Linear equations and eigenvalue problems

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1 Introduction

The eigenvalue problem

$$\mathbf{A}x = \lambda x \tag{1}$$

where $\mathbf{A}$ is a linear transformation on $\mathbb{R}^n$, is equivalent to the problem:

$$(\mathbf{A} - \lambda I)x = 0 \tag{2}$$

By Crammer rule, if $\det(\mathbf{A} - \lambda I) \neq 0$, the equation only has a trivial solution 0. Therefore, in order to get the nontrivial solution, the necessary condition is that

$$\det(\mathbf{A} - \lambda I) = 0 \tag{3}$$

It is not a difficult to calculate the eigenvalue λ. However, for many students, they may have difficulty in finding the eigenvectors respective to eigenvalue λ. In this article, I will introduce a systematic way to find eigenvectors by solving the system of linear equations.

For eigenvalue λ, the solution of $(\mathbf{A} - \lambda I)x = 0$ are eigenvectors.

2 Systems of linear equation

To simply the problem, note $\mathbf{B} = \mathbf{A} - \lambda I$. Due to the condition $\det \mathbf{B} = \det(\mathbf{A} - \lambda I) = 0$, $rank\mathbf{B} < n$. And all solutions of $\mathbf{B}x = 0$ forms a vector subspace ,which is also noted as $\ker \mathbf{B}$. This is quite easy to verify.

Proof $\forall x_1, x_2 \in \ker \mathbf{B}$*, we have* $\mathbf{B}x_1 = 0$ *and* $\mathbf{B}x_2 = 0$*,then*

$$\mathbf{B}(x_1 + x_2) = \mathbf{B}x_1 + \mathbf{B}x_2 = 0 + 0 = 0 \tag{4}$$

implies $(x_1 + x_2) \in \ker \mathbf{B}$*, and*

$$\mathbf{B}(kx_1) = k\mathbf{B}x_1 = k \cdot 0 = 0 \tag{5}$$

so $kx_1 \in \ker \mathbf{B}$ *So* $\ker \mathbf{B}$ *is a linear subspace.*

Since $\det B = 0$, there exists $x \neq 0 \in \ker \mathbf{B}$, therefore $dim \ker \mathbf{B} \geq 1$.

In quantum mechanics, $\mathbf{A}$ is usually a hermitian operator, and $dim \ker(\mathbf{A} - \lambda I)$ often equals to the multiplicity of root λ in equation $\det(\mathbf{A} - \lambda I) = 0$. If the dimension ≥ 2, in quantum mechanics or classical mechanics ,this situation represents the degeneration.

The equation $\mathbf{B}x = 0$, write explicitly

$$\begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0 \tag{6}$$

We note that $dim \ker \mathbf{B} = m$, so the $rank\mathbf{B} = n - m = r$. Use Gauss elimination method, and we can reduce to equations to

$$\begin{cases} b_{11}x_1 + b_{12}x_2 + \cdots b_{1n}x_n = 0 \\ b_{21}x_1 + b_{22}x_2 + \cdots b_{2n}x_n = 0 \\ \vdots \\ b_{r1}x_1 + b_{r2}x_2 + \cdots b_{rn}x_n = 0 \end{cases} \tag{7}$$

By some knowledge of matrix, we can show that

$$rank \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{r1} & \cdots & b_{rn} \end{pmatrix} = r \tag{8}$$

Therefore, we can assume

$$\det \begin{pmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & & \vdots \\ b_{r1} & \cdots & b_{rr} \end{pmatrix} \neq 0$$

otherwise, make permutation of $\{x_1, x_2, \cdots x_n\}$, and switch the column vector of $\mathbf{B}$. Here is the equation

$$\begin{cases} b_{11}x_1 + b_{12}x_2 + \dots b_{1r}x_r = -b_{1,r+1}x_{r+1} - b_{1,r+2}x_{r+2} \cdots - b_{1n}x_n \\ b_{21}x_1 + b_{22}x_2 + \dots b_{2r}x_r = -b_{2,r+1}x_{r+1} - b_{2,r+2}x_{r+2} \cdots - b_{2n}x_n \\ \vdots \\ b_{r1}x_1 + b_{r2}x_2 + \dots b_{rr}x_r = -b_{r,r+1}x_{r+1} - b_{r,r+2}x_{r+2} \cdots - b_{rn}x_n \end{cases} \tag{9}$$

This is an equation of $\{x_1, \cdots, x_r\}$, and $\{x_{r+1}, \cdots, x_n\}$ should be viewed as free variables.

Let $x_{r+1} = 1, x_{r+2} = \cdots = x_{r+n} = 0$, by Crammer rule, we can yield a unique solution $(x_1, x_2, \cdots, x_r)$, and we combine $x_{r+1} = 1, x_{r+2} = \cdots = x_{r+n} = 0$, we can yield a fundamental solution of$\mathbf{B}x = 0$

$$\eta_1 = \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1r} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{10}$$

Then we let $x_{r+2} = 1$, $x_{r+1} = x_{r+3} = \cdots x_n = 0$, and obtain another fundamental solution:

$$\eta_2 = \begin{pmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{1r} \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \tag{11}$$

Similarly, we can obtain $\eta_3, \eta_4, \cdots \eta_{n-r}$.

Obviously,

$$\eta_1 = \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1r} \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \eta_2 = \begin{pmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2r} \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \cdots \eta_{n-r} = \begin{pmatrix} x_{n-r,1} \\ x_{n-r,2} \\ \vdots \\ x_{n-r,r} \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

are $n-r$ linear independent vectors in $\ker \mathbf{B}$. And we have assumed that $dim \ker \mathbf{B} = n-r$; so $\{\eta_1, \cdots \eta_{n-r}\}$ forms a basis of $\ker \mathbf{B}$ i.e $\ker(\mathbf{A} - \lambda I)$. In other words, $\{\eta_1, \cdots \eta_{n-r}\}$ are eigenvectors with respected to eigenvalue λ. And all solution of $\mathbf{B}x = 0$, can be represented by the linear combination of $\eta_1, \cdots \eta_{n-r}$,i.e.

$$x = k_1\eta_1 + \cdots + k_{n-r}\eta_{n-r} \tag{12}$$

If we apply the Gram-Schmidt process, then we can get an orthogonal basis of $\ker(\mathbf{A} - \lambda I)$: $\{\xi_1, \cdots, \xi_{n-r}\}$, where $(\xi_i|\xi_i) = \delta_{ij}$ and $||\xi_i|| = 1$

3 Example

Here is an example how to solve eigenvalue problem of a symmetric matrix.

$$\mathbf{A} = \begin{pmatrix} 4 & -1 & -1 & 1 \\ -1 & 4 & 1 & -1 \\ -1 & 1 & 4 & -1 \\ 1 & -1 & -1 & 4 \end{pmatrix}$$

Find eigenvalue and eigenvectors with respect to the eigenvalue. First of all, solve characteristic equation $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$

$$\det \begin{pmatrix} 4-\lambda & -1 & -1 & 1 \\ -1 & 4-\lambda & 1 & -1 \\ -1 & 1 & 4-\lambda & -1 \\ 1 & -1 & -1 & 4-\lambda \end{pmatrix} = (\lambda - 3)^3(\lambda - 7) = 0$$

So we get two eigenvalues, $\lambda_1 = 3$(3 multiplicity), $\lambda_2 = 7$,then we continue to find their eigenvectors.

For $\lambda_1 = 3$, we can find $\dim \ker(\mathbf{A} - 3\mathbf{I}) = 3$

$$\mathbf{A} - 3\mathbf{I} = \begin{pmatrix} 4-3 & -1 & -1 & 1 \\ -1 & 4-3 & 1 & -1 \\ -1 & 1 & 4-3 & -1 \\ 1 & -1 & -1 & 4-3 \end{pmatrix} \text{Row operation} \Rightarrow \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We get

$$x_1 = x_2 + x_3 - x_4$$

Let $(x_2, x_3, x_4) = (1, 0, 0)$, then get a solution

$$\xi_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Similarly, let $(x_2, x_3, x_4) = (0, 1, 0)$ and $(x_2, x_3, x_4) = (0, 0, 1)$,get another two solutions

$$\xi_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \xi_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

And ξ_1, ξ_2, ξ_3 are eigenvectors with respect to eigenvalue$\lambda_1 = 3$

In the same way

$$\mathbf{A} - 7\mathbf{I} = \begin{pmatrix} -3 & -1 & -1 & 1 \\ -1 & -3 & 1 & -1 \\ -1 & 1 & -3 & -1 \\ 1 & -1 & -1 & -3 \end{pmatrix} \text{Row operation} \Rightarrow \begin{pmatrix} -3 & -1 & -1 & 1 \\ -1 & -3 & 1 & -1 \\ -1 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $x_4 = 1$, then we have a unique solution $(x_1, x_2, x_3) = (1, -1, -1)$,then

$$\xi_4 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

is the eigenvector with respect to$\lambda_2 = 7$