Q1

$I_i \sim Bern(\theta_i)$ 计算时假设θ_i为 $I_i=2$ 为泊努利正例 ~~的概率~~

后验 其他所有参数 似然 先验

$$p(I_i=2 \mid I_{-i}, x, \cdot) \propto p(x \mid I_i, I_{-i}, \cdot) \cdot p(I_i=2)$$

$p(I_i=2) = 1-\pi$ ；π是 $I_i=1$ 的先验概率

正态

$$p(x \mid I_i, I_{-i}, \cdot) = \prod_{j=1}^{n} p(x_j \mid I_j, \cdot) \propto p(x_i \mid I_i, \cdot) = \phi(x_i; \mu_2, \sigma_2^2)$$

吉布斯采样中，采第i个时，其他都可视为定量，所以除了 ith 以外都是常数

$$\therefore\ p(I_i=2 \mid I_{-i}, x, \cdot) \propto (1-\pi)\cdot\phi(x_i; \mu_2, \sigma_2^2)$$

同理 $p(I_i=1 \mid I_{-i}, x, \cdot) \propto \pi\cdot\phi(x_i; \mu_1, \sigma_1^2)$

$$p(I_i=2 \mid I_{-i}, x, \cdot) = C\cdot(1-\pi)\cdot\phi(x_i; \mu_2, \sigma_2^2)$$

$$p(I_i=1 \mid I_{-i}, x, \cdot) = C\cdot\pi\cdot\phi(x_i; \mu_1, \sigma_1^2)$$

$$\because\ p(I_i=2 \mid \sim) + p(I_i=1 \mid \sim) = 1$$

$$\therefore\ p(I_i=2 \mid \sim) = \frac{(1-\pi)\cdot\phi(x_i; \mu_2, \sigma_2^2)}{\pi\cdot\phi(x_i; \mu_1, \sigma_1^2) + (1-\pi)\cdot\phi(x_i; \mu_2, \sigma_2^2)}$$

即后验泊努利 θ_i

2、

点估计

① $V(\hat{\theta}) = V(\frac{1}{n}\sum_{i=1}^{n} x_i)$

$= \frac{1}{n^2} V(\sum_{i=1}^{n} x_i)$

$= \frac{1}{n} V(x_i)$

$= \frac{1}{n} \frac{1}{12} [(\theta+\frac{1}{2}) - (\theta-\frac{1}{2})]^2$

所有x_i独立同分布，和的方差=方差的和

$= \frac{1}{12n}$

② $p(x_i|\theta) = 1$; $p(\theta) \propto 1$

$\therefore p(\theta|x) \propto 1$ $\therefore \theta$后验均匀分布

设 $\theta \sim Uniform(a_0, b_0)$

$x_{min} \geq \theta - \frac{1}{2}$

$x_{max} \leq \theta + \frac{1}{2}$

$x_{max} - \frac{1}{2} \leq \theta \leq x_{min} + \frac{1}{2}$

李永乐里很经典的一道题

$\therefore \theta \sim Uniform(x_{max} - \frac{1}{2}, x_{min} + \frac{1}{2})$

③ 从频率学派角度

$\hat{\theta} = \bar{x} = 1.54$

$V(\hat{\theta}) = \frac{1}{12 \times 3} = \frac{1}{36}$

从贝叶斯角度

$\theta \sim (1.6, 1.63)$

$E(\theta) = 1.615$

$V(\theta) = \frac{1}{12} \cdot 0.03^2$

贝叶斯背景下更精确

3、

(1) $p(x|\theta) \propto \theta^{s}(1-\theta)^{f}$

$p(\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$

$p(\theta|x) \propto \theta^{\alpha+s-1}(1-\theta)^{\beta+f-1}$

$\cancel{\frac{d\ln p(\theta|x)}{d\theta} = (\alpha+s-1)}$

$\ln p(\theta|x) \propto (\alpha+s-1)\ln\theta + (\beta+f-1)\ln(1-\theta)$

$$\frac{d\ln p(\theta|x)}{d\theta} \propto \frac{\alpha+s-1}{\theta} - \frac{\beta+f-1}{1-\theta} = 0$$

$$\tilde{\theta} = \frac{\alpha+s-1}{\alpha+\beta+n-2}$$

(2) 由 L6, P5

$\theta|y \overset{approx}{\sim} N[\tilde{\theta}, J_y^{-1}(\tilde{\theta})]$

$$J_y(\tilde{\theta}) = -\frac{d^2\ln p(\theta|x)}{d\theta^2}\Big|_{\theta=\tilde{\theta}}$$

$$= -\left(-\frac{\alpha+s-1}{\theta^2} - \frac{\beta+f-1}{(1-\theta)^2}\right)\Big|_{\theta=\tilde{\theta}}$$

$$= \frac{\alpha+s-1}{\theta^2} + \frac{\beta+f-1}{(1-\theta)^2}\Big|_{\theta=\tilde{\theta}}$$

$$= \frac{(\alpha+\beta+n-2)^2}{\alpha+s-1} + \frac{(\alpha+\beta+n-2)^2}{\beta+f-1}$$

$$\frac{(a+b)^2}{a} + \frac{(a+b)^2}{b} = \frac{(a+b)^3}{ab}$$

$$= \frac{(\alpha+\beta+n-2)^3}{(\alpha+s-1)(\beta+f-1)}$$

$\cancel{\therefore \theta \sim y}$

$$\therefore \theta|y \overset{approx}{\sim} N\left[\frac{\alpha+s-1}{\alpha+\beta+n-2}, \frac{(\alpha+s-1)(\beta+f-1)}{(\alpha+\beta+n-2)^3}\right]$$

(3) $p(\theta) \propto 1 \longleftarrow$ 相当于 Beta(1,1)

$p(x|\theta) \propto \theta^{s}(1-\theta)^{f}$

$p(\theta|x) \propto \theta^{s}(1-\theta)^{f} \xleftrightarrow{s=2\ f=6} \theta^{2}(1-\theta)^{6}$

$\theta \sim Beta(s+1, f+1) \xLeftarrow{\text{true posterior}}$

$n=8,\ s=2,\ f=6,\ \alpha=1,\ \beta=1$

$\tilde{\theta} = 0.25 \quad -J_y^{-1}(\tilde{\theta}) = \frac{5}{8^3} \approx 0.0098.$

$\theta|y \overset{\text{approx}}{\sim} N(0.25, \frac{5}{8^3})$

(4) same process as above

$\theta \sim Beta(s+1, f+1)$

$\tilde{\theta} = \frac{1}{4} \quad -J_y^{-1}(\tilde{\theta}) = 0.004125$

More samples, more precise approximation.