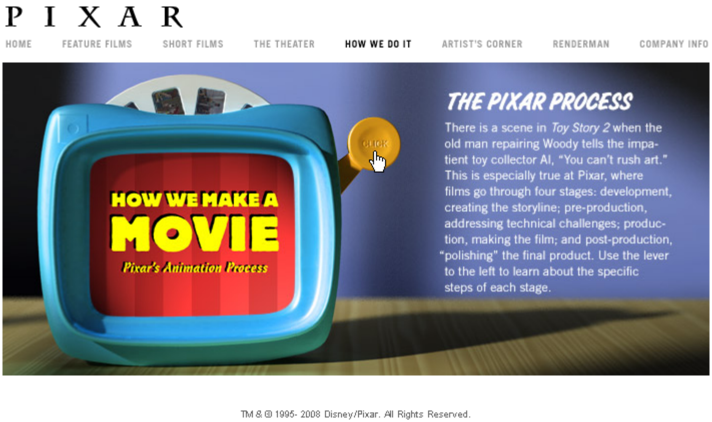
Graphics 2008/2009, period 1

Lecture 2

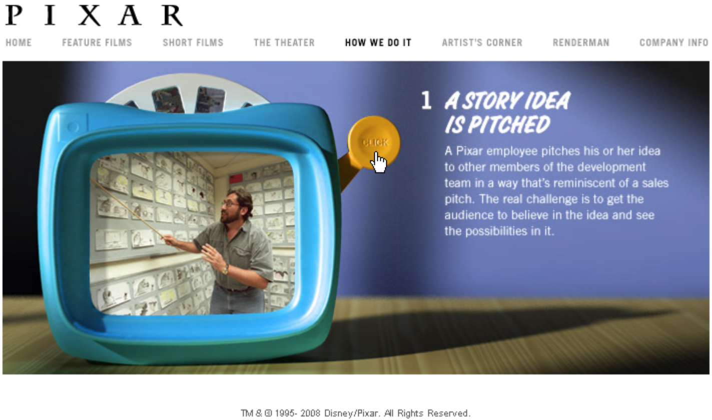
Vectors, curves, and surfaces

Computer graphics example: Pixar



(source: http://www.pixar.com)

Computer graphics example: Pixar



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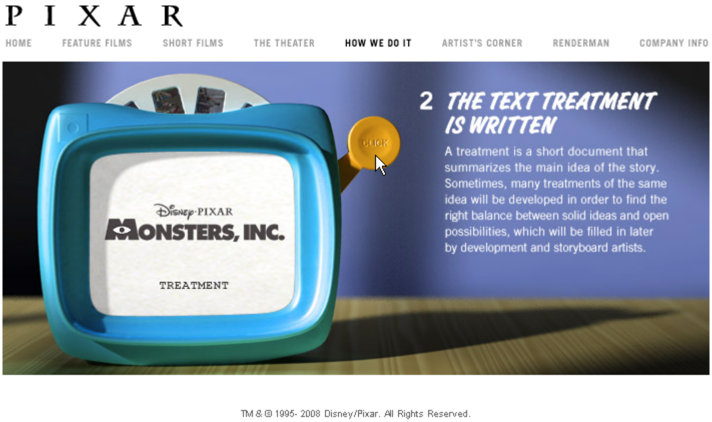
Computer graphics example: Pixar



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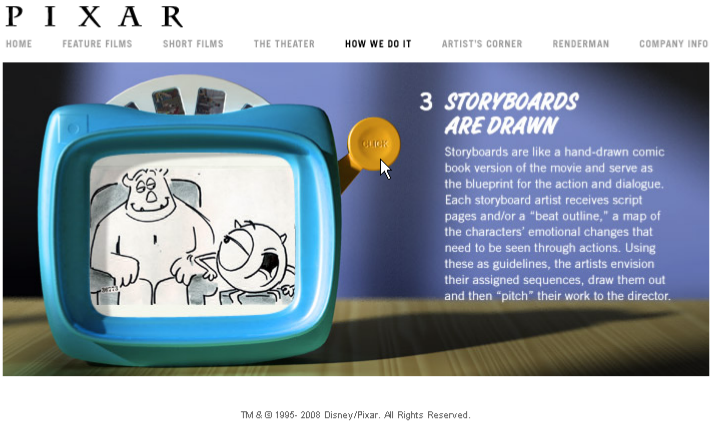
Computer graphics example: Pixar



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Computer graphics example: Pixar

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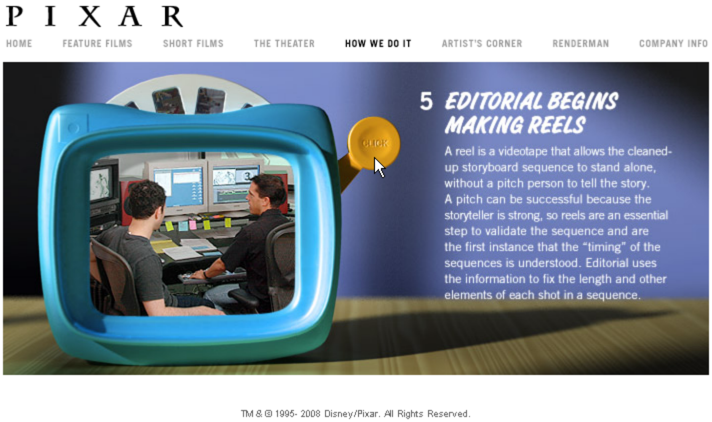
Computer graphics example: Pixar



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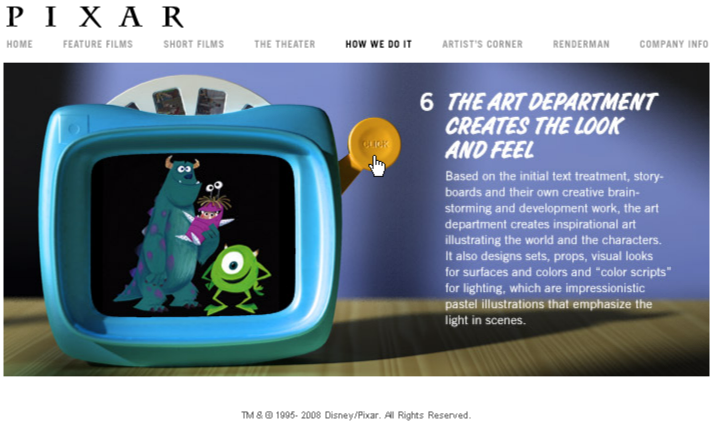
Computer graphics example: Pixar



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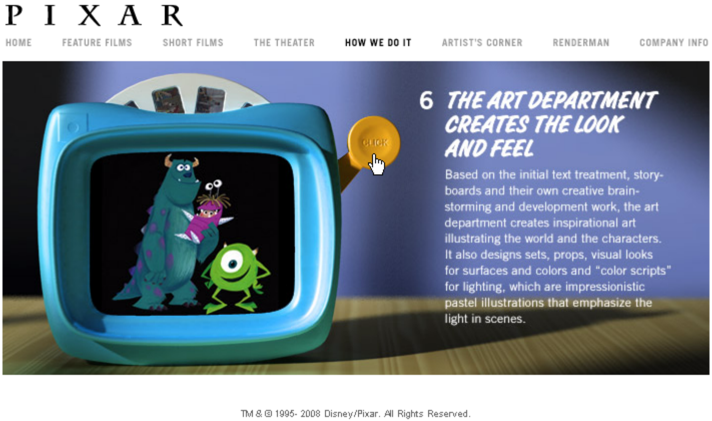
Computer graphics example: Pixar



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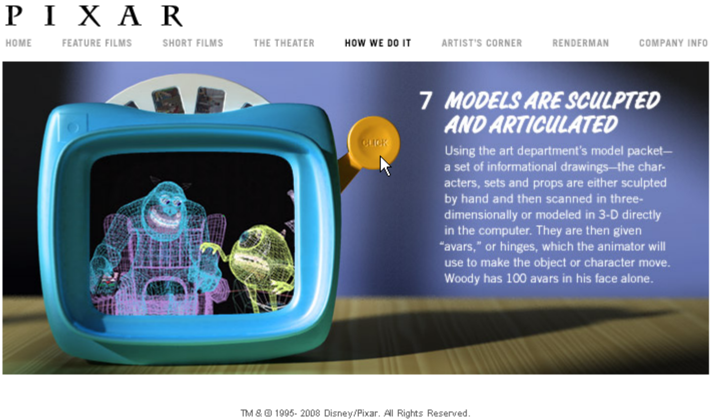
Computer graphics example: Pixar



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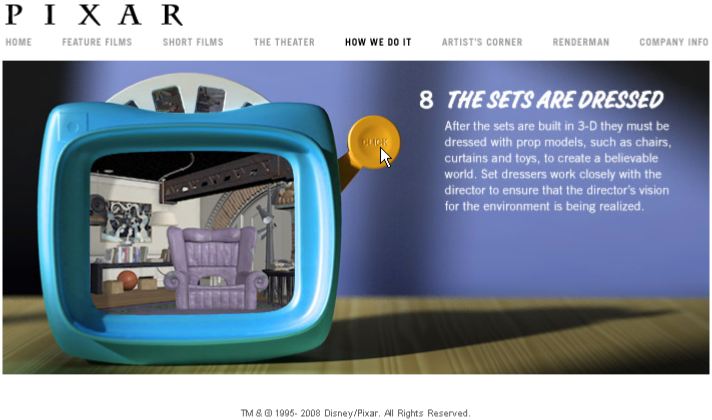
Computer graphics example: Pixar



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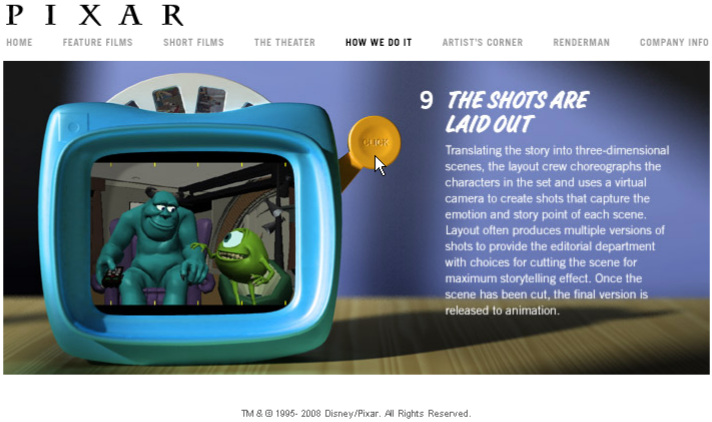
Computer graphics example: Pixar



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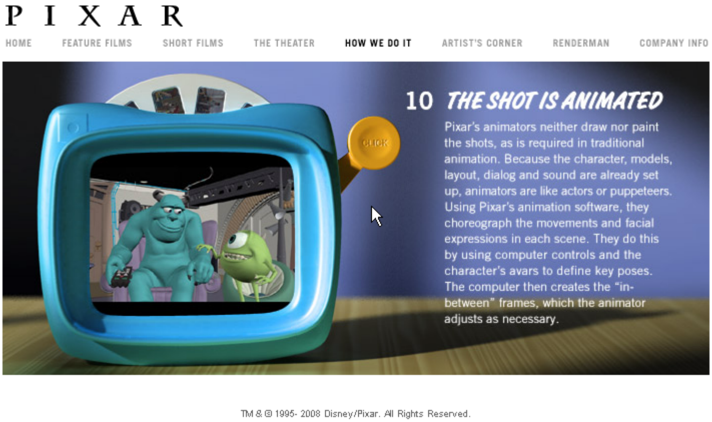
Computer graphics example: Pixar



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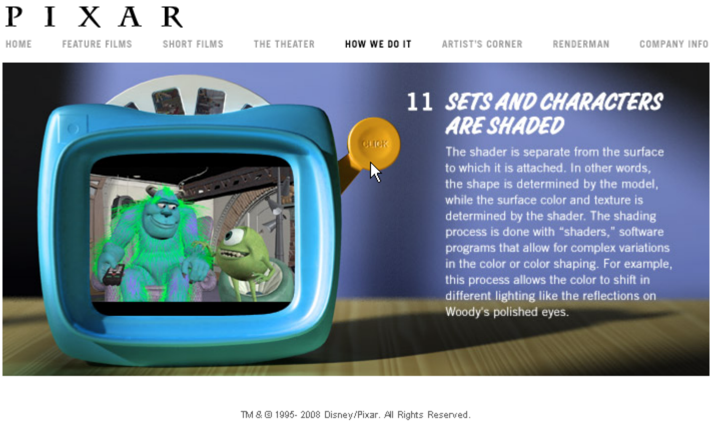
Computer graphics example: Pixar



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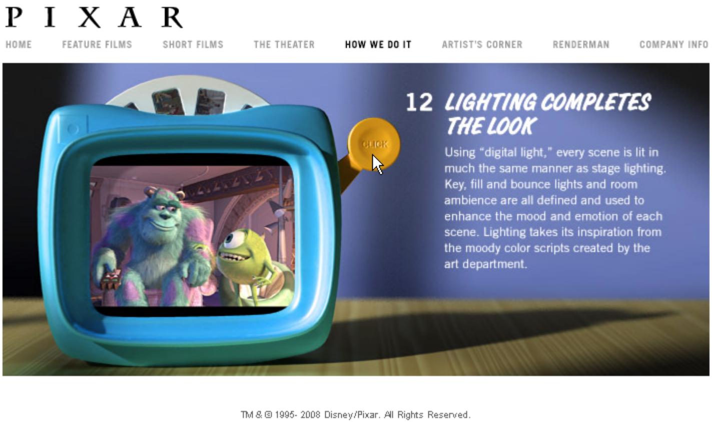
Computer graphics example: Pixar



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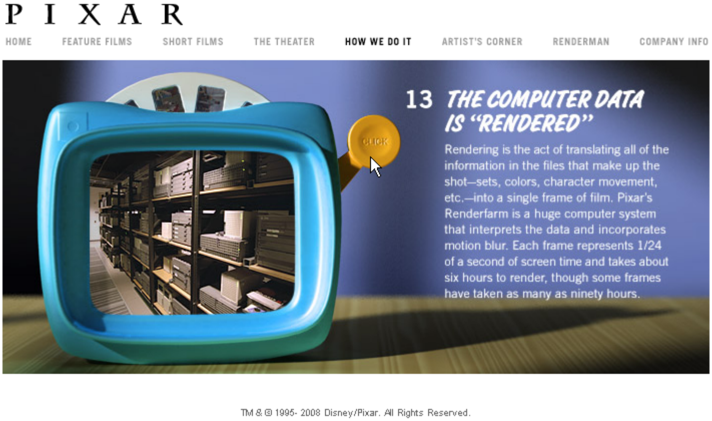

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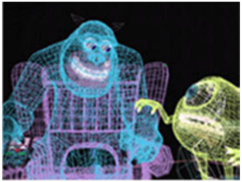


Modeling

Lighting

Rendering

(Animating)

Shading

(source: http://www.pixar.com)

Recap

Computer graphics:

- Modelling (creating 3d virtual worlds)
- Rendering (creating shaded 2d images from 3d models)

Two basic methods:

- Projective methods
- Ray tracing

Last time:

- Lines/rays and spheres
- Intersections between them

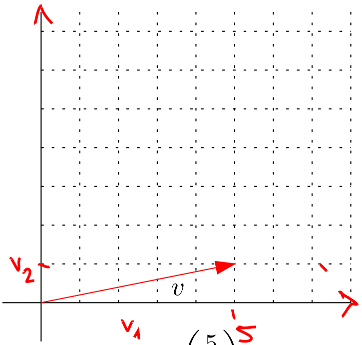
Today: More mathematics to describe objects

Vectors: definition

A vector is specified by a *length* and *direction*

In $\mathbb{R}^d$ it can be defined as an ordered d-tuple $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix}$.

2D



$$v = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Vectors: definition

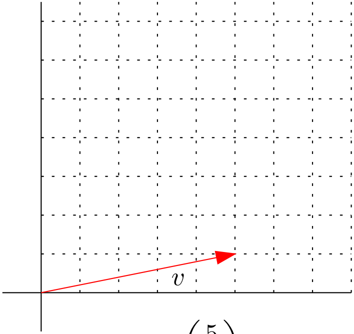
In $\mathbb{R}^3$, for example: $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

or $\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$, or $\begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix}$,

or (v_1, v_2, v_3),

or $(v_1, v_2, v_3)^T$, transposed

or ...

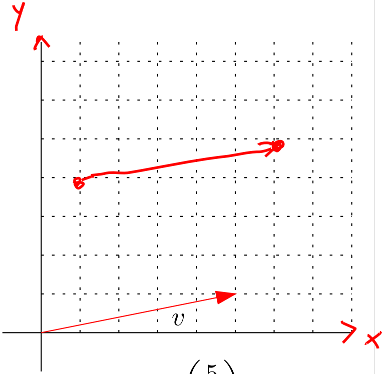


$v = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

Vectors: algebraic interpretation

A 2D vector (x_v, y_v) can be seen as the point (x_v, y_v) in the Cartesian plane.

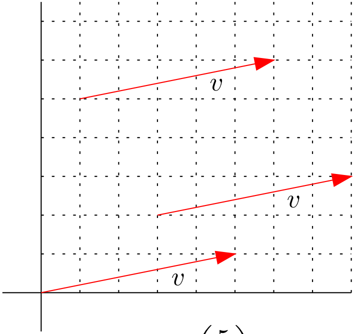
vector ≠ location/position



$$v = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Vectors: geometric interpretation

A 2D vector (x_v, y_v) can be seen as an offset from the origin. Such an offset (arrow) can be translated.



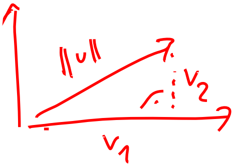
$$v = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

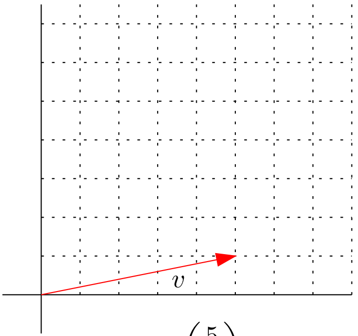
Vectors: length

The Euclidean length of a d-dimensional vector v is

$$||v|| = \sqrt{v_1^2 + v_2^2 + \cdots + v_d^2}$$

2d : Pythagoras $||v||^2 = v_1^2 + v_2^2$





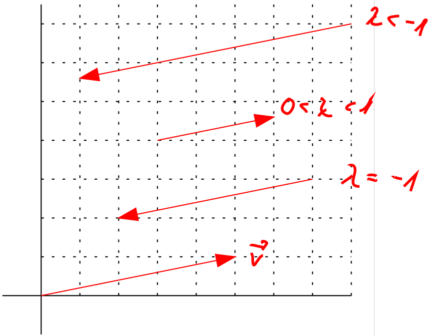
$$v = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Vectors: scalar multiple

A scalar multiple of a d-dimensional vector v is

$$\lambda v = (\lambda v_1, \lambda v_2, \ldots, \lambda v_d)$$

Note that v and λv have the same direction or opposite directions

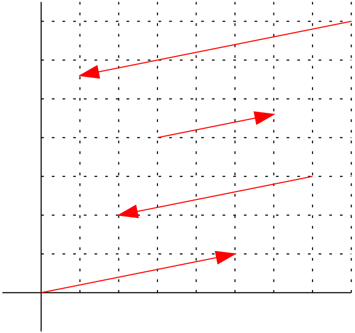


Parallel vectors

Two vectors v_1 and v_2 are parallel if one is a scalar multiple of the other, i.e., there is a λ such that $v_2 = \lambda v_1$.

Note that if one of the vectors is the null vector, then the vectors are considered neither parallel nor not parallel.

length = 0 , direction = undef.



Unit vectors

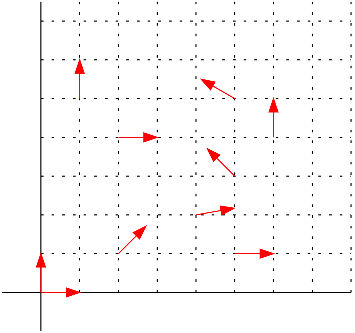
A vector v is a unit vector if $||v|| = 1$.

Normalization

Q: given an arbitrary vector v, how do we find a unit vector parallel to v? Can every vector be normalized?

Null vect.

$$\frac{\vec{v}}{\|\vec{v}\|}$$



Addition of vectors

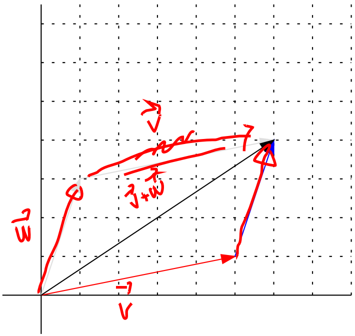
Given two vectors in $\mathbb{R}^d$,
$v = (v_1, v_2, \ldots, v_d)$ and
$w = (w_1, w_2, \ldots, w_d)$

their sum is defined as

$v + w =$
$(v_1 + w_1, v_2 + w_2, \ldots, v_d + w_d)$

$\vec{w} + \vec{v} = (\; \ldots \;)$

Parallelogram Rule



Addition of vectors

$\vec{v} + \vec{w} = \vec{w} + \vec{v}$

Addition of vectors is commutative, as can be seen easily from the geometric interpretation.

Q: show algebraically that vector addition is commutative.

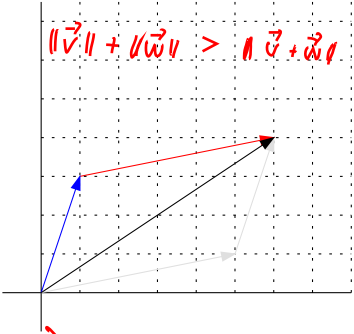
Q: what is the relation between $||v||$, $||w||$, and $||v+w||$?

(not Nullvect.)

$\|\vec{v}\| < \|\vec{v} + \vec{w}\|$

$\|\vec{w}\| < \|\vec{v} + \vec{w}\|$

$\|\vec{v}\| + \|\vec{w}\| > \|\vec{v} + \vec{w}\|$

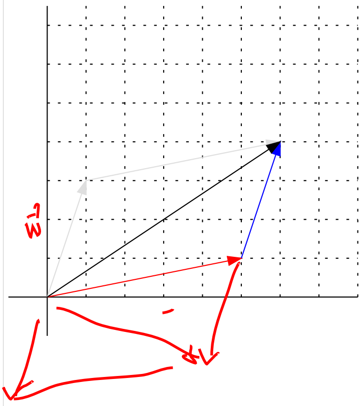


Addition of vectors

Q: How would subtraction be defined?

$$\vec{v} - \vec{w} = \vec{v} + (-1)\vec{w}$$

$$= (v_1 - w_1, \ldots, v_n - w_n)$$

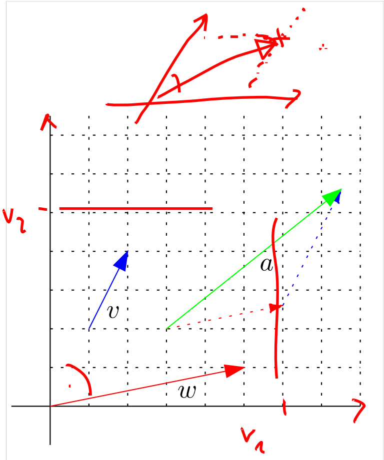


Bases

A 2D vector can be expressed as a combination of any pair of non-parallel vectors. For instance, in the image, $a = 1.5v + 0.6w$.

Such a pair is called linearly independent, and they form a 2D basis.

The extension to higher dimensions is straightforward.



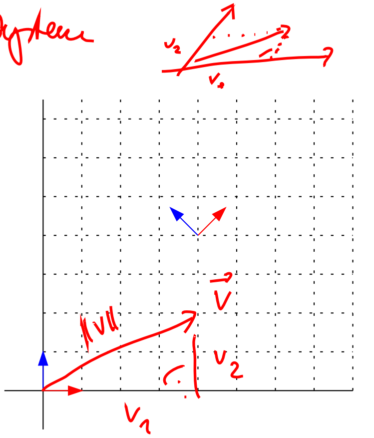
d-dim. space ⇝ d unlin. vectors

Orthonormal basis

Cartesian Coordinates System

Two vectors form an orthonormal basis if (1) they are orthogonal to each other, and (2) they are unit vectors.

The advantage of an orthonormal basis is that lengths of vectors, expressed in the basis, are easy to calculate.



Dot product (inner product, scalar product)

For two vectors $v, w \in \mathbb{R}^d$, the dot product is defined as

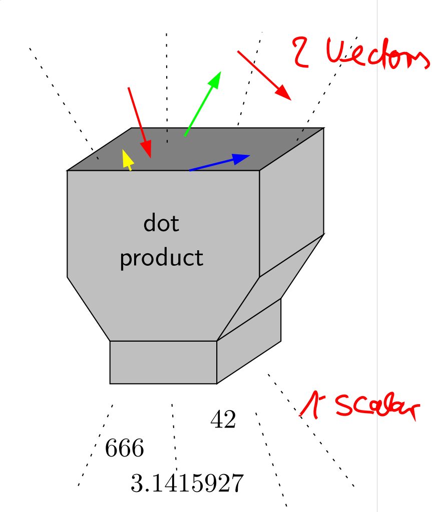
$$v \cdot w = v_1 w_1 + v_2 w_2 + \cdots + v_d w_d,$$

or

$$v \cdot w = \sum_{i=1}^{d} v_i w_i$$

We have that $\cos\theta = \frac{v \cdot w}{||v||\,||w||}$, where θ is the angle between the two vectors.

$\rightarrow \vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\theta$



Dot product (inner product, scalar product)

$$\vec a \cdot \vec b = (x_a \vec x + y_a \vec y) \cdot (x_b \vec x + y_b \vec y) \qquad \vec a = \begin{pmatrix} x_a \\ y_a \end{pmatrix}$$

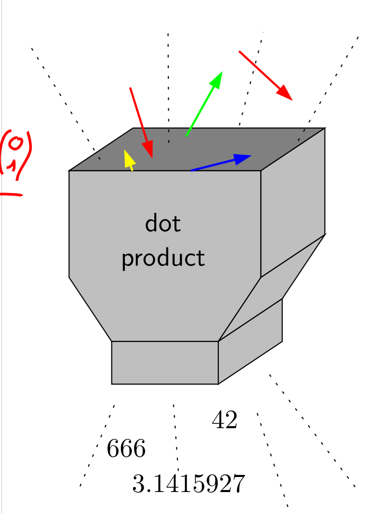
$$= x_a x_b \underbrace{(\vec x \cdot \vec x)}_{1} +$$

$$\dots \left.\begin{array}{l} (\vec x \cdot \vec y) + \\ (\vec y \cdot \vec x) + \end{array}\right\} = 0$$

$$\dots \quad y_a y_b \underbrace{(\vec y \cdot \vec y)}_{1}$$

$$\vec x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= x_a x_b + y_a y_b$$

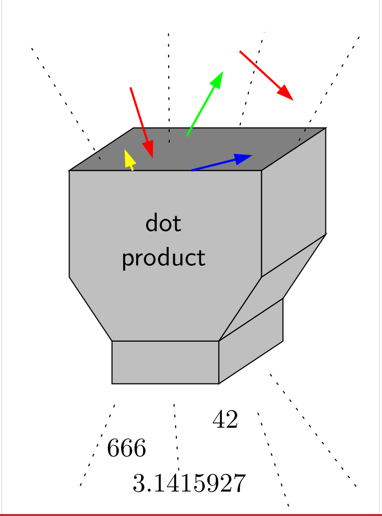


Dot product (inner product, scalar product)

Q: what is the inner product of an arbitrary unit vector with itself? $\cos\Theta \cdot 1 \cdot 1 = 1$

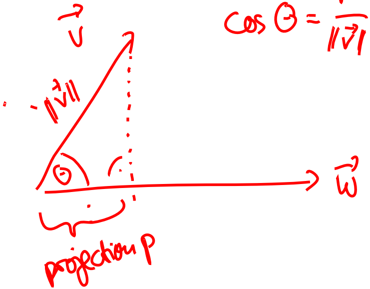
Q: what do we know if for two vectors v and w we have that $v \cdot w = 0$? $\cos\Theta = 0$ $\Theta = 90^\circ$

or Nullvect.

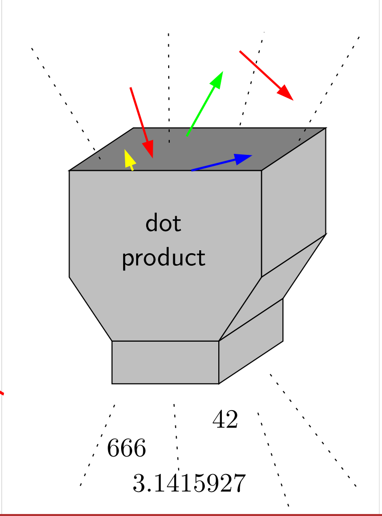


Dot product (inner product, scalar product)

$\cos\Theta = \frac{p}{\|\vec{v}\|}$

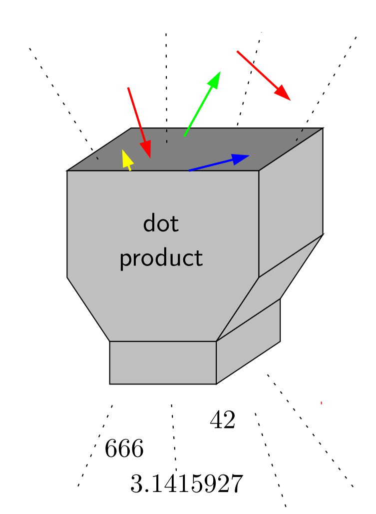


$v \cdot w = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\Theta \quad \leadsto \quad p = \frac{v \cdot w}{\|\vec{w}\|}$



Dot product (inner product, scalar product)

Another use: Calculate the length of the projection of one vector onto another.



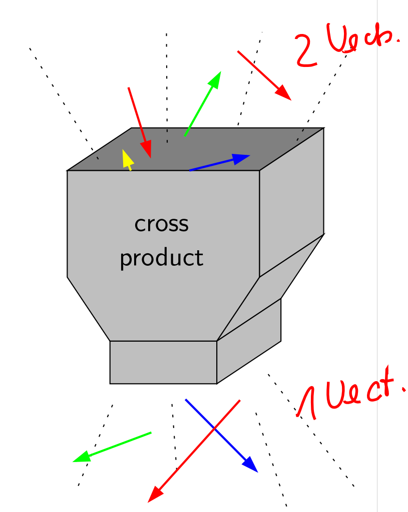
Cross product

3D

For two vectors $v, w \in \mathbb{R}^3$, the cross product is defined as

$$v \times w = \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix}$$

We have that
$||v \times w|| = ||v|| \, ||w|| \sin\theta$,
where θ is the angle between v and w.



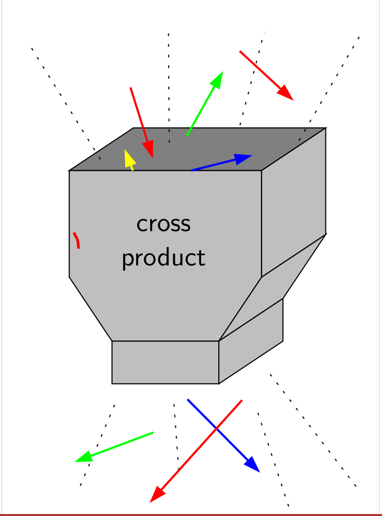
Cross product

Q: show that $v \times w$ is orthogonal to both v and w.

$$\vec{v} \cdot (\vec{v} \times \vec{w}) =$$

$$v_1 v_2 w_3 - v_1 v_3 w_2 + v_2 v_3 w_1 - v_2 v_1 w_3 + v_3 v_1 w_2 - v_3 v_2 w_1 = 0$$

$$\vec{w} \cdot (\ldots) = \ldots = 0$$

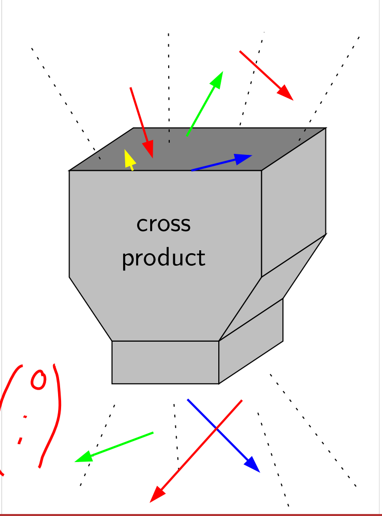


Cross product

Q: what is $v \times v$? Nullvector

Q: what is $v \times w$ if v and w are parallel? Nullvector

$$\vec{w} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{pmatrix} \Rightarrow \vec{v} \times \vec{w} = \begin{pmatrix} \lambda v_2 v_3 - v_3 \lambda v_2 \\ \vdots \\ \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \end{pmatrix}$$



Cross product

1st. coord.: $v_2 w_3 - v_3 w_2$

Q: is it possible or necessary that v and w are orthogonal to form $v \times w$?

unit vect. $\|\vec{v} \times \vec{w}\| = 1 \cdot 1 \cdot \sin 90° = 1$

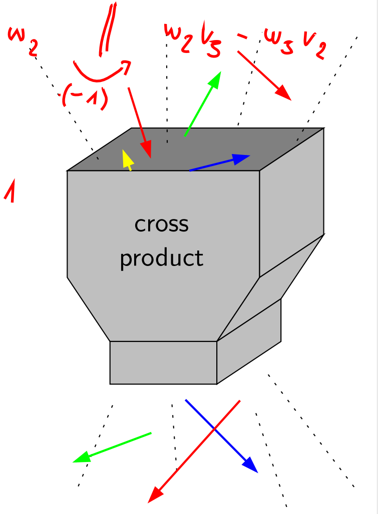
$\vec{v} \times \vec{w}$

$\vec{v}$ $\vec{w}$

$1 \cdot 1 = 1$

$\vec{v} \times \vec{w} \neq \vec{w} \times \vec{v}$

$= -(\vec{w} \times \vec{v})$

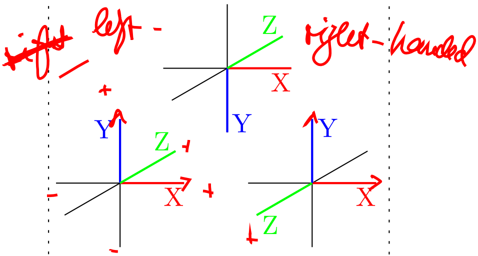


Left- and right-handed systems

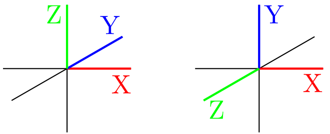
Coordinate systems in 3D come in two flavors: left-handed and right-handed.

There are arguments for both left- and right-handed systems for

- The global system
- The camera system
- Object systems



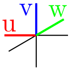
Camera coordinate system



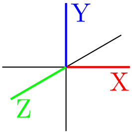
World coordinate system

Coordinate transformations

A frequent operation in graphics is the change from one coordinate system (e.g., the (u, v, w) camera system) to another (e.g., the (x, y, z) global system).



Having orthonormal bases for both systems makes the transformations simpler.

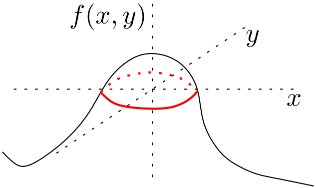


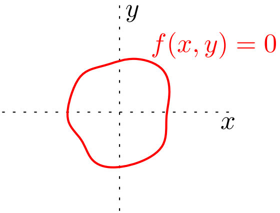
2D Implicit curves

An implicit curve in 2D has the form

$$f(x, y) = 0$$

f maps two-dimensional points to a real value; the points for which this value is 0 are on the curve, while other points are not on the curve.





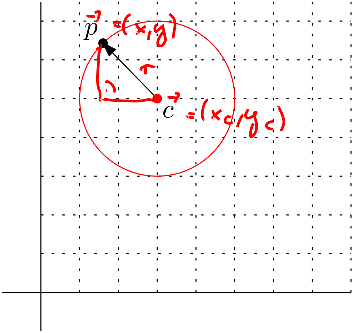
Implicit representation of circles

The implicit representation of a 2D circle with center c and radius r is

$$(x - x_c)^2 + (y - y_c)^2 - r^2 = 0$$

So for any point p that lies on the circle, we have that

$(p - c) \cdot (p - c) - r^2 = 0$, so
$\|p - c\|^2 - r^2 = 0$, which gives
$\|p - c\| = r$.



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$(p - c) \cdot (p - c) - r^2 = 0$, so
$\|p - c\|^2 - r^2 = 0$, which gives
$\|p - c\| = r$.

$(\vec{p} - \vec{c}) \cdot (\vec{p} - \vec{c})$

$= \begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix} \cdot \begin{pmatrix} x - x_c \\ y - y_c \end{pmatrix}$

dot prod.

$= (x - x_c)^2 + (y - y_c)^2$

$\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cdot \cos\theta$

1

$\|\vec{p} - \vec{c}\|^2$

Implicit representation of lines

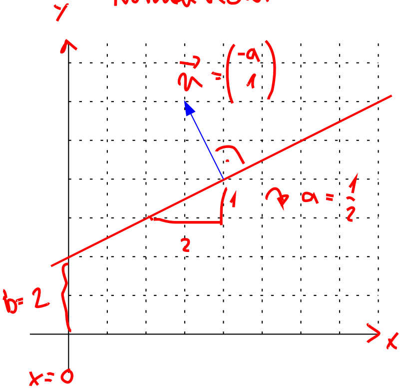
A well-known representation of lines is the slope-intercept form $y = ax + b$ $\quad y = \frac{1}{2}x + 2$

This can easily be converted to $-ax + y - b = 0$.

If $b = 0$, the line intersects the origin, and we have $\boxed{n \cdot p = 0}$, with $\vec{p} = \begin{pmatrix} x \\ y \end{pmatrix}$ and

$$\vec{n} = \begin{pmatrix} -a \\ 1 \end{pmatrix}$$

Normal vector



Implicit representation of lines

$n \cdot p = 0$, with $p = \begin{pmatrix} x \\ y \end{pmatrix}$, $n = \begin{pmatrix} -a \\ 1 \end{pmatrix}$

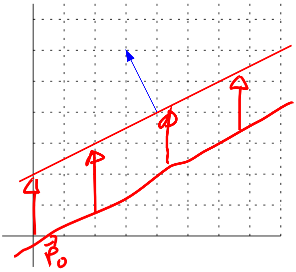
Q: what if the line does not intersect the origin?

$\|\vec{p} - \vec{c}\| = r$

$$\boxed{\vec{n} \cdot (\vec{p} - \vec{p}_0) = 0}$$

$$-a(x - p_{0x}) + (y - p_{0y}) = 0$$

$p_{0x} \doteq 0$ $\qquad$ $p_{0y} \doteq b$ $\qquad$ $\vec{p}_0 = \begin{pmatrix} 0 \\ b \end{pmatrix}$



Implicit representation of lines

Note: The vector $n = \begin{pmatrix} -a \\ 1 \end{pmatrix}$ is orthogonal to the line.

(This is true for the first 2 coefficiants of any implicit line representation.)

Implicit representation of lines

$f(x,y) = ax + by + c = 0$

$\vec{p}_0 = (x_0, y_0)$, $\vec{p}_1 = (x_1, y_1)$

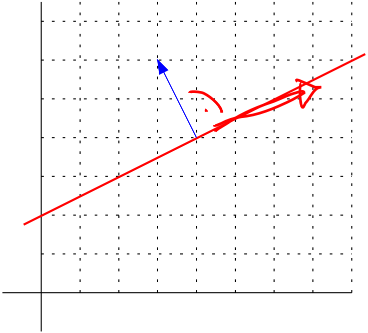
$\vec{p}_1 - \vec{p}_0 \in$ line , $\vec{n} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\vec{n} \cdot (\vec{p}_1 - \vec{p}_0) = \begin{pmatrix} a \\ b \end{pmatrix} \cdot (\ldots)$

$= a(x_1 - x_0) + b(y_1 - y_0)$

$= \ldots \quad + c - c$

$= f(\vec{p}_1) - f(\vec{p}_2) = 0$



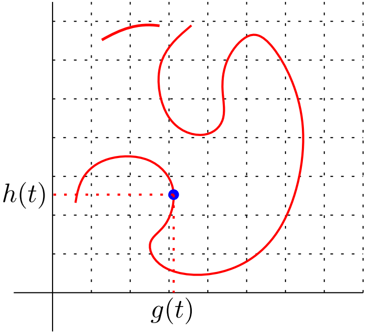
2D parametric curves

A parametric curve is controlled by a single parameter, and has the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} g(t) \\ h(t) \end{pmatrix}$$

time

Parametric representations have some advantages over functions, even if a function would suffice to represent the curve.



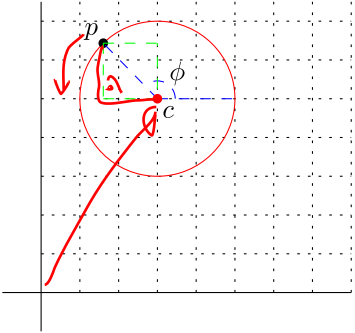
Parametric equation of a circle

The parametric equation of a 2D circle with center c and radius r is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_c + r\cos\phi \\ y_c + \underline{r\sin\phi} \end{pmatrix}$$

$\sin\phi = \frac{y}{r} \rightsquigarrow y = r\sin\phi$

$\cos\phi = \frac{x}{r} \rightsquigarrow x = r \cdot \cos\phi$

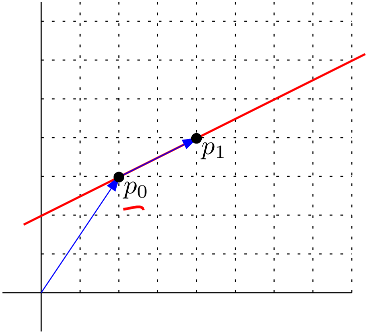


Parametric equation of a line

The parametric equation of a line through the points p_0 and p_1 is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_{p_0} + t(x_{p_1} - x_{p_0}) \\ y_{p_0} + t(y_{p_1} - y_{p_0}) \end{pmatrix}$$

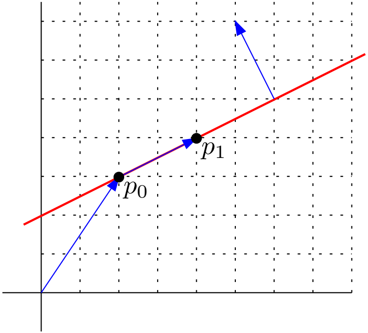
As we have seen before, this can alternatively be written as $p(t) = p_0 + t(p_1 - p_0)$.



Conversion between representations

It is convenient to be able to convert a parametric equation of a line into an implicit equation, and vice versa.

Q: how do we do that?



Conversion between representations

$\vec{n} \cdot (\vec{p} - \vec{p_0})$, $\vec{n} = \begin{pmatrix} -a \\ 1 \end{pmatrix}$, $p_0 = \begin{pmatrix} 0 \\ b \end{pmatrix}$

Q: how do we do that?

- Slope-intercept repr.:
  $y = ax + b$
- Implicit representation:
  $f(x, y) = ax + by + c = 0.$
- Parametric representation (vector form):
  $p(t) = p_0 + t(p_1 - p_0)$

(Note: These are not the same a,b's for the different equations!)