Kálman Filter

Assume that $\alpha_t | y_{1:t-1} \sim N(\hat{\alpha}_{t|t-1}, P_{t|t-1})$

Let $v_t = y_t - \hat{y}_{t|t-1}$

Then $$\begin{pmatrix} \alpha_t \\ v_t \end{pmatrix} \Big| y_{1:t-1} \sim N\left(\begin{pmatrix} \mathbb{E}[\alpha_t | y_{1:t-1}] \\ \mathbb{E}[v_t | y_{1:t-1}] \end{pmatrix}, \begin{pmatrix} Cov(\alpha_t | y_{1:t-1}) & Cov(\alpha_t, v_t | y_{1:t-1}) \\ Cov(\alpha_t, v_t | y_{1:t-1})^T & Cov(v_t | y_{1:t-1}) \end{pmatrix} \right)$$

with:

$\mathbb{E}[\alpha_t | y_{1:t-1}] = \hat{\alpha}_{t|t-1}$ ← From assumption

$$\mathbb{E}[v_t | y_{1:t-1}] = \mathbb{E}[y_t - \hat{y}_{t|t-1} | y_{1:t-1}] = \mathbb{E}[Z\alpha_t + \varepsilon_t - Z\hat{\alpha}_{t|t-1} | y_{1:t-1}]$$
$$= Z\,\mathbb{E}[\alpha_t | y_{1:t-1}] - Z\hat{\alpha}_{t|t-1} = 0.$$

$Cov(\alpha_t | y_{1:t-1}) = \mathbb{E}[(\alpha_t - \hat{\alpha}_{t|t-1})(\alpha_t - \hat{\alpha}_{t|t-1})^T | y_{1:t-1}] = P_{t|t-1}$ ← From assumption

$$Cov(v_t | y_{1:t-1}) = \mathbb{E}[v_t v_t^T | y_{1:t-1}] = \mathbb{E}[(Z\alpha_t + \varepsilon_t - Z\hat{\alpha}_{t|t-1})(Z\alpha_t + \varepsilon_t - Z\hat{\alpha}_{t|t-1})^T | y_{1:t-1}]$$
$$= Z \cdot \mathbb{E}[(\alpha_t - \hat{\alpha}_{t|t-1})(\alpha_t - \hat{\alpha}_{t|t-1})^T | y_{1:t-1}] Z^T + \sigma_\varepsilon^2 = Z P_{t|t-1} Z^T + \sigma_\varepsilon^2$$

$$Cov(\alpha_t, v_t | y_{1:t-1}) = \mathbb{E}[(\alpha_t - \hat{\alpha}_{t|t-1}) v_t^T | y_{1:t-1}] = \mathbb{E}[\alpha_t \cdot (Z\alpha_t + \varepsilon_t - Z\hat{\alpha}_{t|t-1})^T | y_{1:t-1}]$$
$$= \mathbb{E}[\alpha_t (\alpha_t - \hat{\alpha}_{t|t-1})^T | y_{1:t-1}] Z^T = P_{t|t-1} Z^T$$

This gives us

$$\begin{pmatrix} \alpha_t \\ v_t \end{pmatrix} \Big| y_{1:t-1} \sim N\left(\begin{pmatrix} \hat{\alpha}_{t|t-1} \\ 0 \end{pmatrix}, \begin{pmatrix} P_{t|t-1} & P_{t|t-1} Z^T \\ Z P_{t|t-1} & \underbrace{Z P_{t|t-1} Z^T + \sigma_\varepsilon^2}_{= F_{t|t-1}} \end{pmatrix} \right)$$

Using the conditional lemma we get that

$$\alpha_t | v_t, y_{1:t-1} \sim N(\hat{\alpha}_{t|t-1} + P_{t|t-1} Z^T (F_{t|t-1}^{-1}) v_t,\ P_{t|t-1} - P_{t|t-1} Z^T F_{t|t-1}^{-1} Z P_{t|t-1})$$

where $\alpha_t | v_t, y_{1:t-1} = \alpha_t | y_{1:t}$ is the filter distribution.