1 Basics

1.1 List Of Games

- Büchi
- Staiger-Wagner
- weak Parity
- Reachability (E-condition)
- Safety (A-condition)
- Muller
- Parity
- Rabin
- Streett

1.2 List Of Properties

- Determined
  For every node v, either player has a winning strategy.
- Positionally Determined
  For every node v, either player has a positional winning strategy.
- Prefix Independent
  $\forall x \in C^*, \alpha \in C^\omega : \alpha \in \text{Win} \leftrightarrow x\alpha \in \text{Win}$

1.3 Definitions

Definition 1. *A **game graph / arena** is a tuple $G = (V_0, V_1, E, c)$ where $V_0 \cap V_1 = \emptyset$, $E \subseteq V \times V$ where $V = V_0 \cup V_1$, and $c : V \to C$ for a finite set of colors C.*

*A **game** is a pair $\mathcal{G} = (G, Win)$ where G is an arena and $Win \subseteq C^\omega$.*

*A **strategy** for player i is a function $\sigma : V^*V_i \to V$ with $(u, v) \in E$ for all $\sigma(xu) = v$. σ is a **winning strategy** from $v \in V$, if all plays from v that are according to σ are won by player i. σ is **positional** if for all $x, y \in V^*, v \in V$: $\sigma(xv) = \sigma(yv)$.*

2 Memory & Reductions

Definition 2. *A* ***strategy automaton*** *for player 0 in a game* $\mathcal{G}$ *is a tuple* $\mathcal{A} = (M, C, m_{in}, \sigma^u, \sigma^n)$ *with* $\sigma^n : M \times V_0 \to V$ *and* $\sigma^u : M \times C \to M$*. The automaton defines a strategy* $\sigma_{\mathcal{A}}(xv) = \sigma^n(m, v)$ *where* $m = (\sigma^u)^*(m_{in}, x)$*.*

Definition 3. *Let* $\mathcal{G}$ *and* $\mathcal{G}'$ *be games.* $\boldsymbol{\mathcal{G}}$ ***reduces to*** $\boldsymbol{\mathcal{G}'}$ ***with memory*** $\boldsymbol{m}$ *if there is an* $f_{in} : V \to V'$ *such that a player wins from* $v \in V$ *iff that player wins from* $f_{in}(v) \in V'$*. For a winning strategy with memory* n *from* $f_{in}(v)$*, one can compute a winning strategy with memory* $n \cdot m$ *from* v*.*

Definition 4. *Let* $\mathcal{G} = (V_0, V_1, E, c, Win)$ *be a game and let* $\mathcal{A} = (Q, C, q_0, \delta, Acc)$ *be a finite automaton with* $L(\mathcal{A}) = Win$*. The* ***product game*** *is defined as* $\mathcal{G} \times \mathcal{A} = (V_0', V_1', E', c', Acc)$ *with*

- $V_0' = V_0 \times Q$
- $V_1' = V_1 \times Q$
- $E' = \{((u, p), (v, q) \in (V \times Q)^2 \mid (u, v) \in E \text{ and } q = \delta(p, c(u))\}$
- $c'(v, q) = q$

Theorem 1. $\mathcal{G}$ *reduces to* $\mathcal{G} \times \mathcal{A}$ *with memory* $|Q|$*.*

Example Let $\mathcal{A} = (Q, C, q_0, \delta, F)$ be a DFA and let $\mathcal{G} = (G, C^* L(\mathcal{A}C^\omega)$. Then $\mathcal{G}$ is a reachability game. Hence, $\mathcal{G} \times \mathcal{A}$ is determined with memory size $|Q|$.

3 Prefix Dependent Games

3.1 Reachability & Safety

$F \subseteq C$ and $\mathrm{Win} = C^* F C^\omega$ (reachability) or $\mathrm{Win} = (C \setminus F)^\omega$ (safety)

Theorem 2. *Reachability games and safety games are positionally determined. The winning regions and winning strategies can be computed in $\mathcal{O}(|G|)$.*

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3.2 Weak Parity

$C \subseteq \mathbb{N}$ and $\mathrm{Win} = \{\alpha \in C^\omega \mid \max \mathrm{Occ}(\alpha) \text{ is even}\}$.

Theorem 3. *Weak parity games are positionally determined. The winning regions and winning strategies can be computed in $\mathcal{O}(|C| \cdot |G|)$.*

3.3 Staiger-Wagner

$\mathcal{F} \subseteq 2^C$ and $\mathrm{Win} = \{\alpha \in C^\omega \mid \mathrm{Occ}(\alpha) \in \mathcal{F}\}$.

Theorem 4. *Staiger-Wagner games can be reduced to weak parity games with memory $2^{|C|}$.*

Proof. Similar to proof from SWA to WDBA. □

Theorem 5. *For every $n > 0$, there is an arena G_n with $|G_n| \in \mathcal{O}(n)$ and a set $\mathcal{F}_n \subseteq 2^C$ with $|\mathcal{F}_n| \in \mathcal{O}(n)$ such that player 0 has a winning strategy in $(G_n, \mathcal{F}_n)$ but every winning strategy requires memory of size 2^n.*

4 Prefix Independent Games

4.1 Büchi Games

$F \subseteq C$ and $\mathrm{Win} = \{\alpha \in C^\omega \mid \mathrm{Inf}(\alpha) \cap F \neq \emptyset\}$.

Theorem 6. *Büchi games are positionally determined. The winning regions and winning strategies can be computed in polynomial time in $|G|$.*

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