1 Cantor Space & Borel Hierarchy

Definition 1. *The **Cantor space** is the pair* $(\mathbb{B}^\omega, d)$ *with* $d(\alpha,\beta) = \begin{cases} 0 & \text{if } \alpha = \beta \\ 2^{-(\min_n \alpha(n) \neq \beta(n))} & \text{else} \end{cases}$.

Note: The $\frac{1}{2n}$*-neighborhood of* α *is* $\alpha[0,n] \cdot \mathbb{B}^\omega$.

Definition 2. *From the Cantor space we define the **Cantor topology** with open sets* $\mathcal{O} = \{W \cdot \mathbb{B}^\omega \mid W \subseteq \mathbb{B}^*\}$.

Definition 3. *The **Borel hierarchy** is a collection* $\{\Sigma_1, \Pi_1, \Sigma_2, \Pi_2, \dots\}$ *defined as*
$\Sigma_1 = \mathcal{O}$
$\Pi_1 = \mathbb{B}^\omega \setminus \mathcal{O}$
$\Sigma_{n+1} = \{\bigcup_{i \in \mathbb{N}} L_i \mid L_i \in \Pi_n\}$
$\Pi_{n+1} = \{\bigcap_{i \in \mathbb{N}} L_i \mid L_i \in \Sigma_n\}$

Theorem 1.

1. *Every class* Σ_n *or* Π_n *of the Borel hierarchy is closed under finite union and intersection.*
2. *For every* $L \subseteq \mathbb{B}^\omega$*, we have* $L \in \Sigma_n$ *iff* $L^\complement \in \Pi_n$.

Proof.

1. It suffices to show the closure for Σ_n. Then it follows for Π_n from (2). For example, let $L, K \in \Pi_n$, so $L^\complement, K^\complement \in \Sigma_n$, so $L^\complement \cup K^\complement = (L \cap K)^\complement \in \Sigma_n$ and therefore $L \cap K \in \Pi_n$.

   For $n = 0$, this is clear from the definition of $\mathcal{O}$. Let $W_1, W_2 \subseteq \mathbb{B}^*$. Then $W_1 \cdot \mathbb{B}^\omega \cup W_2 \cdot \mathbb{B}^\omega = (W_1 \cup W_2) \cdot \mathbb{B}^\omega$ and $W_1 \cdot \mathbb{B}^\omega \cap W_2 \cdot \mathbb{B}^\omega = (W_1 \cdot \mathbb{B}^* \cap W_2 \cdot \mathbb{B}^*) \cdot \mathbb{B}^\omega$.

   Using an induction argument, consider $L, K \in \Sigma_{n+1}$, so $L = \bigcup\limits_{i \in \mathbb{N}} L_i$ and $K = \bigcup\limits_{i \in \mathbb{N}} K_i$ for $(L_i)_i, (K_i)_i \in \Pi_n$. By induction, $L_i \cup K_i \in \Pi_n$ for all i, and thus $L \cup K = \bigcup\limits_{i \in \mathbb{N}} (L_i \cup K_i) \in \Sigma_{n+1}$.

   For intersection we have $L \cap K = \bigcup\limits_{i,j \in \mathbb{N}} L_i \cap K_j \in \Sigma_{n+1}$.

2. De Morgan law

□

Definition 4. *Let* $f : \mathbb{B}^\omega \to \mathbb{B}^\omega$. f *is **continuous** if for all open sets* $O \in \mathcal{O}$: $f^{-1}(O) \in \mathcal{O}$.

For $L, K \subseteq \mathbb{B}^\omega$*, we write* $K \leq L$ *if there is a continuous function* f *with* $f^{-1}(L) = K$.

Theorem 2. *Let* $f : \mathbb{B}^\omega \to \mathbb{B}^\omega$*. The following three statements are equivalent:*

1. f *is continuous.*
2. $\forall \alpha \in \mathbb{B}^\omega . \forall n \in \mathbb{N} . \exists m \in \mathbb{N} . \forall \beta \in \mathbb{B}^\omega : d(\alpha,\beta) < \frac{1}{2^m} \Rightarrow d(f(\alpha), f(\beta)) \leq \frac{1}{2^n}$
3. $\forall n \in \mathbb{N} . \exists m \in \mathbb{N} . \forall \alpha, \beta \in \mathbb{B}^\omega : d(\alpha,\beta) < \frac{1}{2^m} \Rightarrow d(f(\alpha), f(\beta)) \leq \frac{1}{2^n}$

Proof. **(1) ⇒ (3)**

(3) ⇒ (2): Trivial, since m does not depend on α in general.

(2) ⇒ (1): Let $L = W \cdot \mathbb{B}^\omega \in \mathcal{O}$. Let $U = \{u \in \mathbb{B}^* \mid f(u \cdot \mathbb{B}^\omega) \subseteq L\}$. We claim that $f^{-1}(L) = U \cdot \mathbb{B}^\omega$.

Let $\alpha \in U \cdot \mathbb{B}^\omega$, so $\alpha = u \cdot \beta$ for some $u \in U$. By definition of U, $f(u \cdot \beta) = f(\alpha) \in L$. Therefore, $\alpha \in f^{-1}(L)$.

Let $\alpha \in f^{-1}(L)$, so $f(\alpha) = w\alpha'$ for some $w \in W$. Using the assumption, there is an $m \in \mathbb{N}$ such that for all $\beta \in \mathbb{B}^\omega$ with $d(\alpha, \beta) \leq 2^{-m}$, we have $d(f(\alpha), f(\beta)) \leq 2^{-|w|}$, meaning that $w \sqsubseteq f(\beta)$, so $f(\beta) \in L$.
For all $\beta \in \mathcal{B}^\omega$, we have $f(\alpha[0,m] \cdot \beta) \in L$ by the previous result. This means $\alpha[0,m] \in U$ (by definition of U) and therefore $\alpha \in U \cdot \mathcal{B}^\omega$. ☐

Theorem 3. *If $K \leq L$ and $L \in \Sigma_n$, then $K \in \Sigma_n$. The same is true for Π_n.*

Proof. Let f be a continuous function with $f^{-1}(L) = K$. For $n = 0$, $L \in \mathcal{O}$, so $K \in \mathcal{O}$.

Otherwise, assume the claim is ture for n and let $L \in \Sigma_{n+1}$, so $L = \bigcup\limits_{i \in \mathbb{N}} L_i$ for $L_i \in \Pi_n$. Let $K_i = f^{-1}(L_i)$, so we have $K_i \leq L_i$. By induction this gives us $K_i \in \Pi_n$ for all i and therefore $\bigcup\limits_{i \in \mathbb{N}} K_i \in \Sigma_{n+1}$. It remains to be shown that $K = \bigcup\limits_{i \in \mathbb{N}} K_i$. ☐

Definition 5. *Let $L \subseteq \mathbb{B}^\omega$. L is **complete** for Σ_n if $\forall K \in \Sigma_n : K \leq L$.*

1.1 Relation to Automata

- regular Σ_1 = E-recognizable
- regular Π_1 = A-recognizable
- regular Σ_2 = co-Büchi-recognizable
- regular Π_2 = DBA-recognizable
- boolean combination of Π_2 = NBA-recognizable

2 Gale-Stewart & Wadge

Definition 6. *Let $L \subseteq \mathbb{B}^\omega$. The **Gale-Stewart game** $\Gamma(L)$ is defined as follows: Starting with player 0, two players alternatingly pick bits 0 or 1, resulting in a play $\alpha \in \mathbb{B}^\omega$. Player 0 wins iff $\alpha \in L$.*

Definition 7. *Let $K, L \subseteq \mathbb{B}^\omega$. The **Wadge game** $W(K, L)$ is defined as follows: Starting with player 0, two players alternatingly pick bits 0 or 1, where player 1 also has the option to skip a turn, resulting in a pair (α, β) with $\alpha \in \mathbb{B}^\omega$ and $\beta \in \mathbb{B}^* \cup \mathbb{B}^\omega$.*

Player 1 wins the play (α, β) iff $\beta \in \mathbb{B}^\omega$ and $\alpha \in K \leftrightarrow \beta \in L$.

Theorem 4 (Gale-Stewart)**.** *For $L \in \Sigma_1 \cup \Pi_1$, $\Gamma(L)$ is determined.*

Proof. If $L \in \Sigma_1$, then $L = W \cdot \mathbb{B}^\omega$. A winning strategy for player 0 (if it exists) is the attractor strategy for W.

If $L = (W \cdot \mathbb{B}^\omega)^\complement \in \Pi_1$, then player 1 can play the attractor strategy for W. ☐

Theorem 5 (Martin)**.** *For every set L in the Borel hierarchy, $\Gamma(L)$ is determined.*

Theorem 6. *Let $K, L \subseteq \mathbb{B}^\omega$. Player 1 wins $W(K, L)$ iff $K \leq L$.*

Proof. $\Rightarrow$ Let $\sigma : (\mathbb{B}^* \times \mathbb{B}^*) \to \{0, 1, \varepsilon\}$ be a winning strategy for player 1 in $W(K, L)$. Let $\tau(\alpha)$ be a strategy for player 0 in which they play $\alpha(i)$ in turn i, and for all $\alpha \in \mathbb{B}^\omega$ let $f(\alpha)$ be the play of player 1 if both players play according to $\tau(\alpha)$ and σ respectively. We claim that f is continuous.

$\Leftarrow$ Let $f : \mathbb{B}^\omega \to \mathbb{B}^\omega$ be continuous. ☐

Example Let $L = (0^*1)^\omega$. We claim that L is Π_2-complete. Let $K = \bigcap_{i\in\mathbb{N}} K_i \in \Pi_2$ for $K_i \in \mathcal{O}$, so $K_i = W_i \cdot \mathbb{B}^\omega$. We define a winning strategy for player 1 in $W(K, L)$ which proves the claim.
At the beginning of the game, set a variable $i := 0$. In each turn, let (u, v) be the play up until this point. If $u \notin W_i$, play 0. Otherwise, play 1 and increment i by 1.

Theorem 7. *There is a language $L \subseteq \mathbb{B}^\omega$ such that $\Gamma(L)$ is not determined.*

Proof. Let On be the set of ordinal numbers. Let $\mathcal{S}_0$ and $\mathcal{S}_1$ be the set of strategies for player 0 and 1 respectively in a Gale-Stewart game. Then we have $|\mathcal{S}_0| = |\mathcal{S}_1| = |\mathbb{B}^\omega| = 2^{\aleph_0} =: \kappa$. Let $(f_\alpha)_{\alpha<\kappa}$ and $(g_\alpha)_{\alpha<\kappa}$ be an enumeration of $\mathcal{S}_/$ and $\mathcal{S}_1$ respectively. For $f \in \mathcal{S}_0, g \in \mathcal{S}_1$, we write $\langle f, g\rangle \in \mathbb{B}^\omega$ for the unique play according to f and g.

Our goal is to construct a family of sets $(L_\alpha, M_\alpha)_{\alpha<\kappa} \in \mathbb{B}^\omega \times \mathbb{B}^\omega$ such that for all $\alpha < \kappa$:

1. for all $\beta < \alpha$: $L_\beta \subseteq L_\alpha$ and $M_\beta \subseteq M_\alpha$
2. $L_\alpha \cap M_\alpha = \emptyset$
3. $|L_\alpha| = |M_\alpha| = \alpha$
4. for all $\beta < \alpha$ there is an $f \in \mathcal{S}_0$ such that $\langle f, g_\beta\rangle \in L_\alpha$
5. for all $\beta < \alpha$ there is an $g \in \mathcal{S}_1$ such that $\langle f_\beta, g\rangle \in M_\alpha$

If that is done, set $L := \bigcup_{\alpha<\kappa} L_\alpha$. We claim that $\Gamma(L)$ is not determined. Assume player 0 has a winning strategy $f^* \in \mathcal{S}_0$, so $f^* = f_\alpha$ for some $\alpha < \kappa$. By (5), there must be a $g \in \mathcal{S}_1$ such that $\langle f^*, g\rangle \in M_{\alpha+1}$. Because of (1) and (2), $\langle f^*, g\rangle \notin L$, so this play is won by player 1. Hence, f^* cannot be a winning strategy.

Claim: Sets L_α, M_α as described above exist.

For $\alpha = 0$, set $L_\alpha = M_\alpha = \emptyset$. Otherwise, let $0 < \alpha < \kappa$ be arbitrary. We find plays $\pi_L, \pi_M \in \mathbb{B}^\omega$ such that $L_\alpha = \bigcup_{\beta<\alpha} L_\beta \cup \{\pi_L\}$ and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta \cup \{\pi_M\}$ satisfy the conditions.

Let $P = \bigcup_{\beta<\alpha}(L_\beta \cup M_\beta)$. Let $g \in \mathcal{S}_1$ such that $\pi_L := \langle f_\alpha, g\rangle \notin P$. To see that this is possible, note that $|P| = 2\alpha < \kappa$ because of (3). Analogously, find an $f \in \mathcal{S}_0$ such that $\pi_M := \langle f, g_\alpha\rangle \notin P \cup \{\pi_L\}$. □