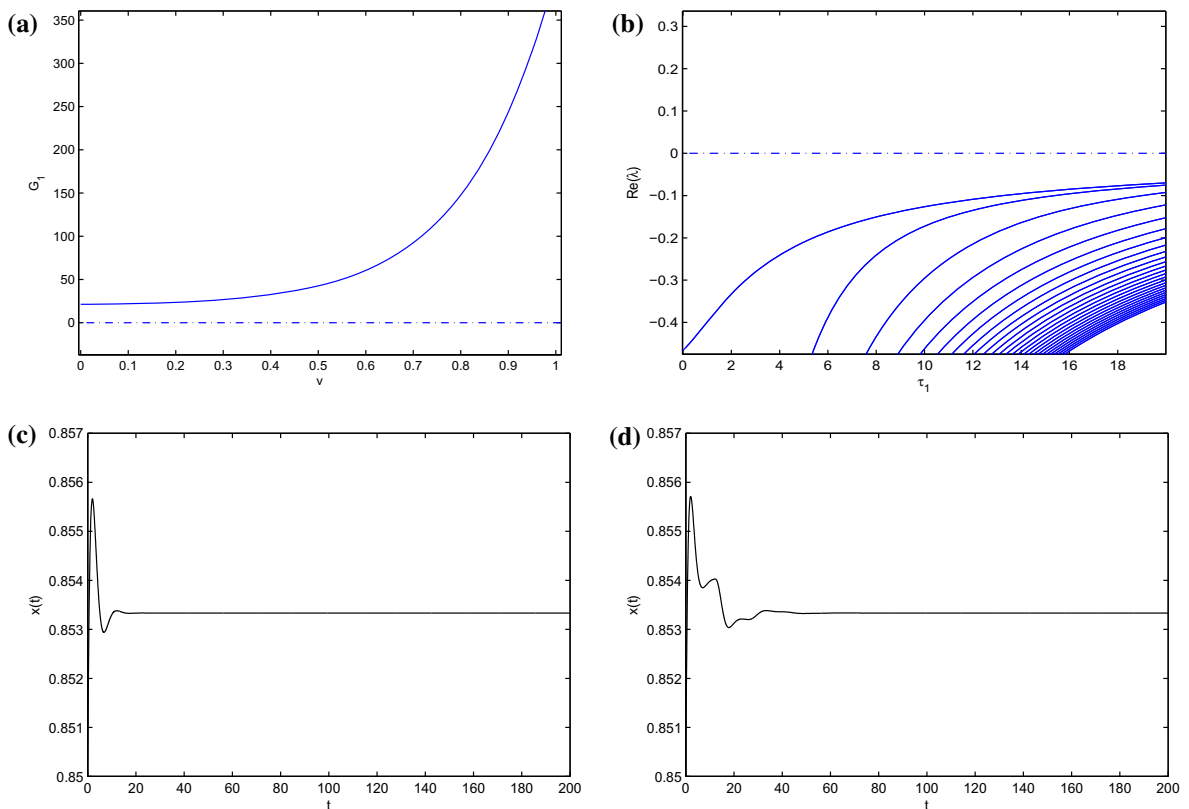


Fig. 1. (a) Graphic of function G_1 , (b) eigenvalue real parts with τ_1 varying, and time histories for the fixed delay (c) $\tau_1 = 1$, and (d) $\tau_1 = 12$. The other parameters are chosen as $a_1 = 1.2$, $a_2 = 2.5$, $a_3 = 1.5$, $a_4 = 1.8$ and $\tau_2 = 0$.

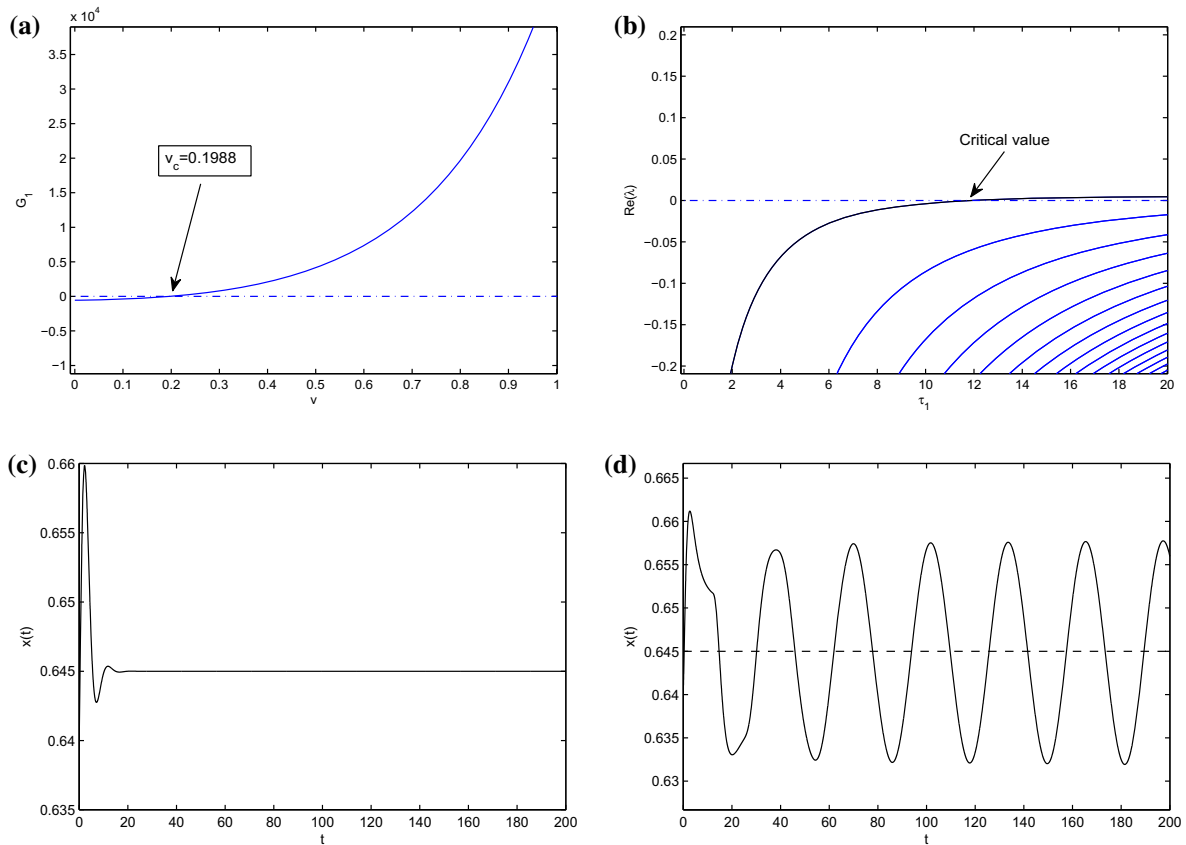


Fig. 2. (a) Graphic of function G_1 , (b) eigenvalue real parts with τ_1 varying, and time histories for the fixed delay (c) $\tau_1 = 1$, and (d) $\tau_1 = 12$. The other parameters are chosen as $a_1 = 1.2$, $a_2 = 2.5$, $a_3 = 1.5$, $a_4 = 4.8$ and $\tau_2 = 0$.

Theorem 4. The following conclusions are true if all roots of Eq. (3.8) have the negative real parts.

- (a) When $G_2(\tau_1, \omega) = 0$ has no positive root, the interior equilibrium point $E^c : (x^c, y^c, z^c)$ of system (2.2) is asymptotically stable for an arbitrary delay τ_2 .
- (b) When $G_2(\tau_1, \omega) = 0$ has at least one positive and simple root ω_c , there exists a critical delay $\tau^c > 0$ such that the interior equilibrium point $E^c : (x^c, y^c, z^c)$ is asymptotically stable for $\tau_2 \in [0, \tau^c)$.
- (c) When $G_2(\tau_1, \omega) = 0$ has at least two positive and simple roots $0 < \omega_1 < \omega_2 < \dots$, there exist a finite number of intervals. If the consumer digestion delay τ_2 is fixed into these intervals, the interior equilibrium point $E^c : (x^c, y^c, z^c)$ is locally asymptotically stable, while unstable if one does not belong to these intervals.

where

$$G_2(\tau_1, \omega) = (f_1(\tau_1, \omega))^2 + (f_2(\tau_1, \omega))^2 - (f_3(\tau_1, \omega))^2 = 0,$$

and

$$f_1(\tau_1, \omega) = (2a_3^2 - a_1a_3a_4 + a_4\omega^2)(a_2^2a_4^2\omega^2 + (2a_3 - 2a_1a_4 + a_2a_4)^2) \\ + (a_3 - a_1a_4)^2((2a_1a_4 - 2a_3 - a_2a_4 + a_2a_3a_4)\omega \sin \omega\tau_1 - (2a_3^2 - 2a_1a_3a_4 + a_2a_3a_4 + a_2a_4\omega^2) \cos \omega\tau_1),$$

$$f_2(\tau_1, \omega) = (a_1a_4 + a_3a_4 - 2a_3)(a_2^2a_4^2\omega^2 + (2a_3 - 2a_1a_4 + a_2a_4)^2)\omega \\ + (a_3 - a_1a_4)^2((2a_3 - 2a_1a_4 + a_2a_4 - a_2a_3a_4)\omega \cos \omega\tau_1 - (2a_3^2 - 2a_1a_3a_4 + a_2a_3a_4 + a_2a_4\omega^2) \sin \omega\tau_1),$$

$$f_3(\tau_1, \omega) = a_3^2(a_2^2a_4^2\omega^2 + (2a_3 - 2a_1a_4 + a_2a_4)^2).$$

Proof. Letting $\lambda = i\omega$ is the simple roots of the characteristic equation (3.5), one obtains

$$(i\omega + a_3)(a_1a_4 - 2a_3 + i\omega a_4)(2a_3 + a_2a_4 - 2a_1a_4 + i\omega a_2a_4) + (a_3 - a_1a_4)^2(i\omega + a_3)e^{-i\omega\tau_1} + a_3^2e^{-i\omega\tau_2}(2a_3 + a_2a_4 - 2a_1a_4 + i\omega a_2a_4) = 0. \quad (4.1)$$

Separating the real and imaginary parts gives

$$\begin{cases} a_3(a_1a_4 - 2a_3)(2a_3 - 2a_1a_4 + a_2a_4) - a_4(2a_3 - 2a_2a_3 + a_2a_3a_4)\omega^2 \\ + (a_2 - 2a_1 + a_1a_2)a_4^2\omega^2 + a_3^2(2a_3 - 2a_1a_4 + a_2a_4)\cos\omega\tau_2 \\ + (a_3 - a_1a_4)^2(a_3\cos\omega\tau_1 + \omega\sin\omega\tau_1) + a_2a_3^2a_4\omega\sin\omega\tau_2 = 0, \\ a_3a_4\omega(6a_1 - 2a_2 - 2a_1a_4 + a_2a_4 + a_1a_2a_4) - 2\omega a_3^2(2 - a_4 + a_2a_4) \\ + \omega a_4^2(a_1a_2 - 2a_1^2 - a_2\omega^2) - (a_3 - a_1a_4)^2(-\omega\cos\omega\tau_1 + a_3\sin\omega\tau_1) \\ + a_2a_3^2a_4\omega\cos\omega\tau_2 - a_3^2(2a_3 - 2a_1a_4 + a_2a_4)\sin\omega\tau_2 = 0. \end{cases} \quad (4.2)$$

Eliminating τ_2 from Eq. (4.2), one has

$$\cos\omega\tau_2 = \frac{f_1(\tau_1, \omega)}{f_3(\tau_1, \omega)}, \quad \sin\omega\tau_2 = \frac{f_2(\tau_1, \omega)}{f_3(\tau_1, \omega)}, \quad (4.3)$$

This implies that

$$G_2(\tau_1, \omega) = (f_1(\tau_1, \omega))^2 + (f_2(\tau_1, \omega))^2 - (f_3(\tau_1, \omega))^2 = 0. \quad (4.4)$$

The equilibrium point is locally asymptotically stable if and only if each of eigenvalues has the negative real parts. Then theorem 4 can be obtained. This completes the proof. \square

Remark. It is interesting to note that introduction of multiple delays corresponding to the consumer-eat-resource and predator-eat-consumer in the food chain system leads to the stability switching of the interior equilibrium point $E^c : (x^c, y^c, z^c)$. The system dynamic near the equilibrium point switches from stable to unstable, and then back to stable when delays increases. It implies that the present of multiple delays either generates or terminates the recurrent bloom or system collapse of the species population in the food chain system. Time delays are helpful to control species populations to the stable coexistence of the steady state $E^c : (x^c, y^c, z^c)$.

Example 2. We consider a specific system model possessed with the fixed parameters $a_1 = 3.2, a_2 = 6.5, a_3 = 2.5, a_4 = 2.2$. Similarly, the fixed values of system parameters imply that the intrinsic growth rate of resource is less than the consumer and predator, which are used in the previous section for the species coexistence destroying but in this section for the coexistence controlling. Firstly, choosing the delay $\tau_1 = 0$, one obtains that $G_2(\tau_1, \omega) = 0$ has no positive root (see Fig. 3(a)). The corresponding eigenvalue with τ_2 varying is shown in Fig. 3(b), which implies that it has the negative real parts for an arbitrary delay τ_2 . The interior equilibrium point $E^c : (x^c, y^c, z^c) = (0.682517, 0.216687, 0.246236)$ of system (2.2) has the delay-independent stability. The time history for the fixed delay $\tau_2 = 1$ and $\tau_2 = 5$ are shown in Figs. 3 (c) and (d), respectively. It is seen that a stable spiral converges to $(0.682517, 0.216687, 0.246236)$.

However, when delay τ_1 reaches to 0.5, the equation $G_2(\tau_1, \omega) = 0$ has two positive and simple roots $\omega_1 = 0.6582, \omega_2 = 1.1817$, as shown in Fig. 4(a). The corresponding eigenvalue with τ_2 varying is shown in Fig. 4(b). There exist a finite number of critical delays. From theorem 4, the system (2.2) exhibits the stability switch when time delay τ_2 increases. In fact, the interior equilibrium point $E^c : (x^c, y^c, z^c) = (0.682517, 0.216687, 0.246236)$ is asymptotically stable when $\tau_2 \in (0, 1.6791)$. Fig. 4(c) illustrates the time history for the fixed delay $\tau_2 = 1$, which implies that the interior equilibrium point is asymptotically stable. When $\tau_2 = 2$, the interior equilibrium point loses its stability and become unstable. The time history is shown in Fig. 4(d). The system obtains the stable state when delay increases and crosses the critical value 4.65833. The time history for the fixed delay $\tau_2 = 5.5$ is illustrated in Fig. 4(e). Similarly, the interior equilibrium point loses its stability if the delay τ_2 increases, which is shown in Fig. 4(f) for the fixed delay $\tau_2 = 9$.

5. Recurrent bloom and the Hopf bifurcation

It follows from above section that multiple delays τ_1 and τ_2 influence the stability of the dynamic behavior near the interior equilibrium point $E^c : (x^c, y^c, z^c)$ in system (2.2). The system dynamic switches from stable to unstable, and then back to stable when delays increases. Time delays are helpful to control species populations to the stable coexistence. However, what is the steady state when the dynamic behavior near the interior equilibrium point $E^c : (x^c, y^c, z^c)$ is unstable? In this section, we illustrate the periodic oscillation of system (2.2) by investigating the Hopf bifurcation in the neighbor of $E^c : (x^c, y^c, z^c)$. The following theorem implies that the species population presents the recurrent bloom induced by the multiple delays.

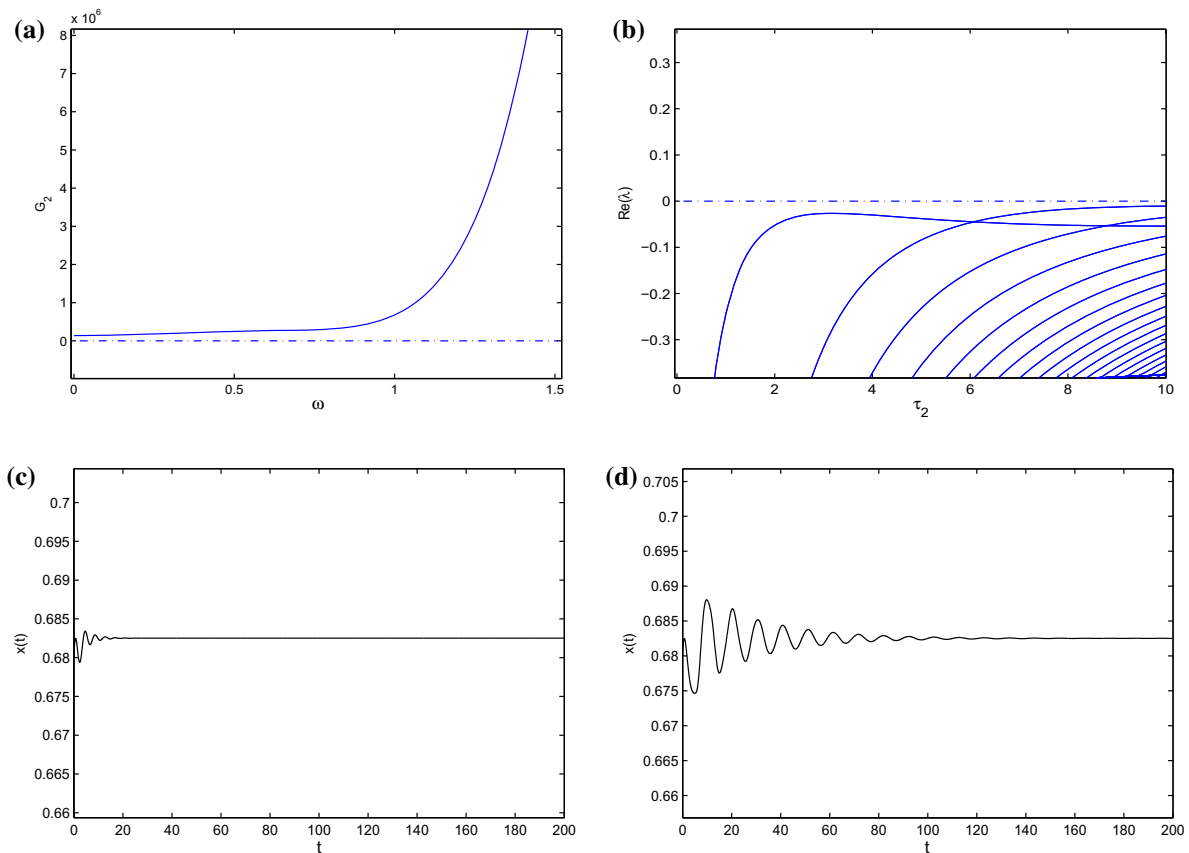


Fig. 3. (a) Graphic of function G_2 , (b) eigenvalue real parts with τ_2 varying, and time histories for the fixed delay (c) $\tau_2 = 1$, and (d) $\tau_2 = 5$. The other parameters are chosen as $a_1 = 3.2$, $a_2 = 6.5$, $a_3 = 2.5$, $a_4 = 2.2$ and $\tau_1 = 0$.

Theorem 5. If all roots of Eq. (3.8) have negative real parts, system (2.2) undergoes a Hopf bifurcation for $\tau_2 = \tau_c$ when the transversality condition $\text{Re}\{\lambda'(\tau_2)|_{\tau_2=\tau_c}\} \neq 0$, holds. That is, it has a branch of periodic solutions bifurcating from the interior equilibrium point $E^c : (x^c, y^c, z^c)$ near the critical delayed value τ_c .

Proof. As a matter of fact, when a system exhibits the Hopf bifurcation, the corresponding eigenvalue must have a pair of purely imaginary roots. It follows from Eq. (4.1) that the characteristic equation (3.5) has a pair of purely imaginary roots when $G_2(\tau_1, \omega) = 0$ has at least one positive and simple root $0 < \omega_1 < \omega_2 < \dots$. Without loss of generality, defining $\omega_c = \max\{\omega_i, i = 1, 2, \dots\}$ and using delay τ_2 as the bifurcation parameter, one has a critical delay value τ_c as following

$$\tau_c = \min \left\{ \frac{(\varphi_c + 2j\pi)}{\omega_c} \right\}, \quad j = 0, 1, 2, \dots, \quad (5.1)$$

where $\varphi_c \in [0, 2\pi)$ and satisfied with

$$\cos(\varphi_c) = \frac{f_1(\tau_1, \omega_c)}{f_3(\tau_1, \omega_c)}, \quad \sin(\varphi_c) = \frac{f_2(\tau_1, \omega_c)}{f_3(\tau_1, \omega_c)}.$$

In addition, to make sure the occurrence of the Hopf bifurcation, it is needed to check the transversality condition. The necessary condition of the Hopf bifurcation existence is that the critical eigenvalue crosses the imaginary axis with non-zero velocity. Differentiating λ with respect to τ_2 in Eq. (3.5) reaches

$$\lambda'(\tau_2) = g_1(\tau_2, \lambda)/g_2(\tau_2, \lambda), \quad (5.2)$$

where

$$g_1 = -\lambda a_3^2 e^{j\tau_1} (2a_3 - 2a_1 a_4 + a_2 a_4 (1 + \lambda)),$$

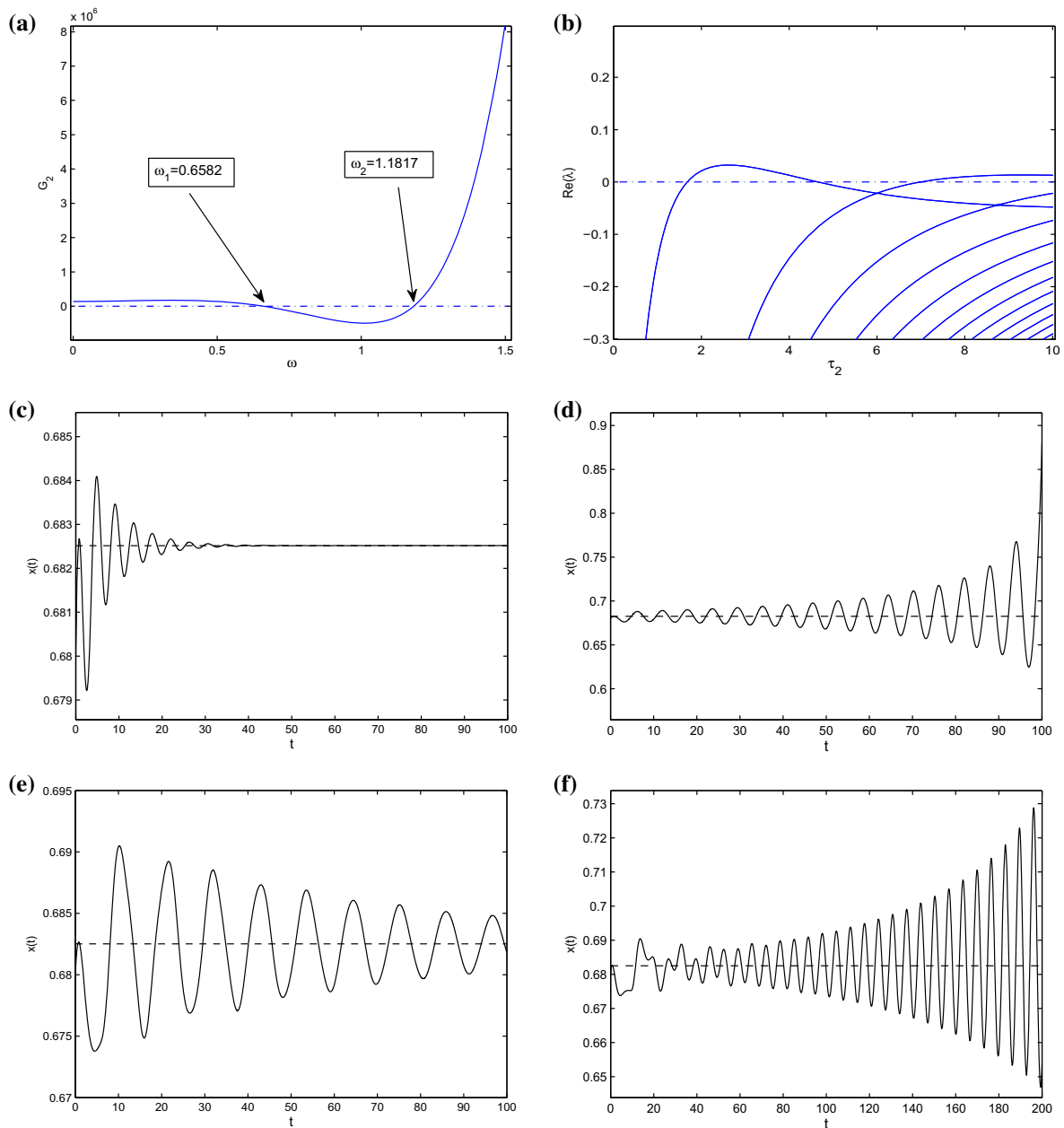


Fig. 4. (a) Graphic of function G_2 , (b) eigenvalue real parts with τ_2 varying, and time histories for the fixed delay (c) $\tau_2 = 1$, (d) $\tau_2 = 2$, (e) $\tau_2 = 5.5$, and (f) $\tau_2 = 9$, respectively. The other parameters are chosen as $a_1 = 3.2$, $a_2 = 6.5$, $a_3 = 2.5$, $a_4 = 2.2$ and $\tau_1 = 0.5$.

$$g_2 = (a_3 - a_1 a_4)^2 (-1 + a_3 \tau_1 + \lambda \tau_1) e^{\lambda \tau_2} + e^{\lambda \tau_1} (2a_3 - 2a_1 a_4) (-e^{\lambda \tau_2} (-2a_3 + a_3 a_4 + a_1 a_4 + 2a_4 \lambda) + a_3^2 \tau_2) \\ + a_2 a_4 e^{\lambda \tau_1} (e^{\lambda \tau_2} (2a_3^2 - a_1 a_4 - a_4 \lambda (2 + 2a_1 + 3\lambda)) + a_3 (2 - a_4 - a_1 a_4 + 4\lambda - 2a_4 \lambda)) - a_3^2 (1 - \tau_2 - \lambda \tau_2))$$

Based on the Hopf bifurcation theorem of the functional differential equation and transversality condition (5.2), one has the theorem 5. This completes the proof. \square

To get the steady state of species population, in the remainder of this section, we study the stability of the bifurcating periodic solution and the direction of the Hopf bifurcation by using the central manifold theorem and normal form method introduced by Hassard et al. [27]. To this end, let $x(t) \rightarrow x(t) - x^c$, $y(t) \rightarrow y(t) - y^c$, $z(t) \rightarrow z(t) - z^c$, and use the Taylor expansion. Then system (2.2) is rewritten as following form

$$\begin{cases} \frac{dx(t)}{dt} = a_{10}x(t) - y(t) - x^2(t), \\ \frac{dy(t)}{dt} = b_{10}y(t) + b_{01}x(t - \tau_1) - z(t) - b_{20}y^2(t) + b_{11}y(t)x(t - \tau_1) - b_{02}x^2(t - \tau_1) + b_{21}y^2(t)x(t - \tau_1) - b_{12}y(t)x^2(t - \tau_1) + b_{03}x^3(t - \tau_1), \\ \frac{dz(t)}{dt} = c_{10}z(t) + c_{01}y(t - \tau_2) - c_{20}z^2(t) + c_{11}z(t)y(t - \tau_2) - c_{02}y^2(t - \tau_2) + c_{21}z^2(t)y(t - \tau_2) - c_{12}z(t)y^2(t - \tau_2) + c_{03}y^3(t - \tau_2), \end{cases} \quad (5.3)$$

where

$$\begin{aligned} a_{10} &= (1 - 2x^c), \quad b_{10} = \frac{a_1x^c - 2a_2y^c}{x^c}, \quad b_{01} = \frac{a_2(y^c)^2}{(x^c)^2}, \quad b_{20} = \frac{a_2}{x^c}, \quad b_{11} = \frac{2a_2y^c}{(x^c)^2}, \quad b_{02} = \frac{a_2(y^c)^2}{(x^c)^3}, \\ b_{21} &= \frac{a_2}{(x^c)^2}, \quad b_{12} = \frac{2a_2y^c}{(x^c)^3}, \quad b_{03} = \frac{a_2(y^c)^2}{(x^c)^4}, \quad c_{10} = \frac{a_3y^c - 2a_4z^c}{y^c}, \quad c_{01} = \frac{a_4(z^c)^2}{(y^c)^2}, \quad c_{20} = \frac{a_4}{y^c}, \quad c_{11} = \frac{2a_4z^c}{(y^c)^2}, \\ c_{02} &= \frac{a_4(z^c)^2}{(y^c)^3}, \quad c_{21} = \frac{a_4}{(y^c)^2}, \quad c_{12} = \frac{2a_4z^c}{(y^c)^3}, \quad c_{03} = \frac{a_4(z^c)^2}{(y^c)^4}, \end{aligned}$$

and $x^c = (a_3 - a_1a_4)/a_2a_4 + 1$, $y^c = (a_1a_4 - a_3)x^c/a_2a_4$, $z^c = (a_3/a_4)y^c$ is the interior equilibrium point of system (2.2).

Regard the delay τ_2 as the Hopf bifurcation parameter, and let $\tau_2 = \tau_c + \varepsilon$, where τ_c is the critical value of the Hopf bifurcation and ε is the unfolding parameter. Then system (5.3) becomes

$$\begin{cases} \frac{dx(t)}{dt} = a_{10}x(t) - y(t) - x^2(t), \\ \frac{dy(t)}{dt} = b_{10}y(t) + b_{01}x(t - \tau_1) - z(t) - b_{20}y^2(t) + b_{11}y(t)x(t - \tau_1) - b_{02}x^2(t - \tau_1) \\ \quad + b_{21}y^2(t)x(t - \tau_1) - b_{12}y(t)x^2(t - \tau_1) + b_{03}x^3(t - \tau_1), \\ \frac{dz(t)}{dt} = c_{10}z(t) + c_{01}y(t - \tau_c - \varepsilon) - c_{20}z^2(t) + c_{11}z(t)y(t - \tau_c - \varepsilon) - c_{02}y^2(t - \tau_c - \varepsilon) \\ \quad + c_{21}z^2(t)y(t - \tau_c - \varepsilon) - c_{12}z(t)y^2(t - \tau_c - \varepsilon) + c_{03}y^3(t - \tau_c - \varepsilon), \end{cases} \quad (5.4)$$

To apply the central manifold reduction, it is necessary to change system (5.4) into a functional differential equation. Let $C \triangleq C([-h, 0], R^3)$ as the Banach space of continuous functions from $[-h, 0]$ to R^3 with the supremum norm, where $h = \max(\tau_1, \tau_c)$. For any $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in C$, we define an operator

$$L(\varepsilon)\phi = \int_{-h}^0 [\delta(\theta, \varepsilon)]\phi(\theta), \quad (5.5)$$

where $\eta : [-h, 0] \rightarrow R^3 \times R^3$ is a real-valued function of bounded variation in $[-h, 0]$ with

$$\eta(\theta, \varepsilon) = \begin{pmatrix} a_{10} & -1 & 0 \\ 0 & b_{10} & -1 \\ 0 & 0 & c_{10} \end{pmatrix} \delta(\theta) + \begin{pmatrix} 0 & 0 & 0 \\ b_{01} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \delta(\theta + \tau_1) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_{01} & 0 \end{pmatrix} \delta(\theta + \tau_c + \varepsilon), \quad (5.6)$$

where $\delta(\theta)$ is Dirac function. For $\phi \in C$, the linear operator defined by system (5.4) generates an infinitesimal generator of the semi-flow of bounded linear operators with

$$A(\varepsilon)\phi = \begin{cases} d\phi(\theta)/d\theta & \theta \in [-h, 0), \\ \int_{-h}^0 [d\eta(\theta, \varepsilon)]\phi(\theta) & \theta = 0, \end{cases} \quad (5.7)$$

and

$$N(\phi, \varepsilon) = \begin{cases} 0 & \theta \in [-h, 0), \\ F(\phi, \varepsilon) & \theta = 0, \end{cases} \quad (5.8)$$

where

$$F(\phi, \varepsilon) = (F_1(\phi, \varepsilon), F_2(\phi, \varepsilon), F_3(\phi, \varepsilon))^T, \quad (5.9)$$

and

$$\begin{cases} F_1(\phi, \varepsilon) = -\phi_1^2(0), \\ F_2(\phi, \varepsilon) = -b_{20}\phi_2^2(0) + b_{11}\phi_2(0)\phi_1(-\tau_1) - b_{02}\phi_1^2(-\tau_1) + b_{21}\phi_2^2(0)\phi_1(-\tau_1) - b_{12}\phi_2(0)\phi_1^2(-\tau_1) + b_{03}\phi_1^3(-\tau_1), \\ F_3(\phi, \varepsilon) = -c_{20}\phi_3^2(0) + c_{11}\phi_3(0)\phi_2(-\tau_c - \varepsilon) - c_{02}\phi_2^2(-\tau_c - \varepsilon) + c_{21}\phi_3^2(0)\phi_2(-\tau_c - \varepsilon) - c_{12}\phi_3(0)\phi_2^2(-\tau_c - \varepsilon) + c_{03}\phi_2^3(-\tau_c - \varepsilon). \end{cases}$$

Thus, system (5.4) is equivalent to the following operator equation

$$\dot{u}_t = A(\varepsilon)u_t + N(u_t, \varepsilon), \quad (5.10)$$

where $u_t(\theta) = u(t + \theta)$, $-h \leq \theta \leq 0$. For $\psi \in C^* = C([0, h], R^3)$, the adjoint operators $A^*(\varepsilon)$ of $A(\varepsilon)$ are given by

$$A^*(\varepsilon)\psi(s) = \begin{cases} -d\psi(s)/ds & s \in (0, h], \\ \int_{-h}^0 d\eta^T(s, \varepsilon)\psi(-s) & s = 0. \end{cases} \quad (5.11)$$

For $\varphi \in C$ and $\psi \in C^*$, we introduce the bilinear form

$$\langle \psi(s), \varphi(\theta) \rangle = \bar{\psi}^T(0)\varphi(0) - \int_{-h}^0 \int_{\xi=0}^{\theta} \bar{\psi}^T(\xi - \theta)d\eta(\theta)\varphi(\xi)d\xi. \quad (5.12)$$

From the above discussion, we know that system (5.4) has a simple pair of pure imaginary eigenvalues $\pm i\omega_c$ for $\varepsilon = 0$. The other eigenvalues exhibit the negative real parts. Therefore, the phase space C can be split into two subspaces as $C = P_\Lambda \oplus Q_\Lambda$, where P_Λ is the two-dimensional central subspace spanned by the basic vectors of the linear operator $A(\varepsilon)$ associated with the eigenvalues $\pm i\omega_c$, and Q_Λ is the complement subspace of P_Λ . We calculate the eigenvector q of A belonging to the eigenvalue $i\omega_c$ and the eigenvector q^* of A^* belonging to the eigenvalue $-i\omega_c$ as following.

$$q(\theta) = (\alpha \quad \beta \quad 1)^T e^{i\omega_c \theta}$$

is the eigenvector of A corresponding to $i\omega_c$, and

$$q^*(s) = \bar{K}(\alpha_1 \quad \beta_1 \quad 1)^T e^{i\omega_c s}$$

is the eigenvector of A^* corresponding to $-i\omega_c$, where \bar{s} denotes the conjugation of s , and

$$\begin{aligned} \alpha &= \frac{e^{i\omega_c \tau_c}(ic_{10} + \omega_c)}{-c_{01}(ia_{10} + \omega_c)}, \beta = \frac{i\omega_c - c_{10}}{c_{01}e^{-i\omega_c \tau_c}}, \alpha_1 = \frac{b_{01}(ic_{10} - \omega_c)}{e^{-i\omega_c \tau_c}\omega_c - ia_{10}}, \beta_1 = c_{10} + i\omega_c, K \\ &= (1 + \alpha\bar{\alpha}_1 + \beta\bar{\beta}_1 + b_{01}e^{-i\omega_c \tau_1}\alpha\bar{\beta}_1\tau_1 + c_{01}e^{-i\omega_c \tau_c}\beta\tau_c)^{-1}. \end{aligned}$$

Moreover, $\langle q^*, q \rangle = 1$ and $\langle q^*, \bar{q} \rangle = 0$.

Following the computation process introduced by Hassard et al. [27] and using the symbolic algorithms similar to that in [28], one obtains the following formulas determined the features of the bifurcating periodic solution. The expressions details of the symbolic functions are provided in Appendix.

$$C_1(0) = \frac{i}{2\omega_c} \left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right) + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{\text{Re}[C_1(0)]}{\text{Re}[\lambda'(\tau_2)]}, \quad \beta_2 = 2\text{Re}[C_1(0)],$$

which determine the quantities of bifurcating periodic solutions reduced on the central manifold at the critical value τ_c . More precisely, μ_2 determines the direction of Hopf bifurcation and β_2 determines the stability of bifurcating periodic solution. Therefore, summarizing the above discussion, we have the following main result.

Theorem 6. If $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau_2 > \tau_c$ ($\tau_2 < \tau_c$). Moreover, β_2 determines the stability of the bifurcating periodic solutions. The bifurcating periodic solutions are stable (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$).

Example 3. We choose the parameter values of system (2.2) as $a_1 = 0.5$, $a_2 = 0.68$, $a_3 = 0.2$, $a_4 = 0.56$, and $\tau_1 = 0.5$ satisfied with $0 < a_i < 1$, $i = 1, \dots, 4$, which implies that the intrinsic growth rate of resource is greater than the consumer and predator. At this time, the interior equilibrium point of system model (2.2) is (0.789916, 0.165949, 0.0592674). Due to the present of time delays, the system dynamic exhibits the recurrent bloom of species population by the Hopf bifurcation. It follows from Eq. (4.1) that the characteristic equation (3.5) has a pair of purely imaginary roots with $\omega_c = 0.204022$. The critical value of delay is $\tau_c = 0.132741$. The formulas determined the quantities of bifurcating periodic solution are respectively $C_1(0) = -5.71133 - 141.686i$, $\mu_2 = 150.992$ and $\beta_2 = -11.4227$, which mean that the Hopf bifurcation is supercritical and the stability of bifurcating periodic solutions is stable. Fig. 5(a) and (b) show the phase diagram and time history of system (2.2) for the delay $\tau_2 = 0.1 < \tau_c$. It follows that a stable spiral converges to the interior equilibrium point (0.789916, 0.165949, 0.0592674), which is locally asymptotically stable. The species population will evolve into the stable coexistence.

Keeping the above-mentioned parameters fixed, and taking time delay $\tau_2 = 0.35 > \tau_c$, one obtains that the interior equilibrium point is unstable. As a matter of fact, the maximum eigenvalues with the negative real parts for the delay $\tau_2 = 0.1$ pass through the imaginary axis and go into the right-half plane when time delay increases from 0.1 to 0.35. There is a bifurcating periodic solution near the equilibrium point $E^c : (x^c, y^c, z^c)$. The system (2.2) has a stable periodic solution, which is shown in Fig. 6(a) and (b). The food chain system (2.2) presents the recurrent bloom of species population due to the presence of multiple delays.

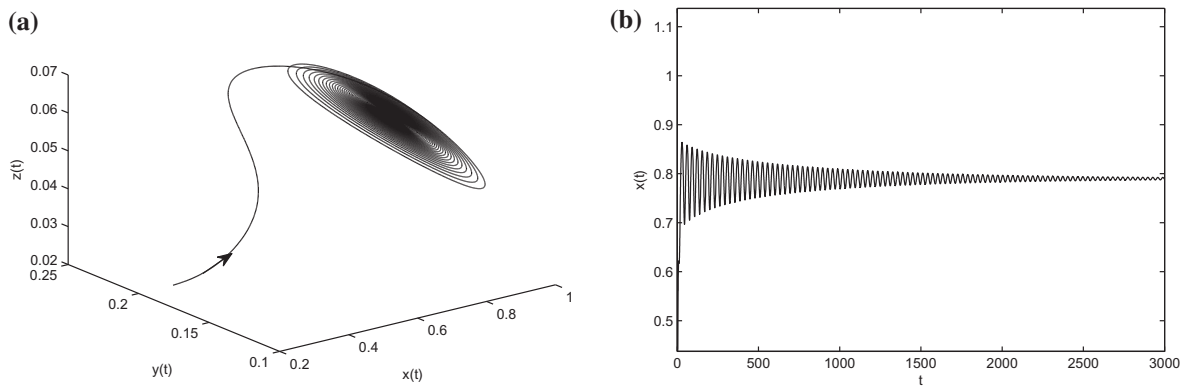


Fig. 5. (a) Phase diagram and (b) time history for the stable equilibrium point of system (2.2) with time delay $\tau_2 = 0.1 < \tau_c$.

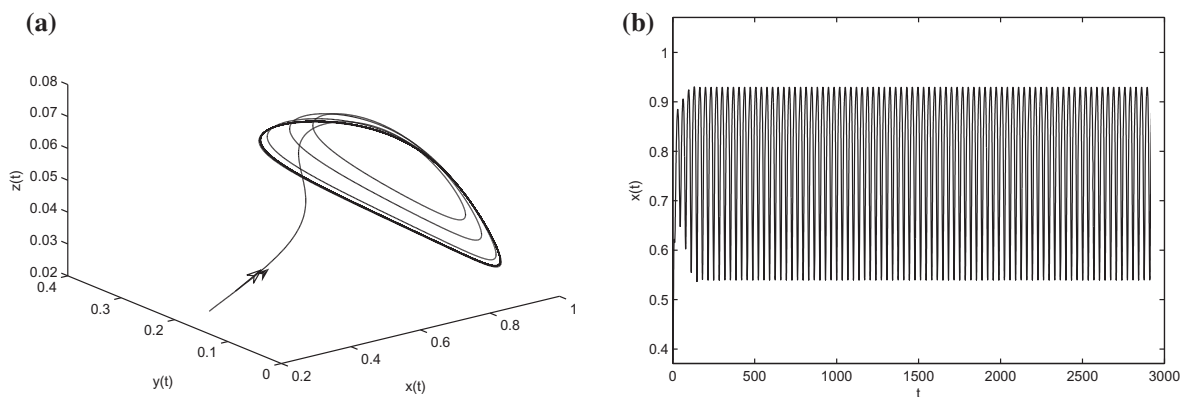


Fig. 6. (a) Phase diagram and (b) time history for the stable periodic solution of system (2.2) with time delay $\tau_2 = 0.35 > \tau_c$.

6. Chaos behavior and the period-doubling bifurcation

As seen above, the stability of the boundary and interior equilibrium points, the existence of the Hopf bifurcation, and the stability of the bifurcating periodic solutions have been studied in the previous discussion, respectively. Time delays may create the stable periodic solution near the interior equilibrium point occurred in the system under consideration. Thus it raises a question: Is the periodic solution persistent for the delay? If a stable periodic solution loses the stability when τ_2 is increased, where does the periodic solution evolve? Therefore, in this section, we perform some numerical simulations to illustrate the dynamical subsequent evolutions for system (2.2). The complex dynamical behaviors including the multiple periodic solution and chaotic motion are observed for the different values of system parameters. System (2.2) leads to the chaos behavior by employing the period-doubling bifurcation.

In what follows, the parameters are fixed as $a_1 = 0.5$, $a_2 = 0.68$, $a_3 = 0.2$, $a_4 = 0.56$, whereas delays τ_1 and τ_2 are considered as the variable parameters. It follows from the fifth section that system (2.2) presents the stable periodic solution for $\tau_1 = 0.5$ and $\tau_2 > \tau_c = 0.132741$. We will investigate the evolution of this periodic attractor with τ_2 increasing. The initial conditions are given by $(x_1(t), x_2(t), x_3(t)) = (0.8, 0.2, 0.06)$ for $-\tau < t \leq 0$, where $\tau = \max(\tau_1, \tau_2)$. In order to locate the multiple periodic solution and chaotic motion, the Poincaré section $\dot{x}_2(t - \tau_1) = 0$ is constructed. If the steady state of system (2.2) is the periodic or multi-periodic solution, there are only one or multiple points in the Poincaré section. However, for a chaotic response, the number of points becomes infinite. The irregular pattern in Poincaré section indicates the existence of a strange attractor.

The detailed Poincaré diagrams in the xy -plane with delay τ_2 increasing are shown in Fig. 7 by using Poincaré section techniques when $\tau_1 = 0.5$ is fixed. The unique point (0.929, 0.0647) appears in Fig. 7(a), which confirms the analytical prediction discussed in the fifth section, where a stable period-1 solution emerges at $\tau_2 = 0.3$. With delay τ_2 increasing to $\tau_2 = 0.4$, the Poincaré diagram exhibits two points (0.932, 0.0622) and (0.942, 0.0537), as shown in Fig. 7(b). System (2.2) presents a period-2 solution. A slight increase of delay τ_2 causes the period-2 motion to lose its stability and give rise to

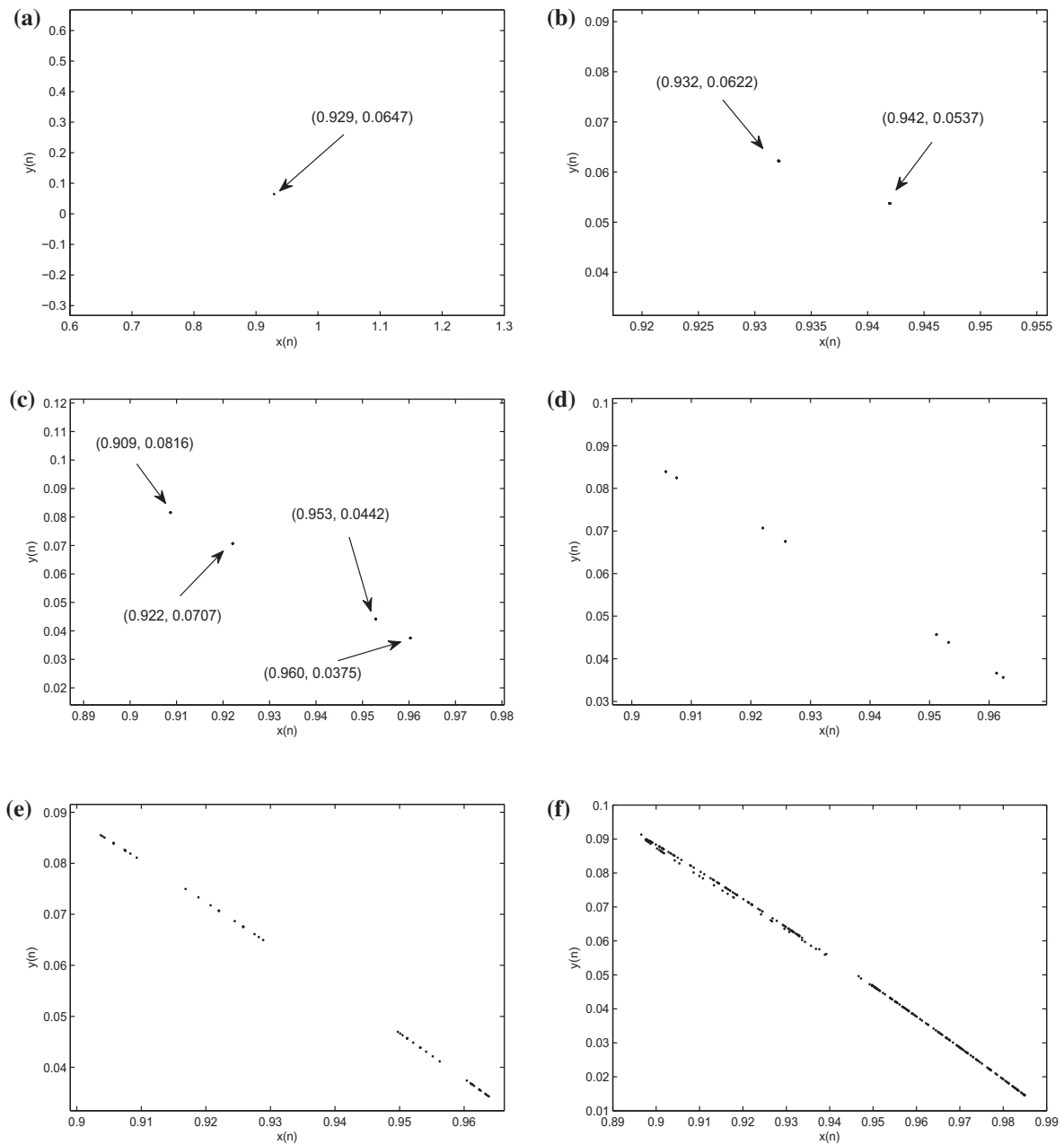


Fig. 7. Poincaré diagrams of system (2.2) in the xy -plane showing the route to chaos with an increase of τ_2 : (a) $\tau_2 = 0.3$ (period-1 attractor), (b) $\tau_2 = 0.4$ (period-2 attractor), (c) $\tau_2 = 0.43$ (period-4 attractor), (d) $\tau_2 = 0.433$ (period-8 attractor), (e) $\tau_2 = 0.435$ (multi-period attractor), (f) $\tau_2 = 0.45$ (chaotic strange attractor).

a period-4 solution, where system (2.2) converges to four points in the Poincaré section (see Fig. 7(c)). When $\tau_2 = 0.433$, the Poincaré diagram with eight different points appears (see Fig. 7(d)). System (2.2) has a period-8 motion. With delay τ_2 increasing, the system dynamic behaviors lose their stability and give rise to a double periodic solution. Fig. 7(e) illustrates the multi-period solution. It follows from Fig. 7(f) that system (2.2) presents a chaotic motion for delay $\tau_2 = 0.45$, which is an irregular pattern in the Poincaré diagram. The bifurcation diagrams as the function of delay τ_2 are shown in Fig. 8 for the fixed delay $\tau_1 = 0.5$ by using the Poincaré section techniques. It follows that the system exhibits period-1, period-2, period-4, period-8, \dots , and chaos solution for the different values of time delays. It is the sequence of period-doubling bifurcations that leads eventually to chaos motion as τ_2 increases.

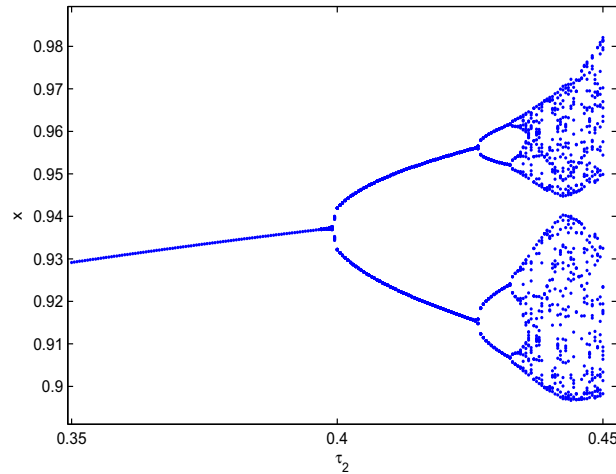


Fig. 8. Bifurcation diagram as a function of the delay τ_2 varied in the range $[0.35, 0.45]$, keeping the other parameters fixed as $a_1 = 0.5$, $a_2 = 0.68$, $a_3 = 0.2$, $a_4 = 0.56$ for the delay value $\tau_1 = 0.5$.

7. Conclusion and discussion

In the last decades, the food chain system with two species, which is also called prey-predator, plant-herbivore or plant-pest system has been studied extensively due to the analytic simplicity. However, the interaction relationship among only two species is uncommonly observed in real ecological system. The food chains composed of three species are often emerging in the different branches of biological system. As matter of fact, the plant-herbivore-parasitoid food chain is extremely important in many field situations. It has been shown that parasitoids may determine the plant fitness by destroying herbivores [29]. In addition, three nutrition relationships between plants, plant-eating insects and corresponding natural enemies have been attracted extensive attention. For example, Looper Caterpillar feeding on tea plant and *Sarcophaga* sp. (natural enemy of Looper Caterpillar) can build a three-species plant-pest-predator ecological system [30]. *Arabidopsis thaliana* plants, *Pieris rapae* caterpillars, and the solitary endoparasitoid *Cotesia rubecula* also form a tritrophic system [31]. In waste treatment process, food chains of waste/nutrient-bacteria-ciliates had got the attention of some scientists [32].

Based on the consideration of digestion time [33], the Leslie–Gower predator–prey systems with time delays were presented in recent years. The local and global existences of periodic solutions were investigated by using the center manifold and the global Hopf bifurcation [34,35]. Two or more delays are often incorporated into the population models for resource regeneration time, maturing time, gestation period, or feedback delay [36,37]. In this paper, the Leslie–Gower type of food chain system with digestion delays is presented, which is consumer–eat–resource and predator–eat–consumer, respectively. The complex dynamical behaviors are illustrated, such as the stable coexistence of species population, the recurrent bloom of species population induced by the digestion delay, and the chaos response obtained by the period-doubling bifurcation. Furthermore, present of multiple delays can control the species population into the stable coexistence. Time delays are helpful to control the species populations into the stable coexistence of corresponding steady state. Some ecological behaviors of the material interchange processing system may be a direct extension of those obtained from studies on the present model.

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Appendix A. The expression of the Hopf bifurcation normal form

$$\begin{aligned} g_{20} &= K(-c_{20} + c_{11}e^{-i\omega_c\tau_c}\beta - c_{02}e^{-2i\omega_c\tau_c}\beta^2 - \alpha^2\bar{\alpha}_1 - b_{02}e^{-2i\omega_c\tau_1}\alpha^2\bar{\beta}_1 + b_{11}e^{-i\omega_c\tau_1}\alpha\bar{\beta}\bar{\beta}_1 - b_{20}\beta^2\bar{\beta}_1), \\ g_{11} &= K(-2c_{20} + c_{11}e^{-i\omega_c\tau_c}\beta - 2\alpha\bar{\alpha}_1 + c_{11}e^{i\omega_c\tau_c}\bar{\beta} - 2c_{02}\beta\bar{\beta} - 2b_{02}\alpha\bar{\alpha}\bar{\beta}_1 + b_{11}e^{i\omega_c\tau_1}\bar{\alpha}\bar{\beta}\bar{\beta}_1 + b_{11}e^{-i\omega_c\tau_1}\alpha\bar{\beta}\bar{\beta}_1 - 2b_{20}\beta\bar{\beta}\bar{\beta}_1), \\ g_{02} &= K(-c_{20} - \bar{\alpha}^2\bar{\alpha}_1 + c_{11}e^{i\omega_c\tau_c}\bar{\beta} - c_{02}e^{2i\omega_c\tau_c}\bar{\beta}^2 - b_{02}e^{2i\omega_c\tau_1}\bar{\alpha}^2\bar{\beta}_1 + b_{11}e^{i\omega_c\tau_1}\bar{\alpha}\bar{\beta}\bar{\beta}_1 - b_{20}\bar{\beta}^2\bar{\beta}_1), \\ g_{21} &= \frac{K}{6\omega_c}(M_1e^{2i\omega_c\tau_c} + M_2e^{i\omega_c\tau_c} + M_3e^{-i\omega_c\tau_c} + M_4e^{-2i\omega_c\tau_c} + N_1e^{2i\omega_c\tau_1} + N_2e^{i\omega_c\tau_1} + N_3e^{-i\omega_c\tau_1} + N_4e^{-2i\omega_c\tau_1} + N_5), \end{aligned}$$

$$\begin{aligned}
M_1 &= -2ic_{02}\bar{\beta}^2\bar{g}_{02}, M_2 = 2ic_{11}\bar{\beta}\bar{g}_{02} + 6ic_{11}\bar{\beta}\bar{g}_{11} + 3ic_{11}\bar{\beta}g_{20} + 6c_{21}\bar{\beta}\omega_c + 3c_{11}\bar{\beta}E_{13}\omega_c, \\
M_3 &= 6ic_{11}\bar{\beta}\bar{g}_{11} - 12ic_{11}\bar{\beta}g_{11} + 3ic_{11}\bar{\beta}g_{20} + 12c_{21}\bar{\beta}\omega_c - 6c_{02}\bar{\beta}E_{12}\omega_c - 12c_{02}\bar{\beta}E_{22}\omega_c + 6c_{11}\bar{\beta}E_{23}\omega_c, \\
M_4 &= 12ic_{02}\bar{\beta}^2g_{11} - 6c_{12}\bar{\beta}^2\omega_c + 3c_{11}E_{12}\omega_c, \\
N_1 &= -2ib_{02}\bar{\alpha}^2\bar{\beta}_1\bar{g}_{02}, N_2 = 2ib_{11}\bar{\alpha}\bar{\beta}\bar{\beta}_1\bar{g}_{02} + 6ib_{11}\bar{\alpha}\bar{\beta}\bar{\beta}_1\bar{g}_{11} + 3ib_{11}\bar{\alpha}\bar{\beta}\bar{\beta}_1g_{20} + 6b_{21}\bar{\beta}^2\bar{\alpha}\bar{\beta}_1\omega_c + 3b_{11}\bar{\alpha}\bar{\beta}_1E_{12}\omega_c, \\
N_3 &= 6ib_{11}\bar{\alpha}\bar{\beta}\bar{\beta}_1\bar{g}_{11} - 12ib_{11}\bar{\alpha}\bar{\beta}\bar{\beta}_1g_{11} + 3ib_{11}\bar{\alpha}\bar{\beta}\bar{\beta}_1g_{20} + 18c_{03}\bar{\beta}^2\bar{\beta}\omega_c + 18b_{03}\bar{\alpha}^2\bar{\alpha}\bar{\beta}_1\omega_c \\
&\quad + 12b_{21}\bar{\alpha}\bar{\beta}\bar{\beta}_1\omega_c - 6b_{02}\bar{\alpha}\bar{\beta}_1E_{11}\omega_c - 12b_{02}\bar{\alpha}\bar{\beta}_1E_{21}\omega_c + 6b_{11}\bar{\alpha}\bar{\beta}_1E_{22}\omega_c, \\
N_4 &= 12ib_{02}\bar{\alpha}^2\bar{\beta}_1g_{11} - 6b_{12}\bar{\alpha}^2\bar{\beta}\bar{\beta}_1\omega_c + 3b_{11}\bar{\beta}\bar{\beta}_1E_{11}\omega_c, \\
N_5 &= -2ic_{20}\bar{g}_{02} - 2i\bar{\alpha}^2\bar{\alpha}_1\bar{g}_{02} - 2ib_{20}\bar{\beta}^2\bar{\beta}_1\bar{g}_{02} - 12ic_{20}\bar{g}_{11} - 12i\bar{\alpha}\bar{\alpha}_1\bar{g}_{11} - 12ic_{02}\bar{\beta}\bar{\beta}\bar{g}_{11} - 12ib_{02}\bar{\alpha}\bar{\alpha}\bar{\beta}_1\bar{g}_{11} \\
&\quad - 12ib_{20}\bar{\beta}\bar{\beta}\bar{\beta}_1\bar{g}_{11} + 12ic_{20}\bar{g}_{11} + 12i\bar{\alpha}^2\bar{\alpha}_1g_{11} + 12ib_{20}\bar{\beta}^2\bar{\beta}_1g_{11} - 6ic_{20}\bar{g}_{20} - 6i\bar{\alpha}\bar{\alpha}_1\bar{g}_{20} \\
&\quad - 6ic_{02}\bar{\beta}\bar{\beta}_1g_{20} - 6ib_{02}\bar{\alpha}\bar{\alpha}\bar{\beta}_1g_{20} - 6ib_{20}\bar{\beta}\bar{\beta}\bar{\beta}_1g_{20} - 12c_{12}\bar{\beta}\bar{\beta}\omega_c - 12\bar{\alpha}\bar{\beta}\bar{\alpha}\bar{\beta}_1\omega_c - 6\bar{\alpha}\bar{\alpha}_1E_{11}\omega_c \\
&\quad - 6b_{20}\bar{\beta}\bar{\beta}_1E_{12}\omega_c - 6c_{20}E_{13}\omega_c - 12\bar{\alpha}\bar{\alpha}_1E_{21}\omega_c + 6b_{11}\bar{\beta}\bar{\beta}_1E_{21}\omega_c + 6c_{11}E_{22}\omega_c - 12b_{20}\bar{\beta}\bar{\beta}_1E_{22}\omega_c - 12c_{20}E_{23}\omega_c,
\end{aligned}$$

$$\begin{aligned}
\begin{pmatrix} E_{11} \\ E_{12} \\ E_{13} \end{pmatrix} &= \begin{pmatrix} 2i\omega_c - a_{10} & 1 & 0 \\ -b_{01}e^{-2i\omega_c\tau_1} & 2i\omega_c - b_{10} & 1 \\ 0 & -c_{01}e^{-2i\omega_c\tau_c} & 2i\omega_c - c_{10} \end{pmatrix}^{-1} \begin{pmatrix} -\alpha^2 \\ -b_{02}e^{-2i\omega_c\tau_1}\alpha^2 + b_{11}e^{-i\omega_c\tau_1}\alpha\bar{\beta} - b_{20}\bar{\beta}^2 \\ -c_{20} + c_{11}e^{-i\omega_c\tau_c}\bar{\beta} - c_{02}e^{-2i\omega_c\tau_c}\bar{\beta}^2 \end{pmatrix}, \\
\begin{pmatrix} E_{21} \\ E_{22} \\ E_{23} \end{pmatrix} &= \begin{pmatrix} -a_{10} & 1 & 0 \\ -b_{01} & -b_{10} & 1 \\ 0 & -c_{01} & -c_{10} \end{pmatrix}^{-1} \begin{pmatrix} -2\alpha\bar{\alpha} \\ -2b_{02}\alpha\bar{\alpha} + b_{11}e^{i\omega_c\tau_1}\bar{\alpha}\bar{\beta} + b_{11}e^{-i\omega_c\tau_1}\alpha\bar{\beta} - 2b_{20}\bar{\beta}\bar{\beta} \\ -2c_{20} + c_{11}e^{-i\omega_c\tau_c}\bar{\beta} + c_{11}e^{i\omega_c\tau_c}\bar{\beta} - 2c_{02}\bar{\beta}\bar{\beta} \end{pmatrix}.
\end{aligned}$$

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