Self-adjoint operator

Wu Yuhan

Oct. 4, 2020

1 Preparation

In this article, we mainly discuss linear operators in Hilbert space.

Let's review these two concepts

1. Hilbert space
2. Linear operator

Firstly, hilbert space $\mathbf{X}$ is an infinite dimensional vector space equipped with inner product $\langle \cdot | \cdot \rangle$

And, $\mathcal{A}$ is a linear operator, iff $\forall u, v \in \mathbf{X}$ and $k \in \mathbf{F}$

1. $\mathcal{A}(ku) = k\mathcal{A}u$
2. $\mathcal{A}(u+v) = \mathcal{A}u + \mathcal{A}v$

2 Definition of adjoint operator

Motivation : $\mathcal{A}$ is a linear operator in Hilbert space. Then suppose that there is a linear operator $\mathcal{A}^*$ such that

$$\langle \mathcal{A}^* u | v \rangle = \langle u | \mathcal{A} v \rangle \tag{1}$$

Let's start. First step, observe the equation of $w \in \mathbf{X}$, and $u \in \mathbf{X}$ is fixed

$$\langle u | \mathcal{A} v \rangle = \langle w | v \rangle \quad \forall v \in \mathbf{X} \tag{2}$$

Proposition 1 *If the equation*

$$\langle u | \mathcal{A} v \rangle = \langle w | v \rangle \quad \forall v \in \mathbf{X} \tag{3}$$

has a solution, then the solution is unique.

Proof *Suppose there are two solutions w_1 and w_2 that satisfy the equation. Then*

$$\langle w_1 | v \rangle = \langle u | \mathcal{A} v \rangle = \langle w_2 | v \rangle$$

And we can yield,

$$\langle w_1 | v \rangle = \langle w_2 | w \rangle \implies \langle w_1 - w_2 | v \rangle = 0 \forall u \in \mathbf{X}$$

Here we can choose $v = w_1 - w_2$, then

$$\langle w_1 - w_2 | w_1 - w_2 \rangle = 0 \implies ||w_1 - w_2|| = 0 \implies w_1 = w_2$$

Therefore $\mathcal{A}^*$ is well-defined by $\mathcal{A}^* u = w$,then let's check the property of $\mathcal{A}^*$

Proposition 2 *1. $\mathcal{A}^* : D(\mathcal{A}^*) \subseteq \mathbf{X} \to \mathbf{X}$ is linear.*

2. $(\alpha \mathcal{A})^* = \bar{\alpha} \mathcal{A}^*$

First of all, I need to introduce a very simple, but extremely important lemma : *Variational lemma.*

To emphasize its importance, I will note it as a theorem.

Theorem 1 *Variational lemma: $u, v \in \mathbf{X}$, $\mathbf{X}$ is a Hilbert space, and if*

$$\langle w|u\rangle = \langle w|v\rangle \quad \forall w \in \mathbf{X}$$

then

$$u = v$$

The proof is very simple, just apply the same procedure in proof of proposition above.

Proof

$$\begin{aligned}
&\langle w|u\rangle = \langle w|v\rangle \quad \forall w \in \mathbf{X} \\
&\langle w|u-v\rangle = 0 \quad \forall w \in \mathbf{X} \\
&Let\ w = u - v \\
&\langle u-v|u-v\rangle = 0 \implies ||u-v|| = 0 \\
&\implies u = v
\end{aligned}$$

QED

To prove $\mathcal{A}^*$ is a linear operator, we need to check:

$$\mathcal{A}^*(a_1v_1 + a_2v_2) = a_1\mathcal{A}^*v_1 + a_2\mathcal{A}^*v_2$$

By Variational lemma, we only need to prove

$$\langle \mathcal{A}^*(a_1v_1 + a_2v_2)|u\rangle = \langle a_1\mathcal{A}^*v_1 + a_2\mathcal{A}^*v_2|u\rangle \quad \forall u \in \mathbf{X}$$

Proof *By definition of adjoint operator:*

$$\begin{aligned}
\langle \mathcal{A}^*(a_1v_1 + a_2v_2)|u\rangle &= \langle a_1v_1 + a_2v_2|\mathcal{A}u\rangle \\
&= a_1\langle v_1|\mathcal{A}u\rangle + a_2\langle v_2|\mathcal{A}u\rangle \\
&= a_1\langle \mathcal{A}^*v_1|u\rangle + a_2\langle \mathcal{A}^*v_2|u\rangle \\
&= \langle a_1\mathcal{A}^*v_1 + a_2\mathcal{A}^*v_2|u\rangle
\end{aligned}$$

QED

Then let's finish another proof:$(\alpha\mathcal{A})^* = \bar{\alpha}\mathcal{A}^*$

Proof

$$\langle (\alpha\mathcal{A})^*v|u\rangle = \langle v|\alpha\mathcal{A}u\rangle = \alpha\langle v|\mathcal{A}u\rangle = \alpha\langle \mathcal{A}^*v|u\rangle = \langle \bar{\alpha}\mathcal{A}^*v|u\rangle$$

QED

Lat's have a look on adjoint operator in finite space.

Example 1 $\mathbf{C}^n$ *is finite dimensional vector space, choose an orthogonal basis $\{e_i\}$,$i = 1, \cdots, n$. $\mathcal{A}$ is a linear operator on $\mathbf{C}^n$ and it has a matrix representation $\mathbf{A}$,i.e.*

$$\mathcal{A}(e_1, \cdots, e_n) = (e_1, \cdots, e_n)\mathbf{A} \quad i.e.\mathcal{A}e_i = \sum_j A_{ji}e_j$$

Similarly, the adjoint operator $\mathcal{A}^*$ has a matrix representation $\mathbf{A}^*$. Specifically, $\mathcal{A}^*e_i = \sum_j A^*_{ji}e_j$ Therefore

$$\begin{aligned}
\langle \mathcal{A}^*e_i|e_j\rangle &= \langle e_i|\mathcal{A}e_j\rangle \\
\langle \sum_k A^*_{ki}e_k|e_j\rangle &= \langle e_i|\sum_k A_{kj}e_k\rangle \\
\sum_k \bar{A^*_{ki}}\langle e_k|e_j\rangle &= \sum_k A_{kj}\langle e_i|e_k\rangle \\
\overline{A^*_{ji}} &= A_{ij}
\end{aligned}$$

To conclude

$$\overline{\mathbf{A}}^{\mathbf{T}} = \mathbf{A}^* \tag{4}$$

3 Self-adjoint operator and skew-adjoint operator

3.1 Self-adjoint operator

In vector space $\mathbf{X}$, linear operator $\mathcal{A}$ is self-adjoint operator, iff $\mathcal{A} = \mathcal{A}^*$,i.e. $\forall u, v \in \mathbf{X}$,

$$\langle \mathcal{A}u|v\rangle = \langle u|\mathcal{A}v\rangle \tag{5}$$

In many cases, self-adjoint operator is also known as Hermitian operator. Here is an example of self-adjoint operator.

Example 2 *Integral operator. Suppose $A : [a, b] \times [a, b] \to \mathbf{C}$ is a continuous function. Define*

$$(\mathcal{A}u)(x) = \int_a^b A(x, y)u(y)dy \quad \forall x \in [a, b] \tag{6}$$

and set $\mathbf{X} = \mathbf{L_2}([a, b])$, $u \in \mathbf{X}$.

Our task is to find the adjoint operator of the integral operator.

$$\begin{aligned}
\langle v|\mathcal{A}u\rangle &= \int_a^b \overline{v(x)}(\int_a^b A(x, y)u(y)dy)dx \\
&= \int_a^b \int_a^b \overline{v(x)}A(x, y)u(y)dxdy \quad \textit{Fubini Thm} \\
&= \int_a^b u(y)(\int_a^b A(x, y)\overline{v(x)}dx)dy \\
&= \int_a^b (\int_a^b \overline{\overline{A(y, x)}v(y)}dy)u(x)dx \quad \textit{exchange x,and y} \\
&= \langle A^* v|u\rangle
\end{aligned}$$

where $\mathcal{A}^*v$ are defined as $\mathcal{A}^*v = \int_a^b \overline{A(y, x)}v(y)dx$.Therefore, if $A(x, y) = \overline{A(y, x)}$, then $\mathcal{A} = \mathcal{A}^*$.

To conclude, the integral operator $\mathcal{A}$ is a self-adjoint operator, iff the integral kernel $A(x, y)$ satisfies

$$A(x, y) = \overline{A(y, x)}$$

3.2 Skew-adjoint operator

Define, $\mathcal{A}$ is a skew-adjoint operator, iff $\mathcal{A} = -\mathcal{A}^*$, i.e. for all $u, v \in \mathbf{X}$, $\langle v|\mathcal{A}u\rangle = -\langle \mathcal{A}v|u\rangle$. Let's have a look on differential operator:

Example 3 *Let's consider $\mathbf{X} = C_0^\infty(\mathbf{R}) \cap L_2^{\mathbf{C}}(\mathbf{R})$, which means $\forall u \in \mathbf{X}$, $u^{(n)}(\pm\infty) = 0$,for all n, and $\int_{\mathbf{R}} |u|^2 < \infty$. We define the differential operator $\mathcal{A}u(x) = u'(x)$*

$$\begin{aligned}
\langle v|\mathcal{A}u\rangle &= \int_{\mathbf{R}} \bar{v}u'dx \\
&= \bar{v}u|_{-\infty}^{+\infty} - \int_{\mathbf{R}} \overline{v'}udx \quad \textit{integral by part} \\
&= -\langle \mathcal{A}v|u\rangle
\end{aligned}$$

Conclude, differential operator $\mathcal{A}$ is a skew-adjoint operator

Proposition 3 *If $\mathcal{A}$ is a self-adjoint operator, then $i\mathcal{A}$ is a skew-adjoint operator. Conversely, id $\mathcal{A}$ is a skew-adjoint operator, then $i\mathcal{A}$ is a self-adjoint operator.*

This can be easily verified by $(\alpha\mathcal{A})^* = \overline{\alpha}\mathcal{A}^*$.

The momentum operator $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$ is a self-adjoint operator. In other words, $\hat{p}$ is a Hermitian operator.