2024 Online Physics Olympiad: Invitational Contest

Theoretical Examination

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Instructions for Theoretical Exam

The theoretical examination consists of 4 long answer questions over 2 full days from August 30, 12:00AM UTC to September 1, 12:00 AM UTC.

- **The team leader should submit their final solution document in this google form. We don't anticipate a tie, but in the rare circumstance that there is one, the time you submit will be used to break it.**

- If you wish to request a clarification, please use this form. To see all clarifications, view this document.

- Participants are given a Google Form where they are allowed to submit up 50 MB of data for each problem solution. It is recommended that participants write their solutions in LaTeX. However, handwritten solutions (or a combination of both) are accepted too. If participants have more than one photo of a handwritten solution (`jpg, png`, etc), it is required to organize them in the correct order in a `pdf` before submitting. If you wish a premade LaTeX template, we have made one for you here.

- Since each question is a long answer response, participants will be judged on the quality of your work. To receive full points, participants need to show their work, including deriving equations. As a general rule of thumb, any common equations (such as the ones in the IPhO formula sheet) can be cited without proof.

- Remember to state any approximations made and which system of equations were solved after every step. Explicitly showing every step of algebra is not necessary. Participants may leave all final answers in symbolic form (in terms of variables) unless otherwise specified. Be sure to state all assumptions.

Problems

- **T1: Penned Particles**

- **T2: Bouncy Bubble**

- **T3: Stellar Shaping**

- **T4: Hot Solids**

List of Constants

- Proton mass, $m_p = 1.67 \cdot 10^{-27}$ kg
- Neutron mass, $m_n = 1.67 \cdot 10^{-27}$ kg
- Electron mass, $m_e = 9.11 \cdot 10^{-31}$ kg
- Avogadro's constant, $N_0 = 6.02 \cdot 10^{23}$ mol^{-1}
- Universal gas constant, $R = 8.31$ J/(mol $\cdot$ K)
- Boltzmann's constant, $k_B = 1.38 \cdot 10^{-23}$ J/K
- Electron charge magnitude, $e = 1.60 \cdot 10^{-19}$ C
- 1 electron volt, 1 eV $= 1.60 \cdot 10^{-19}$ J
- Speed of light, $c = 3.00 \cdot 10^8$ m/s
- Universal Gravitational constant,
$$G = 6.67 \cdot 10^{-11}\ (\text{N} \cdot \text{m}^2)/\text{kg}^2$$
- Solar Mass
$$M_\odot = 1.988 \cdot 10^{30}\ \text{kg}$$
- Acceleration due to gravity, $g = 9.8$ m/s^2
- 1 unified atomic mass unit,
$$1\ \text{u} = 1.66 \cdot 10^{-27}\ \text{kg} = 931\ \text{MeV/c}^2$$
- Planck's constant,
$$h = 6.63 \cdot 10^{-34}\ \text{J} \cdot \text{s} = 4.41 \cdot 10^{-15}\ \text{eV} \cdot \text{s}$$
- Permittivity of free space,
$$\epsilon_0 = 8.85 \cdot 10^{-12}\ \text{C}^2/(\text{N} \cdot \text{m}^2)$$
- Coulomb's law constant,
$$k = \frac{1}{4\pi\epsilon_0} = 8.99 \cdot 10^9\ (\text{N} \cdot \text{m}^2)/\text{C}^2$$
- Permeability of free space,
$$\mu_0 = 4\pi \cdot 10^{-7}\ \text{T} \cdot \text{m/A}$$
- Magnetic constant,
$$\frac{\mu_0}{4\pi} = 1 \cdot 10^{-7}\ (\text{T} \cdot \text{m})/\text{A}$$
- 1 atmospheric pressure,
$$1\ \text{atm} = 1.01 \cdot 10^5\ \text{N/m}^2 = 1.01 \cdot 10^5\ \text{Pa}$$
- Wien's displacement constant, $b = 2.9 \cdot 10^{-3}$ m $\cdot$ K
- Stefan-Boltzmann constant,
$$\sigma = 5.67 \cdot 10^{-8}\ \text{W/m}^2/\text{K}^4$$

1 Penned Particles

A Penning trap is a device used to store charged particles using static magnetic and electric fields. In this problem we will investigate the motion of an ion inside the trap.

1.1

(a) The trap is a cylinder, parallel to the z-axis, with the origin at the center. Inside, the electric potential is $V = V_0\frac{z^2-r^2}{2d^2}$, where d is the characteristic dimension of the trap. In order to generate the quadrupole field inside, there are two sets of electrodes: two endcaps and the ring electrode, which are held at potential difference V_0, and are solids of revolution. Refer to part (e) for a diagram. Let the minimum distance between endcaps be $2z_0$, and the smallest inside diameter of the ring be $2r_0$.

- Take a cross section parallel to the z-axis through the origin. What are the equations of the cross section of the ring and endcap electrodes?
- Express d in terms of r_0 and z_0.

(b) The magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$ is homogeneous inside the trap. Suppose we have a particle with charge q and mass m. Assume its speed is nonrelativistic, and neglect energy loss from radiation. Throughout the rest of the problem, assume q is positive.

- The z-axis motion is simple harmonic. Find the angular frequency ω_z.
- Write the differential equation for the motion in the xy-plane.
- Suppose $\omega_z = 0$. Solve the differential equation, and find the angular frequency of the motion ω_c. This is the *cyclotron* frequency.

Typically, $\omega_c \gg \omega_z$. Assume this for the rest of the problem.

(c) The motion of the electron in the xy-plane consists of two separate uniform circular motions overlaid on top of each other. One is the cyclotron motion and the other is the *magnetron* motion. Find expressions for the angular frequencies of the cyclotron motion and the magnetron motion, in terms of ω_z and ω_c.

1.2

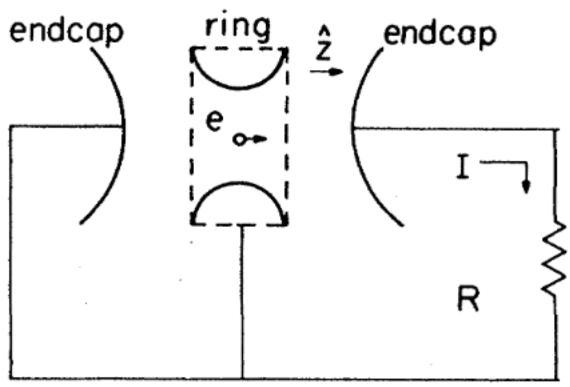
(d) We will now consider the effects of radiation. Typically, the magnetron motion has a much lower frequency than the cyclotron motion, so the decay of the magnetron motion is negligible. The power radiated by an accelerating particle is:

$$P = \frac{q^2a^2}{6\pi\varepsilon_0 c^3}.$$

- The energy of the orbit decays as e^{-t/γ_c}. Find γ_c.
- Now consider the radiation damping of the axial motion. The energy of the oscillation decays as e^{-t/γ_z}. Find γ_z.

(e) For an electron at typical ω_c, γ_c is quite small, allowing for easy damping. However, γ_z is much larger, and for a proton, radiation damping is insignificant. In order to cool large particles, a circuit is used instead. We first consider axial damping.

The oscillations of the ion induce image charges in the electrode, which can be interpreted as a current I. See the following circuit:



You may ignore the quadrupole potential in this part.

- There will be a potential difference of IR between the endcap and the ring (as well as the other endcap). This will produce an electric field $E\hat{\mathbf{z}}$ proportional to I inside the trap. Find E, up to a constant factor κ, which depends on the geometry of the electrodes. *Hint*: if the endcaps are infinite flat planes, κ is equal to 1.

- Consider the power lost through the resistor. Use this to derive the force on the ion, $f = -m\zeta\dot{z}$. Write an expression for ζ.

(f) To conclude, we will consider how to cool the magnetron motion (decrease its radius).

- Find the total energy of the magnetron motion. Assume $z = 0$.

The process works as follows. We shine photons of energy $\hbar(\omega_z + \omega_m)$, which interact with the ion. Let the quantum numbers of the z motion and the magnetron motion be k and l respectively. Then, the cooling transition is from $(k, l) \to (k + 1, l - 1)$, and the heating transition is from $(k, l) \to (k - 1, l + 1)$. Using quantum mechanics, we can derive that these happen at rates proportional to $(k+1)l$ and $k(l+1)$ respectively. The magnetron motion will be cooled until $l = k$, at which point it will be in equilibrium, and there is no long term change in temperature.

- We will now derive the equilibrium energy of the magnetron motion. Assume that at equilibrium, the axial and magnetron motions are at temperatures T_z and T_m respectively. As we continue to shine photons, consider the change in entropy. Use this to derive T_m in terms of ω_m, ω_z, and T_z.

2 Bouncy Bubble

In this problem, we will investigate the interaction between fast oscillations and gradual changes in a physical system.

2.1

A large volume of incompressible, non-viscous liquid with density ρ is kept at temperature T_c and pressure P_c. A spherical bubble consisting of N particles of ideal gas with temperature $T_0 > T_c$ is introduced into the liquid. Neglect surface tension and any heat transfer between the liquid and the gas.

(a) Find the equilibrium radius R_0 of the bubble.

The bubble's radius is perturbed slightly from equilibrium and its oscillations are observed; the gas remains near thermal equilibrium at all times. Assume that the motion of the liquid is laminar and radial, and that the density of the gas is negligible compared to ρ. You may express future answers in terms of R_0.

(b) Find the frequency ω of the bubble's small oscillations.

2.2

Now, assume that the interface between the gas and the liquid has thermal conductance per unit area κ. Then, because of heat loss, the bubble will shrink over time, approaching a final radius R_f. The shrinkage is slow enough that the kinetic energy of the liquid can be neglected.

(c) If the bubble starts at radius R_0, find the approximate time τ until it shrinks to radius $(R_0 + R_f)/2$. Express your answer to the lowest order in the quantity $\alpha = R_0/R_f - 1$.

(d) Next, the bubble starts off oscillating with amplitude $R_0\delta_0$ around radius R_0, where $\delta_0 \ll 1$; assume that the oscillations are much faster than the shrinkage. Find the time-averaged final radius R'_f of the bubble, to the lowest order in δ_0. Qualitatively explain the reason for any difference between R'_f and R_f.

(e) Given the situation in part (d), find the approximate time τ' until the bubble's time-averaged radius shrinks to $(R_0 + R'_f)/2$, to the lowest orders in $\alpha' = R_0/R'_f - 1$ and δ_0.

3 Stellar Shaping

In this problem, we investigate the formation of stellar systems.

3.1

Consider a cloud of dust of radius R of mass M with particles of mass m, all held at a constant temperature T. Assume that $kT \gg GMm/R$; i.e. the particles are far enough apart such that gravitational interactions are nearly negligible.

(a) What is the expected value and variance of the angular momentum of one particle in the $\hat{x}$ direction?

(b) What is variance in the total angular momentum of the cloud, $\langle L^2 \rangle$?

3.2

Suppose some density fluctuations occur, which leads this cloud of gas into gravitational collapse. Now, we must take gravitational interaction into account; assume that the cloud remains at thermal equilibrium and that the total energy of the cloud remains constant.

(c) Assume that the cloud remains spherically symmetric. Find the distribution of densities $\rho(r)$.

(d) What is the new radius of the cloud, R'?

(e) Find the angular velocity ω of the cloud, assuming that the cloud rotates uniformly. Take the total angular momentum of the cloud to be $\sqrt{\langle L^2 \rangle}\,\hat{\mathbf{z}}$, which you found in part (b).

(f) What is the new temperature of the cloud?

3.3

The nebula is not at its most stable state because of the high angular velocity. Suppose that the part of the cloud that reaches beyond a critical density limit ρ_c collapses and begins forming a star.

(g) Find the initial radius of collapse, R_c, and the mass of the star M_s. Assume the radius of the star is a lot smaller than R_c.

For the last two parts, we will assume that the gravitational potential is quadratic, $U = \frac{1}{2}k(x^2 + y^2 + z^2)$, and the angular velocity of the particles is ω. Leave answers in terms of the variables given in this part.

(h) Suppose all the leftover material, some N particles at temperature T', begins to settle into a gas. What is the expected value for r^2, the distance of these particles to the axis of rotation, once they reach their most stable state?

(i) What is the variance in the orbital inclination for this leftover material–that eventually begins to form asteroids and planets?

4 Hot Solids

In this problem, we investigate a one-dimensional model of atoms in a solid. Assume the atoms are point masses of mass m connected by springs with spring constant κ and rest length a, and the total rest length of the chain is L.

4.1

First, assume that the mass is spread continuously throughout the chain (in other words, a is very small). Here, longitudinal waves have the same speed for all values of the angular frequency ω and wavenumber k.

(a) Find this speed of sound in the solid, v, up to a dimensionless constant.

Now, we get rid of this assumption and solve fully.

(b) Find a dispersion relation (a relationship between ω and k) for the chain of atoms if a is not required to be small. Use this result to find the dimensionless constant from part (a).

4.2

We can use the above results to find the heat capacity of the chain. To do so, treat each possible frequency ω as its own quantum harmonic oscillator (QHO) with a particle of mass m moving in a potential defined by $V(x) = \frac{1}{2}m\omega^2x^2$. Each of these harmonic oscillators is at thermal equilibrium, and the total energy of the chain is the sum of the contributions from each frequency. You may find the following integrals useful:

$$\int_0^\infty \frac{x}{e^x - 1}\,dx = \frac{\pi^2}{6}, \quad \int_0^\infty \frac{x^3}{e^x - 1}\,dx = \frac{\pi^4}{15}$$

(c) First, derive the energy levels of a quantum harmonic oscillator by using the *WKB approximation*:

$$\oint p(x)\,dx = 2\pi\hbar(n + 1/2) \tag{1}$$

Here, $p(x)$ is the momentum of the particle as a function of position and the integral is across one classical period.

(d) Using the model from part (a), derive the total energy and heat capacity as a function of the temperature T. (Your result only needs to hold for $\beta\hbar\omega_{avg} \gg 1$, with $\beta = 1/k_BT$.) Assume that the atoms at either end of the chain must remain fixed in place.

(e) Using the dispersion relation from part (b), find the energy and heat capacity to the next order in T.

(f) Above we assumed $\beta\hbar\omega_{avg} \gg 1$. Why do our results fail for high T?

4.3

When the mass-energy of a particle is small compared to its energy level, relativistic corrections are required. The relativistic energy levels of a particle in a harmonic oscillator potential are given by:

$$E_n = mc^2\left(-1 + \sqrt{1 + \frac{2\hbar\omega}{mc^2}\left(n + \frac{1}{2}\right)}\right) \tag{2}$$

(g) Use the given energy levels to find the total energy and heat capacity of the chain where each particle is moving relativistically; you may assume that the dispersion relation is linear as in part (d). Give your answer to the lowest order in $\hbar\omega/mc^2$.