Graphics 2008/2009, period 1

Lecture 4

Matrices

$m \times n$ matrices

The system of m linear equations in n variables $x_1, x_2, \ldots, x_n$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

can be written as a matrix equation by $Ax = b$, or in full

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$m \times n$ matrices

The matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

has m rows and n columns, and is called an $m \times n$ matrix.

Special cases

A square matrix (for which $m = n$) is called a diagonal matrix if all elements a_{ij} for which $i \neq j$ are zero.

If all elements a_{ii} are one, then the matrix is called an identity matrix, denoted with I_m (depending on the context, the subscript m may be left out).

$$\underline{\underline{I}} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$a_{ij} = 0, \quad j \neq i$

and

$a_{ij} = 1, \quad i = j$

If all matrix entries are zero, then the matrix is called a zero matrix, denoted with $\underline{\underline{0}}$.

Matrix addition

For two matrices A and B, we have $A + B = C$, with
$\boxed{c_{ij} = a_{ij} + b_{ij}}$:

$a_{22} + b_{22} = c_{22} = 16$

$$\begin{pmatrix} 1 & 4 \\ 2 & \underline{5} \\ 3 & 6 \end{pmatrix} + \begin{pmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{pmatrix} = \begin{pmatrix} 8 & 14 \\ 10 & 16 \\ 12 & 18 \end{pmatrix}$$

Q: what are the conditions for the dimensions of the matrices A and B? $m_A = m_B$, $n_A = n_B$

Q: how do we define subtraction?

$c_{ij} = a_{ij} - b_{ij}$

Matrix multiplication

Multiplying a matrix with a scalar is defined as follows: $cA = B$, with $b_{ij} = ca_{ij}$. For example,

$$2\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{pmatrix}$$

Matrix multiplication

Multiplying two matrices is a bit more involved.

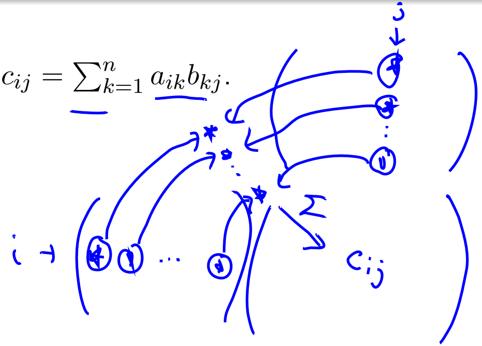
We have $AB = C$ with $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$. For example,

$$\begin{pmatrix} 6 & 5 & 1 & -3 \\ -2 & 1 & 8 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 2 & 2 \\ 37 & 5 & 16 \end{pmatrix}$$

$(m_A \times n)$ $(n \times n_B)$ $i = j = 2$

Matrix multiplication

$AB = C$ with $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.



Q: what are the conditions for the dimensions of A and B?

Q: what are the dimensions of C?

$(m_A \times n_A), (m_B \times n_B)$

$(m_A \times n_B)$

Properties of matrix multiplication

Matrix multiplication is associative and distributive over addition:

$$
\begin{array}{lrcl}
\text{assoc.:} & (\underline{AB})C & = & A(\underline{BC}) \\
\text{distrib.} \left\{ \right. & A(B+C) & = & AB+AC \\
 & (A+B)C & = & AC+BC
\end{array}
$$

However, matrix multiplication is not commutative:
in general, $\underline{AB} \neq \underline{BA}$.

Properties of matrix multiplication

Matrix multiplication is not commutative:
in general, $AB \neq BA$.

$n_A = m_B$, $m_A \neq n_B$

$(m_A \times n_A)(m_B \times n_B)$

$\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}5 & 6\\ 7 & 8\end{pmatrix}$ 19

$\begin{pmatrix}5 & 6\\ 7 & 8\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ 23

Also: if $AB = AC$, it doesn't necessarily follow that $B = C$ (even if A is not the null matrix).

$\begin{pmatrix}1 & -1\end{pmatrix}$ $\begin{pmatrix}1\\ 1\end{pmatrix}$ $\begin{pmatrix}2\\ 2\end{pmatrix}$ ... $\begin{pmatrix}300\\ 300\end{pmatrix}$

0 0 0

Identity and zero matrix revisited

Identity matrix I_m:

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$I \cdot A = A \cdot I = A$

Zero matrix 0: $0 \cdot A = 0 = A \cdot 0$

Transposed matrices

The transpose A^T of an $m \times n$ matrix A is an $n \times m$ matrix that is obtained by interchanging the rows and columns of A: $i \leftrightarrow j$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

Vector: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x\ y\ z)^T$

Transposed matrices

Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Transposed matrices

For the transpose of the product of two matrices we have

$$(AB)^T = B^T A^T$$

Note: $m_A = n_B$

$AB = (n_A \times m_A) \cdot (n_B \times m_B) \; (\ldots c_{ij} \ldots)$

$c_{ij} = a_{i1} b_{1j} + a_{i2} \cdot b_{2j} + \ldots + a_{i m_A} \cdot b_{n_B j}$ //

$(AB)^T$

$\boxed{c_{ij} \leftrightarrow c_{ji}}$

$B^T \cdot A^T = (m_B \times n_B) \cdot (n_A \times m_A) \; (\ldots)$

$c_{ij} = b_{i1} a_{1j} + \ldots + b_{i n_B} \cdot a_{n_A j}$ //

✓

The dot product revisited

If we regard (column) vectors as matrices, we see that the inproduct of two vectors can be written as $u \cdot v = u^T v$:

dot prod. (vect.)

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}4\\5\\6\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\end{pmatrix} \begin{pmatrix}4\\5\\6\end{pmatrix} = (32)$$

matrix multipl.

(A 1×1 matrix is simply a number, and the brackets are omitted.)

Vector: $(1 \times n)$

$(n \times 1)$

Inverse matrices

$(n \times n)$

The inverse of a matrix A is a matrix A^{-1} such that $\boxed{AA^{-1} = I.}$

Only square matrices possibly have an inverse.

Note that the inverse of A^{-1} is A, so we have $AA^{-1} = A^{-1}A = I$

Gaussian elimination

Matrices are a convenient way of representing systems of linear equations:

$$\boxed{A \cdot \vec{x} = \vec{b}}$$

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

$(m \times n)$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = \vec{b} \qquad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

If such a system has a unique solution, it can be solved with Gaussian elimination.

Gaussian elimination

Permitted operations in Gaussian elimination are

- interchanging two rows. ✓
- multiplying a row with a (non-zero) constant. ✓
- adding a multiple of another row to a row. ✓

$$a_{i1} x_1 + a_{i2} x_2 + \dots + a_{in} x_n = b_i \quad \Big| \cdot c$$

Gaussian elimination

Matrices are not necessary for Gaussian elimination, but very convenient, especially augmented matrices. The augmented matrix corresponding to the system of equations on the previous slides is

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

Gaussian elimination: example

Suppose we want to solve the following system:

$$
\begin{aligned}
1\,x + y + 2z &= 17 = 0 \\
2x + y + z &= 15 = 0 \\
1\,x + 2y + 3z &= 26 = 0
\end{aligned}
$$

Q: what is the geometric interpretation of this system?

$$
\begin{aligned}
1x + 0\cdot y + 0\cdot z &= b_1 \\
y &= b_2 \\
z &= b_3
\end{aligned}
$$

Gaussian elimination: example

Applying the rules in a clever order, we get

$$A\ \left(\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 2 & 1 & 1 & 15 \\ 1 & 2 & 3 & 26 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 0 & -1 & -3 & -19 \\ 0 & 1 & 1 & 9 \end{array}\right) \rightsquigarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 17 \\ 0 & 1 & 3 & 19 \\ 0 & 1 & 1 & 9 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 3 & 19 \\ 0 & 0 & -2 & -10 \end{array}\right) \rightsquigarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 3 & 19 \\ 0 & 0 & 1 & 5 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

Rules:

1. exchange rows
2. multipl. row w. scalar
3. add multipl. of 1 row to another

Gaussian elimination: example

The interpretation of the last augmented matrix $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right)$

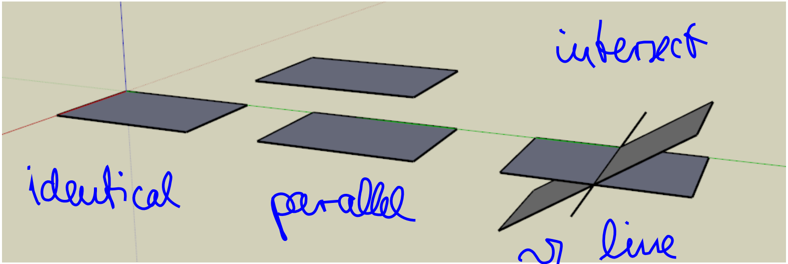
is the very convenient system of linear equations $\boxed{\begin{array}{ccc} x & = & 3 \\ y & = & 4 \\ z & = & 5 \end{array}}$

Q: what is the geometric interpretation of this solution?

Geom. interpretations: intersection of 2 planes

Q: what is the geometric interpretation of this system?

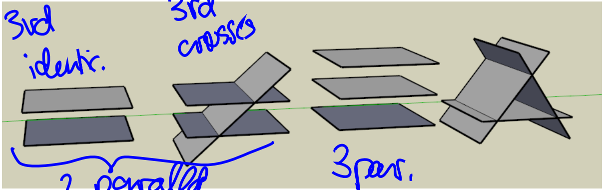
1st: intersection of 2 planes



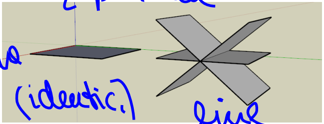
#solutions:

∞ 0 ∞

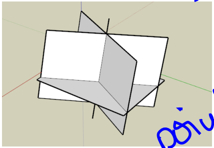
Geom. interpretations: intersection of 3 planes



no sol.



∞ solutions



1 sol.

Gaussian elimination

Q: what if a system can not be reduced to a triangular form?
Q: does this always mean that their is no valid solution?

$$\begin{matrix} 1x & & \\ & 1y & 0 \\ 0 & & 1z \end{matrix} = \left(\quad \right) = p?$$

$$0x + 0 \cdot y + 0 \cdot z = b_1 \neq 0$$

$$0x + 0y + 0z = 0$$

Gaussian elimination

Note: In the literature you also find a slightly different procedure

1st step: *"Put 0's in the lower triangle ..."*
2nd step: *".. then work your way back up"*

$$\left(\begin{array}{ccccc|c} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \end{array}\right) \rightsquigarrow \left(\begin{array}{ccccc|c} * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * \end{array}\right)$$

$\cdot \frac{1}{*}$

Back-ward step

0

Forward step

Gaussian elimination: inverting matrices

The same procedure can also be used to invert matrices.

$$A \cdot A^{-1} = I = A^{-1} \cdot A$$

$$A \cdot x = B \xrightarrow[A^{-1}]{} \underbrace{A^{-1} \cdot A}_{I} x = A^{-1} B \rightsquigarrow x = A^{-1} \cdot B$$

$$Ax = I \cdot B \longleftrightarrow I \cdot x = A^{-1} \cdot B$$

Gaussian elimination: inverting matrices

Example:

$$\overset{A}{\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right)} \; -1 \leadsto \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right) \; 1/2 \leadsto$$

③

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 2 & 5 & 0 & 0 & 1 \end{array}\right) \begin{array}{c} -1 \\ \\ -2 \end{array} \leadsto \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 3 & 1 & -1 & 1 \end{array}\right) \; 1/3 \leadsto$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{array}\right) \; -1 \leadsto \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{6} & -\frac{1}{6} & -\frac{2}{6} \\ 0 & 1 & 0 & -\frac{5}{6} & \frac{5}{6} & -\frac{2}{6} \\ 0 & 0 & 1 & \frac{2}{6} & -\frac{2}{6} & \frac{2}{6} \end{array}\right)$$

$I \qquad A^{-1}$

Gaussian elimination: inverting matrices

The last augmented matrix tells us that the inverse of

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 5 \end{pmatrix}$$

equals

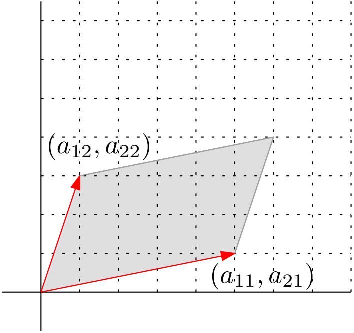
$$A^{-1} = \frac{1}{6} \begin{pmatrix} 7 & -1 & -2 \\ -5 & 5 & -2 \\ 2 & -2 & 2 \end{pmatrix}$$

Determinants

The determinant of a matrix is the signed volume spanned by the column vectors. The determinant $det\,A$ of a matrix A is also written as $|A|$. For example,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$det\,A = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

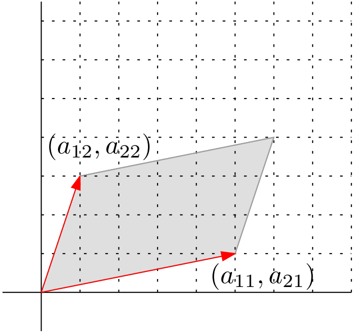


Determinants: geometric interpretation

$|a_1\ a_2| = -|a_2\ a_1|$

In 2D: $detA$ is the oriented area of the parallelogram defined by the 2 vectors.

$$det\ A = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$



In 3D: $detA$ is the oriented area of the parallelepiped defined by the 3 vectors.

Computing determinants

Determinants can be computed as follows:

The determinant of a matrix is the sum of the products of the elements of any row or column of the matrix with their cofactors

If only we knew what cofactors are. . .

Cofactors

Take a deep breath...

The cofactor of an entry a_{ij} in an $n \times n$ matrix a is the determinant of the $(n-1) \times (n-1)$ matrix A' that is obtained from A by removing the i-th row and the j-th column, multiplied by -1^{i+j}.

Right: long live recursion!

Q: what is the bottom of this recursion?

$|a_{xy}| = a_{xy}$

Cofactors

Example: for a 4×4 matrix A, the cofactor of the entry a_{13} is

$$\begin{array}{cc} & \begin{array}{cccc} \phantom{a_{11}} & \phantom{a_{12}} & j=3 & \phantom{a_{14}} \end{array} \\ i=1 & A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \end{array}$$

4×4

$$a^c_{13} = + \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} \cdot (-1)^{1+3}$$

3 × 3

and $|A| = a_{12}a^c_{12} + a_{22}a^c_{22} + a_{32}a^c_{32} + a_{42}a^c_{42}$

Determinants and cofactors

Example

$(-1)^{i+j}$ a^c_{12}

$$\begin{vmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix} = 0\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} - 1\begin{vmatrix} 3 & 5 \\ 6 & 8 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ 6 & 7 \end{vmatrix}$$

$$= 0(32-35) - 1(24-30) + 2(21-24)$$

$$= 0.$$

$$\begin{vmatrix} 3 & 4 \\ 6 & 7 \end{vmatrix}$$

$+3 \cdot 7 - 4 \cdot 6 = 21 - 24$

Systems of linear equations and determinants

Consider our system of linear equations again:

$$\begin{aligned} x_1 + \cancel{y}\,x_2 + 2\cancel{z}\,x_3 &= 17 \\ 2x_1 + y + z &= 15 \\ x + 2y + 3z &= 26 \end{aligned}$$

Such a system of n equations in n unknowns can be solved by using determinants.

In general, if we have $Ax = b$, then $x_i = \frac{|A^i|}{|A|}$. where A^i is obtained from A by replacing the i-th column with b.

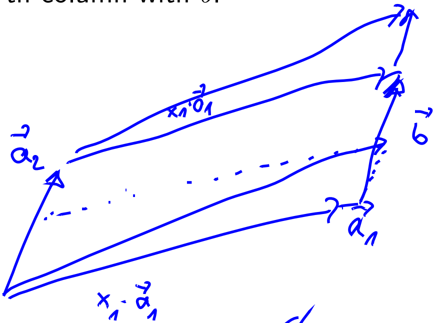
Systems of linear equations and determinants

In general, if we have $Ax = b$, then $x_i = \frac{|A^i|}{|A|}$. where A^i is obtained from A by replacing the i-th column with b.

$$a_{11} x_1 + a_{12} x_2 = b_1$$
$$a_{21} x_1 + a_{22} x_2 = b_2$$

$$|x_1 \vec{a}_1 \; \vec{a}_2| = |\vec{b} \; \vec{a}_2|$$

$$x_1 \cdot \underbrace{|\vec{a}_1 \; \vec{a}_2|}_{A} = \underbrace{|\vec{b} \; \vec{a}_2|}_{A^1}$$



Determinants and systems of linear equations

So for our system

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

$$\begin{aligned} x + y + 2z &= 17 \\ 2x + y + z &= 15 \\ x + 2y + 3z &= 26 \end{aligned} \qquad \begin{pmatrix} 17 \\ 15 \\ 26 \end{pmatrix} = \vec{b}$$

we have

$$x = \frac{\begin{vmatrix} 17 & 1 & 2 \\ 15 & 1 & 1 \\ 26 & 2 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} 1 & 17 & 2 \\ 2 & 15 & 1 \\ 1 & 26 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}} \qquad z = \frac{\begin{vmatrix} 1 & 1 & 17 \\ 2 & 1 & 15 \\ 1 & 2 & 26 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}}$$

Determinants and inverse matrices

Determinants can also be used to compute the inverse A^{-1} of an invertible matrix A:

$$\boxed{A^{-1} = \frac{1}{|A|}\tilde{A}}$$

where $\tilde{A}$ is the adjoint of A, which is the transpose of the cofactor matrix of A.

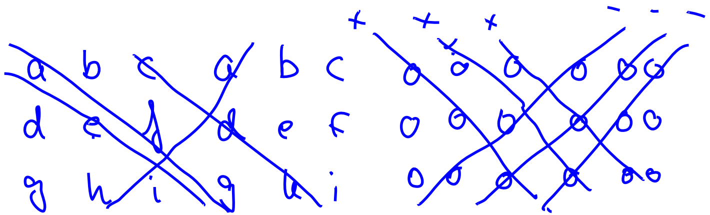
The cofactor matrix of A is obtained from A by replacing every entry a_{ij} by its cofactor a^c_{ij}.

Computing determinants differently

3x3 matrices:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = + a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - \dots$$

$$= \underbrace{(aei + bfg + cd\overset{h}{\cancel{g}})}_{+} - \underbrace{(hfa + idb + gec)}_{-}$$



Computing determinants differently

3x3 matrices:

$$\begin{aligned}\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - \dots \\ &= (aei + bfg + cd\overset{h}{\cancel{g}}) - (hfa + idb + gec)\end{aligned}$$

2x2 matrices:

$+ \qquad -$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Note: This does not generalize to higher dimensions!

Computing determinants differently

Determinants can also be computed by a method that resembles Gaussian elimination:

1 - if A' is obtained from A by exchanging two rows, then $|A'| = -|A|$.
2 - if A' is obtained from A by multiplying a row with a scalar c, then $|A'| = c|A|$.
3 - if A' is obtained from A by adding a multiple of one row of A to another, then $|A'| = |A|$.
- $|I| = 1$
- if A contains any row or column with only zeros, then $|A| = 0$

det A

|I|

0, det A = 0

Computing determinants differently

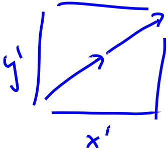
Example

$$\begin{vmatrix}0&1&2\\3&4&5\\6&7&8\end{vmatrix} \overset{2}{=} \frac{1}{2}\begin{vmatrix}0&1&2\\6&8&10\\6&7&8\end{vmatrix} = \frac{1}{2}\begin{vmatrix}0&1&2\\6&7&8\\6&7&8\end{vmatrix} =$$

$$\frac{1}{2}\underbrace{\begin{vmatrix}0&1&2\\0&0&0\\6&7&8\end{vmatrix}}_{\det \to 0} = \frac{1}{2}\cdot 0 = 0.$$

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$



$x' = 2 \cdot x$

$y' = 2 \cdot y$