Mathematical Exercises 1: problem 1

Let $x = (x_1, \dots, x_n)$, where $n = 30$ and $s = 3$

$p(\theta) = 1 \implies \theta \sim Beta(1,1)$

a) $p(x|\theta) = \theta^s \cdot (1-\theta)^f = \theta^3 \cdot (1-\theta)^{27}$

$p(\theta|x) \propto \theta^3 \cdot (1-\theta)^{27}$, i.e. $\theta|x \sim Beta(\alpha + s = 4, \beta + f = 28)$

b) $p(s|\theta) = \binom{n}{s} \cdot \theta^s \cdot (1-\theta)^f \propto \theta^3 \cdot (1-\theta)^{27}$.

Hence, same inference as in a)

c) $p(n|\theta) = \binom{n-1}{s-1} \theta^s \cdot (1-\theta)^f \propto \theta^3 \cdot (1-\theta)^{27}$

Hence, same inference as in a)

Mathematical Exercises 1: problem 2

Let $x = (x_1, \dots, x_n)$

$$p(x_1, \dots, x_n \mid \theta, \sigma^2) \propto \prod_{i=1}^{n} \exp\left[-\frac{1}{2\sigma^2}\underbrace{(x_i - \theta)^2}_{\theta^2 - 2x_i\theta + x_i^2}\right] \propto \prod_{i=1}^{n} \exp\left[-\frac{1}{2\sigma^2}\cdot\theta^2 + \frac{x_i}{\sigma^2}\cdot\theta\right]$$

$$= \exp\left[-\frac{n}{2\sigma^2}\cdot\theta^2 + \frac{\sum_{i=1}^{n} x_i}{\sigma^2}\cdot\theta\right]$$

We need to show that $\theta \mid x \sim N(\mu_n, \tau_n^2)$. Then,

$$p(\theta \mid x) \propto \exp\left[-\frac{1}{2\tau_n^2}\cdot\underbrace{(\theta - \mu_n)^2}_{\theta^2 - 2\mu_n\theta + \mu_n^2}\right] \propto \exp\left[-\frac{1}{2\tau_n^2}\cdot\theta^2 + \frac{\mu_n}{\tau_n^2}\cdot\theta\right]$$

a) $p(\theta) \propto c \implies p(\theta \mid x) \propto p(x \mid \theta)$

Solve for τ_n^2: $\quad -\frac{n}{2\sigma^2} = -\frac{1}{2\tau_n^2} \implies \tau_n^2 = \frac{\sigma^2}{n}$

Solve for μ_n: $\quad \frac{\sum_{i=1}^{n} x_i}{\sigma^2} = \frac{\mu_n}{\tau_n^2} \implies \mu_n = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$

b) $\theta \sim N(\mu_0, \tau_0^2) \implies p(\theta) \propto \exp\left[-\frac{1}{2\tau_0^2}\cdot\theta^2 + \frac{\mu_0}{\tau_0^2}\cdot\theta\right]$

$$p(\theta \mid x) \propto p(x \mid \theta)\cdot p(\theta) \propto \exp\left[\left(-\frac{n}{2\sigma^2} - \frac{1}{2\tau_0^2}\right)\cdot\theta^2 + \left(\frac{\sum_{i=1}^{n} x_i}{\sigma^2} + \frac{\mu_0}{\tau_0^2}\right)\cdot\theta\right]$$

Solve for τ_n^2: $\quad -\frac{n}{2\sigma^2} - \frac{1}{2\tau_0^2} = -\frac{1}{2\tau_n^2} \implies \frac{1}{\tau_n^2} = \frac{n}{\sigma^2} + \frac{1}{\tau_0^2}$

Solve for μ_n: $\quad \frac{\sum_{i=1}^{n} x_i}{\sigma^2} + \frac{\mu_0}{\tau_0^2} = \frac{\mu_n}{\tau_n^2} \implies$

$$\implies \mu_n = \underbrace{\frac{n/\sigma^2}{1/\tau_n^2}}_{w}\cdot\bar{x} + \underbrace{\frac{1/\tau_0^2}{1/\tau_n^2}}_{1-w}\cdot\mu_0$$

Mathematical Exercises 1: problem 3

a) Let $x = (x_1, \dots, x_{10})$

$$\theta | x \sim N(\mu_n, \tau_n^2)$$

We have that $n = 10$, $\sigma^2 = 1$, $\bar{x} = 1.873$, $\mu_0 = 0$, $\tau_0^2 = 5$

Use the formulas for μ_n and τ_n^2 on slide 4, lecture 2.

- $w = \dfrac{10/1}{10/1 + \frac{1}{5}}$, $\mu_n = w \cdot 1.873 + (1-w) \cdot 0 = 1.84$
- $\dfrac{1}{\tau_n^2} = \dfrac{10}{1} + \dfrac{1}{5} \Rightarrow \tau_n^2 = 0.098$

b) Let $y = (y_1, \dots, y_{10})$

$$p(\theta | x, y) \propto p(x, y | \theta) \cdot p(\theta) = p(y | \cancel{x}, \theta) \cdot \underbrace{p(x|\theta) \cdot p(\theta)}_{\propto\, p(\theta|x)}$$

Hence, use $p(\theta|x)$ as the prior before obtaining the second sample y.

Use the formulas for μ_n and τ_n^2 again.

- $\mu_n = \dfrac{\overset{n}{10}/\overset{\sigma^2}{2}}{\frac{10}{2} + \frac{1}{0.098}} \cdot \underbrace{0.582}_{\bar{y}} + \dfrac{\frac{1}{0.098}}{\frac{10}{2} + \frac{1}{0.098}} \cdot 1.84 = 1.43$
- $\dfrac{1}{\tau_n^2} = \dfrac{10}{2} + \dfrac{1}{0.098} \Rightarrow \tau_n^2 = 0.066$

c) $$p(\theta | x, y, z) \propto p(x, y, z | \theta) \cdot p(\theta) = \underbrace{p(z | \cancel{x}, \cancel{y}, \theta)}_{p(z|\theta)} \cdot \underbrace{p(x, y | \theta) \cdot p(\theta)}_{\propto\, p(\theta | x, y)}$$

Hence, $p(\theta | x, y)$ is the prior before obtaining the third sample z.

The likelihood $p(z|\theta)$ can be written as

$$P(Z_1 = z_1, \dots, Z_8 = z_8, Z_9 > 3, Z_{10} > 3 \mid \theta) = P(Z_1 = z_1, \dots, Z_8 = z_8 \mid \theta)$$

$$\cdot P(Z_9 > 3, Z_{10} > 3 | \theta) = p(z_1, \dots, z_8 | \theta) \cdot \prod_{j=1}^{2} \boxed{P(Z_j > 3 | \theta)} \rightarrow 1 - \Phi\left[\frac{3 - \theta}{\sqrt{3}}\right]$$

σ_z^2

Mathematical Exercises 1: problem 4

$$p(\theta) \propto \theta^{\alpha-1} e^{-\beta\theta}$$

$$p(x|\theta) = \prod_{i=1}^{n} p(x_i|\theta) = \prod_{i=1}^{n} \theta e^{-\theta x_i} = \theta^n \cdot e^{-\sum_{i=1}^{n} x_i \cdot \theta}$$

$$p(\theta|x) \propto p(x|\theta) \cdot p(\theta) \propto \theta^n \cdot e^{-\sum_{i=1}^{n} x_i \cdot \theta} \cdot \theta^{\alpha-1} e^{-\beta\theta} = \theta^{\alpha+n-1} e^{-(\beta+\sum_{i=1}^{n} x_i)\theta}$$

Hence, $\theta|x \sim \text{Gamma}\left(\alpha+n, \beta + \sum_{i=1}^{n} x_i\right)$, i.e.

the gamma prior for θ is the conjugate prior.