

Reinforcement Learning China Summer School



RLChina 2020

Game Theory Basics



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Aug 4, 2020

Outline

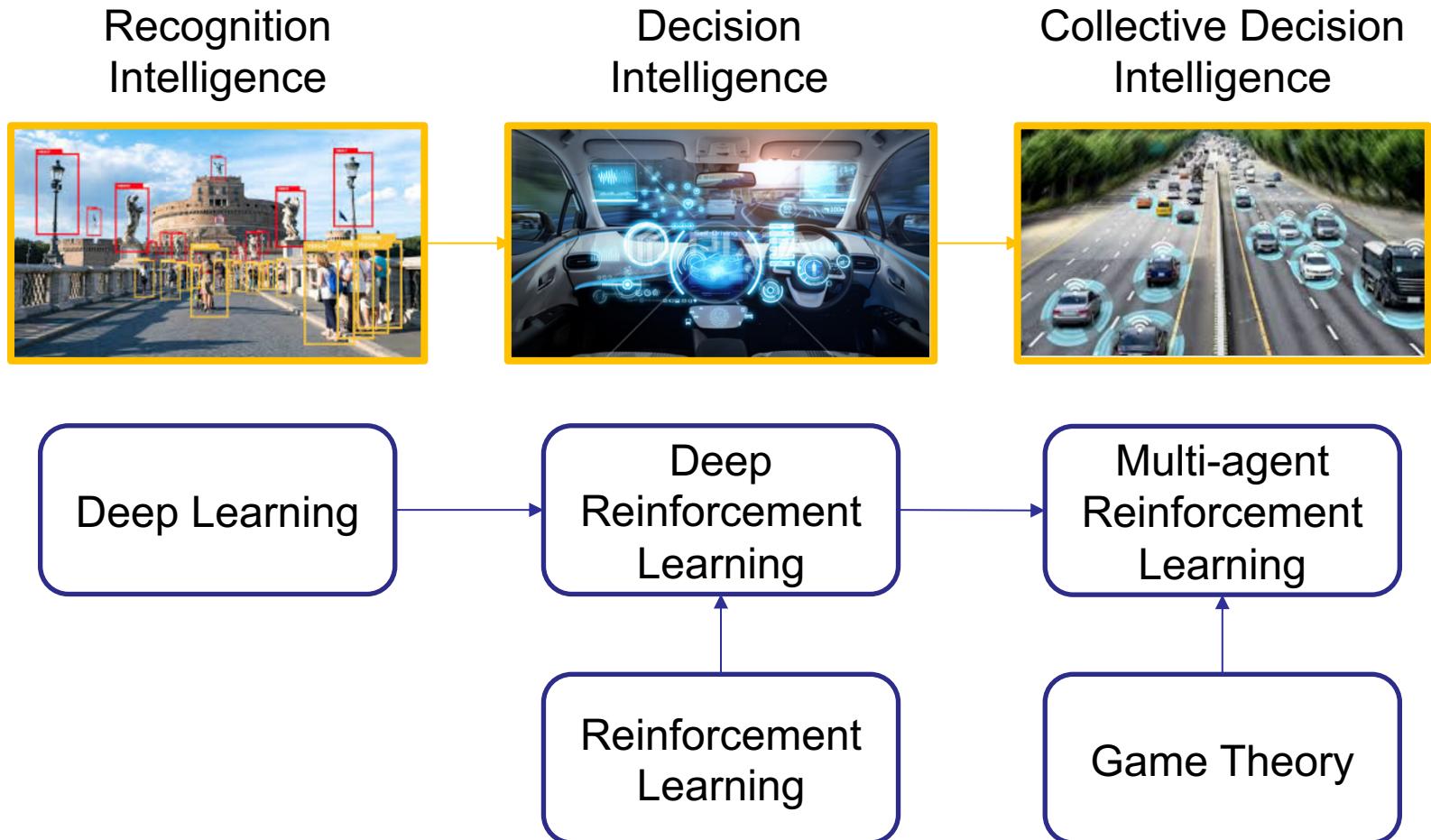
- Motivation and Normal-form game
- Extensive-form Game and Imperfect Information
- Bayesian Game and Incomplete Information
- Nash Equilibrium and Variants
- Theoretical Results of Nash Equilibrium
- Repeated Game and Learning Methods
- Alternate Solution Concepts and Evolutionary Game Theory

Outline

- **Motivation and Normal-form game**
- Extensive-form Game and Imperfect Information
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- Repeated Game and Learning Methods
- Alternate Solution Concepts and Evolutionary Game Theory

Collective Decision Intelligence

- Progress of Artificial Intelligence



Games in Reality



Rock, Scissors, Paper



Auction



Chess



Poker

History of Game Theory

1934, Stackelberg,
Stackelberg Equilibrium[1]

1950, Nash,
Mixed Nash Equilibrium[2]

1967, Harsanyi,
Bayesian Nash Equilibrium
in Bayesian game[5]

1994,
Papadimitriou,
PPAD[8]

1951, Brown,
Fictitious Play in Repeated game[3]

1965, Selten,
Subgame Perfect Equilibrium in
Extensive-form Game[4]

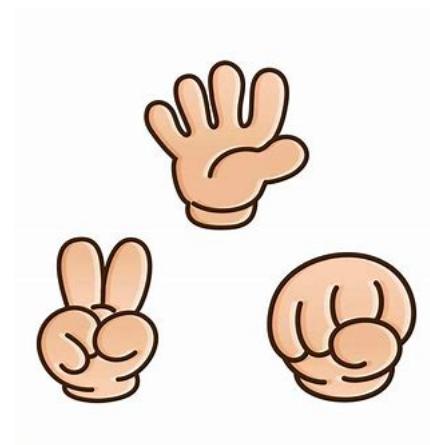
1973, Smith & Price,
Evolutional Game Theory[6]

1974, Aumann,
Correlated Equilibrium[7]

Till now, 18 game theorists received Nobel Prize in Economics!

Elements of Game

- Players $N = \{1, 2, \dots, n\}$
 - $N = \{1, 2\}$
- Strategies (actions) $A = A_1 \times A_2 \times \dots \times A_n$
 - $A_1 = \{R, S, P\}$
 - $A_2 = \{R, S, P\}$
- Payoff (utility) $u = (u_1, u_2, \dots, u_n)$
 - $u_1: A_1 \times A_2 \rightarrow \mathbb{R}$
 - $u_2: A_1 \times A_2 \rightarrow \mathbb{R}$



Normal-form Game

- Payoff Matrix

		Column Player Actions		
		R	S	P
Row Player Actions	R	0, 0	1, -1	-1, 1
	S	-1, 1	0, 0	1, -1
	P	1, -1	-1, 1	0, 0

- More than 2 players

		p_1	p_2	p_3
Joint Actions	R, R, R	0	-1	1
	R, R, S	1	1	0

Rationality of Players

- Self-interested
 - Preference over game outcome
 - E.g. (paper, rock) is better than (rock, paper) for row player
- Utility
 - Utility of (paper, rock) is 1
 - Utility of (rock, paper) is -1
- Objective
 - Act to maximize (expected) utility

Pure Strategy and Mixed Strategy

Matching Pennies

	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

- Pure Strategy
 - $a_1 \in A_1 = \{Heads, Tails\}$
 - $a_2 \in A_2 = \{Heads, Tails\}$
- Mixed Strategy: Probability Distribution over Pure Strategy
 - $a_1 = (x_H, x_T)$, $x_H \in [0,1]$, $x_T \in [0,1]$, $x_H + x_T = 1$
 - $a_2 = (y_H, y_T)$, $y_H \in [0,1]$, $y_T \in [0,1]$, $y_H + y_T = 1$
- Expected Utility
 - $EU_1 = x_H y_H u_1(H, H) + x_H y_T u_1(H, T) + x_T y_H u_1(T, H) + x_T y_T u_1(T, T)$
 - $EU_2 = x_H y_H u_2(H, H) + x_H y_T u_2(H, T) + x_T y_H u_2(T, H) + x_T y_T u_2(T, T)$
- Example
 - $a_1 = (0.1, 0.9), a_2 = (0.3, 0.7)$
 - $EU_1 = 0.32, EU_2 = -0.32$

Classic Games

- Zero-sum Game
 - $u_1(a) + u_2(a) = 0, \forall a \in A$
- Cooperative Game
 - $u_i(a) = u_j(a), \forall a \in A, i, j \in N$
- Coordination Game
 - Multiple Nash Equilibria Exist
- Social Dilemma [9]
 - Everyone suffers in an NE

Matching Pennies

	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

Road Selection

	Left	Right
Left	1, 1	0, 0
Right	0, 0	1, 1

Battle of Sex

	Party	Home
Party	10, 5	0, 0
Home	0, 0	5, 10

Prisoner's Dilemma

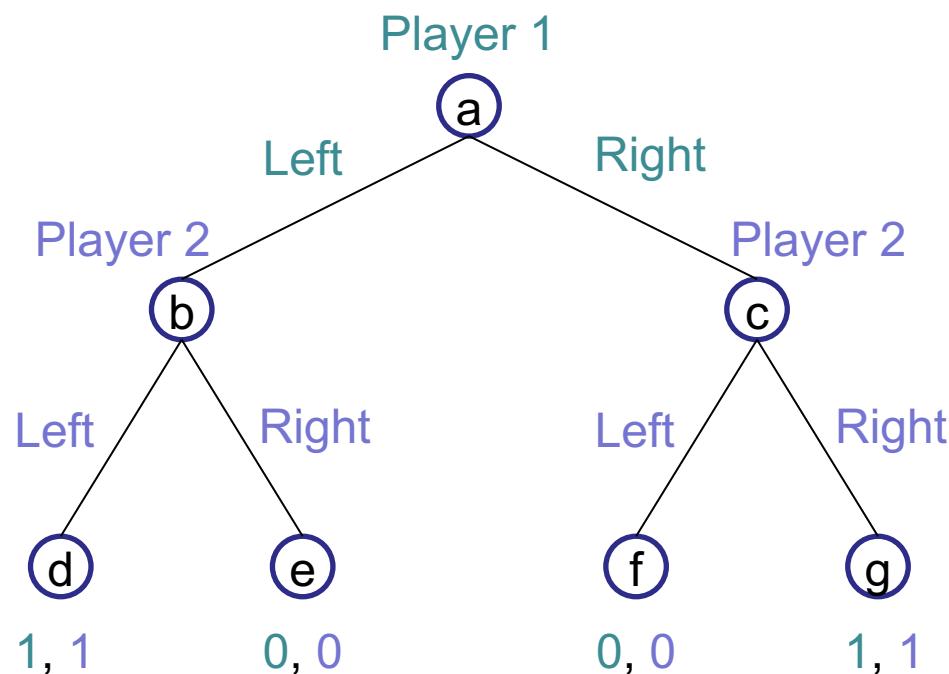
	Cooperate	Defect
Cooperate	2, 2	0, 3
Defect	3, 0	1, 1

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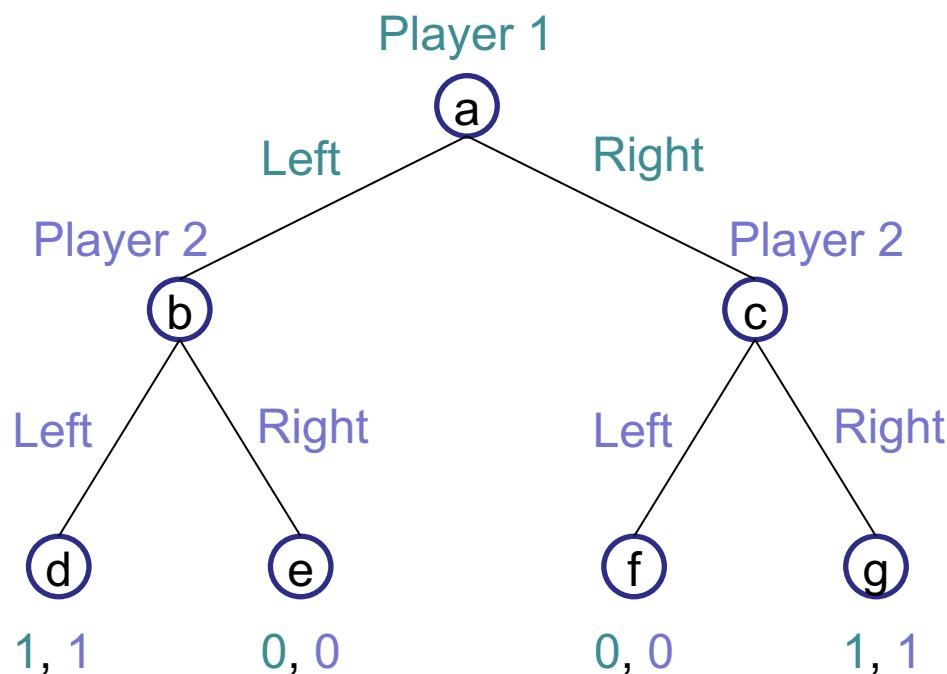
Extensive-form Game

- Game Tree
 - Node: decision point for a specified player
 - Edge: action decided by the player
 - Leaf: outcome of the game with payoff



Strategies in Extensive-form Game

- Strategy Space
 - Player 1: {Left, Right}
 - Player 2: {(Left, Left), (Left, Right), (Right, Left), (Right, Right)}
 - action in every node



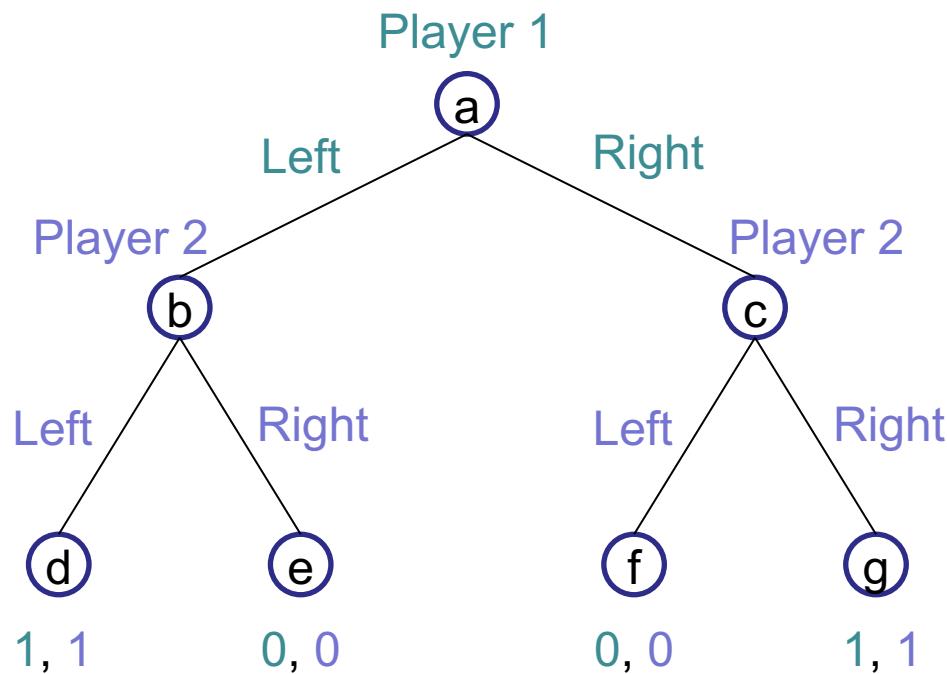
Extensive-form vs. Normal-form

- Equivalent Normal-form Game

single
step/state
 \Updownarrow
static

	(Left, Left)	(Left, Right)	(Right, Left)	(Right, Right)
Left	1, 1	1, 1	0, 0	0, 0
Right	0, 0	1, 1	0, 0	1, 1

multiple
step/state
 \Updownarrow
dynamic



完美信息: 博弈的参与者在行动时知道过去所有人的所有行动

Imperfect Information

- Imperfect Information Game
 - Some historical actions are invisible by other players

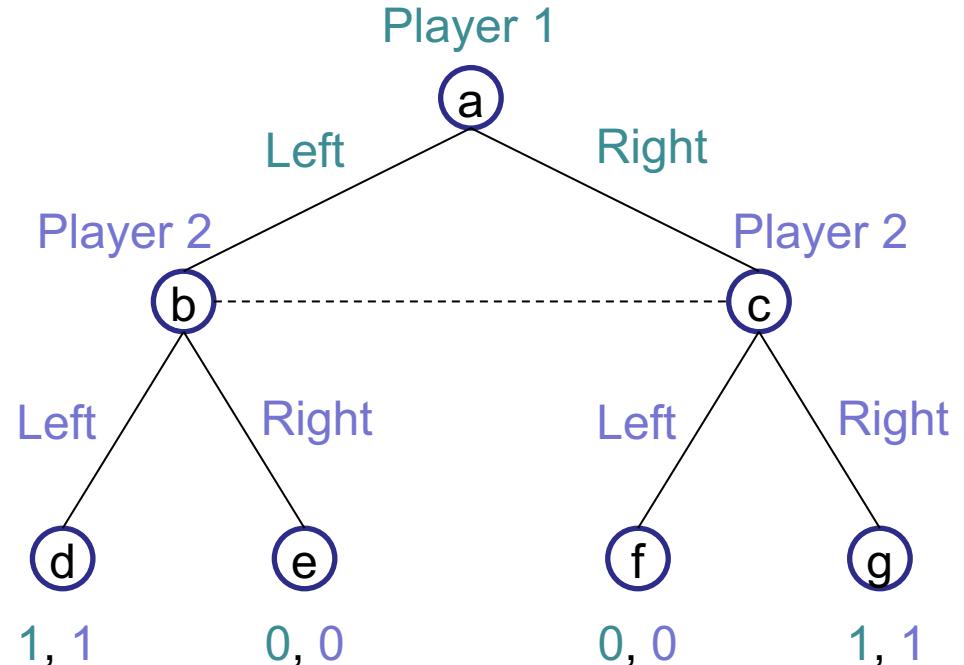
- Information Set

- A set containing undistinguishable states, e.g. {b, c} is an information set for player 2

- Strategy Space

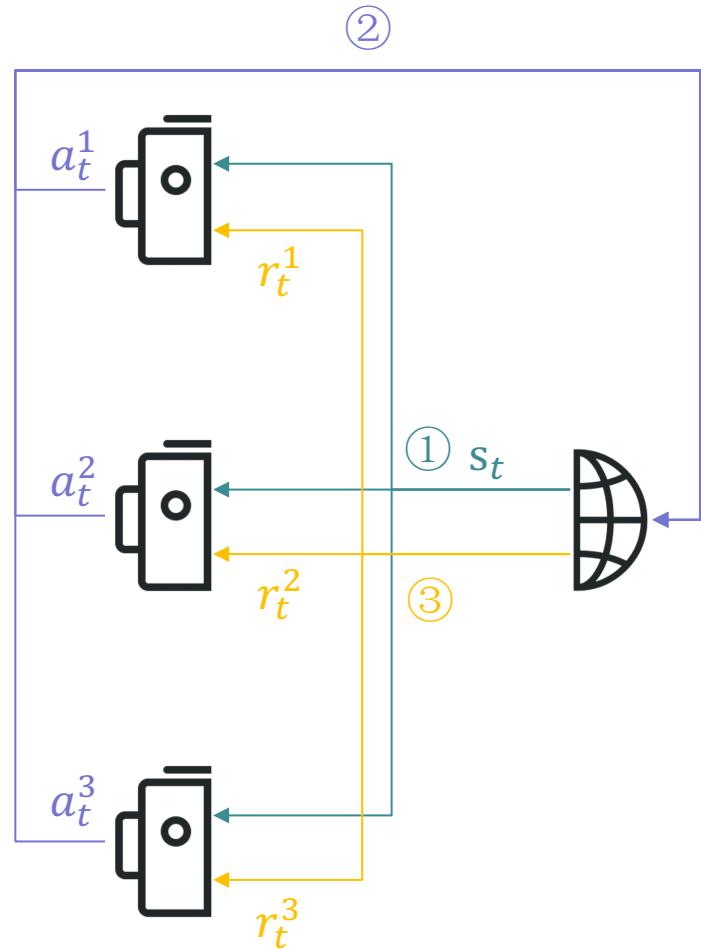
- Player 1: {Left, Right}
- Player 2: {Left, Right}

action in every information set



Markov Game (or Stochastic Game)

- Game Definition
 - State space S
 - Action space $A = A_1 \times A_2 \times \dots \times A_n$
 - Transition function $p: S \times A \rightarrow S$
 - Reward function $r: S \times A \rightarrow \mathbb{R}$
- Behavioral Strategy
 - Policy $\pi_i: S \times A_i \rightarrow [0,1]$
- Properties
 - Simultaneous action (Normal-form)
 - Multiple step/state (Extensive-form)
 - Immediate reward
 - Randomness
 - Cycle



Interaction at time-step t

Summary of Strategy Representation

	Static Game (Single Step/state)	Dynamic Game (Multiple step/state)
Pure Strategy	$a_i \in A_i$	$\pi_i: S \rightarrow A_i$ or $\pi_i \in A_i^S$
Mixed Strategy	$a_i: A_i \rightarrow [0,1]$	$\pi_i: A_i^S \rightarrow [0,1]$
Behavioral Strategy	$a_i: A_i \rightarrow [0,1]$	$\pi_i: S \times A_i \rightarrow [0,1]$

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Motivation: Auction

- Game Definition

- Players has private value v_1, v_2
- Players decide biddings b_1, b_2
- Player i with higher bidding b_i has utility $v_i - b_i$
- The other player has utility 0

- Uncertainty of Private Value

- $v_1 = 4, v_2 = 4$
- $b_1 \in \{1,3\}, b_2 \in \{2,4\}$

	$b_2 = 2$	$b_2 = 4$
$b_1 = 1$	0, 2	0, 0
$b_1 = 3$	1, 0	0, 0

- $v_1 = 4, v_2 = 5$
- $b_1 \in \{1,3\}, b_2 \in \{2,4\}$

	$b_2 = 2$	$b_2 = 4$
$b_1 = 1$	0, 3	0, 1
$b_1 = 3$	1, 0	0, 1

- Players don't know the exact payoff matrix of the game!

Incomplete Information

完全信息: 参与者总是知道彼此所有相关信息

如果每个局中人对其他局中人的特征和支付函数有准确的了解，这称为完备信息 complete information, 否则称为不完备信息 incomplete information

- Recall the Elements of a Game
 - Players $N = \{1, 2, \dots, n\}$
 - Action space $A = A_1 \times A_2 \times \dots \times A_n$
 - Payoff functions $u = (u_1, u_2, \dots, u_n)$
- Incomplete Information Game
 - Players know: N and A
 - Players don't completely know: u
 - Criteria: whether players have private information when game starts
- Example
 - Auction
 - Mahjong
 - Werewolves of Miller's Hollow

Bayesian Game

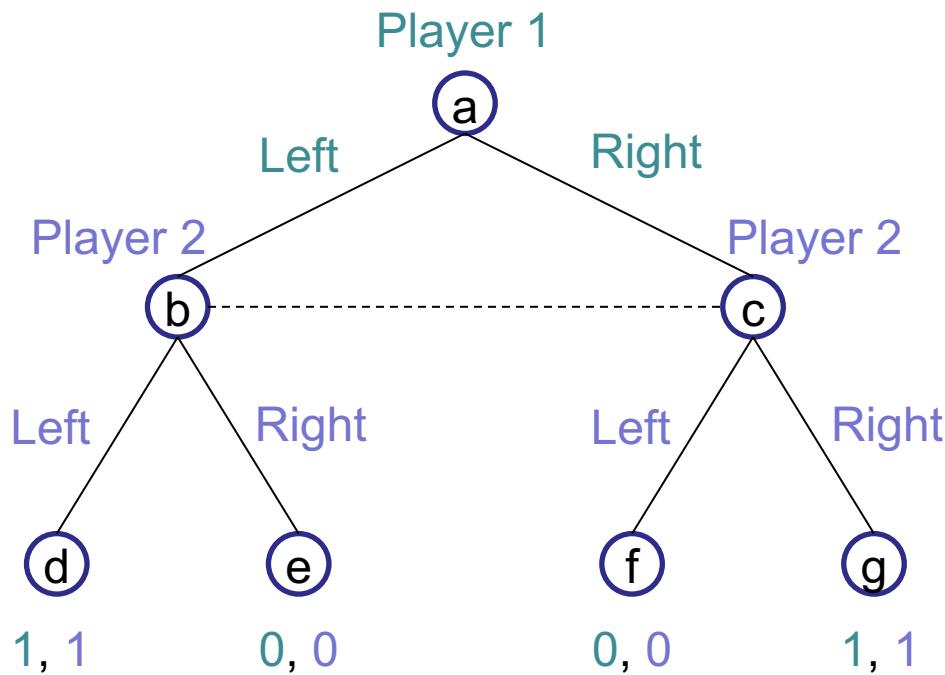
- Basic Idea
 - Payoff function p_i is unknown, but the distribution of p_i is known
- Elements of Bayesian Game
 - Players $N = \{1, 2, \dots, n\}$, Action space $A = A_1 \times A_2 \times \dots \times A_n$
 - Player type space $\Theta = \Theta_1 \times \Theta_2 \times \dots \times \Theta_n$
 - Distribution over types $d: \Theta \rightarrow [0, 1]$
 - Payoff functions $u = (u_1, u_2, \dots, u_n)$, $u_i: \Theta \times A \rightarrow \mathbb{R}$
- Strategy
 - Pure strategy $\pi_i: \Theta_i \rightarrow A_i$
 - Mixed strategy $\pi_i: \Theta_i \times A_i \rightarrow [0, 1]$
- Example
 - $\Theta_1 = \{4\}, \Theta_2 = \{4, 5\}$
 - $d(4, 4) = 0.3, d(4, 5) = 0.7$

	$b_2 = 2$	$b_2 = 4$
$b_1 = 1$	0, 3	0, 1
$b_1 = 3$	1, 0	0, 1

	$b_2 = 2$	$b_2 = 4$
$b_1 = 1$	0, 2	0, 0
$b_1 = 3$	1, 0	0, 0

Dynamic Bayesian Game

- Belief System in Imperfect Information Extensive-form Game
 - Distribution over the states in an information set $b_i: S \rightarrow [0,1]$
- Strategy
 - Pure strategy $\pi_i: S \rightarrow A_i$
 - Behavioral strategy $\pi_i: S \times A_i \rightarrow [0,1]$



Summary of Game Representation

完美信息：博弈的参与者在行动时知道过去所有人的所有行动

完全信息：参与者总是知道彼此所有相关信息

		Complete	Incomplete
Static		Normal-form Game, e.g. Prisoner's Dilemma	Bayesian Game, e.g. Auction
Dynamic	Perfect	Extensive-form Game, e.g. Chess	Texas Hold'em Poker
	Imperfect	StarCraft	Mahjong



Dynamic Bayesian game

- Harsanyi Transformation
 - Incomplete Information → Imperfect Information
 - Introduce a nature player who decides the type of each player

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Game Solution Reasoning

- Best Response (BR)
 - Given $a_{-i} \in A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n$
 - a_i is best response to $a_{-i} \Leftrightarrow u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i}), \forall a'_i \in A_i$
 - Dominant Strategy (DS)
 - a_i is dominant strategy \Leftrightarrow Given any a_{-i} , a_i is best response
 - Example

Prisoner's Dilemma

	Cooperate (C)	Defect (D)
Cooperate (C)	2, 2	0, 3
Defect (D)	3, 0	1, 1

$3 > 2$, D is BR to C $1 > 0$, D is BR to D

$1 > 0$, D is BR to D

D is DS

Game Solution Concept: Nash Equilibrium

- Definition
 - A joint strategy (or strategy profile) $a \in A$ is a Nash Equilibrium $\Leftrightarrow a_i$ is best response to a_{-i} holds for every player i
- Example

Matching Pennies

	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

Battle of Sex

	Party	Home
Party	10, 5	0, 0
Home	0, 0	5, 10

Road Selection

	Left	Right
Left	1, 1	0, 0
Right	0, 0	1, 1

Prisoner's Dilemma

	Cooperate	Defect
Cooperate	2, 2	0, 3
Defect	3, 0	1, 1

Pareto Optimality vs. Nash Equilibrium

- Pareto Optimality (PO)
 - A joint strategy (or strategy profile) $a \in A$ achieves Pareto optimality $\Leftrightarrow u_i(a) \geq u_i(a'), \forall a' \in A$ holds for every player i
 - A Pareto optimality is not necessarily a Nash equilibrium
 - A Nash equilibrium is not necessarily a Pareto optimality

Chicken

	C	D
C	3, 3	1, 4
D	4, 1	0, 0

Stag Hunt

	C	D
C	3, 3	0, 2
D	2, 0	1, 1

Prisoner's Dilemma

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

C-D is PO and NE

D-D is NE but not PO

C-C is PO but not NE

Mixed-Strategy Nash Equilibrium

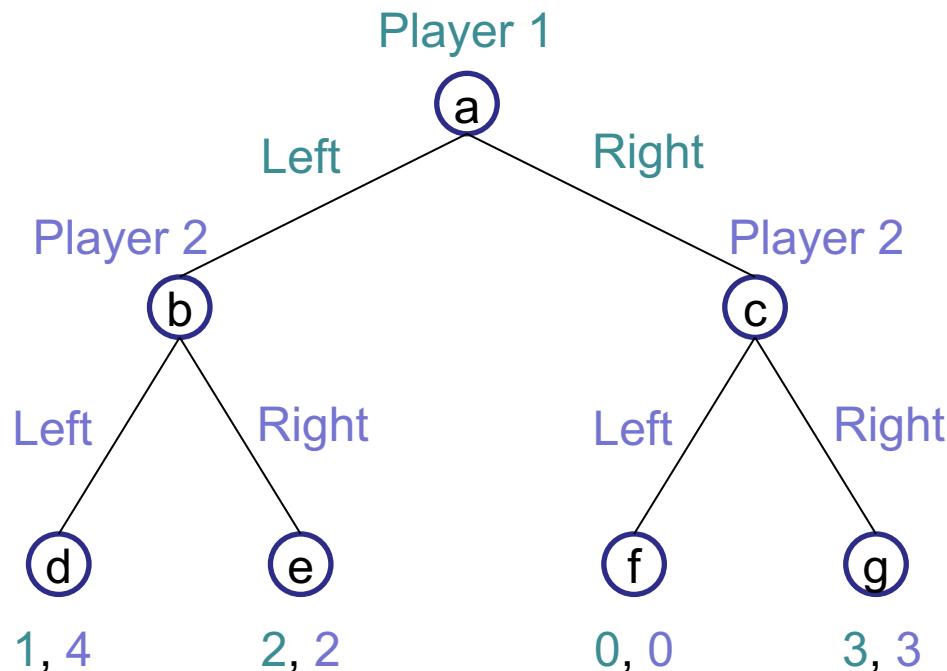
- Definition
 - A mixed-strategy profile $(a_1, a_2, \dots, a_n), a_i \in \text{PD}(A_i)$ is a Nash Equilibrium $\Leftrightarrow a_i$ is best response to a_{-i} holds for every player i
- Example (Rock-Scissors-Paper)
 - $a_1 = (1/3, 1/3, 1/3), a_2 = (1/3, 1/3, 1/3)$
 - $EU_1(a_1, a_2) = 0 \geq EU_1(a_1', a_2) = 0, \forall a_1' \in A_1$
 - $EU_2(a_1, a_2) = 0 \geq EU_2(a_1, a_2') = 0, \forall a_2' \in A_2$

	$1/3$	$1/3$	$1/3$
$1/3$	R	$0, 0$	$1, -1$
$1/3$	S	$-1, 1$	$0, 0$
$1/3$	P	$1, -1$	$-1, 1$

Nash Equilibrium in Extensive-form Game

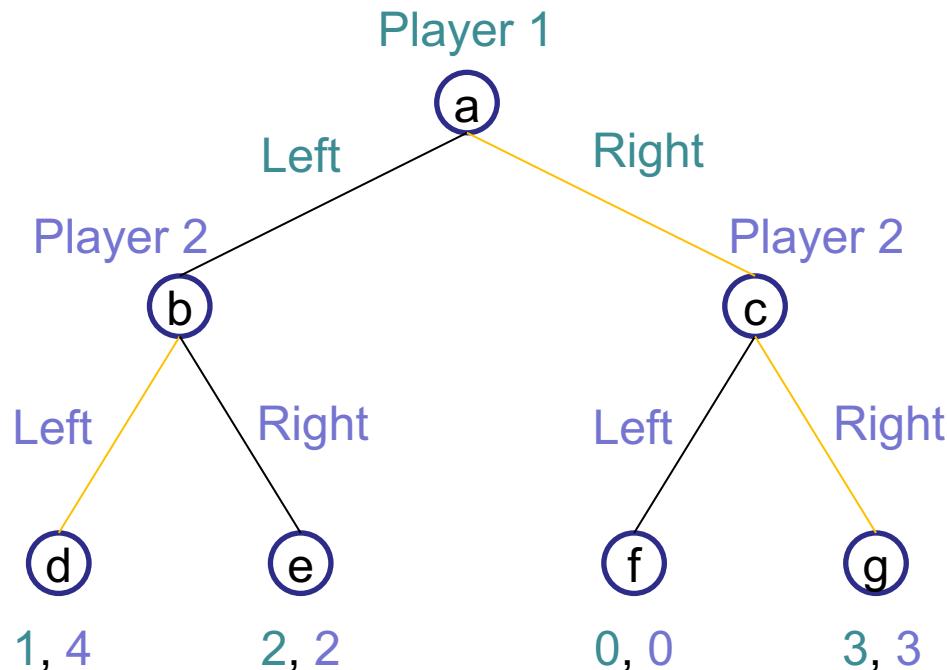
- Incredible Threat

	(Left, Left)	(Left, Right)	(Right, Left)	(Right, Right)
Left	1, 4 ?	1, 4	2, 2	2, 2
Right	0, 0	3, 3	0, 0	3, 3 ?



Subgame Perfect Nash Equilibrium (SPNE)

- Definition
 - An NE is SPNE \Leftrightarrow the NE holds in every subgame
- Solution
 - Backward induction: Right - (Left, Right)



Bayesian Nash Equilibrium

- Recall Bayesian Game
 - Player type space $\Theta = \Theta_1 \times \Theta_2 \times \dots \times \Theta_n$
 - Distribution over types $d: \Theta \rightarrow [0,1]$
 - Payoff functions $u = (u_1, u_2, \dots, u_n)$, $u_i: \Theta \times A \rightarrow \mathbb{R}$
- Strategy in Bayesian Game
 - Pure strategy $\pi_i: \Theta_i \rightarrow A_i$
 - Mixed strategy $\pi_i: \Theta_i \times A_i \rightarrow [0,1]$
- Bayesian Nash Equilibrium (BNE)
 - Assume each player i knows her own type $\theta_i \in \Theta_i$
 - Set expected utility $\mathbb{E}[u_i | \pi, \theta] = \sum_{a \in A} (\prod_{j \in N} \pi_j(\theta_j, a_j) u_i(a, \theta))$
 - π is BNE $\Leftrightarrow \pi_i \in \operatorname{argmax}_{\pi'_i} \sum_{\theta_{-i} \in \Theta_{-i}} d(\theta_i, \theta_{-i}) \mathbb{E}[u_i | \pi'_i, \pi_{-i}, \theta_i, \theta_{-i}]$
holds for each player i with her own type θ_i

Bayesian Nash Equilibrium: Example

- Auction
 - $A_1 = \{1,3\}, A_2 = \{2,4\}, \Theta_1 = \{4\}, \Theta_2 = \{4,5\}, d(4,4) = 0.3, d(4,5) = 0.7$
- Strategy
 - $\pi_1(4,1) = x, \pi_1(4,3) = 1 - x$
 - $\pi_2(4,2) = y_1, \pi_2(4,4) = 1 - y_1$
 - $\pi_2(5,2) = y_2, \pi_2(5,4) = 1 - y_2$
- Equilibrium
 - $\mathbb{E}[u_1 | \pi_1, \pi_2, 4, 4] = (1 - x)y_1$
 - $\mathbb{E}[u_1 | \pi_1, \pi_2, 4, 5] = (1 - x)y_2$
 - $\mathbb{E}[u_2 | \pi_1, \pi_2, 4, 4] = 3xy_1 + x(1 - y_1) + (1 - x)(1 - y_1)$
 - $\mathbb{E}[u_2 | \pi_1, \pi_2, 4, 5] = 2xy_2$
 - (x, y_1, y_2) satisfies $x = \operatorname{argmax}_x 0.3(1 - x)y_1 + 0.7(1 - x)y_2$ and $y_1 = \operatorname{argmax}_{y_1} 3xy_1 + x(1 - y_1) + (1 - x)(1 - y_1)$ and $y_2 = \operatorname{argmax}_{y_2} 2xy_2$

$v_2 = 4$	$b_2 = 2$	$b_2 = 4$
$b_1 = 1$	0, 3	0, 1
$b_1 = 3$	1, 0	0, 1

$v_2 = 5$	$b_2 = 2$	$b_2 = 4$
$b_1 = 1$	0, 2	0, 0
$b_1 = 3$	1, 0	0, 0

Perfect Bayesian (Nash) Equilibrium

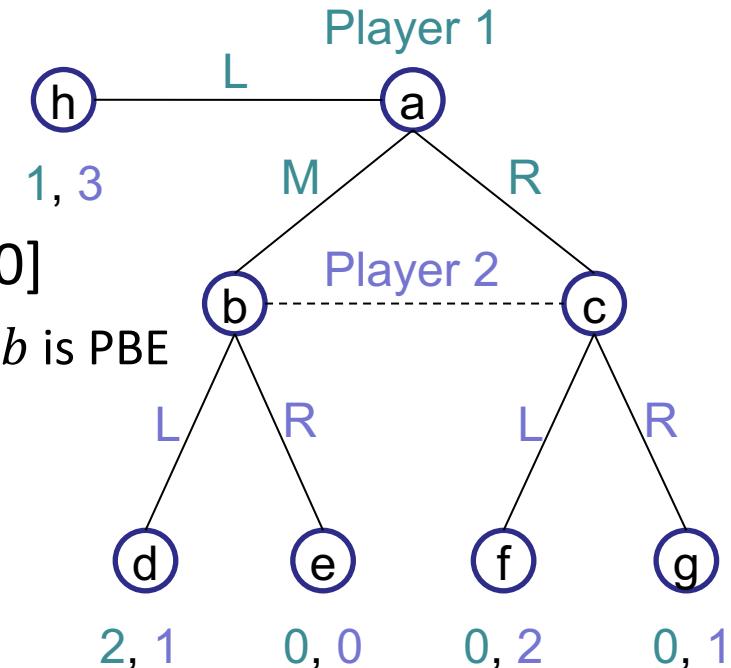
- Motivation
 - SPNE is not enough for some imperfect information game
 - Example: (L, R) is an SPNE but is incredible
- Recall Dynamic Bayesian Game

- Belief function $b_i: S \rightarrow [0,1]$
- Behavioral strategy $\pi_i: S \times A_i \rightarrow [0,1]$

- Perfect Bayesian Equilibrium (PBE) [10]

- A strategy profile π with a belief system b is PBE
- **Sequential rationality**
 - Each player has best expected utility in each information set following b and π

- **Consistency of beliefs with Strategies**
 - Beliefs b are correct according to strategies π



Summary of Nash Equilibrium

		Complete	Incomplete
Static		Nash Equilibrium	Bayesian Nash Equilibrium
Dynamic	Perfect	Subgame Perfect Nash Equilibrium	Perfect Bayesian Nash Equilibrium
	Imperfect		

- Harsanyi Transformation
 - Incomplete Information → Imperfect Information
 - Introduce a nature player who decides the type of each player

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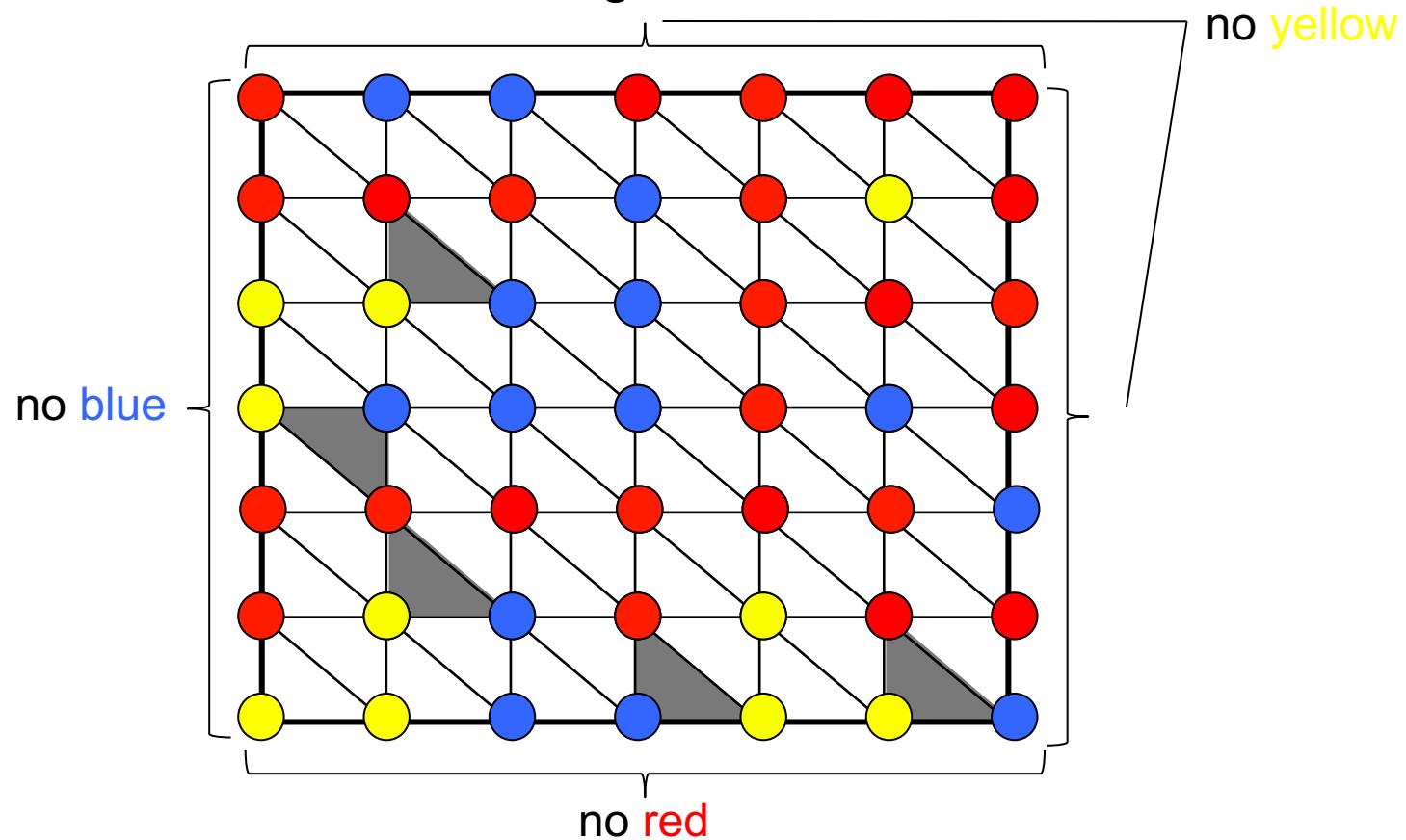
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Existence of Nash Equilibrium

- Nash's Theorem [2]
 - Every finite game has a mixed-strategy Nash equilibrium.
 - Proof: apply Brouwer's fixed point theorem.
- Brouwer's Fixed Point Theorem
 - Let D be a convex, compact subset of the Euclidean space. If $f: D \rightarrow D$ is continuous, then there exists $x \in D$ such that $f(x) = x$.
 - Proof: apply Sperner's lemma.
- Sperner's Lemma
 - Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

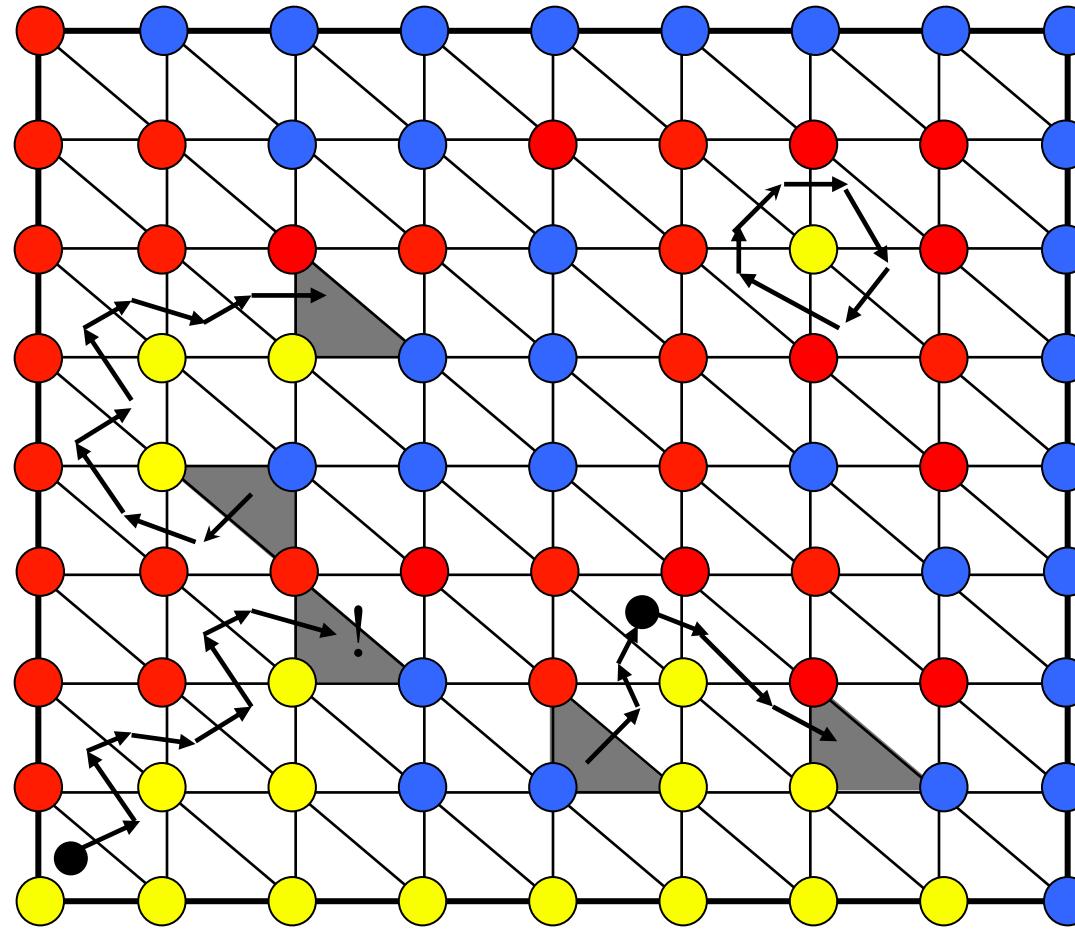
Sperner's Lemma (2-D)

- Lemma
 - Color the boundary using three colors in a legal way. There exists odd number of tri-chromatic triangles.



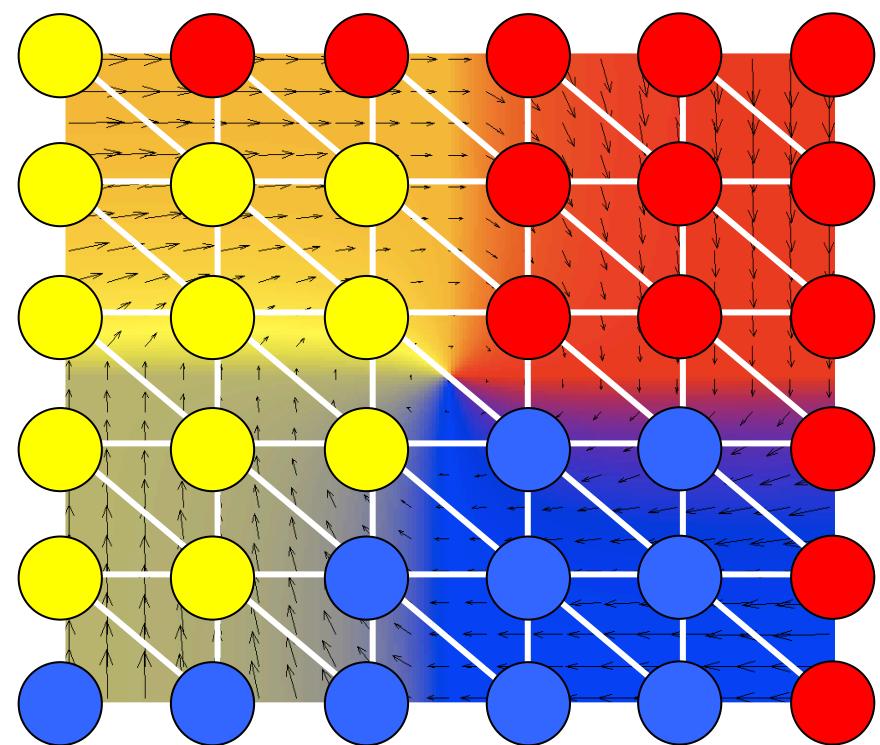
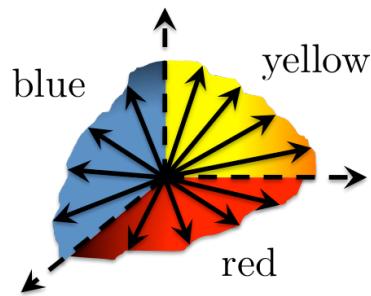
Proof of Sperner's Lemma

- Proof Sketch



Sperner's Lemma to Brouwer's Theorem

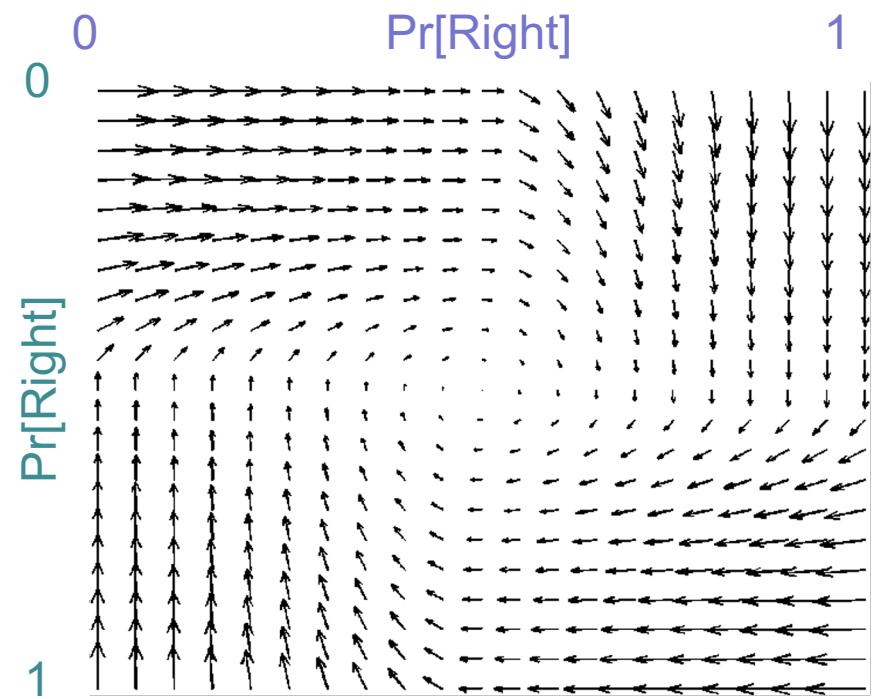
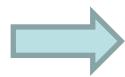
- Proof Sketch



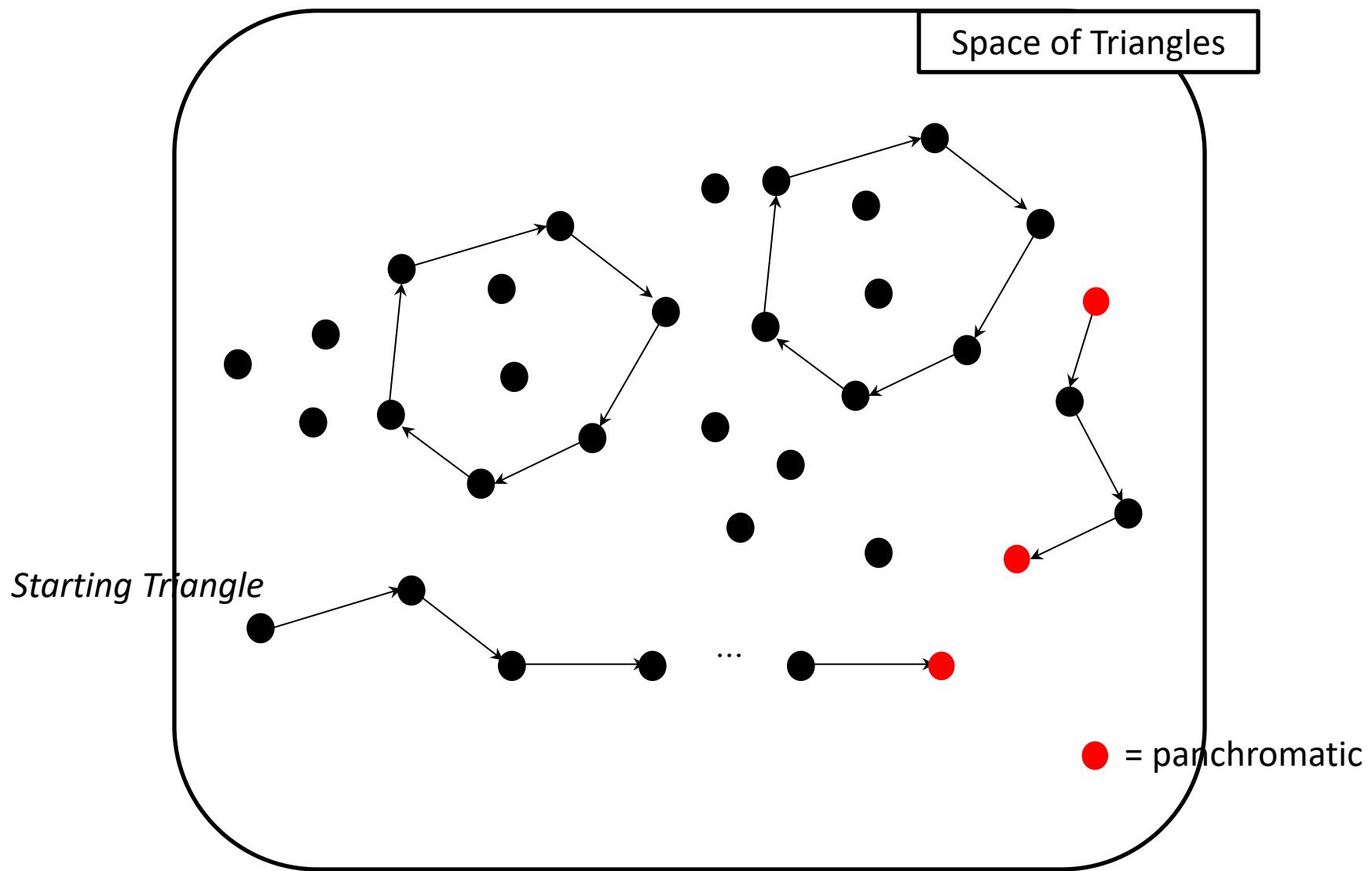
Brouwer's Theorem to Nash's Theorem

- Proof Sketch

	Left	Right
Left	1, -1	-1, 1
Right	-1, 1	1, -1

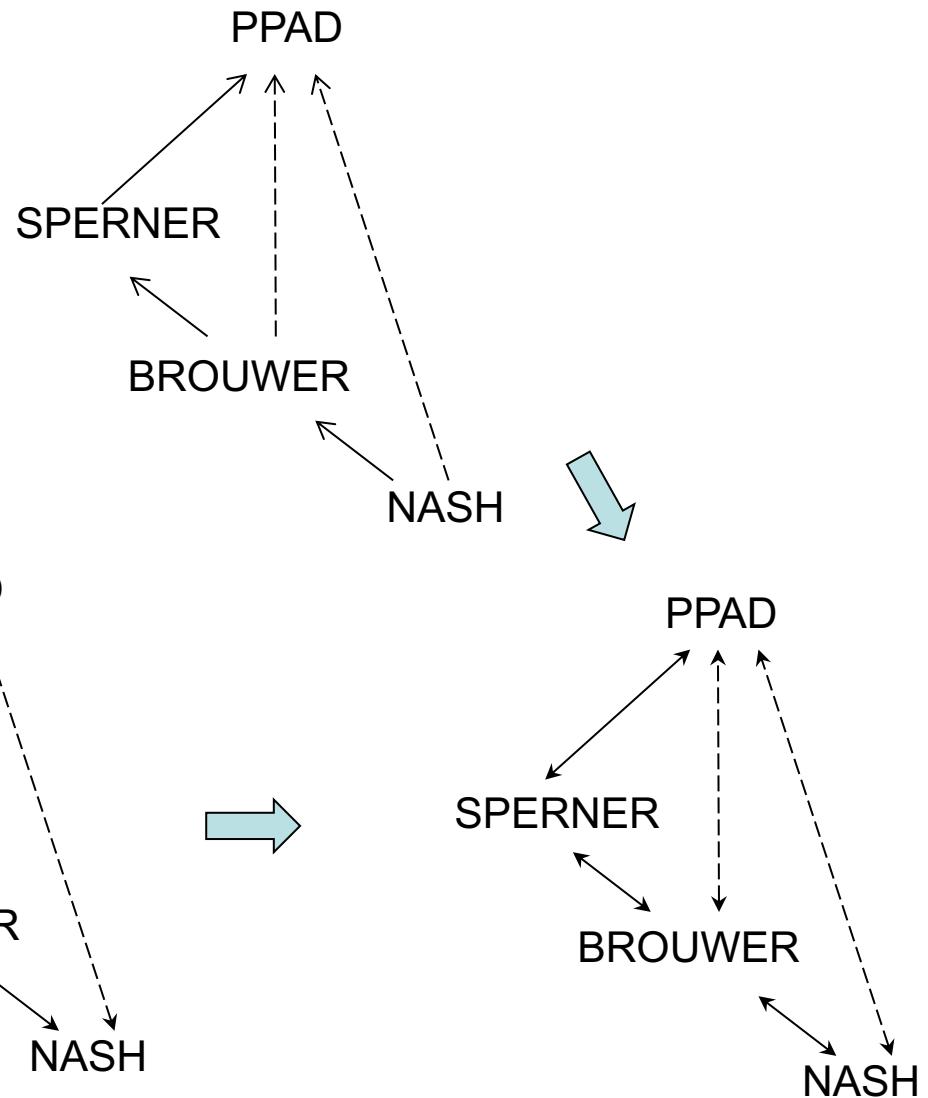


End-of-the-Line to Sperner's Lemma



PPAD Complexity Class

- PPAD-complete [11,12,13]
 - End-of-the-line
 - Sperner, Brouwer, Nash
- Computational Complexity
 - Poly-time algorithm not found

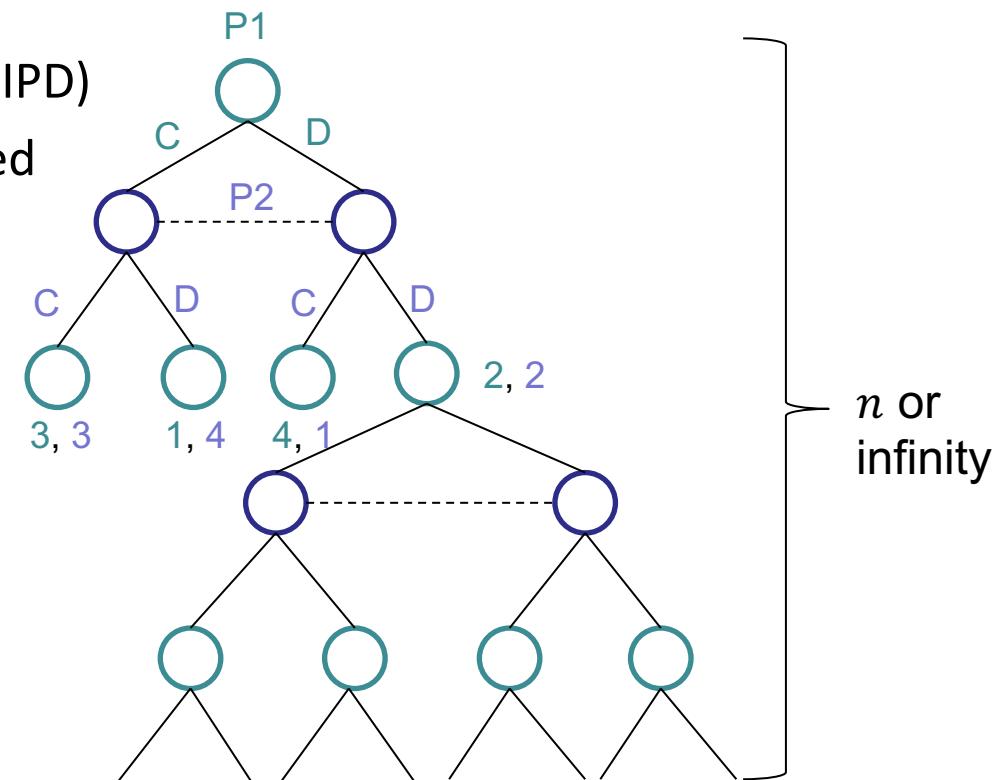


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Repeated Game

- Definition
 - A normal-form game is played over and again by the same players
 - The game repeated in each period is referred to as the **stage game**
- Example
 - Iterated Prisoners' Dilemma (IPD)
 - Reward: average or discounted
 - Memory: perfect recall



Memory

- Historical Behavior
 - At stage t , the action profile is a_t
 - Each player remembers the action profiles at last k stages
 - We say the players have k -memory
- Relation to Markov Game
 - Memory is regarded as state

1-memory

	1	2	3	...	m
P1	C	C	D	...	D
P2	D	D	C	...	C
...					
Pn	D	C	D	...	C

 as state

k-memory

	1	2	3	...	m
P1	C	C	D	...	D
P2	D	D	C	...	C
...					
Pn	D	C	D	...	C

 as state

Tit-for-tat

- Idea [14]
 - The Tit-for-tat strategy copies what the other player previously choose.
 - Nice: start by cooperating.
 - Clear: be easy to understand and adapt to.
 - Provocable: retaliate against anti-social behavior.
 - Forgiving: cooperate when faced with pro-social play.

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

Prisoner's Dilemma

	1	2	3	4	...
P1	C	C	C	C	...
P2	C	C	C	C	...

cooperate by playing strategy (C,C)

$$\text{payoff} = 2 + 2\gamma + 2\gamma^2 + 2\gamma^3 + \dots = 2 \frac{\gamma^n - 1}{\gamma - 1} = \frac{2}{1 - \gamma}$$

a player deviates to defecting (D)

$$\text{payoff} = 3 + 0\gamma + 3\gamma^2 + 0\gamma^3 + \dots = \frac{3}{1 - \gamma^2}$$

	1	2	3	4	...
P1	C	D	C	D	...
P2	D	C	D	C	...

Win-stay, lose-shift

- Idea [15]
 - Repeat** if it was rewarded by 2 or 3
 - Shift** if it was punished by 0 or 1
 - Advantage: tolerant, one round of mutual defection followed by a return to cooperation
 - Disadvantage: fares poorly against inveterate defectors

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

Prisoner's Dilemma

	1	2	3	4	...
P1	C	D	C	C	...
P2	D	D	C	C	...

P2 deviates to defecting(D) initially

$$\text{Payoff} > \frac{3}{1 - \gamma^2}$$

	1	2	3	4	...
P1	C	C	D	C	...
P2	C	D	D	D	...

P2 is an inveterate defectors

$$\begin{aligned} \text{payoff}_1 &= 2 + 0\gamma + 1\gamma^2 + 0\gamma^3 + \dots = \frac{1\gamma}{1 - \gamma^2} + 1 \\ &\wedge \\ \text{payoff}_2 &= 2 + 3\gamma + 1\gamma^2 + 3\gamma^3 + \dots = \frac{1 + 3\gamma}{1 - \gamma^2} + 1 \end{aligned}$$

Strategies in Iterated Prisoner's Dilemma

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

Prisoner's Dilemma

Action profile at time t	Player 1 strategies at time $t + 1$ with 1-memory																
$(a_1, a_2) = (C, C)$	C	C	C	C	C	C	C	C	D	D	D	D	D	D	D	D	D
$(a_1, a_2) = (C, D)$	C	C	C	C	D	D	D	C	C	C	C	D	D	D	D	D	D
$(a_1, a_2) = (D, C)$	C	C	D	D	C	C	D	D	C	C	D	D	C	C	D	D	D
$(a_1, a_2) = (D, D)$	C	D	C	D	C	D	C	D	C	D	C	D	C	D	C	D	D



Tit-for-tat strategy



Win-stay, lose-shift strategy

Folk Theorem

- Game Setting
 - n -player infinitely-repeated game $G = (N, A, u)$ with average reward
- Enforceable
 - A payoff profile r is **enforceable** if $r_i \geq v_i$, $v_i = \min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_{-i}, s_i)$
- Feasible
 - A payoff profile r is **feasible** if there exist rational, non-negative values α_a such that for all i , we can express r_i as $\sum_{a \in A} \alpha_a u_i(a)$, with $\sum_{a \in A} \alpha_a = 1$
- Folk Theorem
 - r is **feasible** and **enforceable** $\Rightarrow r$ is the payoff in some Nash equilibrium

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

Prisoner's Dilemma

$$(v_1, v_2) = (1, 1)$$

(-1, -1) is not enforceable, not feasible

(0.5, 2) is not enforceable, **feasible**

(5, 5) is **enforceable**, not feasible

(2, 2) is **enforceable, feasible**

Fictitious Play

- Definition
 - Each player plays a best response to **assessed** strategy of the opponent and observe the opponent's actual play and update **beliefs**.

Matching Pennies

	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

Round	1's action	2's action	1's beliefs	2's beliefs
0			(1.5, 2)	(2, 1.5)
1	T	T	(1.5, 3)	(2, 2.5)
2	T	H	(2.5, 3)	(2, 3.5)
3	T	H	(3.5, 3)	(2, 4.5)
4	H	H	(4.5, 3)	(3, 4.5)
...

Convergence of Fictitious Play

- Fictitious Play → Convergence
 - Each of the following are sufficient conditions for the empirical frequencies of play to converge in fictitious play:
 - The game is zero sum;
 - The game is solvable by iterated elimination of strictly dominated strategies;
 - The game is a potential game;
 - The game is $2 \times n$ and has generic payoffs.
- Convergence → Nash Equilibrium
 - If the empirical distribution of each player's strategies converges in fictitious play, then it converges to a Nash equilibrium.
- Results in Extensive-form Game with Imperfect Information
 - Fictitious self-play converges to approximate Nash equilibrium [16]
 - AlphaStar for StarCraft [17]

No-regret Learning

- Regret
 - Let a^t be the action profile played at time t
 - Regret of player i for not playing action a'_i at time t is $R^t(a'_i) = u_i(a'_i, a_{-i}^t) - u_i(a^t)$
 - Regret cumulated from time 1 to T is $CR^T(a'_i) \sum_{t=1}^T R^t(a'_i)$
- Regret Matching
 - At each time step, each action is chosen with probability proportional to its cumulated regret: $\sigma_i^{t+1}(a_i) = \frac{CR^t(a_i)}{\sum_{a'_i \in A_i} CR^t(a'_i)}$
 - Converge to correlated equilibrium
- No-regret learning in Extensive-form Game
 - Counterfactual Regret Minimization (CFR)
 - DeepStack for Texas Hold'em poker [18]

Two Views of Repeated Game

- A Special Case of Markov Game
 - Consider repeated normal-form game
 - Stage can be regarded as step in Markov game
 - **State** can be defined by **memory (historical action profiles)**
- A General Framework for All Kinds of Game
 - Extensive-form, Bayesian, Markov games can also be repeated
 - Theorems for repeated game can be extended to any-form games
 - Repeated Markov game is the setting for **reinforcement learning**

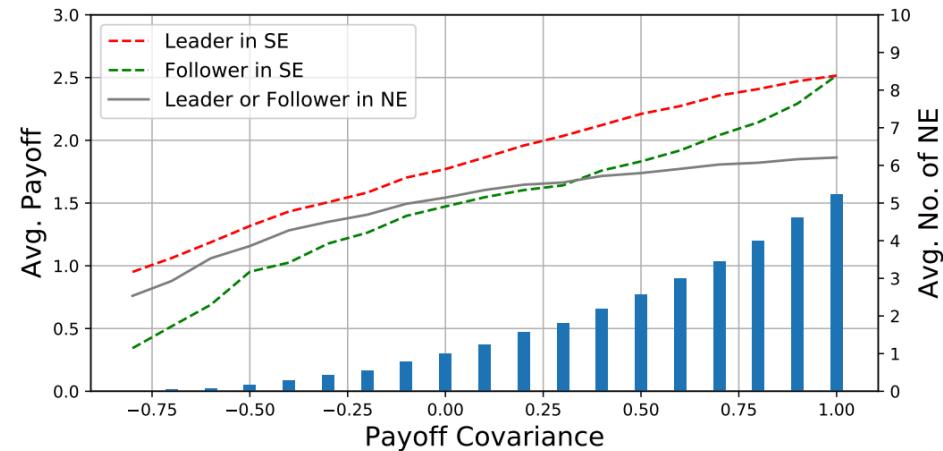
Outline

- Motivation and Normal-form game
- Extensive-form Game and Imperfect Information
- Bayesian Game and Incomplete Information
- Nash Equilibrium and Variants
- Theoretical Results of Nash Equilibrium
- Repeated Game and Learning Methods
- **Alternate Solution Concepts and Evolutionary Game Theory**

Stackelberg Equilibrium

- Stackelberg Game
 - A **leader** moves first
 - The **follower(s)** move after the leader
- Equilibrium
 - Perfect subgame Nash equilibrium
- Compared with Nash Equilibrium
 - Order is good in highly cooperative games
 - Bi-level Actor-critic RL [19]

	X	Y	Z
A	20, 15	0, 0	0, 0
B	30, 0	10, 5	0, 0
C	0, 0	0, 0	5, 10



Correlated Equilibrium

- Motivation
 - Equilibrium selection
- Basic Idea
 - Introduce a **public signal**
 - Sample from a probability distribution over **action profiles**
 - Each player is informed with her own action
 - No player has incentive to deviate
- Example
 - $\Pr[(\text{Party}, \text{Party})] = 0.5$
 - $\Pr[(\text{Home}, \text{Home})] = 0.5$
 - $\Pr[(\text{Home}, \text{Party})] = 0$
 - $\Pr[(\text{Party}, \text{Home})] = 0$

Battle of Sex

	Party	Home
Party	10, 5	0, 0
Home	0, 0	5, 10

Evolutional Game Theory

- Motivation
 - Nash equilibrium is static, the dynamic of strategy is not described
 - Players are not fully rational
- Basic Idea
 - Strategy is inherent and player can not select strategy by herself
 - Player with high payoff is has more chance to be reproduced
- Evolutionary Stable Strategy (ESS)
 - If almost every member of the population follows a strategy, no mutant (that is, an individual who adopts a novel strategy) can successfully invade.

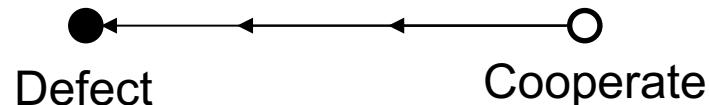
Replicator Dynamics

- Definition

- $\dot{x}_i = x_i[f_i(x) - \varphi(x)]$, $\varphi(x) = \sum_{j=1}^n x_j f_j(x)$
- x is distribution of types(strategies) over the population
- $f_i(x)$ is the fitness for type i in population x
- $\varphi(x)$ is the average fitness of the population

	C	D
C	2, 2	0, 3
D	3, 0	1, 1

Prisoner's Dilemma



Replicator Dynamics: Experiment

- On a local interaction model [20]
 - $T = 2.8$, $R = 1.1$, $P = 0.1$, and $S = 0$ **All Defect**



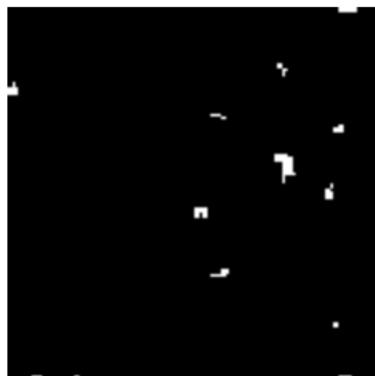
Generation 1



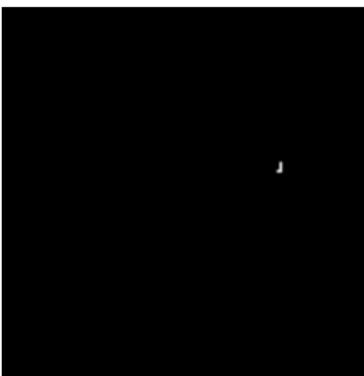
Generation 2



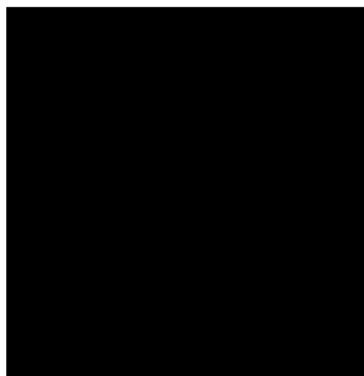
Generation 3



Generation 4



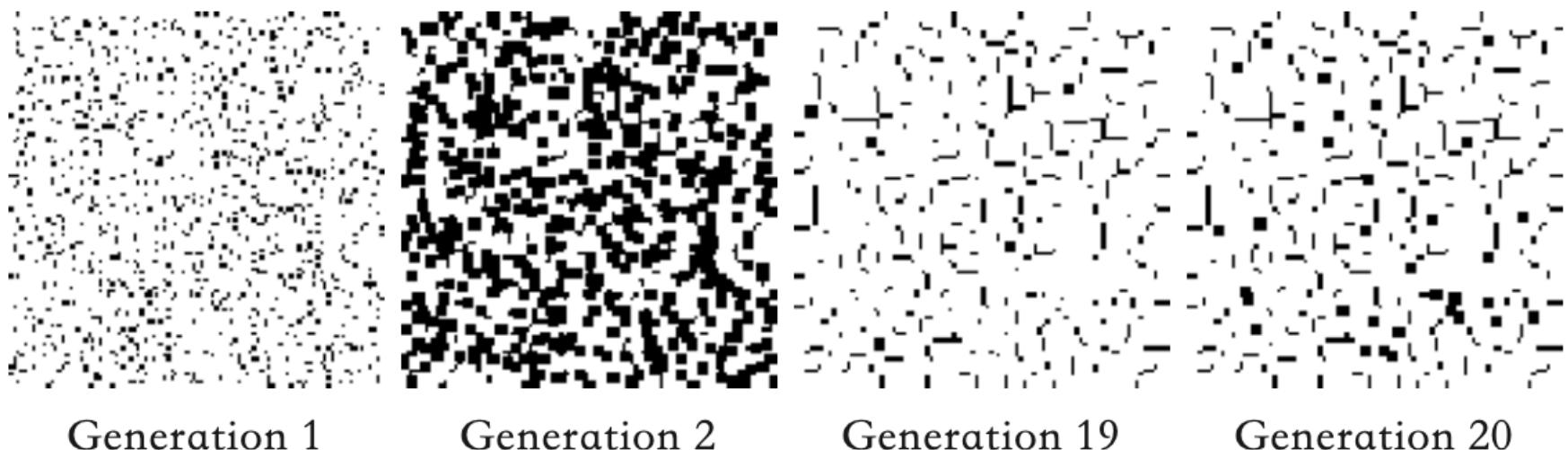
Generation 5



Generation 6

Replicator Dynamics: Experiment

- On a local interaction model
 - $T = 1.2, R = 1.1, P = 0.1$, and $S = 0$ **Cooperate**



Replicator Dynamics: Experiment

- On a local interaction model
 - $T = 1.61$, $R = 1.01$, $P = 0.01$, and $S = 0$ **Chaotic**



Generation 1



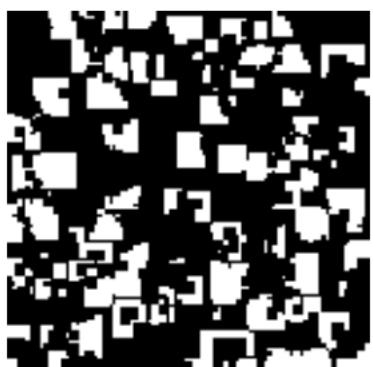
Generation 3



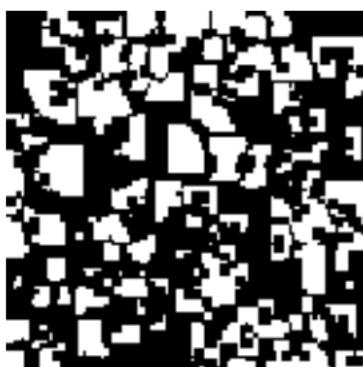
Generation 5



Generation 7



Generation 9



Generation 11

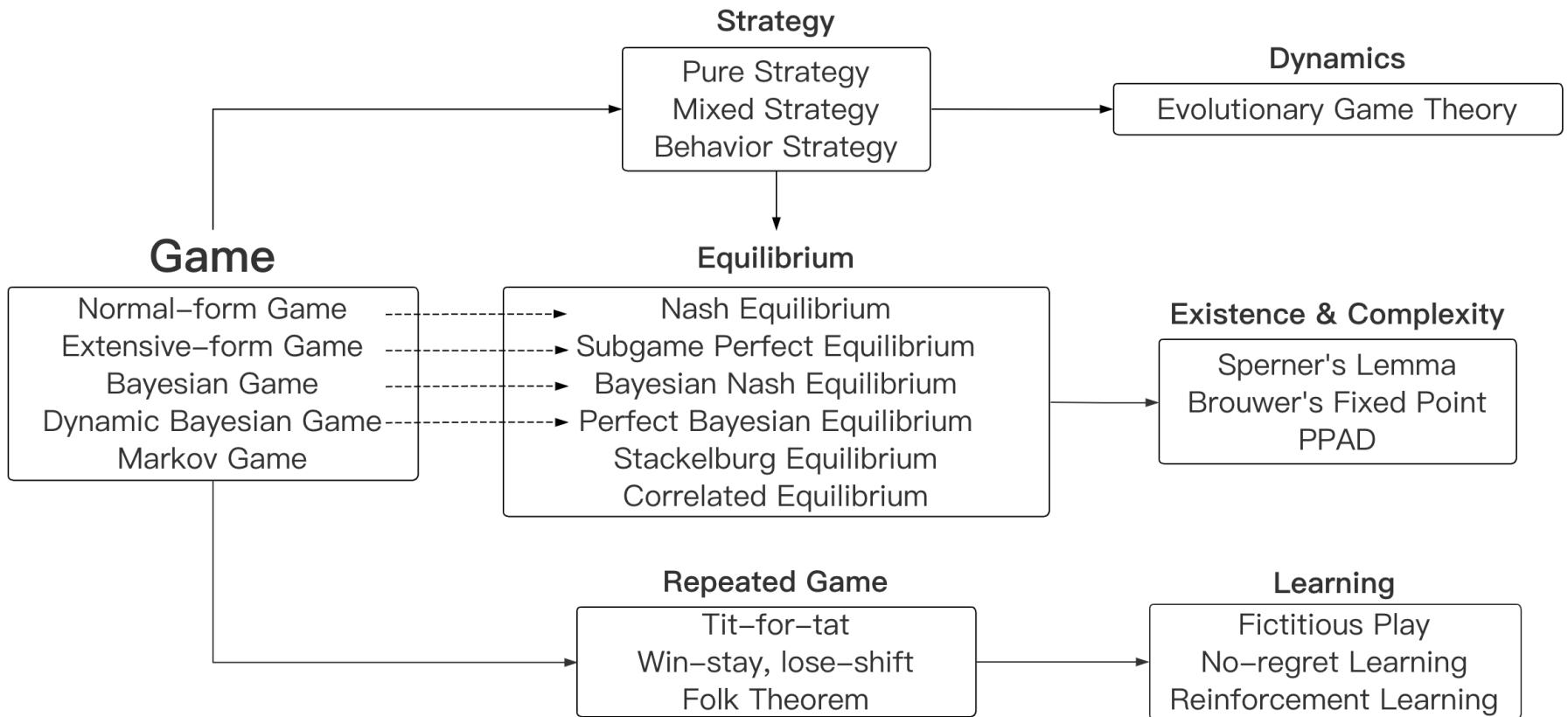


Generation 13



Generation 15

Summary



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