(a) T1

$$p(y) = \int_\theta p(y|\theta) \cdot p(\theta)\, d\theta$$

$$p(x|\theta) = \theta^s (1-\theta)^f$$

$$p(\theta) = \frac{1}{B(\alpha,\beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}$$

$$p(y) = \frac{1}{B(\alpha,\beta)} \int_\theta \theta^{\alpha+s-1} (1-\theta)^{\beta+f-1} d\theta$$

$$= \frac{B(\alpha+s, \beta+f)}{B(\alpha,\beta)}$$

(b)

$$\ln p(y) = \ln p(y|\hat{\theta}) + \ln p(\hat{\theta}) + \frac{1}{2}|J^{-1}_{\hat{\theta},y}| + \frac{p}{2}\ln(2\pi)$$

We only have one parameter θ so $p=1$

$$\hat{\theta} = \frac{\alpha+s-1}{\alpha+\beta+n-2} \qquad J^{-1}_{\hat{\theta},y} = \frac{(\alpha+s-1)(\beta+f-1)}{(\alpha+\beta+n-2)^3}$$

$$\ln p(y|\hat{\theta}) = \ln(\hat{\theta}^s (1-\hat{\theta})^f), \quad \ln p(\hat{\theta}) = \frac{\hat{\theta}^{\alpha-1}(1-\hat{\theta})^{\beta-1}}{B(\alpha,\beta)}$$

(c)

T2

(1)

$$p(y_i|\theta) = (1-\theta)^{y_i}\theta$$

$$p(y|\theta) = (1-\theta)^{\sum y_i}\theta^{n}$$

$$p(\theta) = \frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}, \quad B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$p(y) = \int_\theta p(y|\theta)\cdot p(\theta)\, d\theta$$

$$= \frac{1}{B(\alpha,\beta)}\cdot\int_\theta \theta^{\alpha+n-1}(1-\theta)^{\sum y_i+\beta-1}\, d\theta$$

$$= \frac{B(\alpha+n, \beta+\sum y_i)}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+n)\Gamma(\beta+\sum_{i=1}^{n} y_i)\cdot\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\beta+n+\sum_{i=1}^{n} y_i)}$$

(2)

$$p(y_i|\theta) = \frac{\theta^{y_i}e^{-\theta}}{y_i!}$$

$$p(y|\theta) = \frac{\theta^{\sum y_i}\cdot e^{-n\theta}}{\prod_{i=1}^{n} y_i!}$$

$$p(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}$$

$$p(y) = \int_\theta p(y|\theta)\cdot p(\theta)\, d\theta$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)\cdot\prod_{i=1}^{n} y_i!}\cdot\int_\theta \theta^{\alpha+\sum y_i-1}\cdot e^{-(n+\beta)\theta}\, d\theta$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)\cdot\prod_{i=1}^{n} y_i!}\cdot\left(\frac{(n+\beta)^{\alpha+\sum y_i}}{\Gamma(\alpha+\sum y_i)}\right)^{-1} = \frac{\Gamma(\alpha+\sum_{i=1}^{n} y_i)\cdot\beta^\alpha}{\Gamma(\alpha)\cdot(n+\beta)^{(\alpha+\sum y_i)}}\cdot\frac{1}{\prod_{i=1}^{n} y_i!}$$

T3

(1) $$p(\theta) = \frac{\alpha \cdot \beta^{\alpha}}{\theta^{\alpha+1}} \cdot I(\theta \geq \beta)$$

$$p(x|\theta) = \frac{1}{\theta^{n}} \cdot \prod_{i=1}^{n} I(\theta \geq x_i)$$

$$= \frac{1}{\theta^{n}} \cdot I(\theta \geq x_{max})$$

$$p(\theta|x) \propto \frac{1}{\theta^{n+\alpha+1}} \cdot I[\theta \geq \max(\beta, x_{max})]$$

$\therefore$ $\theta|x \sim Pareto(n+\alpha, \bar{\beta})$, where $\bar{\beta} = \max(\beta, x_{max})$

(2) $$p(x_{n+1}|x_{1:n}) = \int_0^{\infty} p(x_{n+1}|\theta) \cdot p(\theta|x_{1:n})\, d\theta$$

$$= \int_0^{\infty} \frac{1}{\theta} \cdot I(\theta \geq x_{n+1}) \cdot \frac{(n+\alpha)\bar{\beta}^{(n+\alpha)}}{\theta^{n+\alpha+1}}\, d\theta$$

$$= (n+\alpha) \cdot \bar{\beta}^{(n+\alpha)} \cdot \int_0^{\infty} \frac{1}{\theta^{n+\alpha+2}} \cdot I(\theta \geq x_{n+1}, \theta \geq \bar{\beta})\, d\theta$$

i) $x_{n+1} \leq \bar{\beta}$

$$p(x_{n+1}|x_{1:n}) = (n+\alpha) \cdot \int_0^{\infty} \frac{(n+\alpha+1) \cdot \bar{\beta}^{(n+\alpha+1)}}{\theta^{n+\alpha+2} \cdot \bar{\beta} \cdot (n+\alpha+1)}\, d\theta$$

$$= \frac{n+\alpha}{(n+\alpha+1)\bar{\beta}}$$

$\therefore$ $x_{n+1} \sim \frac{n+\alpha}{n+\alpha+1} \cdot Uniform(0, \bar{\beta})$

ii) $x_{n+1} \geq \bar{\beta}$

$$p(x_{n+1}|x_{1:n}) = \frac{n+\alpha}{(n+\alpha+1) \cdot x_{n+1}}$$