Using tensor product to study $\frac{1}{2}$-spin particles

Wu Yuhan

2020 Nov. 3

1 Introduction

This article will introduce an application of tensor product in quantum mechanics. Two-particle problem can be very confusing for most of students who study quantum mechanics for first time. If we treat this problem with tensor product, it will be much easier to understand. However, understanding the concept of tensor product is even more challenging. If you are unfamiliar with tensor product, you can read my another article *Construction of tensor product*.

To simplify the problem, we assume that 2 particles are independent; in other words, they do not interact with each other. By applying tensor product, we can view them in one system. Firstly, we start from how to construct the tensor product of two state space. Then we will study the tensor product of operators. Finally, we will study an important example: the total spin.

2 Some preparation

2.1 Construction of tensor product

As I have introduced in *Construction of tensor product*, the tensor product $\otimes$ should be viewed as a bilinear map. And $|\phi\rangle$ can be viewed as a linear functional of its dual space (for more information, reader can read my article *Dual space and inner product*). For example:

$$|\phi\rangle(\langle\Phi|) := \langle\Phi|\phi\rangle \tag{1}$$

By applying a tensor product, we can construct a bilinear functional. For example, $|\phi\rangle \in \mathbf{V_1}$ and $|\psi\rangle \in \mathbf{V_2}$; and their dual space $\langle\Phi| \in \mathbf{V_1}^*$ and $\langle\Psi| \in \mathbf{V_2}^*$. So the bilinear functional are defined as

$$|\phi\rangle \otimes |\psi\rangle(\langle\Phi|, \langle\Psi|) = \langle\Phi|\phi\rangle\langle\Psi|\psi\rangle \tag{2}$$

This suggests that we can just multiply the wave function in many cases. Similarly, we can also construct $\langle\Phi| \otimes \langle\Psi|$.

2.2 One-particle situation

Suppose $\mathbf{V_1}$ is a state space of one $\frac{1}{2}$-spin particle. We choose the eigenvectors of $\mathcal{S}_z$,$\{|+\rangle, |-\rangle\}$ as base, where $\mathcal{S}_z = \frac{\hbar}{2}\sigma_z$ We have

$$\begin{aligned}
\sigma_z|+\rangle &= |+\rangle \\
\sigma_z|-\rangle &= -|-\rangle \\
\sigma_x|+\rangle &= |-\rangle \\
\sigma_x|-\rangle &= |+\rangle
\end{aligned}$$

And it is easy to verify that σ_x has eigenvalue 1 and -1. $\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ is the eigenvector of σ_x belonging to 1; and $\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$ belongs to -1.

3 Tensor product of state space and operators

3.1 Tensor product of state space

In this section, we shall use indices 1 and 2 to distinguish two particles. We assume their state spaces are $\mathbf{V_1}$ and $\mathbf{V_2}$ respectively, and their tensor product is $\mathbf{V} = \mathbf{V_1} \otimes \mathbf{V_2}$.

Since $\{|i;\pm\rangle\}$,i=1,2, are the bases of $\mathbf{V_1}$ and $\mathbf{V_2}$ respectively. By the property of tensor product, $\{|i;\pm\rangle \otimes |j;\pm\rangle\}$, i,j=1,2 is the base of $\mathbf{V_1} \otimes \mathbf{V_2}$

Then, we turns to the usual notation

$$\begin{aligned}
|{+}{+}\rangle &= |1;+\rangle \otimes |2;+\rangle \\
|{+}{-}\rangle &= |1;+\rangle \otimes |2;-\rangle \\
|{-}{+}\rangle &= |1;-\rangle \otimes |2;+\rangle \\
|{-}{-}\rangle &= |1;-\rangle \otimes |2;-\rangle
\end{aligned}$$

4 Tensor product of operators

Suppose $\mathcal{A}$ is an operator on $\mathbf{V_1}$, and $\mathcal{B}$ is an operator on $\mathbf{V_2}$. For arbitrary $\alpha \in \mathbf{V_1}$, $\beta \in \mathbf{V_2}$, the operators in $\mathbf{V_1} \otimes \mathbf{V_2}$ is defined by

$$\mathcal{A} \otimes \mathcal{B}(\alpha \otimes \beta) = \mathcal{A}\alpha \otimes \mathcal{B}\beta \tag{3}$$

Here are some examples on 2 $\frac{1}{2}$-spin particles system.

$$\begin{aligned}
\mathcal{S}_{1z} \otimes \mathcal{S}_{2z}|{+}{+}\rangle &= \mathcal{S}_{1z} \otimes \mathcal{S}_{2z}(|1;+\rangle \otimes |2;+\rangle) \\
&= \mathcal{S}_{1z}|1;+\rangle \otimes \mathcal{S}_{2z}|2;+\rangle \\
&= \frac{\hbar}{2}|1;+\rangle \otimes \frac{\hbar}{2}|2;+\rangle \\
&= \frac{\hbar^2}{4}|1;+\rangle \otimes |2;+\rangle \\
&= \frac{\hbar^2}{4}|{+}{+}\rangle
\end{aligned}$$

Similarly we can calculate:

$$\begin{aligned}
\mathcal{S}_{1z} \otimes \mathcal{S}_{2z}|{+}{-}\rangle &= -\frac{\hbar^2}{4}|{+}{-}\rangle \\
\mathcal{S}_{1z} \otimes \mathcal{S}_{2z}|{-}{+}\rangle &= -\frac{\hbar^2}{4}|{-}{+}\rangle \\
\mathcal{S}_{1z} \otimes \mathcal{S}_{2z}|{-}{-}\rangle &= \frac{\hbar^2}{4}|{-}{-}\rangle
\end{aligned}$$

Another example $\mathcal{S}_{1z} \otimes \mathcal{S}_{1x}$

$$\begin{aligned}
\mathcal{S}_{1z} \otimes \mathcal{S}_{2x}|{+}{+}\rangle &= \mathcal{S}_{1z} \otimes \mathcal{S}_{2x}(|1;+\rangle \otimes |2;+\rangle) \\
&= \mathcal{S}_{1z}|1;+\rangle \otimes \mathcal{S}_{2x}|2;+\rangle \\
&= \frac{\hbar}{2}|1;+\rangle \otimes \frac{\hbar}{2}|2;-\rangle \\
&= \frac{\hbar^2}{4}|{+}{-}\rangle
\end{aligned}$$

We can calculate in the same way:

$$\begin{aligned}
\mathcal{S}_{1z} \otimes \mathcal{S}_{2x}|+-\rangle &= \frac{\hbar^2}{4}|++\rangle \\
\mathcal{S}_{1z} \otimes \mathcal{S}_{2x}|-+\rangle &= -\frac{\hbar^2}{4}|--\rangle \\
\mathcal{S}_{1z} \otimes \mathcal{S}_{2x}|--\rangle &= -\frac{\hbar^2}{4}|-+\rangle
\end{aligned}$$

5 Total spin

5.1 Total spin operator

To simplify the problem, we firstly discuss the spin on z-direction.
The total spin operator is defined as

$$\mathcal{S}_z = \mathcal{S}_{1z} \otimes I_2 + I_1 \otimes s_{2z} \tag{4}$$

So, let's investigate it:

$$\begin{aligned}
\mathcal{S}_z|++\rangle &= (\mathcal{S}_{1z} \otimes I_2 + I_1 \otimes \mathcal{S}_{2z})(|1;+\rangle \otimes |2;+\rangle) \\
&= \mathcal{S}_{1z}|1;+\rangle \otimes I_2|2;+\rangle + I_1|1;+\rangle \otimes \mathcal{S}_{2z}|2;+\rangle \\
&= \frac{\hbar}{2}|1;+\rangle \otimes |2;+\rangle + |1;+\rangle \otimes \frac{\hbar}{2}|2;+\rangle \\
&= \frac{\hbar}{2}|++\rangle + \frac{\hbar}{2}|++\rangle \\
&= \hbar|++\rangle
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}_z|+-\rangle &= (\mathcal{S}_{1z} \otimes I_2 + I_1 \otimes s_{2z})(|1;+\rangle \otimes |2;-\rangle) \\
&= \mathcal{S}_{1z}|1;+\rangle \otimes I_2|2;-\rangle + I_1|1;+\rangle \otimes s_{2z}|2;-\rangle \\
&= \frac{\hbar}{2}|1;+\rangle \otimes |2;-\rangle + |1;+\rangle \otimes (-\frac{\hbar}{2})|2;-\rangle \\
&= 0
\end{aligned}$$

Similarly, $\mathcal{S}_z|-+\rangle = 0$ and $\mathcal{S}_z|--\rangle = -\hbar$

5.2 Total spin of $\mathcal{S}_x$

Since $\mathcal{S}_x = \frac{\hbar}{2}\sigma_x$, we can study the property of σ_x, where

$$\sigma_x = \sigma_{1x} \otimes I_2 + I_1 \otimes \sigma_{2x} \tag{5}$$

$$\begin{aligned}
\sigma_x|++\rangle &= (\sigma_{1x} \otimes I2 + I_1 \otimes \sigma_{2x})(|1;+\rangle \otimes |2;+\rangle) \\
&= \sigma_{1x}|1;+\rangle \otimes I_2|2;+\rangle + I_1|1;+\rangle \otimes \sigma_{2x}|2;+\rangle \\
&= |1;-\rangle \otimes |2;+\rangle + |1;+\rangle \otimes |2;-\rangle \\
&= |-+\rangle + |+-\rangle
\end{aligned}$$

We can calculate in the same way:

$$\begin{aligned}
\sigma_x|+-\rangle &= |--\rangle + |++\rangle \\
\sigma_x|-+\rangle &= |++\rangle + |--\rangle \\
\sigma_x|--\rangle &= |+-\rangle + |-+\rangle
\end{aligned}$$

5.3 The eigenvalue problem of σ_x

We can note that

$$|{+}{+}\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad |{+}{-}\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad |{-}{+}\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad |{-}{-}\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \tag{6}$$

σ_x can be written in a matrix form:

$$\sigma_x = \begin{pmatrix}0&1&1&0\\1&0&0&1\\1&0&0&1\\0&1&1&0\end{pmatrix} \tag{7}$$

Let's solve the eigenvalue problem of σ_x

$$\det(\sigma_x - \lambda I) = 0 \implies \lambda^2(\lambda-2)(\lambda+2) = 0 \tag{8}$$

So σ_x has eigenvalue 0(2 multiplicity),2 and -2.

For $\lambda = 0$, $\dim ker\sigma_x = 2$, we can find 2 linear independent eigenvectors:

$$\xi_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} \quad \xi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} \tag{9}$$

For $\lambda = 2$, $dim\ker(\sigma_x - 2I) = 1$, so we can find an eigenvector

$$\xi_3 = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \tag{10}$$

Similarly, for $\lambda = -2$,

$$\xi_4 = \frac{1}{2}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} \tag{11}$$

5.4 Some discussions

First of all, we need to review some results of one particle system. The eigenvector of σ_x,

$$|+\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle_z + |-\rangle_z) \tag{12}$$

$|+\rangle_z$ and $|-\rangle_z$ are eigenvectors of σ_z. And

$$|-\rangle_x = \frac{1}{\sqrt{2}}(|+\rangle_z - |-\rangle_z) \tag{13}$$

Now let's study the total spin σ_x, in two particles system.

$$\begin{aligned}
|{+}{+}\rangle_x &= |1;+\rangle_x \otimes |2;+\rangle_x \\
&= \frac{1}{\sqrt{2}}(|1;+\rangle_z + |1;-\rangle_z) \otimes \frac{1}{\sqrt{2}}(|2;+\rangle_z + |2;-\rangle_z) \\
&= \frac{1}{2}(|1;+\rangle_z \otimes |2;+\rangle_z + |1;+\rangle_z \otimes |2;-\rangle_z + |1;-\rangle_z \otimes |2;+\rangle_z + |1;-\rangle_z \otimes |2;-\rangle_z) \\
&= \frac{1}{2}(|{+}{+}\rangle_z + |{+}{-}\rangle_z + |{-}{+}\rangle_z + |{-}{-}\rangle_z) \\
&= \xi_3
\end{aligned}$$

$$
\begin{aligned}
|{+}{-}\rangle_x &= |1;+\rangle_x \otimes |2;-\rangle_x \\
&= \frac{1}{\sqrt{2}}(|1;+\rangle_z + |1;-\rangle_z) \otimes \frac{1}{\sqrt{2}}(|2;+\rangle_z - |2;-\rangle_z) \\
&= \frac{1}{2}(|{+}{+}\rangle + |{-}{+}\rangle - |{+}{-}\rangle - |{-}{-}\rangle) \\
&= \frac{1}{\sqrt{2}}(\xi_1 + \xi_2)
\end{aligned}
$$

$$
\begin{aligned}
|{-}{+}\rangle_x &= |1;-\rangle_x \otimes |2;+\rangle_x \\
&= \frac{1}{\sqrt{2}}(|1;+\rangle_z - |1;-\rangle_z) \otimes \frac{1}{\sqrt{2}}(|2;+\rangle_z + |2;-\rangle_z) \\
&= \frac{1}{2}(|{+}{+}\rangle_z + |{+}{-}\rangle_z - |{-}{+}\rangle_z - |{-}{-}\rangle_z) \\
&= \frac{1}{\sqrt{2}}(\xi_2 - \xi_1)
\end{aligned}
$$

$$
\begin{aligned}
|{-}{-}\rangle_x &= |1;-\rangle_x \otimes |2;-\rangle_x \\
&= \frac{1}{\sqrt{2}}(|1;+\rangle_z - |1;-\rangle_z) \otimes \frac{1}{\sqrt{2}}(|2;+\rangle_z - |2;-\rangle_z) \\
&= \frac{1}{2}(|{+}{+}\rangle_z + |{-}{-}\rangle_z - |{+}{-}\rangle_z - |{-}{+}\rangle_z) \\
&= \xi_4
\end{aligned}
$$