PP1

1a) Derive $\hat{y}_{t+2|t}$ for AR(p)

$$\hat{y}_{t+2|t} \triangleq \mathbb{E}[y_{t+2} | y_{1:t}]$$

$$= \Big/\!\!\Big/ \quad y_{t+2} = \underbrace{a_1 y_{t+1} + a_2 y_t + \dots + a_p y_{t+2-p}}_{\text{deterministic given } y_{1:t}} + \underbrace{\varepsilon_{t+2}}_{\text{mean zero}} \quad \Big/\!\!\Big/$$

$$= a_1 \underbrace{\mathbb{E}[y_{t+1} | y_{1:t}]}_{:= \hat{y}_{t+1|t}} + a_2 y_t + \dots + a_p y_{t+2-p}$$

where $\hat{y}_{t+1|t} = a_1 y_t + \dots + a_p y_{t+1-p}$

1b) $\hat{y}_{t+k|t} = \mathbb{E}[y_{t+k} | y_{1:t}]$

$$= \mathbb{E}\left[a_1 y_{t+k-1} + \cdots + a_p y_{t+k-p} \,|\, y_{1:t} \right]$$

$$= \sum_{j=1}^{p} a_j \underbrace{\mathbb{E}[y_{t+k-j} | y_{1:t}]}$$

$$= \begin{cases} \hat{y}_{t+k-j|t} & \text{if } k > j \\ y_{t+k-j} & \text{if } k \leq j \end{cases}$$

Hence, if we define

$\hat{y}_{s|t} := y_s$ for $s \leq t$ we get the recursive expression

$$\hat{y}_{t+k|t} = \sum_{j=1}^{p} a_j \hat{y}_{t+k-j|t}$$

1c) From 1b, we know that the prediction $\hat{y}_{n+k|n}$ is given by a linear combination of the previous p prediction, where if $n+k-j \leq n$ we replace $\hat{y}_{n+k-j|n}$ with the observed value y_{n+k-j}.

Pseudo-code:

$$\varphi_{n+1} = (y_n \; \cdots \; y_{n-p+1})^T$$

For $k = 1, \ldots, m$

$$\hat{y}_{n+k|n} = \theta^T \varphi_{n+k}$$

$$\varphi_{n+k+1} \leftarrow \begin{pmatrix} \hat{y}_{n+k|n} \\ [\varphi_{n+k}]_{1:p-1} \end{pmatrix}$$

2a) $y_t = a y_{t-1} + \varepsilon_t$

$\mu(t) = \mathbb{E}[y_t] = \mathbb{E}[a y_{t-1} + \varepsilon_t]$

$= a \mu(t-1)$

Since $\mu(1) = \mathbb{E}[y_1] = c$ we get

$\mu(t) = a^{t-1} c$

b) $Var(y_1) = 0$ since y_1 is a deterministic constant.

$$Var(y_t) = Var(\underbrace{a y_{t-1} + \varepsilon_t}_{\text{independent}})$$

$$= a^2 Var(y_{t-1}) + \sigma_\varepsilon^2$$

$$= a^2 \left[a^2 Var(y_{t-2}) + \sigma_\varepsilon^2\right] + \sigma_\varepsilon^2$$

$$= a^4 Var(y_{t-2}) + a^2 \sigma_\varepsilon^2 + \sigma_\varepsilon^2$$

$$= \ldots = \sigma_\varepsilon^2 \sum_{j=1}^{t-1} a^{2(j-1)}$$

c) No, if $a \neq 1$ then the mean function is not constant

$$\mu(t) \neq \mu(t-1)$$

and if $a=1$, then the variance is not constant,

$$Var(y_t) = (t-1)\,\sigma_\varepsilon^2$$

d) Mean:

If $|a|<1$, $\mu(t) \to 0$ at an exponential rate as $t \to \infty$

Variance:

When $t \to \infty$ we get a geometric series

$$\sigma_\varepsilon^2 \sum_{j=1}^{\infty} a^{2(j-1)} = \sigma_\varepsilon^2 \frac{1}{1-a^2} \quad (\text{if } |a|<1)$$

For $|a| \geq 1$, $|\mu(t)| \to \infty$ and $Var(y_t) \to \infty$

3a) $y_t = y_{t-1} + \varepsilon_t$

$= y_{t-2} + \varepsilon_{t-1} + \varepsilon_t$

$= \dots = y_1 + \varepsilon_2 + \dots + \varepsilon_t$

$$= y_1 + \underbrace{\sum_{s=2}^{t} \varepsilon_s}_{\text{mutually independent}}$$

b) $$\mathbb{E}[y_t] = \underbrace{\mathbb{E}[y_1]}_{=c} + \sum_{s=2}^{t} \underbrace{\mathbb{E}[\varepsilon_s]}_{=0}$$

$$= c$$

$$\mathrm{Var}(y_t) = \mathrm{Var}(y_1) + \sum_{s=2}^{t} \mathrm{Var}(\varepsilon_s)$$

$$= (t-1)\sigma_\varepsilon^2$$

c) Mean is constant and variance increases linearly.