System of linear equations and inverse matrix

Wu Yuhan

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1 Introduction

The linear system equations can be written into a matrix form. By linearity, the solutions satisfy the superposition principle. Therefore, to solve the equation $\mathbf{A}x = b$,where $\mathbf{A}$ is a n-order matrix, we can solve the equation $\mathbf{A}x = \epsilon_i$,(where ϵ_i are base vectors), then apply the superposition principle to obtain the final solution. In this process, we can find the inverse of the matrix $\mathbf{A}$, if $\det \mathbf{A} \neq 0$. Finally, we can derive the Crammer's rule and show that the equation $\mathbf{A}x = b$ has a unique solution iff $\det \mathbf{A} \neq 0$.

2 Superposition principle of the solutions

Observe the equation, $\mathbf{A}x = b$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \tag{1}$$

Write it in matrix form:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Superposition principle

By the linearity of matrix $\mathbf{A}$, we can easily prove the following:
If $\mathbf{A}x_1 = b_1$ and $\mathbf{A}x_2 = b_2$, then

$$\mathbf{A}(x_1 + x_2) = \mathbf{A}x_1 + \mathbf{A}x_2 = b_1 + b_2 \tag{2}$$

$$\mathbf{A}(kx_1) = k\mathbf{A}x_1 = kb_1 \tag{3}$$

Then for any $b = \sum b_i \epsilon_i \in \mathbb{R}^n$, where $\{\epsilon_i\}$ is the basis of $\mathbb{R}^n$, fixed $\mathbf{A}$, we only need to firstly solve the equation $\mathbf{A}x = \epsilon_i$ respectively and get the solution x_i. Then apply the superposition principle and get the solution. For example, $\mathbf{A}x = \epsilon_1$

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{4}$$

Solve this equation and we can get a solution

$$x_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}$$

Similarly, apply the same procedure to $\epsilon_2, \epsilon_3 \cdots, \epsilon_n$, we can yield

$$x_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix} \quad x_3 = \begin{pmatrix} x_{13} \\ x_{23} \\ \vdots \\ x_{n3} \end{pmatrix} \cdots x_n = \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}$$

As we know $b = \sum b_i \epsilon_i$, we can obtain the solution $x = \sum b_i x_i$. This can be easily verified by the linearity of $\mathbf{A}$:

$$\mathbf{A}(x) = \mathbf{A}(\sum b_i x_i) = \sum b_i \mathbf{A}(x_i) = \sum b_i \epsilon_i = b \tag{5}$$

Comment: if $\mathbf{A}$ remain unchanged and change b into $b' = \sum b'_i \epsilon_i$, we can get the solution immediately by $x' = \sum b'_i x_i$. The good thing is that we don't have to solve the equation again. This is extremely useful in numerical computation.

3 Inverse matrix of A

From the previous section, we have derived that $\mathbf{A}x_1 = \epsilon_1$, $\mathbf{A}x_2 = \epsilon_2 \cdots \mathbf{A}x_n = \epsilon_n$.

We combine the column vectors $x_1, x_2 \cdots x_n$ into a matrix:

$$(x_1, x_2, \cdots x_n) = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n_2} & \cdots & x_{nn} \end{pmatrix} \quad \text{note as}\mathbf{X}$$

Here we can easily verify that $\mathbf{X}$ is the inverse matrix of$\mathbf{A}$.

$$\begin{aligned} \mathbf{AX} &= \mathbf{A}(x_1, x_2, \cdots, x_n) \\ &= (\mathbf{A}x_1, \mathbf{A}x_2, \cdots, \mathbf{A}x_n) \\ &= (\epsilon_1, \epsilon_2, \cdots, \epsilon_n) \\ &= \mathbf{I} \end{aligned}$$

However, we have missed a big problem. We didn't prove the existence of solution x_i. Now our next job is to investigate it. Before that we need some preparation.

3.1 Determinant

Firstly, the determinant of matrix $\mathbf{A}$ can be expanded by row or column elements for example

$$\det \mathbf{A} = \sum a_{1i} A_{1i}$$

and A_{ij} is called the cofactor of a_{ij}, which refers to the determinant of the matrix that $\mathbf{A}$'s ith row elements and jth column elements are eliminated.

Another important property of determinant is that if any two of the column vectors or row vectors are same, then the determinant is zero.

3.2 Formula of solution and inverse matrix

If readers have a good foundation in mathematical derivation, you can use Gauss elimination method and apply the property of determinant to obtain the result below. But now, We assert that

$$x_1 = \frac{1}{\det \mathbf{A}} \begin{pmatrix} A_{11} \\ A_{12} \\ \vdots \\ A_1 n \end{pmatrix} \tag{6}$$

is the solution of $Ax = \epsilon_1$, where A_{ij} is the cofactor of a_{ij}. Let's check it.

$$\mathbf{A}X_1 = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \cdot \frac{1}{\det \mathbf{A}} \begin{pmatrix} A_{11} \\ \vdots \\ A_{1n} \end{pmatrix} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} \sum a_{1i}A_{1i} \\ \vdots \\ \sum a_{ni}A_{1i} \end{pmatrix}$$

Notice that:

$$\sum_i a_{1i}A_{1i} = \det \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \det \mathbf{A} \tag{7}$$

and

$$\sum_i a_{ki}A_{1i} = \det \begin{pmatrix} a_{k1} & a_{k2} & \cdots & a_{kn} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = 0 \tag{8}$$

because the 1th and kth row elements are same.

Therefore

$$\mathbf{A}x_1 = \frac{1}{\det A} \begin{pmatrix} \det \mathbf{A} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \epsilon_1 \tag{9}$$

We can check $\mathbf{A}x_2, \mathbf{A}x_3, \cdots, \mathbf{A}x_n$ in the same way. By this, we proved that

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}^{-1} = (x_1, x_2, \cdots, x_n) = \frac{1}{\det \mathbf{A}} = \begin{pmatrix} A_{11} & \cdots & A_{n1} \\ \vdots & & \vdots \\ A_{1n} & \cdots & A_{nn} \end{pmatrix} \tag{10}$$

So far, we have derived the formula of inverse matrix if $\mathbf{A}$ is invertible, i.e. $\det \mathbf{A} \neq 0$. In my opinion, if you are required to find the inverse matrix of a known matrix, I recommend you to calculate it by solving the linear function instead of using the inverse matrix formula. There are two reasons. Firstly, they are actually in the same complexity. Second, you can easily check your results step by step. It is very likely to make mistakes when you are using formula to calculate, and you will find it difficult to debug it.

4 Crammer rule

With the inverse matrix formula, it is an easy job to prove the Crammer rule. Equation $\mathbf{A}x = b$,

$$x = \mathbf{A}^{-1}b = \frac{1}{\det \mathbf{A}} \begin{pmatrix} A_{11} & \cdots & A_{n1} \\ \vdots & & \vdots \\ A_{1n} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \frac{1}{\det \mathbf{A}} \begin{pmatrix} \sum_i A_{1i}b_i \\ \vdots \\ \sum_i A_{ni}b_i \end{pmatrix} \tag{11}$$

Observe that

$$\sum_i A_{1i}b_i = \det \begin{pmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{pmatrix} \tag{12}$$

and

$$\sum_i A_{ki}b_i = \det \begin{pmatrix} a_{11} & \cdots & a_{1,k-1} & b_1 & a_{1,k+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,k-1} & b_2 & a_{2,k+1} & \cdots & a_{2n} \\ \vdots & & & \vdots & & & \vdots \\ a_{n1} & \cdots & a_{n,k-1} & b_n & a_{n,k+1} & \cdots & a_{nn} \end{pmatrix} \tag{13}$$