

Decision Theory Assignment1

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2022-10-01

Assignment1

Question 1

The first question concerns a bank official concerned about the rate at which the five tellers working at the bank provide service for their customers. The official feels that the tellers all work at about the same speed; serving either 30, 40 or 50 customers per hour. Of these three estimates, 40 customers per hour is deemed to be twice as likely as the other two, which are assumed to be equally likely. To test this theory the official observes the five tellers for a two-hour period, during which the tellers serve 380 customers.

It should then be shown that the posterior probabilities for the three possible speeds are approximately 0.000045, 0.99996 and 0.00000012 respectively.

Answer1

From the example exists in the course slides, we can assume bank server into Poisson process. There will be λ customers being severed per hour averagely. And we can also set duration of time t (hours) and in total r customers are severed. The posterior probability can be written as:

$$P(\lambda|r, t) \propto \frac{(\lambda t)^r e^{-\lambda t}}{r!} P(\lambda)$$

And for each λ , we can have:

$$P(\lambda_i|r, t) = \frac{\frac{(\lambda_i t)^r e^{-\lambda_i t}}{r!} P(\lambda_i)}{\sum_{j=1}^3 \frac{(\lambda_j t)^r e^{-\lambda_j t}}{r!} P(\lambda_j)}$$

Where $P(\lambda_j)$ is the prior of each λ_j .

Note that t^r and $r!$ and be canceled since both numerator and denominator have these components.

$$P(\lambda_i|r, t) = \frac{\lambda_i^r e^{-\lambda_i t} P(\lambda_i)}{\sum_{j=1}^3 \lambda_j^r e^{-\lambda_j t} P(\lambda_j)}, i = 1, 2, 3$$

From the description, we know that:

$$\lambda_1 = 30 * 5 = 150, P(\lambda_1) = 0.25$$

$$\lambda_2 = 40 * 5 = 200, P(\lambda_2) = 0.5$$

$$\lambda_3 = 50 * 5 = 250, P(\lambda_3) = 0.25$$

$$t = 2 \text{ and } r = 380$$

When we do the calculation, we can change $t = 120(\text{mins})$ and $\lambda_i = \lambda_i/60$ to avoid overflow

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r = 380
t = 120

l1 = 150/60
p1 = 0.25

l2 = 200/60
p2 = 0.5

l3 = 250/60
p3 = 0.25

n1 = l1^r*exp(-t*l1)*p1
n2 = l2^r*exp(-t*l2)*p2
n3 = l3^r*exp(-t*l3)*p3
cons = n1+n2+n3
n1 = n1/cons
n2 = n2/cons
n3 = n3/cons
df = data.frame(lambda = c(30,40,50),pro = c(n1,n2,n3))
knitr::kable(df,format = "pandoc")

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lambda	pro
30	0.0000448
40	0.9999550
50	0.0000001

Question 2

Assume that data follows a beta distribution with parameters a and b , i.e. the probability density function is $f(y|a, b) = y^{a-1}(1-y)^{b-1}/B(a, b)$ where $B(a, b)$ is the beta function. This could be the case when data consists of the analysed purity (in percent) of a narcotic substance in a number of seized packages.

- a) Show – by using the properties of distributions belonging to the exponential class –that the probability density function of a conjugate family to the family of beta distributions can be expressed as

$$f'(a, b|\gamma, \sigma, \theta) \propto \frac{e^{\gamma a + \sigma b}}{B(a, b)^\theta}$$

- b) Find the corresponding expression for the probability density function of the posterior distribution when a sample of size n has been obtained

Answer(a)

We can rewrite formula in the question in :

$$f(y|a, b) = e^{(a-1)\ln y + (b-1)\ln(1-y) - \ln B(a, b)} = e^{a\ln y + b\ln(1-y) - \ln y - \ln(1-y) - \ln B(a, b)}$$

And from the course slides, we can ‘decompose’ the formula in the form of exponential family with:

$$A_1(\theta) = a$$

$$A_2(\theta) = b$$

$$B_1(y) = \ln y$$

$$B_2(y) = \ln(1 - y)$$

$$C(y) = -\ln y - \ln(1 - y)$$

$$D(\theta) = -\ln B(a, b)$$

By using the property:

the prior should be :

$$\exp(A_1(\theta) * \alpha_1 + A_2(\theta) * \alpha_2 + D(\theta) * \alpha_3)$$

We can plug in our notation above, the probability density function of a conjugate family to the family of beta distributions can be expressed as:

$$\frac{e^{a*\alpha_1 + b*\alpha_2}}{B(a, b)^{\alpha_3}}$$

Answer(b)

The likelihood can be written as

$$L(y_{1:n}|a, b) \propto \exp(a * \sum_{i=1}^n \ln y_i + b \sum_{i=1}^n \ln (1 - y_i) - \sum_{i=1}^n \ln (y_i - y_i^2) - n \ln B(a, b))$$

From the question (a), we can have the prior(now we use the notation $\alpha_1 \rightarrow \gamma$, $\alpha_2 \rightarrow \delta$ and $\alpha_3 \rightarrow \theta$)

$$p(a, b) = \exp(a * \gamma + b * \delta - \theta * \ln B(a, b))$$

So, the posterior

$$q(a, b|y_{1:n}, \gamma, \delta, \theta) \propto \exp(a * (\gamma + \sum_{i=1}^n \ln y_i) + b * (\delta + \sum_{i=1}^n \ln (1 - y_i)) - \sum_{i=1}^n \ln (y_i - y_i^2) - \ln B(a, b)^{\theta+n})$$