1 Cantor Space & Borel Hierarchy

Definition 1. *The **Cantor space** is the pair* $(\mathbb{B}^\omega, d)$ *with* $d(\alpha,\beta) = \begin{cases} 0 & \text{if } \alpha = \beta \\ 2^{-(\min_n \alpha(n) \neq \beta(n))} & \text{else} \end{cases}$.

Note: The $\frac{1}{2n}$*-neighborhood of* α *is* $\alpha[0,n] \cdot \mathbb{B}^\omega$.

Definition 2. *From the Cantor space we define the **Cantor topology** with open sets* $\mathcal{O} = \{W \cdot \mathbb{B}^\omega \mid W \subseteq \mathbb{B}^*\}$.

Definition 3. *The **Borel hierarchy** is a collection* $\{\Sigma_1, \Pi_1, \Sigma_2, \Pi_2, \dots\}$ *defined as*
$\Sigma_1 = \mathcal{O}$
$\Pi_1 = \mathbb{B}^\omega \setminus \mathcal{O}$
$\Sigma_{n+1} = \{\bigcup_{i\in\mathbb{N}} L_i \mid L_i \in \Pi_n\}$
$\Pi_{n+1} = \{\bigcap_{i\in\mathbb{N}} L_i \mid L_i \in \Sigma_n\}$

Theorem 1.

1. *Every class* Σ_n *or* Π_n *of the Borel hierarchy is closed under finite union and intersection.*
2. *For every* $L \subseteq \mathbb{B}^\omega$*, we have* $L \in \Sigma_n$ *iff* $L^\complement \in \Pi_n$.

Proof.

1. It suffices to show the closure for Σ_n. Then it follows for Π_n from (2). For example, let $L, K \in \Pi_n$, so $L^\complement, K^\complement \in \Sigma_n$, so $L^\complement \cup K^\complement = (L \cap K)^\complement \in \Sigma_n$ and therefore $L \cap K \in \Pi_n$.

   For $n = 0$, this is clear from the definition of $\mathcal{O}$. Let $W_1, W_2 \subseteq \mathbb{B}^*$. Then $W_1 \cdot \mathbb{B}^\omega \cup W_2 \cdot \mathbb{B}^\omega = (W_1 \cup W_2) \cdot \mathbb{B}^\omega$ and $W_1 \cdot \mathbb{B}^\omega \cap W_2 \cdot \mathbb{B}^\omega = (W_1 \cdot \mathbb{B}^* \cap W_2 \cdot \mathbb{B}^*) \cdot \mathbb{B}^\omega$.

   Using an induction argument, consider $L, K \in \Sigma_{n+1}$, so $L = \bigcup\limits_{i\in\mathbb{N}} L_i$ and $K = \bigcup\limits_{i\in\mathbb{N}} K_i$ for $(L_i)_i, (K_i)_i \in \Pi_n$. By induction, $L_i \cup K_i \in \Pi_n$ for all i, and thus $L \cup K = \bigcup\limits_{i\in\mathbb{N}} (L_i \cup K_i) \in \Sigma_{n+1}$.

   For intersection we have $L \cap K = \bigcup\limits_{i,j\in\mathbb{N}} L_i \cap K_j \in \Sigma_{n+1}$.

2. De Morgan law

□

Definition 4. *Let* $f : \mathbb{B}^\omega \to \mathbb{B}^\omega$. f *is **continuous** if for all open sets* $O \in \mathcal{O}$: $f^{-1}(O) \in \mathcal{O}$.
For $L, K \subseteq \mathbb{B}^\omega$*, we write* $K \leq L$ *if there is a continuous function* f *with* $f^{-1}(L) = K$.

Theorem 2. *Let* $f : \mathbb{B}^\omega \to \mathbb{B}^\omega$*. The following three statements are equivalent:*

1. f *is continuous.*
2. $\forall \alpha \in \mathbb{B}^\omega. \forall n \in \mathbb{N}. \exists m \in \mathbb{N}. \forall \beta \in \mathbb{B}^\omega : d(\alpha,\beta) < \frac{1}{2^m} \Rightarrow d(f(\alpha), f(\beta)) \leq \frac{1}{2^n}$
3. $\forall n \in \mathbb{N}. \exists m \in \mathbb{N}. \forall \alpha, \beta \in \mathbb{B}^\omega : d(\alpha,\beta) < \frac{1}{2^m} \Rightarrow d(f(\alpha), f(\beta)) \leq \frac{1}{2^n}$

Proof. **(1) ⇒ (3)**
(3) ⇒ (2): Trivial, since m does not depend on α in general.
(2) ⇒ (1): Let $L = W \cdot \mathbb{B}^\omega \in \mathcal{O}$. Let $U = \{u \in \mathbb{B}^* \mid f(u \cdot \mathbb{B}^\omega) \subseteq L\}$. We claim that $f^{-1}(L) = U \cdot \mathbb{B}^\omega$.
Let $\alpha \in U \cdot \mathbb{B}^\omega$, so $\alpha = u \cdot \beta$ for some $u \in U$. By definition of U, $f(u \cdot \beta) = f(\alpha) \in L$. Therefore, $\alpha \in f^{-1}(L)$.

Let $\alpha \in f^{-1}(L)$, so $f(\alpha) = w\alpha'$ for some $w \in W$. Using the assumption, there is an $m \in \mathbb{N}$ such that for all $\beta \in \mathbb{B}^\omega$ with $d(\alpha, \beta) \leq 2^{-m}$, we have $d(f(\alpha), f(\beta)) \leq 2^{-|w|}$, meaning that $w \sqsubseteq f(\beta)$, so $f(\beta) \in L$.
For all $\beta \in \mathcal{B}^\omega$, we have $f(\alpha[0, m] \cdot \beta) \in L$ by the previous result. This means $\alpha[0, m] \in U$ (by definition of U) and therefore $\alpha \in U \cdot \mathcal{B}^\omega$. ☐

Theorem 3. *If $K \leq L$ and $L \in \Sigma_n$, then $K \in \Sigma_n$. The same is true for Π_n.*

Proof. Let f be a continuous function with $f^{-1}(L) = K$. For $n = 0$, $L \in \mathcal{O}$, so $K \in \mathcal{O}$.

Otherwise, assume the claim is ture for n and let $L \in \Sigma_{n+1}$, so $L = \bigcup_{i \in \mathbb{N}} L_i$ for $L_i \in \Pi_n$. Let $K_i = f^{-1}(L_i)$, so we have $K_i \leq L_i$. By induction this gives us $K_i \in \Pi_n$ for all i and therefore $\bigcup_{i \in \mathbb{N}} K_i \in \Sigma_{n+1}$. It remains to be shown that $K = \bigcup_{i \in \mathbb{N}} K_i$. ☐

Definition 5. *Let $L \subseteq \mathbb{B}^\omega$. L is **complete** for Σ_n if $\forall K \in \Sigma_n : K \leq L$.*

1.1 Relation to Automata

- regular Σ_1 = E-recognizable
- regular Π_1 = A-recognizable
- regular Σ_2 = co-Büchi-recognizable
- regular Π_2 = DBA-recognizable
- boolean combination of Π_2 = NBA-recognizable

2 Gale-Stewart & Wadge

Definition 6. *Let $L \subseteq \mathbb{B}^\omega$. The **Gale-Stewart game** $\Gamma(L)$ is defined as follows: Starting with player 0, two players alternatingly pick bits 0 or 1, resulting in a play $\alpha \in \mathbb{B}^\omega$. Player 0 wins iff $\alpha \in L$.*

Definition 7. *Let $K, L \subseteq \mathbb{B}^\omega$. The **Wadge game** $W(K, L)$ is defined as follows: Starting with player 0, two players alternatingly pick bits 0 or 1, where player 1 also has the option to skip a turn, resulting in a pair (α, β) with $\alpha \in \mathbb{B}^\omega$ and $\beta \in \mathbb{B}^* \cup \mathbb{B}^\omega$.*

Player 1 wins the play (α, β) iff $\beta \in \mathbb{B}^\omega$ and $\alpha \in K \leftrightarrow \beta \in L$.

Theorem 4 (Gale-Stewart)**.** *For $L \in \Sigma_1 \cup \Pi_1$, $\Gamma(L)$ is determined.*

Theorem 5. *If $L \in \Sigma_1$, then $L = W \cdot \mathbb{B}^\omega$. A winning strategy for player 0 (if it exists) is the attractor strategy for W.*

If $L = (W \cdot \mathbb{B}^\omega)^\complement \in \Pi_1$, then player 1 can play the attractor strategy for W.

Theorem 6 (Martin)**.** *For every set L in the Borel hierarchy, $\Gamma(L)$ is determined.*

Theorem 7. *Let $K, L \subseteq \mathbb{B}^\omega$. Player 1 wins $W(K, L)$ iff $K \leq L$.*

Proof. $\Rightarrow$ Let $\sigma : (\mathbb{B}^* \times \mathbb{B}^*) \to \{0, 1, \varepsilon\}$ be a winning strategy for player 1 in $W(K, L)$. Let $\tau(\alpha)$ be a strategy for player 0 in which they play $\alpha(i)$ in turn i, and for all $\alpha \in \mathbb{B}^\omega$ let $f(\alpha)$ be the play of player 1 if both players play according to $\tau(\alpha)$ and σ respectively. We claim that f is continuous.

$\Leftarrow$ Let $f : \mathbb{B}^\omega \to \mathbb{B}^\omega$ be continuous. ☐

Example Let $L = (0^*1)^\omega$. We claim that L is Π_2-complete. Let $K = \bigcap_{i \in \mathbb{N}} K_i \in \Pi_2$ for $K_i \in \mathcal{O}$, so $K_i = W_i \cdot \mathbb{B}^\omega$. We define a winning strategy for player 1 in $W(K, L)$ which proves the claim.
At the beginning of the game, set a variable $i := 0$. In each turn, let (u, v) be the play up until this point. If $u \notin W_i$, play 0. Otherwise, play 1 and increment i by 1.

Theorem 8. *There is a language $L \subseteq \mathbb{B}^\omega$ such that $\Gamma(L)$ is not determined.*