Dirac equation:A tensor product equation

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1 Introduction

This article is aimed to use tensor product to represent the Dirac equation, then I will show a detailed solution of the free particle. When I first learn Dirac equation, I am quite confused about the notation, such as $\alpha \cdot \mathbf{p}$, and how do spinor operator α interact with momentum operator $\mathbf{p}$? By using tensor product representation, we can avoid this ambiguity, and we can clear see how these operators act on wave functions. In this way, the operator $\alpha \cdot p$ should be view as tensor product of two operator

$$\alpha \cdot \mathbf{p} = \alpha_1 \otimes p_1 + \alpha_2 \otimes p_2 + \alpha_3 \otimes p_3$$

In this article, if we use Dirac representation, the α_i can be represented by $\sigma_1 \otimes \sigma_i$, and the wave function can be represented by

$$\psi = \chi_1 \otimes \chi_2 \otimes \phi_p$$

To read this article, the reader should be familiar with tensor product and eigenvalue problem. If you are unfamiliar with these topics, I highly recommend you to read my another article *Construction of tensor product* and *Linear equations and eigenvalue problems* first.

2 Derivation of Dirac equation

The relativistic energy $E = p^2c^2 + m_0^2c^4$, so we assume the square of Hamiltonian operator

$$\mathcal{H}^2 = c^2\mathbf{p}^2 + m_0^2c^4 \tag{1}$$

To solve $\mathcal{H}$, we assume

$$\mathcal{H} = c\alpha \cdot \mathbf{p} + \beta m_0 c^2 \tag{2}$$

Therefore,

$$\begin{aligned}\mathcal{H}^2 = (c\alpha \cdot \mathbf{p})^2 &= c^2[\alpha_1^2p_1^2 + \alpha_2^2p_2^2 + \alpha_3^2p_3^2 + (\alpha_1\alpha_2 + \alpha_2\alpha_1)p_1p_2 + (\alpha_1\alpha_3 + \alpha_3\alpha_1)p_1p_3 + (\alpha_2\alpha_3 + \alpha_3\alpha_2)p_2p_3] \\ &+ m_0c^3[(\alpha_1\beta + \beta\alpha_1)p_1 + (\alpha_2\beta + \beta\alpha_2)p_2 + (\alpha_3\beta + \beta\alpha_3)p_3] + \beta^2 m_0 c^4 \\ &= c^2\mathbf{p}^2 + m_0^2c^4\end{aligned}$$

Compare the coefficients, then we have,

1. $\alpha_i\alpha_j + \alpha_j\alpha_i = 0, i \neq j$
2. $\alpha_i\beta + \beta\alpha_i\beta = 0$,i=1,2,3
3. $\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \beta^2$

If we use Dirac representation:

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} I & \\ & -I \end{pmatrix} \tag{3}$$

where σ_i is Pauli matrix and I is identity matrix. We can easily check the property above by matrix multiplication. Before continuing to next step, let's review some property of tensor product.

3 Some properties of tensor product

Assume that $\mathbf{V}_1$ and $\mathbf{V}_2$ are two linear spaces with bases $\{\alpha_i\}$,$i = 1, 2, \cdot, n$ and $\{\beta_j\}, j = 1, \cdot, m$. $\mathcal{A}$ is a linear operator on $\mathbf{V}_1$, with matrix representation $\mathbf{A} = (a_{ij})_{n\times n}$, which means $\mathcal{A}\alpha_i = \sum_k a_{ki}\alpha_k$.
$\mathcal{B}$ is a linear operator on $\mathbf{V}_2$ with matrix representation $\mathbf{B} = (b_ij)_{m\times m}$, which means $\mathcal{B}\beta_j = \sum_l b_{lj}\beta_l$

We can mix two vector space by tensor product, $\mathbf{V}_1 \otimes \mathbf{V}_2$ with basis $\alpha_i \otimes \beta_j, i = 1, \cdots, n$ and $j = 1, \cdots, m$. $\mathcal{A} \otimes \mathcal{B}$ is a linear operator on $V_1 \otimes V_2$.

$$\begin{aligned} \mathcal{A} \otimes \mathcal{B}(\alpha_i \otimes \beta_j) &= \mathcal{A}\alpha_i \otimes \mathcal{B}\beta_j \\ &= \sum_k a_{ki}\alpha_k \otimes \sum_l b_l j \beta_l \\ &= \sum_{kl} a_{ki} b_{lj} \alpha_k \otimes \beta_l \end{aligned}$$

If we rearrange the basis, $\alpha_1 \otimes \beta_1, \alpha_1 \otimes \beta_2, \cdots, \alpha_1 \otimes \beta_m$,$\alpha_2 \otimes \beta_1, \alpha_2 \otimes \beta_2, \cdots, \alpha_2 \otimes \beta_m$,$cdots\ \alpha_n \otimes \beta_1, \alpha_n \otimes \beta_2, \cdots, \alpha_n \otimes \beta_m$.

Then we can find the matrix representation of $\mathcal{A} \otimes \mathcal{B}$ is

$$\begin{pmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & & \vdots \\ a_{n1}\mathbf{B} & a_{n2}\mathbf{B} & \cdots & a_{nn}\mathbf{B} \end{pmatrix}$$

4 Tensor product representation

4.1 Motivation

Let's have a look on α_i

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \sigma_i = \sigma_1 \otimes \sigma_i$$

and β

$$\beta = \begin{pmatrix} I & \\ & -I \end{pmatrix} = \sigma_3 \otimes I$$

4.2 Pauli matrix

Before we check the availability of this representation, let's review the property of Pauli matrix.

$$\sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

It is easily to verify

1. $\sigma_1^2 = \sigma_2^2\sigma_3^2 = I$
2. $\sigma_i\sigma_j + \sigma_j\sigma_i = 0$, if $i \neq j$
3. $\sigma_i\sigma_j = \sum_k \epsilon_{ijk}\sigma_k$, if $i \neq j$, ϵ_{ijk} is Levi-Civita tensor.

4.3 Check

$$\alpha_i^2 = (\sigma_1 \otimes \sigma_i)(\sigma_1 \otimes \sigma_i) = \sigma_1\sigma_1 \otimes \sigma_i\sigma_i = I \otimes I$$
$$\beta^2 = (\sigma_3 \otimes I)(\sigma_3 \otimes I) = \sigma_3^2 \otimes I^2 = I \otimes I$$

For $i \neq j$

$$\begin{aligned}\{\alpha_i, \alpha_j\} = \alpha_i\alpha_j + \alpha_j\alpha_i &= (\sigma_1 \otimes \sigma_i)(\sigma_1 \otimes \sigma_j) + (\sigma_1 \otimes \sigma_j)(\sigma_1 \otimes \sigma_i) \\ &= \sigma_1^2 \otimes \sigma_i\sigma_j + \sigma_1^2 \otimes \sigma_j\sigma_i \\ &= I \otimes \sigma_i\sigma_j + \sigma_j\sigma_i \\ &= I \otimes 0 = 0\end{aligned}$$

Then

$$\begin{aligned}\{\alpha_i, \beta\} = \alpha_i\beta + \beta\alpha_i &= (\sigma_1 \otimes \sigma_i)(\sigma_3 \otimes I) + (\sigma_3 \otimes I)(\sigma_1 \otimes \sigma_i) \\ &= \sigma_1\sigma_3 \otimes \sigma_i I + \sigma_3\sigma_1 \otimes I\sigma_i \\ &= (\sigma_1\sigma_3 + \sigma_3\sigma_1) \otimes \sigma_i \\ &= 0 \otimes \sigma_i = 0\end{aligned}$$

Therefore, the tensor product representation is well defined.

5 Solve free Dirac equation

The Hamiltonian $\mathcal{H} = c\alpha \cdot \mathbf{p} + \beta m_0 c^2$

5.1 Wave function space

To solve the Dirac equation, we firstly need to study the space of wave function.

First of all, we analyze the operator α_i,

$$\alpha_i = \sigma_1 \otimes \sigma_i$$

so the correspond vector space is the tensor product of two spin space $\mathbf{V}_{s1} \otimes \mathbf{V}_{s2}$, where $\mathbf{V}_{s1}$ and $\mathbf{V}_{s2}$ are two 2-dimensional vector space, with bases $\{\chi_{1+}, \chi_{1-}\}$ and $\{\chi_{2+}, \chi_{2-}\}$.

Then another important step is that we should combine the spin wave function space and momentum wave function space, $\mathbf{V}_p$, by tensor product.

$$\mathbf{V}_\psi = \mathbf{V}_{s1} \otimes \mathbf{V}_{s2} \otimes \mathbf{V}_p \tag{4}$$

Therefore, the wave function should be written in form

$$\psi = \chi_1 \otimes \chi_2 \otimes \phi_p \tag{5}$$

Here we use index 1,2 to distinguish two spin space wave function. So, α_i should be viewed as $\sigma_1 \otimes \sigma_i \otimes I_p$, p_i should be viewed as $I_1 \otimes I_2 \otimes p_i$. Therefore,

$$\alpha_i p_i = \sigma_1 \otimes \sigma_i \otimes p_i \tag{6}$$

and

$$\beta = \sigma_3 \otimes I_2 \otimes I_p \tag{7}$$

The Hamiltonian can be written as,

$$\begin{aligned}\mathcal{H} &= c\alpha \cdot \mathbf{p} + \beta m_0 c^2 \\ &= c(\sigma_1 \otimes \sigma_1 \otimes p_1 + \sigma_1 \otimes \sigma_2 \otimes p_2 + \sigma_1 \otimes \sigma_3 \otimes p_3) + m_0 c^2 \sigma_3 \otimes I_2 \otimes I_p\end{aligned}$$

5.2 The base to wave function space

To simplify the problem, we assume the momentum $\mathbf{p}$ is fixed, then we have the momentum eigenfunction ϕ_p. So, the wave function space V_ψ will be reduced to $\mathbf{V}_{s1} \otimes \mathbf{V}_{s2} \otimes span\{\phi_p\}$,

$$Dim\mathbf{V}_\psi = Dim\mathbf{V}_{s1} \cdot Dim\mathbf{V}_{s2} \cdot Dim\{\phi_p\} = 2 \cdot 2 \cdot 1 = 4 \tag{8}$$

Then we can choose the base of wave function space.

$$\psi_1 = \chi_{1+} \otimes \chi_{2+} \otimes \phi_p \tag{9}$$
$$\psi_2 = \chi_{1+} \otimes \chi_{2-} \otimes \phi_p \tag{10}$$
$$\psi_3 = \chi_{1-} \otimes \chi_{2+} \otimes \phi_p \tag{11}$$
$$\psi_4 = \chi_{1-} \otimes \chi_{2-} \otimes \phi_p \tag{12}$$

It is very easy to verify this is an orthogonal basis. For example,

$$\begin{aligned}\langle\psi_1|\psi_1\rangle &= \langle\chi_{1+} \otimes \chi_{2+} \otimes \phi_p|\chi_{1+} \otimes \chi_{2+} \otimes \phi_p\rangle \\ &= \langle\chi_{1+}|\chi_{1+}\rangle\langle\chi_{2+}|\chi_{2+}\rangle\langle\phi_p|\phi_p\rangle \\ &= 1 \cdot 1 \cdot 1 = 1\end{aligned}$$

and

$$\begin{aligned}\langle\psi_1|\psi_2\rangle &= \langle\chi_{1+} \otimes \chi_{2+} \otimes \phi_p|\chi_{1+} \otimes \chi_{2-} \otimes \phi_p\rangle \\ &= \langle\chi_{1+}|\chi_{1+}\rangle\langle\chi_{2+}|\chi_{2-}\rangle\langle\phi_p|\phi_p\rangle \\ &= 1 \cdot 0 \cdot 1 = 0\end{aligned}$$

Therefore, we can let

$$\psi_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \psi_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad \psi_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad \psi_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

5.3 Matrix representation of $\mathcal{H}$

$$\begin{aligned}\mathcal{H}\psi_1 &= [c(\sigma_1 \otimes \sigma_1 \otimes p_1 + \sigma_1 \otimes \sigma_2 \otimes p_2 + \sigma_1 \otimes \sigma_3 \otimes p_3) + m_0c^2\sigma_3 \otimes I_2 \otimes I_p]\chi_{1+} \otimes \chi_{2+} \otimes \phi_p \\ &= c(\sigma_1\chi_{1+} \otimes \sigma_1\chi_{2+} \otimes p_1\phi_p + \sigma_1\chi_{1+} \otimes \sigma_2\chi_{2+} \otimes p_2\phi_p + \sigma_1\chi_{1+} \otimes \sigma_3\chi_{2+} \otimes p_3\phi_p) \\ &+ m_0c^2\sigma_3\chi_{1+} \otimes I_2\chi_{2+} \otimes I_p\phi_p \\ &= c(\chi_{1-} \otimes \chi_{2-} \otimes p_1\phi_p + \chi_{1+} \otimes i\chi_{2-} \otimes p_2\phi_p + \chi_{1-} \otimes \chi_{2+} \otimes p_3\phi_p) + m_0c^2\chi_{1+} \otimes \chi_{2+} \otimes \phi_p \\ &= m_0c^2\psi_1 + cp_3\psi_3 + c(p_1 + ip_2)\psi_4\end{aligned}$$

Similarly, we can calculate

$$\begin{aligned}\mathcal{H}\psi_2 &= m_0c^2\psi_2 + c(p_1 - ip_2)\psi_3 - cp_3\phi_4 \\ \mathcal{H}\psi_3 &= cp_3\psi_1 + c(p_1 + ip_2)\psi_2 - m_0c^2\psi_3 \\ \mathcal{H}\psi_4 &= c(p_1 - ip_2)\psi_1 - cp_3\psi_2 - m_0c^2\psi_4\end{aligned}$$

So the matrix representation of $\mathcal{H}$ is

$$\mathbf{H} = \begin{pmatrix} m_0c^2 & 0 & cp_3 & c(p_1+ip_2) \\ 0 & m_0c^2 & c(p_1-ip_2) & cp_3 \\ cp_3 & c(p_1+ip_2) & -m_0c^2 & 0 \\ c(p_1-ip_2) & -cp_3 & 0 & -m_0c^2 \end{pmatrix}$$

5.4 Eigenvalue problem

Let's now consider the eigenvalue problem.

$$\mathcal{H}\phi = E\phi$$

We need to calculate the $\det(\mathcal{H} - IE)$

$$\mathcal{H} - IE = \begin{pmatrix} m_0c^2 - E & 0 & cp_3 & c(p_1+ip_2) \\ 0 & m_0c^2 - E & c(p_1-ip_2) & cp_3 \\ cp_3 & c(p_1+ip_2) & -m_0c^2 - E & 0 \\ c(p_1-ip_2) & -cp_3 & 0 & -m_0c^2 - E \end{pmatrix}$$

To calculate the determinant, we need to use a trick.

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det\begin{pmatrix} A - BD^{-1}C & B \\ 0 & D \end{pmatrix} = \det(AD - BD^{-1}CD) = \det(AD - BC)$$

when C and D are commutable.

Here

$$A = \begin{pmatrix} m_0c^2 - E & 0 \\ 0 & m_0c^2 - E \end{pmatrix} \quad B = C = \begin{pmatrix} cp_3 & c(p_1 + ip_2) \\ c(p_1 - ip_2) & -cp_3 \end{pmatrix} \quad D = \begin{pmatrix} -m_0c^2 - E & 0 \\ 0 & -m_0c^2 - E \end{pmatrix}$$

And

$$AD = \begin{pmatrix} m_0c^2 - E & 0 \\ 0 & m_0c^2 - E \end{pmatrix}\begin{pmatrix} -m_0c^2 - E & 0 \\ 0 & -m_0c^2 - E \end{pmatrix} = \begin{pmatrix} E^2 - m_0^2c^4 & 0 \\ 0 & E^2 - m_0^2c^4 \end{pmatrix} \tag{13}$$

$$BC = \begin{pmatrix} cp_3 & c(p_1 + ip_2) \\ c(p_1 - ip_2) & -cp_3 \end{pmatrix}^2 = \begin{pmatrix} c^2p^2 & 0 \\ 0 & c^2p^2 \end{pmatrix} \tag{14}$$

where $p^2 = p_1^2 + p_2^2 + p_3^2$

Therefore,

$$AD - BC = \begin{pmatrix} E^2 - m_0^2c^4 - c^2p^2 & 0 \\ 0 & E^2 - m_0^2c^4 - c^2p^2 \end{pmatrix} \tag{15}$$

In conclude,

$$\det(\mathcal{H} - IE) = \det\begin{pmatrix} E^2 - m_0^2c^4 - c^2p^2 & 0 \\ 0 & E^2 - m_0^2c^4 - c^2p^2 \end{pmatrix} = (E^2 - m_0^2c^4 - c^2p^2)^2 \tag{16}$$

Here we get two eigenvalues $E = \sqrt{m_0^2c^4 + c^2p^2}$ and $E = -\sqrt{m_0^2c^4 + c^2p^2}$, both are double degeneration, because they both are roots with double multiplications. So their correspond eigenspaces are two-dimensional. To make the notation simple, we note $E_0 = \sqrt{m_0^2c^4 + c^2p^2}$. Now, let's find their correspond eigen-state.

For $E = \sqrt{m_0^2c^4 + c^2p^2}$, the linear equation $\mathcal{H} - IE = 0$ can be reduced to,

$$\begin{cases} p_3x_1 + c(p_1 + ip_2)x_2 - (m_oc^2 + E_0)x_3 = 0 \\ c(p_1 - ip_2)x_1 - cp_3x_2 - (m_0c^2 + E_0)x_4 = 0 \end{cases} \tag{17}$$

Here we set x_1 and x_2 as free variables, get the fundamental solution:

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{cp_3}{m_0c^2+E_0} \\ \frac{c(p_1-ip_2)}{m_0c^2+E_0} \end{pmatrix} \quad \xi_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_1+ip_2)}{m_0c^2+E_0} \\ \frac{-cp_3}{m_0c^2+E_0} \end{pmatrix}$$

For $E = -\sqrt{m_0^2c^4 + c^2p^2}$, linear equation $\mathcal{H} - IE = 0$ can be reduced to

$$\begin{cases} p_3x_3 + (m_0c^2 + E_0)x_1 + c(p_1 + ip_2)x_4 = 0 \\ (m_0c^2 + E_0)x_2 + c(p_1 - ip_2)x_3 - cp_3x_4 = 0 \end{cases}$$

We set x_3 and x_4 as free variables, get the fundamental solution:

$$\xi_3 = \begin{pmatrix} \frac{-cp_3}{m_0c^2+E_0} \\ \frac{-c(p_1-ip_2)}{m_0c^2+E_0} \\ 1 \\ 0 \end{pmatrix} \quad \xi_4 = \begin{pmatrix} \frac{-c(p_1+ip_2)}{m_0c^2+E_0} \\ \frac{cp_3}{m_0c^2+E_0} \\ 0 \\ 1 \end{pmatrix}$$

Now, we have solved the eigenvalue problem. There are two eigenvalues $E = +E_0$ and $E = -E_0$. For $E = +E_0$, there are two eigenvectors ξ_1 and ξ_2. For $E = -E_0$, there are two eigenvectors ξ_3 and ξ_4.

6 $\sigma \cdot \mathbf{p}$ Representation

If $\sigma \cdot \mathbf{p}$ and $\mathcal{H}$ are commutable, then they have the same eigenvectors. So, we can use the eigenvectors of $\sigma \cdot \mathbf{p}$ to represent $\mathcal{H}$, then we will solve the eigenvalue problem of $\mathcal{H}$. Here, $\sigma \cdot \mathbf{p}$ should be viewed as $(\sum_{i=1}^{3} I_1 \otimes \sigma_i \otimes p_i)$

6.1 $[\sigma\cdot\mathbf{p},\mathcal{H}]=0$

To check $[\sigma\cdot\mathbf{p},\mathcal{H}]=0$, we only need to check $[\sigma\cdot\mathbf{p},\alpha\cdot\mathbf{p}]=0$ and $[\sigma\cdot\mathbf{p},\beta]=0$

First step, $[\sigma\cdot\mathbf{p},\alpha\cdot\mathbf{p}]=0$

$$\begin{aligned}
[\sigma\cdot\mathbf{p},\alpha\cdot\mathbf{p}] &= [\sum_{i=1}^{3} I_1\otimes\sigma_i\otimes p_i, \sum_{j=1}^{3}\sigma_1\otimes\sigma_j\otimes p_j] \\
&= \sum_{i,j}\{I_1\sigma_1\otimes\sigma_i\sigma_j\otimes p_ip_j - \sigma_1 I_1\otimes\sigma_j\sigma_i\otimes p_jp_i\} \\
&= \sum_{i,j}\sigma_1\otimes[\sigma_i\sigma_j-\sigma_j\sigma_i]\otimes p_ip_j \\
&= \sum_{i,j}\sigma_1\otimes 2\sum_k \epsilon_{ijk}\sigma_k\otimes p_ip_j \\
&= 0
\end{aligned}$$

To verify the last step, we fix k=1. $\epsilon_{ij1}=1$ for i=2 and j=3; $\epsilon_{ij1}=-1$ for i=3,j=2; and $\epsilon_{ij1}=0$, otherwise.

$$\sum_{i,j}\sigma_1\otimes 2\epsilon_{ij1}\sigma_k\otimes p_ip_j = \sigma_1\otimes 2\sigma_1\otimes p_1p_2 + \sigma_1\otimes -2\sigma_2\otimes p_1p_2 = 0$$

Second step,

$$\begin{aligned}
[\sigma\cdot\mathbf{p},\beta] &= [\sum_{i=1}^{3} I_1\otimes\sigma_i\otimes p_i, \sigma_3\otimes I_2\otimes T_p] \\
&= \sum_{i=1}^{3}\{I_1\sigma_3\otimes\sigma_i I_2\otimes p_i I_p - \sigma_3 I_1\otimes I_2\sigma_i\otimes I_p p_i\} \\
&= 0
\end{aligned}$$

6.2 Eigenvalue problem of $\sigma\cdot\mathbf{p}$

Review the base,

$$\begin{aligned}
\psi_1 &= \chi_{1+}\otimes\chi_{2+}\otimes\phi_p \\
\psi_2 &= \chi_{1+}\otimes\chi_{2-}\otimes\phi_p \\
\psi_3 &= \chi_{1-}\otimes\chi_{2+}\otimes\phi_p \\
\psi_4 &= \chi_{1-}\otimes\chi_{2-}\otimes\phi_p
\end{aligned}$$

We want to find the eigenvector of $\sigma\cdot\mathbf{p}$

$$\begin{aligned}
(\sigma\cdot\mathbf{p})\psi_1 &= I_1\chi_{1+}\otimes\sigma_1\chi_{2+}\otimes p_1\phi_p + I_1\chi_{1+}\otimes\sigma_2\chi_{2+}\otimes p_2\phi_p + I_1\chi_1\otimes\sigma_3\chi_{2+}\otimes p_3\phi_p \\
&= p_3\psi_1 + (p_1+ip_2)\psi_2 \\
&= p[\cos\theta\psi_1 + \sin\theta e^{i\varphi}\psi_2]
\end{aligned}$$

Here we take polar coordinates, $\mathbf{p}=p\mathbf{n}$, where $\mathbf{n}=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)$

Similarly, we calculate

$$\begin{aligned}
(\sigma\cdot\mathbf{p})\psi_2 &= p[\sin\theta e^{-i\varphi}\psi_1 - \cos\theta\psi_2] \\
(\sigma\cdot\mathbf{p})\psi_3 &= p[\cos\theta\psi_3 + \sin\theta e^{i\varphi}\psi_4] \\
(\sigma\cdot\mathbf{p})\psi_4 &= p[\sin\theta e^{-i\varphi}\psi_3 - \cos\theta\psi_4]
\end{aligned}$$

The matrix representation of $\sigma\cdot\mathbf{p}$ is

$$\sigma\cdot\mathbf{p} = p\begin{pmatrix} \cos\theta & \sin\theta e^{i\varphi} & & \\ \sin\theta e^{-i\varphi} & -\cos\theta & & \\ & & \cos\theta & \sin\theta e^{i\varphi} \\ & & \sin\theta e^{-i\varphi} & -\cos\theta \end{pmatrix} \tag{18}$$

To simplify the problem, we only solve the eigenvalue problem of $\sigma \cdot \mathbf{n}$

$$\det(\sigma \cdot \mathbf{n} - \lambda I) = \det{}^{2} \begin{pmatrix} \cos\theta - \lambda & \sin\theta e^{i\varphi} \\ \sin\theta e^{-i\varphi} & -\cos\theta - \lambda \end{pmatrix} = (\lambda-1)^2(\lambda+1)^2 \tag{19}$$

So we get two eigenvalue $\lambda_1 = 1$ (double root) and $\lambda_2 = -1$(double root)

Now let's find out the eigenvector.

For $\lambda = 1$, the linear equation $\sigma \cdot \mathbf{p} - \lambda_1 I = 0$,can be reduced to

$$\sin\theta e^{-i\varphi} x_1 - (\cos\theta + 1)x_2 = 0$$
$$\sin\theta e^{-i\varphi} x_3 - (\cos\theta + 1)x_4 = 0$$

Set x_1 and x_3 as free variable, we can get eigenvector,

$$|n,+\rangle_1 = \begin{pmatrix} 1 \\ \tan\frac{\theta}{2}e^{-i\varphi} \\ 0 \\ 0 \end{pmatrix} \quad |n,+\rangle_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \tan\frac{\theta}{2}e^{-i\varphi} \end{pmatrix}$$

For $\lambda = -1$, apply the same procedure, we can find the eigenvector,

$$|n,-\rangle_1 = \begin{pmatrix} \tan\frac{\theta}{2}e^{i\varphi} \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad |n,-\rangle_2 = \begin{pmatrix} 0 \\ 0 \\ \tan\frac{\theta}{2}e^{i\varphi} \\ -1 \end{pmatrix}$$

6.3 Eigenvalue problem of $\mathcal{H}$

Here we will study the eigenvalue problem of $\mathcal{H}$ under the basis of $\{|n,\pm\rangle_{1,2}\}$

First of all, let's calculate $\mathcal{H}|n,+\rangle_1$. This is quite complicate, so we can use a trick.

$$\mathcal{H} = \begin{pmatrix} m_0c^2 I & cp\sigma\cdot\mathbf{n} \\ cp\sigma\cdot\mathbf{n} & -m_0c^2 I \end{pmatrix} \tag{20}$$

where I is identity Matrix, and

$$\sigma \cdot \mathbf{n} = \begin{pmatrix} \cos\theta & \sin\theta e^{i\varphi} \\ \sin\theta e^{-i\varphi} & -\cos\theta \end{pmatrix} \tag{21}$$

Then $|n,+\rangle_1$ and $|n,+\rangle_2$ can be expressed as

$$|n,+\rangle_1 = \begin{pmatrix} |n,+\rangle \\ 0 \end{pmatrix} \quad |n,+\rangle_2 = \begin{pmatrix} 0 \\ |n,+\rangle \end{pmatrix}$$

where,

$$|n,+\rangle = \begin{pmatrix} 1 \\ \tan\frac{\theta}{2}e^{-i\varphi} \end{pmatrix}$$

and $|n,+\rangle$ is the eigenvector of $\sigma \cdot \mathbf{p}$ correspond to eigenvalue $+1$. Therefore,

$$\mathcal{H}|n,+\rangle_1 = \begin{pmatrix} m_0c^2 I & p\sigma\cdot\mathbf{n} \\ p\sigma\cdot\mathbf{n} & -m_0c^2 I \end{pmatrix} \begin{pmatrix} |n,+\rangle \\ 0 \end{pmatrix} = \begin{pmatrix} m_0c^2 I|n,+\rangle \\ cp\sigma\cdot\mathbf{p}|n,+\rangle \end{pmatrix} = m_0c^2|n,+\rangle_1 + cp|n,+\rangle_2$$

Similarly, we can yield

$$\mathcal{H}|n,+\rangle_2 = cp|n,+\rangle_1 - m_0c^2|n,+\rangle_2$$
$$\mathcal{H}|n,-\rangle_1 = m_0c^2|n,-\rangle_1 - cp|n,-\rangle_2$$
$$\mathcal{H}|n,-\rangle_2 = -cp|n,-\rangle_1 - m_0c^2|n,-\rangle_2$$

Let

$$|n,+\rangle_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad |n,+\rangle_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad |n,-\rangle_1 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad |n,-\rangle_2 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

We can represent $\mathcal{H}$ in a matrix form,

$$\mathcal{H} = \begin{pmatrix} m_0c^2 & cp & & \\ cp & -m_0c^2 & & \\ & & m_0c^2 & -cp \\ & & -cp & -m_0c^2 \end{pmatrix}$$

Let's solve the eigenvalue problem.

$$\det(\mathcal{H} - IE) = \det\begin{pmatrix} m_0c^2 - E & cp & & \\ cp & -m_0c^2 - E & & \\ & & m_0c^2 - E & -cp \\ & & -cp & -m_0c^2 - E \end{pmatrix} = 0 \implies (E^2 - m_0^2c^4 - c^2p^2)^2 = 0 \tag{22}$$

Therefore, we get two eigenvalue $E_1 = \sqrt{m_0^2c^4 + c^2p^2}$ (double root) and $E_2 = -\sqrt{m_0^2c^4 + c^2p^2}$

For $E_1 = \sqrt{m_0^2c^4 + c^2p^2}$, the linear equation $\mathcal{H} - E_1 I = 0$ can be reduced to

$$\begin{aligned} &cpx_1 - (m_0c^2 + E_0)x_2 = 0 \\ &-cpx_3 - (m_0c^2 + E_0)x_4 = 0 \end{aligned}$$

Set x_1 and x_3 as free variables, we can derive the eigenvectors.

$$\xi_1' = \begin{pmatrix} 1 \\ \frac{cp}{m_0c^2+E_0} \\ 0 \\ 0 \end{pmatrix} \quad \xi_2' = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{-cp}{m_0c^2+E_0} \end{pmatrix}$$

Similarly, for $E = -E_0$, we can derive the eigenvectors.

$$\xi_3' = \begin{pmatrix} \frac{-cp}{m_0c^2+E_0} \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \xi_4' = \begin{pmatrix} 0 \\ 0 \\ \frac{cp}{m_0c^2+E_0} \\ 1 \end{pmatrix}$$

6.4 Discussion

By $\sigma \cdot \mathbf{p}$ representation, we can have a clear understanding of the eigenvalue problem of $\mathcal{H}$.

For $E = +E_0$

$$\xi_1' = \begin{pmatrix} 1 \\ \frac{cp}{m_0c^2+E_0} \\ 0 \\ 0 \end{pmatrix} = |n,+\rangle_1 + \frac{cp}{m_0c^2 + E_0}|n,+\rangle_2 = \begin{pmatrix} |n,+\rangle \\ \frac{cp}{m_0c^2+E_0}|n,+\rangle \end{pmatrix}$$

$$\xi_2' = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \frac{-cp}{m_0c^2+E_0} \end{pmatrix} = |n,-\rangle_1 - \frac{cp}{m_0c^2 + E_0}|n,-\rangle_2 = \begin{pmatrix} |n,-\rangle \\ \frac{-cp}{m_0c^2+E_0}|n,-\rangle \end{pmatrix}$$

For $E = -E_0$

$$\xi_3' = \begin{pmatrix} \frac{-cp}{m_0c^2+E_0} \\ 1 \\ 0 \\ 0 \end{pmatrix} = \frac{-cp}{m_0c^2 + E_0}|n,+\rangle_1 + |n,+\rangle_2 = \begin{pmatrix} \frac{-cp}{m_0c^2+E_0}|n,+\rangle \\ |n,+\rangle \end{pmatrix}$$

$$\xi_4' = \begin{pmatrix} 0 \\ 0 \\ \frac{cp}{m_0 c^2 + E_0} \\ 1 \end{pmatrix} = \frac{cp}{m_0 c^2 + E_0} |n, -\rangle_1 + |n, -\rangle_2 = \begin{pmatrix} \frac{cp}{m_0 c^2 + E_0} |n, -\rangle \\ |n, -\rangle \end{pmatrix}$$

These formulas are quite popular in many advanced quantum mechanics books.