Mathematical Exercises 2

Problem 1

a) $\sigma^2 \sim \text{Inv-}\chi^2(\nu_0, \sigma_0^2)$, θ is known
$X = (x_1, \dots, x_n)$

Prior: $$p(\sigma^2) \propto \frac{\exp\left[\frac{-\nu_0\sigma_0^2}{2\sigma^2}\right]}{\sigma^{2(1+\nu_0/2)}}$$

Likelihood:
$$p(x|\sigma^2,\theta) = \prod_{i=1}^{n} \frac{1}{(2\pi\sigma^2)^{1/2}} \cdot \exp\left[-\frac{1}{2\sigma^2}(x_i-\theta)^2\right]$$
$$\propto \frac{1}{(\sigma^2)^{n/2}} \cdot \exp\left[-\frac{ns^2}{2\sigma^2}\right], \quad \text{where} \quad s^2 = \frac{\sum_{i=1}^{n}(x_i-\theta)^2}{n}$$

Posterior:
$$p(\sigma^2|x,\theta) \propto p(x|\sigma^2,\theta)\, p(\sigma^2) \propto \frac{\exp\left[\frac{-(\nu_0\sigma_0^2+ns^2)}{2\sigma^2}\right]}{\sigma^{2(1+\underbrace{\frac{\nu_0}{2}+\frac{n}{2}}_{(\nu_0+n)/2})}},$$

i.e, $\sigma^2|x,\theta \sim \text{Inv-}\chi^2(\nu_n, \sigma_n^2)$

$\nu_n = \nu_0 + n$

$$\nu_n\sigma_n^2 = \nu_0\sigma_0^2 + ns^2 \iff \sigma_n^2 = \frac{\nu_0\sigma_0^2+ns^2}{\nu_n} = \frac{\nu_0\sigma_0^2+ns^2}{\nu_0+n}$$

b) Non-informative prior: let $\nu_0 \to 0 \Rightarrow p(\sigma^2) \propto \frac{1}{\sigma^2}$, $n=3$, $\theta=1$

$$s^2 = \frac{\sum_{i=1}^{3}(x_i-\theta)^2}{n} = 1.68$$

So, $\sigma^2|x,\theta \sim \text{Inv-}\chi^2(3, 1.68)$

Problem 2 , Posterior predictive distribution of x_{n+1} , given $x_{1:n}$:

a)

$$p(x_{n+1}|x_{1:n}) = \int p(x_{n+1}|\theta) \cdot p(\theta|x_{1:n}) \, d\theta$$

$$p(x_{n+1}|\theta) = \theta^{x_{n+1}} \cdot (1-\theta)^{1-x_{n+1}}$$

$$\theta|x_{1:n} \sim \text{Beta}(\alpha+s, \beta+f) \text{ , so } p(\theta|x_{1:n}) = \frac{\Gamma(\alpha+s+\beta+f)}{\Gamma(\alpha+s)\Gamma(\beta+f)} \cdot \theta^{\alpha+s-1} \cdot (1-\theta)^{\beta+f-1}$$

$$p(x_{n+1}|x_{1:n}) = \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+s)\,\Gamma(\beta+f)} \cdot \underbrace{\int \theta^{\overbrace{x_{n+1}+\alpha+s}^{\alpha'}-1}(1-\theta)^{\overbrace{1-x_{n+1}+\beta+f}^{\beta'}-1} d\theta}_{= B(\alpha',\beta') = \frac{\Gamma(\alpha')\,\Gamma(\beta')}{\underbrace{\Gamma(\alpha+\beta+n+1)}_{(\alpha+\beta+n)\Gamma(\alpha+\beta+n)}}}$$

$$= \frac{\Gamma(\alpha')\,\Gamma(\beta')}{\Gamma(\alpha+s)\,\Gamma(\beta+f)(\alpha+\beta+n)}$$

So, $x_{n+1} = 1 \Rightarrow p(x_{n+1}=1|x_{1:n}) = \dfrac{\Gamma(\alpha+s+1)}{\Gamma(\alpha+s)(\alpha+\beta+n)} = \dfrac{\alpha+s}{\alpha+\beta+n}$

$x_{n+1} = 0 \Rightarrow p(x_{n+1}=0|x_{1:n}) = \dfrac{\alpha+\beta+n-(\alpha+s)}{\alpha+\beta+n} = \dfrac{\beta+f}{\alpha+\beta+n}$

b) $\theta \sim \text{Beta}(\alpha=1, \beta=1)$, $n = 10$, $s = 8$

$$Pr(x_{11}=1|x_{1:10}) = \frac{1+8}{1+1+10} = 0.75 \Rightarrow Pr(x_{11}=0|x_{1:10}) = 0.25$$

$$\left.\begin{aligned} EU_{bring} &= 0.75 \cdot 20 + 0.25 \cdot 10 = 17.5 \\ EU_{leave} &= 0.75 \cdot 50 - 0.25 \cdot 50 = 25.0 \end{aligned}\right\} \text{leave the umbrella}$$

c) Change the values for α and β and calculate new EU_{bring} and EU_{leave}