Dual space and inner product

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1 Introduction

When I read *The principle of quantum mechanics*, Dirac introduced the bra vector by dual space instead of inner product. Hence, I decide to investigate the relationship between them.

In this article, the traditional dual space $\mathbf{V}^*$ will be firstly presented; then I will show that every linear function can be represented by inner product,for example

$$\exists\alpha \in \mathbf{V}, f(\beta) = \langle\alpha|\beta\rangle, \forall\beta \in \mathbf{V} \tag{1}$$

Then, we will continue to define a linear transformation $\mathcal{A}^*$on $\mathbf{V}^*$ by $\mathcal{A}^*(\alpha_i^*)(\beta) = \alpha_i^*(\mathcal{A}(\beta))$, and show its relation with $\mathcal{A}$. By this process, readers can have a closer view on hermitian conjugate. Also, it can help us to have a better understanding of hermitian operator in quantum mechanics.

2 Linear function and dual space

2.1 Linear function

Definition 1 *Linear function:* $\mathbf{V}$*is a linear space, eg.* $\mathbf{R^n}$*, and* $f : \mathbf{V} \to \mathbf{F}$*, if for all* $\alpha, \beta \in \mathbf{V}$ *and* $a, b \in \mathbf{F}$*,* $f(a\alpha + b + \beta) = af(\alpha) + bf(\beta)$*,then* f *is a linear function.*

Here we want to show that all linear functions $\{f : \mathbf{V} \to \mathbf{F}\ \}$ form a linear space V^*=$\{f$:$\mathbf{V} \to \mathbf{F}$,and f is linear$\}$,let's check it:

Suppose $f_1, f_2 \in \mathbf{V}*$, we need to verify that $f_1 + f_2 \in \mathbf{V}^*$, and $kf_1 \in \mathbf{V}^*$

$$\begin{aligned}(f_1 + f_2)(a\alpha + b\beta) &= f_1(a\alpha + b\beta) + f_2(a\alpha + b\beta)\\ &= af_1(\alpha) + bf_1(\beta) + af_2(\alpha) + bf_2(\alpha)\\ &= a(f_1 + f_2)(\alpha) + b(f_1 + f_2)(\beta)\end{aligned}$$

Similarly, we can also prove that $kf_1 \in \mathbf{V}^*$; therefore, $\mathbf{V}^*$ is also a linear space.

2.2 Dual space

Now our task is to find the basis of $\mathbf{V}^*$.

Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_n$ are orthonormal basis of vector space of $\mathbf{V}$. Then we define a mapping σ : $\mathbf{V} \to \mathbf{V}^*, \sigma(\alpha_i) = \alpha_i^*$, with property:

$$\alpha_i^*(\alpha_j) = \delta_{ij}$$

Here α_i^* is a linear function, and the motivation of the definition is $\langle\alpha_i|\alpha_j\rangle = \delta_{ij}$, and we want to check $\{\alpha_i^*\}$ is the basis of $\mathbf{V}^*$

Firstly, let's have a look on α_i^* , suppose $\beta = \sum_{i=1}^{n} b_i\alpha_i \in \mathbf{V}$

$$\alpha_i^*(\beta) = \alpha_i^*(\sum_{j=1}^{n}\alpha_j) = \sum_{j=1}^{n} b_j\alpha_i^*(\alpha_j) = \sum_{j=1}^{n} b_j\delta_{ij} = b_i \tag{2}$$

Then, $\forall f \in \mathbf{V}^*, \forall \beta = \sum_{i=1}^n b_i \alpha_i \in \mathbf{V}$

$$\begin{aligned} f(\beta) &= \sum_{i=1}^n b_i f(\alpha_i) \\ &= \sum_{i=1}^n \alpha_i^*(\beta) f(\alpha_i) \\ &= \sum_{i=1}^n f(\alpha_i) \alpha_i^*(\beta) \end{aligned}$$

Since β is an arbitrary vector in $\mathbf{V}$, then :

$$f = \sum_{i=1}^n f(\alpha_i) \alpha_i^* \tag{3}$$

We can conclude that $\{\alpha_i^*\}$ form a basis of $\mathbf{V}^*$

2.3 Double duality

Let's consider a dual space of the dual space $\mathbf{V}^*$.

$$\mathbf{V} \to \mathbf{V}^* \to \mathbf{V}^{**}$$

and,

$$\alpha_i \to \alpha_i^* \to \alpha_i^{**}$$

with property:

$$\alpha_i^*(\alpha_j) = \delta_{ij} \quad \alpha_i^{**}(\alpha_j^*) = \delta_{ij}$$

Therefore,

$$\alpha_i^{**}(\alpha_j^*) = \alpha_i^*(\alpha_j) = \delta_{ij} \tag{4}$$

Suppose, $\alpha = \sum_{i=1}^n a_i \alpha_i, \quad \beta = \sum_{j=1}^n b_j \alpha_j$

$$\begin{aligned} \alpha^{**}(\beta^*) = \alpha^{**}(\sum_{j=1}^n b_j \alpha_j^*) &= \sum_{i=1}^n a_i \alpha_i^{**}(\sum_{j=1}^n b_j \alpha_j^*) \\ &= \sum_{i,j=1}^n a_i b_j \alpha_i^{**}(\alpha_j^*) \\ &= \sum_{i,j=1}^n a_i b_j \delta_{ij} \\ &= \sum_{j=1}^n a_j b_j \end{aligned}$$

Im the same way, we calculate:

$$\begin{aligned} \beta^*(\alpha) &= \sum_{j=1}^n b_j \alpha_j^*(\sum_{i=1}^n a_i \alpha_i) \\ &= \sum_{i,j=1}^n a_i b_j \alpha_j^*(a_i) \\ &= \sum_{i,j=1}^n a_i b_j \delta_{ij} \\ &= \sum_{j=1}^n a_j b_j \end{aligned}$$

Hence,

$$\alpha^{**}(\beta^*) = \beta^*(\alpha) \tag{5}$$

and we can conclude$\mathbf{V} = \mathbf{V}^{**}$

3 Inner product and Dual space

In this section, we are going to study the relation between dual space and inner product.

3.1 Linear function represented by inner product

Firstly, we need to assume that the linear space $\mathbf{V}$ is equipped with inner product $\langle\cdot|\cdot\rangle$, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ form an orthonormal basis of $\mathbf{V}$

As the definition of the basis of dual space $\mathbf{V}^*$, $\alpha_i^*(\alpha_j) = \delta_{ij} = \langle\alpha_i|\alpha_j\rangle$, we can hypothesis that for any linear function in $\mathbf{V}^*$, there exists a unique vector α, such that $\forall\beta \in \mathbf{V}$, and

$$f(\beta) = \langle\alpha|\beta\rangle \tag{6}$$

Let's prove it. To avoid confusion, we firstly complete the proof in real vector space,i.e $\mathbf{F} = \mathbf{R}$

As we hypothesis above, $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*$ are orthonormal bases of $\mathbf{V}$ and $\mathbf{V}^*$, with the property $\alpha_i^*(\alpha_j) = \delta_{ij} = \langle\alpha_i|\alpha_j\rangle$. Suppose that $\beta = \sum_{i=1}^n b_i\alpha_i$, and multiplied by $\langle\alpha_j|$ from left side, we can yield:

$$\langle\alpha_j|\beta\rangle = \sum_{i=1}^n b_i\langle\alpha_j|\alpha_i\rangle = \sum_{i=1}^n b_i\delta_{ij} = b_j \tag{7}$$

Then,

$$\begin{aligned} f(\beta) = f(\sum_{j=1}^n b_j\alpha_j) &= \sum_{j=1}^n f(\alpha_j) \\ &= \sum_{j=1}^n \langle\alpha_j|\beta\rangle f(\alpha_j) \\ &= \langle\sum_{j=1}^n f(\alpha_j)\alpha_j|\beta\rangle \\ &= \langle\alpha|\beta\rangle \quad \text{,where} \quad \alpha = \sum_{j=1}^n f(\alpha_j)\alpha_j \end{aligned}$$

Then let's discuss it in complex vector space, i.e.$\mathbf{F} = \mathbf{C}$, and we have inner product:

$$\langle\alpha|\beta\rangle = \overline{\langle\beta|\alpha\rangle}$$

and,

$$\begin{aligned} \langle a\alpha|\beta\rangle &= \bar{a}\langle\alpha|\beta\rangle \\ \langle\alpha|b\beta\rangle &= b\langle\alpha|\beta\rangle \end{aligned}$$

In the similar way,

$$\begin{aligned} f(\beta) &= \sum_{j=1}^n \langle\alpha_j|\beta\rangle f(\alpha_j) \\ &= \langle\sum_{j=1}^n \overline{f(\alpha_j)}\alpha_j|\beta\rangle \\ &= \langle\alpha|\beta\rangle \quad \text{, where} \quad \alpha = \sum_{j=1}^n \overline{f(\alpha_j)}\alpha_j \end{aligned}$$

3.2 Generalization to infinite dimensional space

In the last section, we discuss the problem in a finite dimensional space; now we want to generalize it to infinite dimensional vector space. And this is the famous Riesz theorem. This part mainly refers to *Applied functional analysis Vol.1*, written by Eberhard Zeidler.

Firstly, hypothesis that $\mathbf{X}$ is a Banach space equipped with inner product $\langle\cdot|\cdot\rangle$ and norm $||u||^2 = \langle u|u\rangle$. Then, $\forall f \in \mathbf{X}$, there exists a unique $v \in \mathbf{X}$, such that $f(u) = \langle v|u\rangle, \forall u \in \mathbf{X}$ and, $||f|| = ||v||$,where $||f|| = \sup_{||u||=1} |f(u)|$

1st step, the uniqueness of v. Assume $\exists v_0, v_1$ satisfy the condition, such that:

$$f(u) = \langle v_0|u\rangle = \langle v_1|u\rangle \quad \forall u \in \mathbf{X}$$

$\Longrightarrow \langle v_0 - v_1|u\rangle = 0 \quad \forall u \in \mathbf{X}$, let $u = v_0 - v_1$, then $\langle v_0 - v_1|v_0 - v_1\rangle = ||v_0 - v_1||^2 = 0 \Longrightarrow \quad v_0 = v_1$

Second step, existence of v

For any $f \in \mathbf{X}^*$, set $N(f) = \{u \in \mathbf{X}, f(u) = 0\}$ and $N(f)$ is a subspace of $\mathbf{X}$. Suppose $N(f) \neq \mathbf{X}$, or otherwise $f = 0$.

Therefore, we can find $u_0 \in N(f)^\perp$ with $u_0 \neq 0$, and this is the key to proof.To be honest, it is not an easy job to find such element u_0, and I can write another article to show how to find u_0

Assume $f(u_0) = 1$, otherwise take $\bar{u_0} = \frac{u_0}{f(u_0)}$, then $f(\bar{u_0}) = 1$ Then, we have

$$f(u - f(u)u_0) = f(u) - f(u)f(u_0) = 0 \quad \forall u \in \mathbf{X} \tag{8}$$

So $u - f(u)u_0 \in N(f)$, by decomposition theorem, there exists $w \in N(f)$ such that

$$u - f(u)u_0 = w \Longrightarrow u = f(u)u_0 + w \tag{9}$$

Multiply u_0 from left side, and notice that $u_0 \in N(f)^\perp$,

$$\langle u_0|u\rangle = f(u)\langle u_0|u_0\rangle$$

Therefore,

$$f(u) = \langle \frac{u_0}{\langle u_0|u_0\rangle}|u\rangle \quad \text{with} \quad v = \frac{u_0}{\langle u_0|u_0\rangle} \tag{10}$$

Last step $||f|| = ||v||$,let $||u|| = 1$

$|f(u)| = |\langle v|u\rangle| \leqslant ||u|| \cdot ||v||, \forall ||u|| = 1 \Longrightarrow ||f|| \leqslant ||v||$

Then observe $f(\frac{v}{||v||}) = \langle v|\frac{v}{||v||}\rangle = ||v|| \Longrightarrow ||f|| \geqslant ||v||$, to conclude

$$||f|| = ||v|| \tag{11}$$

4 Linear transformation on $\mathbf{V}^*$

4.1 Introduction of $\mathcal{A}^*$

$\mathcal{A}$ is a linear transformation on $\mathbf{V}$, and we want to define a linear transformation on $\mathbf{V}^*$

Define: $f \in \mathbf{V}^*$, $\mathcal{A}^*f = f\mathcal{A}$, i.e. $\mathcal{A}^*f(\alpha) = f\mathcal{A}(\alpha) = f(\mathcal{A}\alpha)$.

Now, we need to prove $\mathcal{A}^*$ is a linear transformation on $\mathbf{V}^*$ First step, $\mathcal{A}^*f = f\mathcal{A} \in \mathbf{V}^*$

$$\begin{aligned} f\mathcal{A}(\alpha + \beta) &= f(\mathcal{A}\alpha + \mathcal{A}\beta) \\ &= f(\mathcal{A}\alpha) + f(\mathcal{A}\beta) \\ &= f\mathcal{A}(\alpha) + f\mathcal{A}(\beta) \\ &= \mathcal{A}^*f(\alpha) + \mathcal{A}^*f(\beta) \end{aligned}$$

and

$$\mathcal{A}^*f(k\alpha) = f(\mathcal{A}(k\alpha)) = f(k\mathcal{A}\alpha) = kf(\mathcal{A}\alpha) = k\mathcal{A}^*f(\alpha)$$

Therefore, $\mathcal{A}^*f$ is a linear function, i.e. $\mathcal{A}^*f \in \mathbf{V}^*$

Second step, we need to verify that $\mathcal{A}*$ is a linear transformation in $\mathbf{V}^*$,$\forall f, g \in \mathbf{V}^*$,and $\alpha \in \mathbf{V}$

$$\mathcal{A}^*(f + g)(\alpha) = (f + g)(\mathcal{A}\alpha) = f(\mathcal{A}\alpha) + g(\mathcal{A}\alpha) = (\mathcal{A}^*f + \mathcal{A}^*g)(\alpha) \tag{12}$$

$$\mathcal{A}^*(kf)(\alpha) = kf(\mathcal{A}\alpha) = (k\mathcal{A}^*f)(\alpha) \tag{13}$$

Therefore, $\mathcal{A}^*$ is a linear transformation on $\mathbf{V}^*$

4.2 Matrix representation of $\mathcal{A}^*$

Firstly, let's review the matrix representation of the linear transformation in $\mathbf{V}$. Suppose $\mathcal{A}$ is a linear transformation on $\mathbf{V}$, and its matrix representation is $\mathbf{A}$, whose matrix elements are $\mathbf{A_{ij}}$. Then, we have

$$\mathcal{A}(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\alpha_1, \alpha_2, \cdots, \alpha_n)\mathbf{A}, i.e. \quad \mathcal{A}\alpha_i = \sum_{j=1}^{n} \mathbf{A_{ji}}\alpha_j \tag{14}$$

Similarly, assume $\{\alpha_1^*, \alpha_2^*, \cdots, \alpha_n^*\}$ is a basis of $\mathbf{V}^*$, $\mathcal{A}^*$ is a linear linear transformation on V^* and $\mathbf{A}$ is the matrix representation with elements $\mathbf{A_{ij}}$

$$\mathcal{A}^*(\alpha_1^*, \alpha_2^*, \cdots, \alpha_n^*) = (\alpha_1^*, \alpha_2^*, \cdots, \alpha_n^*)\mathbf{A}^*, i.e. \quad \mathcal{A}^*\alpha_i^* = \sum_{j=1}^{n} \mathbf{A_{ji}^*}\alpha_j^* \tag{15}$$

Let's begin from real vector space, recall that $\forall f \in \mathbf{V}^*$, $f = \sum_{j=1}^{n} f(\alpha_j)\alpha_j^*$,

$$\begin{aligned}
\mathcal{A}^*(\alpha_i^*) &= \alpha_i^*\mathcal{A} \\
&= \sum_{j=1}^{n}(\alpha_i^*(\mathcal{A}(\alpha_j)))\alpha_j^* \\
&= \sum_{j=1}^{n}\alpha_i^*(\sum_{k=1}^{n}\mathbf{A_{kj}}\alpha_k)\alpha_j^* \\
&= \sum_{j,k=1}^{n}\mathbf{A_{kj}}\alpha_i^*(\alpha_k)\alpha_j^* = \sum_{j,k=1}^{n}\mathbf{A_{kj}}\delta_{ik}\alpha_j^* \\
&= \sum_{j=1}^{n}\mathbf{A_{ij}}\alpha_j^* = \sum_{j=1}^{n}\mathbf{A_{ji}^*}\alpha_j^*
\end{aligned}$$

Then, we get $\mathbf{A_{ij}} = \mathbf{A_{ji}^*}$, which means

$$\mathbf{A}^* = \mathbf{A}^{\mathrm{T}} \tag{16}$$

Now, let's study it in complex vector space:

Recall that the linear function can be represented by inner product, so the element α_i^* in $\mathbf{V}^*$ can be viewed as a bra vector $\langle\alpha_i|$, and α_j in $\mathbf{V}$ is a ket vector $|\alpha_j\rangle$. For example, $\mathcal{A}^*(\alpha_i^*) = \langle\sum_{j=1}^{n} A_{ji^*}\alpha_j^*|$, let's rewrite the formula above:

$$\begin{aligned}
\mathcal{A}^*(\alpha_i^*) &= \alpha_i^*\mathcal{A} \\
&= \sum_{j=1}^{n}(\alpha_i^*(\mathcal{A}(\alpha_j)))\alpha_j^* \\
&= \sum_{j=1}^{n}\langle\alpha_i|\mathcal{A}\alpha_j\rangle\langle\alpha_j| \\
&= \sum_{j,k=1}^{n}\langle\alpha_i|\sum_{k=1}^{n}\mathbf{A_{kj}}\alpha_{\mathbf{k}}\rangle\langle\alpha_j| \\
&= \sum_{j,k=1}^{n}\mathbf{A_{kj}}\langle\alpha_i|\alpha_k\rangle\langle\alpha_j| = \sum_{j,k=1}^{n}\mathbf{A_{kj}}\delta_{ik}\langle\alpha_j| \\
&= \sum_{j=1}^{n}\mathbf{A_i j}\langle\alpha_j| \\
&= \langle\sum_{j=1}^{n}\overline{\mathbf{A_{ij}}}\alpha_j|
\end{aligned}$$

Therefore, $\overline{\mathbf{A_{ij}}} = \mathbf{A^*_{ji}}$, which means

$$\mathbf{A}^* = \overline{\mathbf{A}}^{\mathrm{T}} \tag{17}$$

This is why many math text books using symbol" $*$ ", to represent hermitian conjugate. For example, $\mathbf{A}$ is a Hermitian matrix, iff $\mathbf{A} = \mathbf{A}^*$, i.e. $\mathbf{A_{ij}} = \overline{\mathbf{A_{ji}}}$