TSSL: Exercise session 1

Multi-step predictor of AR(p) model

In the lectures we have only discussed how a time series model can be used for making one-step-ahead predictions. That is, predicting y_{t+1} conditionally on all values up to time t. However, in many practical applications it is of interest to use the model to predict further into the future. For instance, in an epidemiological application we might wish to predict the state of a disease one month from now, even though the observations (disease cases, say) are recorded daily.

In this exercise we will generalize the 1-step prediction for an AR(p) model to a general k-step prediction.

1. Assume that $\{y_t : t = 1, 2, \dots\}$ is generated by an AR(p) process.

   (a) Derive an expression for the 2-step-ahead predictor, which we define as as the conditional mean of the process at time $t + 2$, knowing all values up to time t,

   $$\hat{y}_{t+2|t} := \mathbb{E}[y_{t+2}|y_{1:t}]. \tag{1}$$

   (b) Similarly, derive an expression for the k-step-ahead predictor,

   $$\hat{y}_{t+k|t} := \mathbb{E}[y_{t+k}|y_{1:t}]. \tag{2}$$

   (c) Assume that we have observed the process $y_{1:n}$ up to some fixed time point n. Based on these observations, we wish to predict the process m steps into the future, i.e., we wish to compute

   $$\hat{y}_{t|n} \qquad \text{for } t = n + 1, \dots, n + m.$$

   Based on the expressions for the k-step predictor above, write a few lines of pseudo-code detailing how this can be done.

Basic properties of AR(1)

2. Consider an AR(1) process, $y_t = ay_{t-1} + \varepsilon_t$ where ε_t is white noise with variance σ_ε^2.

   (a) Assume that the initial state is deterministic, $y_1 = c \neq 0$. Compute the mean function $\mu(t) = \mathbb{E}[y_t]$ for all values of $t \geq 1$.

   (b) Compute an expression for the marginal variance $\mathrm{Var}(y_t)$ for arbitrary value of t.

   (c) Is the process *stationary*?

   (d) What can be said about the mean function and the marginal variance as $t \to \infty$?

3. An AR(1) process with $a = 1$ is referred to as a *random walk*. Assume that $\{y_t : t = 1, 2, \dots\}$ is given by a random walk with $y_1 = c$ (deterministic).

   (a) Derive an expression for y_t at arbitrary t involving *only* noise terms ε_s at different time points s and the initial value c.

   (b) Based on this expression, what is the mean function $\mu(t)$ and the marginal variance $\mathrm{Var}(y_t)$ of the process? Compare with the expressions derived in the previous exercise.