# 1 Infinite Computations

**Theorem 1.** *NBA-recognizable languages are closed under complement.*

*Proof.* Let $\mathcal{A} = (Q, \Sigma, q_0, \Delta, F)$ be an NBA. We define the transition profile of a word $w \in \Sigma^*$ as a finite directed graph $\mathrm{t}(w) = (Q, E, E_F)$ with $E = \{(p, q) \in Q \times Q \mid \mathcal{A} : p \xrightarrow{w} q\}$ and $E_F = \{(p, q) \in Q \times Q \mid \mathcal{A} : p \underset{F}{\overset{w}{\rightarrow}} q\}$. Then we define $\mathrm{TP} = \{\mathrm{t}(w) \mid w \in \Sigma^*\}$ and the transition profile automaton $\mathrm{TPA}_F = (\mathrm{TP}, \Sigma, \mathrm{t}(\varepsilon), \delta_{\mathrm{TPA}}, F)$ with $\delta_{\mathrm{TPA}}(\mathrm{t}(u), a) = \mathrm{t}(ua)$.

For $t \in \mathrm{TP}$, let $U_t = \{u \in \Sigma^+ \mid \mathrm{t}(u) = t\}$. These sets are regular, as they are accepted by the NFA $\mathrm{TPA}_{\{t\}}$.

Let $\bar{t} = t_0 t_1 \cdots \in \mathrm{TP}^\omega$ be an infinite sequence of transition profiles. We call $\bar{t}$ accepting if there are $q_1 \cdots \in Q^\omega$ such that every $t_i$ has an edge $(q_i, q_{i+1})$ and infinitely many of these edges are labeled by $F$. Let $\mathrm{NTP} = \{(t_0, t_1) \in \mathrm{TP} \times \mathrm{TP} \mid t_0 t_1^\omega \text{ is non-accepting}\}$.

**Claim** : $L(\mathcal{A})^\complement = \bigcup_{(t_0,t_1)\in\mathrm{NTP}} U_{t_0} U_{t_1}^\omega$. Then one can construct an NBA for $L(\mathcal{A})^\complement$.

For every $\alpha \in \Sigma^\omega$, let $\alpha = u_0 u_1 \cdots \in (\Sigma^*)^\omega$ be a factorization of $\alpha$ into finite words and let $\bar{t} = \mathrm{t}(u_0)\mathrm{t}(u_1)\ldots$. Then $\bar{t}$ is accepting iff $\alpha \in L(\mathcal{A})$.

In particular, this is true for $\mathrm{t}(u_1) = \mathrm{t}(u_2) = \ldots$, so it remains to be shown that every word $\alpha$ has such a factorization.

**Ramsey's Theorem**: Let $C$ be a finite set, $X$ with $|X| = \aleph_0$, $E = \{(x, y) \in X \times X \mid x \neq y\}$, and $f : E \to C$. Then there is an infinite $Y \subseteq X$ such that $|f(E \cap (Y \times Y))| = 1$.

For a $\alpha \in \Sigma^\omega$, let $X = \mathbb{N}$ and $C = \mathrm{TP}$. For $i < j$ we set $f(j, i) = f(i, j) = \mathrm{t}(\alpha[i, j])$. Using Ramsey's theorem, there is an infinite $Y \subseteq \mathbb{N}$ such that $f(E \cap (Y \times Y)) = \{t\}$. Let $Y = \{i_j \mid j > 0\}$ such that $i_j < i_{j+1}$ and $i_0 := 0$. We define $u_j = \alpha[i_j, i_{j+1}]$. Then $\alpha = u_0 u_1 u_2 \ldots$ and $\mathrm{t}(u_1) = \mathrm{t}(u_2) = \ldots$. □

**Theorem 2.** *For every $\varphi \in LTL$, one can construct an equivalent GBA with $\mathcal{O}(2^{|\varphi|})$ states.*

*Proof.* Let $\mathrm{cl}(\varphi) \subseteq \mathrm{LTL}$ be the set of all sub-formulas in $\varphi$ (including $\varphi$ itself). We define a $\varphi$-expansion of a word $\alpha \in (\mathbb{B}^n)^\omega$ as a function $\beta : \mathbb{N} \times \mathrm{cl}(\varphi) \to \mathbb{B}$ as below. The GBA then is $\mathcal{A} = (Q, \mathbb{B}^n, q_0, \Delta, F)$.

- $Q = \{q_0\} \cup 2^{\mathrm{cl}}(\varphi)$
- $F = (F_\psi)_{\psi \in \mathrm{Ucl}}$ where $\mathrm{Ucl} = \{\psi \in \mathrm{cl}(\varphi) \mid \psi = \psi_1 U \psi_2\}$
- $\Delta : q_0 \xrightarrow{a} \Phi$ with $\varphi \in \Phi$ according to $\varphi$-expansion rules
- $\Delta : \Phi \xrightarrow{a} \Psi$ according to $\varphi$-expansion rules

$\beta$ is a $\varphi$-extension of $\alpha$ if it satisfies the following local conditions

- $\beta(i, p_j) = a_j$ where $\alpha(i) = (a_1, \ldots, a_n)$.
- $\beta(i, \neg\psi) = 1$ iff $\beta(i, \psi) = 0$.

- $\beta(i, \psi \wedge \vartheta) = 1$ iff $\beta(i, \psi) = \beta(i, \vartheta) = 1$.
- $\beta(i, \psi \vee \vartheta) = 1$ iff $\beta(i, \psi) = 1$ or $\beta(i, \vartheta) = 1$.
- $\beta(i, X\psi) = 1$ iff $\beta(i+1, \psi) = 1$.
- $\beta(i, \psi U \vartheta) = 1$ iff $\beta(i, \vartheta) = 1$ or $[\beta(i, \psi_1) = 1$ and $\beta(i+1, \psi U \vartheta) = 1]$

and the following global condition

$$\forall i \quad \beta(i, \psi U \vartheta) = 1 \ \rightarrow \ \exists k \geq i \ \beta(k, \vartheta) = 1$$

The local conditions are checked in the transitions of $\Delta$. The global transition is checked by the acceptance sets $F_\psi$. $\square$

**Theorem 3** (Landweber)**.** *Let $\mathcal{A} = (Q, \Sigma, q_0, \delta, \mathcal{F})$ be a DMA.*

1. *$L(\mathcal{A})$ is DBA-recognizable iff $\mathcal{F}$ is closed under super loops.*
2. *$L(\mathcal{A})$ is E-recognizable iff $\mathcal{F}$ is closed under reachable loops.*

*Proof.* $\square$

**Theorem 4.** *NBAs, S1S, S1S$_0$, and $\exists$S1S have the same expressive power.*

*Proof.* **NBA $\Rightarrow$ $\exists$S1S** Let $\mathcal{A} = (Q, \mathbb{B}^n, 1, \Delta, F)$ be the NBA with $Q = \{1, \ldots, m\}$. We encode a run of $\mathcal{A}$ on a word with a formula.

$$\varphi_{\mathcal{A}}(X_1, \ldots, X_n) = \exists Y_1 \ldots \exists Y_m \mathrm{Part}(\overline{Y}) \wedge Y_1(0) \wedge \mathrm{Trans}(\overline{X}, \overline{Y}) \wedge \mathrm{Fin}(\overline{Y})$$

$$\mathrm{Part}(Y_1, \ldots, Y_m) = \forall x \left( \bigvee_{i=1}^{m} Y_i x \wedge \bigwedge_{i=1}^{m} \bigwedge_{j \neq i} \neg Y_i x \vee \neg Y_j x \right)$$

$$\mathrm{Trans}(\overline{X}, \overline{Y}) = \forall x \bigvee_{\tau \in \Delta} \psi_\tau(x, \overline{X}, \overline{Y})$$

$$\psi_{(p,a,q)}(x, X_1, \ldots, X_n, Y_1, \ldots, Y_m) = Y_p x \wedge Y q(x+1) \wedge X_a x$$

$$\mathrm{Fin}(Y_1, \ldots, Y_m) = \forall x \exists y \left( x < y \wedge \bigvee_{q \in F} Y_q y \right)$$

**$\exists$S1S $\Rightarrow$ S1S** trivial

**S1S $\Rightarrow$ S1S$_0$** Let $\varphi \in$ S1S. We define an equivalent $\varphi' \in$ S1S$_0$ inductively.

- Boolean operators stay the same.
- $\exists x \psi(x) \mapsto \exists X_x(\mathrm{Sing}(X_x) \wedge \psi'(X_x))$
- $Xx \mapsto X_x \subseteq X$
- $x + 1 = y \mapsto \mathrm{Succ}(X_x, X_y)$

**S1S$_0$ ⇒ NBA** $\exists$, $\vee$, $\neg$ correspond to projection, union, and complement of automata respectively. Therefore it suffices to give NBAs for the three base formulas.

- $X_1 \subseteq X_2$: $Q = F = \{q_0\}$, $\Delta = \{(q_0, a, q_0) \mid a \neq (1\ 0)\}$.
- $\mathrm{Sing}(X)$: $Q = \{q_0, q_f\}$, $F = \{q_f\}$, $\Delta = \{(q_0, 0, q_0), (q_f, 0, q_f), (q_0, 1, q_f)\}$.
- $\mathrm{Succ}(X_1, X_2)$: $Q = \{q_0, q_1, q_2\}$, $F = \{q_2\}$,
  $\Delta = \{(q_0, (0\ 0), q_0), (q_2, (0\ 0), q_2), (q_0, (1\ 0), q_1), (q_1, (0\ 1), q_2)\}$.

The complexity of this construction is $2 \uparrow\uparrow |\varphi|$.

□

**Theorem 5** (McNaughton)**.** *A language is NBA-recognizable iff it is DMA-recognizable.*

*Proof.* $\Leftarrow$ NBA with $L(\mathcal{A}) = \bigcup_{F \in \mathcal{F}} \left( \bigcap_{q \in F} L(\mathcal{A}_q) \cap \bigcap_{q \notin F} \overline{L(\mathcal{A}_q)} \right)$ where $\mathcal{A}_q$ is $\mathcal{A}$ starting in $q$.

$\Rightarrow$ □

**Theorem 6.** *For each Muller automaton, one can construct an equivalent parity automaton. The construction preserves determinism.*

*Proof.* Let $\mathcal{A} = (Q, \Sigma, q_0, \Delta, \mathcal{F})$ be a Muller automaton with $Q = \{q_0, \ldots, q_{n-1}\}$. We define the LAR automaton $\mathcal{A}_{\mathrm{LAR}} = (\mathrm{LAR}(Q), \Sigma, [q_0 q_1 \ldots q_{n-1}, 1], \Delta_{\mathrm{LAR}}, c_{\mathrm{LAR}})$ as follows:

- $\mathrm{LAR}(Q)$ contains an ordering of $Q$ and a so called hit-marker: $[p_1 \ldots p_n, h]$ with $1 \leq h \leq n$. This means that the most recently seen state was $p_1$ which was at position $h$ in the list before.
- $c_{\mathrm{LAR}}([p_1 \ldots p_n, h]) := \begin{cases} 2h & \text{if } P \in \mathcal{F} \\ 2h - 1 & \text{if } P \notin \mathcal{F} \end{cases}$ where $P = \{p_1, \ldots, p_h\}$.
- $\Delta_{\mathrm{LAR}} : [p_1 \ldots p_n, h] \xrightarrow{a} \mathrm{up}([p_1 \ldots p_n, h], p')$ for all $(p_1, a, p') \in \Delta$, where $\mathrm{up}([p_1 \ldots p_n, h], p') = [p' p_1 \ldots p_{i-1} p_{i+1} \ldots p_n, i]$.

**Claim** : Let $\rho \in Q^\omega$ be a run of $\mathcal{A}$ on some word $\alpha$ and let $\rho' \in (\mathrm{LAR}(Q))^\omega$ be the corresponding run of $\mathcal{A}_{\mathrm{LAR}}$. Then in $\rho'$, the hit marker $h$ is greater than $|\mathrm{Inf}(\rho)|$ only finitely often; and the hit segment $\{p_1, \ldots, p_h\}$ equals $\mathrm{Inf}(\rho)$ infinitely often. □

**Theorem 7.** *For each ABA, one can construct an equivalent NBA with states at most* $3^{|Q|}$.

*Proof.* Let $\mathcal{A} = (Q, \Sigma, q_0, \delta, F)$ be an ABA. We define $\mathcal{A}' = (\{0,1,2\}^Q, \Sigma, f_0, \Delta, \{0,2\}^Q)$ with $f_0(q) = \begin{cases} 1 & \text{if } q = q_0 \\ 0 & \text{else} \end{cases}$ and $\Delta$ as described below.

A state is a function $f : Q \to \{0,1,2\}$. Consider a run-tree of $\mathcal{A}$ on some word and all states that are "active" on one level in the tree. $f(q)$ is 0 for all states which are not active and 1 or 2 for all others. A state is assigned to 1 by default, which just means that it is active in some path. It is assigned 2 if "recently" on all paths it is active on, a final state in $F$ was seen. Recently means that it occurred since the last time that an entire level in the run tree was made up of final states.

Formally, we have $(f, a, g) \in \Delta$ for all $g : Q \to \{0,1,2\}$ which satisfy the following:

- The active states need to be passed on, i.e. for all $q \in Q$: if $f(q) \in \{1,2\}$ then there must be an $X_q \subseteq Q$ with $X_q \models \delta(q, a)$ such that $g(X_q) \subseteq \{1,2\}$.
- A state is assigned 2 if it is final, i.e. $g(q) = 2$ if $q \in F$.
- Otherwise, a 1 overwrites a 2 for succession, i.e. for all $p \in P$, if there is a $q$ with $f(q) = 1$ and $p \in X_q$, then also $g(p) = 1$.
- If all states are marked with a 2, $\mathcal{A}'$ reached a final state. We reset the values to $g(p) = 2$ iff $p \in F$.

□

## 2 Tree Automata

**Theorem 8.** *Let $\mathcal{A} = (Q, \Sigma, q_0, \Delta, F)$ be a TWA. There is an NTA $\mathcal{A}'$ of size exponential in $|Q|$ that recognizes $T(\mathcal{A})$.*

*Proof.* Let $\sim_{T(\mathcal{A})}$ be the usual equivalence relation, i.e. $t_1 \sim_{T(\mathcal{A})} t_2$ iff $\forall s \in S_\Sigma : s \cdot t_1 \in T(\mathcal{A}) \leftrightarrow s \cdot t_2 \in T(\mathcal{A})$. We define a relation $\sim \subseteq T_\Sigma \times T_\Sigma$ such that $\text{index}(\sim_{T(\mathcal{A})}) \leq \text{index}(\sim) \leq 2^{|Q|^2 \cdot m+1}$, where $m$ is the maximal rank in $\Sigma$.

Let $t_0 \in T_\Sigma$ and $a_m \in \Sigma_m$ be arbitrary. For every $t \in T_\Sigma$ and $1 \leq i \leq m$, we define $t^{(i)} = a_m(t_0, t_0, \ldots, t, \ldots, t_0)$, meaning the $i$-th subtree below the root is $t$. Further, we define a relation $B_t^i \subseteq Q \times Q$ with $(p, q) \in B_t^i$ iff there is a run segment $\rho$ of $\mathcal{A}$ on $t^{(i)}$, such that the run begins at the root of $t$, never leaves that subtree until the end. Meaning, $\rho = (p, i)(q_1, iu_1) \ldots (q_n, iu_n)(q, \varepsilon)$.

Finally, let $t_1 \sim t_2$ iff $t_1 \in T(\mathcal{A}) \leftrightarrow t_2 \in T(\mathcal{A})$ and $\forall i : B_{t_1}^i = B_{t_2}^i$.

Idea: $(p, q) \in B_t^i$ if $\mathcal{A}$ can enter $t$ as $i$-th child with state $p$ and after some while leaves it again with state $q$.

**Claim** : Let $t_1 \sim t_2$. Then $t_1 \sim_{T(\mathcal{A})} t_2$.

Let $s \in S_\Sigma$. Due to the symmetric definition of $\sim$, it suffices to show that $t_1 \in T(\mathcal{A})$ implies $t_2 \in T(\mathcal{A})$, so let $t_1 \in T(\mathcal{A})$. If $s = \circ$, then $s \cdot t_1 = t_1 \in T(\mathcal{A})$. By definition of $\sim$, this implies $s \cdot t_2 = t_2 \in T(\mathcal{A})$.

Otherwise $s \neq \circ$. Let $\rho_1\rho_2\rho_3$ be an accepting run of $\mathcal{A}$ on $s \cdot t_1$ such that $\rho_1$ only stays outside of $t_1$ and $\rho_2$ only stays inside of $t_1$. Since $B_{t_1}^i = B_{t_2}^i$, there is a run segment of $\mathcal{A}$ on $t_2$ which enters and exits the tree with the same states as $\rho_2$ does, meaning it can replace $\rho_2$ in the accepting run. Repeating this procedure gives an accepting run of $\mathcal{A}$ on $s \cdot t_2$, so $t_2 \in T(\mathcal{A})$.

**Notes** on the construction: each state in the NTA corresponds to a list of $Q$-states that $\mathcal{A}$ had when visiting this node, together with the direction from which it was coming. That can be used to check the correctness of a run. □

**Theorem 9.** *A language of finite trees $T \subseteq T_\Sigma$ can be recognized by an NTA iff it can be described by a regular tree expression.*

*Proof.* $\Rightarrow$ Let $\mathcal{A} = (Q, \Sigma, \Delta, F)$ be an NTA for $T$. For $R, I \subseteq Q, q \in Q$, we define $T(R, I, q) \subseteq T_{\Sigma \cup C_R}$ (where $C_R = \{c_p \mid p \in R\}$) as the set of all trees on which $\mathcal{A}$ has a run that only uses states in $I$ and ends in $q$. We can inductively define regular expressions for $T(R, I, q)$. Then $T(\mathcal{A}) = \bigcup_{q \in F} T(\emptyset, Q, q)$.

For all $R \subseteq Q, q \in Q$, $T(R, \emptyset, q)$ contains only trees of height 0 or 1, so it is a finite union of singular languages.

$$T(R, I \cup \{i\}, q) = T(R, I, q) + T(R \cup \{i\}, I, q) \cdot^{c_i} (T(R \cup \{i\}, I, i))^{*_{c_i}} \cdot^{c_i} T(R, I, i)$$

$\Leftarrow$ Show by induction that a regular expression $r$ can be transformed to a NTA $\mathcal{A}_r$.

- If $r = t \in T_{\Sigma \cup C}$, then there is an automaton $\mathcal{A}_t$ with $T(\mathcal{A}_t) = \{t\}$.
- If $r = s + t$, then $\mathcal{A}_r$ is the union automaton of $\mathcal{A}_s$ and $\mathcal{A}_\sqcup$.
- If $r = s \cdot^c t$, then $\mathcal{A}_r = (Q_s \cup Q_t, \Sigma \cup C, \Delta_r, F_s)$, where $\Delta_r$ simulates $\mathcal{A}_s$ and $\mathcal{A}_t$ but the final transitions in $\mathcal{A}_t$ are replaced by initial transitions to $\mathcal{A}_s$.
- If $r = s^{+_c}$, let $\mathcal{A}_r = (Q_s, \Sigma \cup C, \Delta_r, F_s)$, where $\Delta_r$ simulates $\mathcal{A}_s$ and allows "restarts" when a final state could be reached.

$\square$

**Theorem 10.** *Let $D$ be an EDTD. We can construct a NUTA $\mathcal{A}$ with $T(D)^{\complement} = T(\mathcal{A})$ in polynomial time in $|D|$.*

*Proof.* Let $D = (\Sigma', P, S^{(1)})$. For $a^{(i)} \in \Sigma'$, let $a^{(i)} \to r_{a,i}$ be the according rule in $P$, where $r_{a,i}$ is a regular expression that can be transformed to a DFA in polynomial time. If $b \in \Sigma$ occurs in $r_{a,i}$ then it must have a unique type $j$ (since $D$ is single-typed). We call $b^{(j)}$ the $b$-successor of $a^{(i)}$. If $b$ does not occur in $r_{a,i}$, we say that the $b$-successor is $b^{\perp}$. Furthermore, we assume $r_{a,\perp} = \varepsilon$ for all $a$.

We define a typing function $f : \mathrm{dom}_t \to \{\perp, 1, \ldots, k\}$. We assign $f(\varepsilon) = \begin{cases} 1 & \text{if } \mathrm{val}_t(\varepsilon) = S \\ \perp & \text{else} \end{cases}$.
For the other nodes, let $u$ be a node with parent $v$. We call $\mathrm{val}_t(v) = a$ and $\mathrm{val}_t(u) = b$. Then there is a unique $i \in \{\perp, 1, \ldots, k\}$ such that $b^{(i)}$ is the $b$-successor of $a^{f(v)}$. We set $f(u) := i$.

**Claim**: $t \in T(D)$ iff $\forall v \in \mathrm{dom}_t : f(v) \neq \perp$ and $a_1^{f(v1)} \ldots a_m^{f(vm)} \in L(r_{a,f(v)})$, where $a = \mathrm{val}_t(v)$ and $a_j = \mathrm{val}_t(vj)$. (without proof)

Using this claim, we can provide an automaton $\mathcal{A} = (Q, \Sigma, \Delta, F)$ that accepts a tree iff it violates the constraint.

- $Q = \Sigma \times \{\perp, 1, \ldots, k\} \times \{0, 1\}$, where the last component denotes whether a violation was found.
- $\Delta = \{(L_{a,i,x}, a, (a, i, x)) \mid a \in \Sigma, i \in \{\perp, 1, \ldots, k\}, x \in \{0, 1\}\}$
  Let $\alpha = (a_1, i_1, x_1) \ldots (a_m, i_m, x_m) \in Q^m$ and $w := a_1^{(i_1)} \ldots a_m^{(i_m)} \in (\Sigma')^m$.
  - succ $:\Leftrightarrow$ for all $1 \leq j \leq m$, $a_j^{(i_j)}$ is the $a_j$-successor of $a^{(i)}$.
  - $\mathrm{sat}_0 :\Leftrightarrow w \in L(r_{a,i})$ and for all $1 \leq j \leq m$, $x_j = 0$ and $i_j \neq \perp$.
  - $\mathrm{sat}_1 :\Leftrightarrow w \notin L(r_{a,i})$ or there is a $1 \leq j \leq m$ such that $x_j = 1$ or $i_j = \perp$.

  Then $\alpha \in L_{a,i,x}$ iff succ and $\mathrm{sat}_x$ hold.
- $F = \{(S, 1, 1)\} \cup \{(a, i, x) \mid a \neq S\}$, meaning either a violation was found or the starting symbol was not $S$.

$\square$

**Theorem 11.** *The class of DTWA-recognizable languages is closed under complement.*

*Proof.* Let $\mathcal{A} = (Q, \Sigma, q_0, \delta, \{q_f\})$ be a DTWA that only moves to $q_f$ at the root. The **backwards configuration graph** $\mathrm{BCG}(\mathcal{A}, t)$ is defined as a tree over $Q \times \mathrm{dom}_t$ with root $(q_f, \varepsilon)$. For a node $(q, u)$, the children are all $(p, v)$ such that $(p, v) \to_{\mathcal{A}} (q, u)$. We define $\overline{\mathcal{A}}$ in a way that it performs DFS on the BCG of the input tree and accepts iff the node $(q_0, \varepsilon)$ is found.

For that, let $\prec \subseteq (Q \times \mathrm{Dir})^2$ be an arbitrary linear order on $Q \times \mathrm{Dir}$. We set $\overline{\mathcal{A}} = (\overline{Q}, \Sigma, q_0, \overline{\delta}, q_f)$ with $\overline{Q} = \{q_0, q_f\} \cup \{\langle p, (q, d)\rangle \mid p, q \in Q, d \in \mathrm{Dir}\}$. The behavior of $\overline{\delta}$ is described below. Let $\langle q, (q', d)\rangle$ be a state.

**Case 1** : In the ordering $\prec$, $(\hat{q}, \hat{d})$ is the next largest element after $(q', d)$. (for $q_0$ we also consider this case with the $\prec$-minimal pair.)

**Case 2** : $(q', d)$ □

**Theorem 12.** *Let $T \subseteq T_\Sigma$. $T$ is regular iff fcns($T$) is regular.*

*Proof.* $\Rightarrow$ Let $\mathcal{A} = (Q, \Sigma, \Delta, F)$ be a NUTA with $T(\mathcal{A}) = T$. Wlog we assume that $\mathcal{A}$ is normalized. For every transition $\tau = (L_{a,q}, a, q) \in \Delta$, let $\mathcal{B}_{a,q} = (P_{a,q}, Q, p^0_{a,q}, \Delta_{a,q}, F_{a,q})$ be a NFA with $L(\mathcal{B}_{a,q}) = L_{a,q}$. We define $\mathcal{A}_{\mathrm{fcns}} = (Q_{\mathrm{fcns}}, \Gamma, \Delta_{\mathrm{fcns}}, F_{\mathrm{fcns}})$ so that $T(\mathcal{A}_{\mathrm{fcns}}) = \mathrm{fcns}(T)$.

- $Q_{\mathrm{fcns}} = \{q_f, q_\#\} \cup \bigcup_{a \in \Sigma, q \in Q} P_{a,q}$
- $F_{\mathrm{fcns}} = \{q_f\}$
- $\Delta_{\mathrm{fcns}}$:
  - $(\#, q_\#)$
  - For all $p \in \bigcup_{a \in \Sigma, q \in Q} F_{a,q}$: $(\#, p)$
  - For all $a \in \Sigma, q \in F$: $(p^0_{a,q}, q_\#, a, q_f)$
  - For all $a \in \Sigma, p \in P, p' \in P, q \in Q$ with $(p, q', p') \in \bigcup_{b \in \Sigma, q' \in Q} \Delta_{b,q'}$: $(p^0_{a,q}, p', b, p)$

Via induction on $t_1 \dots t_n$, one can show that $p \in \Delta^*_{\mathrm{fcns}}(\mathrm{fcns}(t_1 \dots t_n))$ iff there are $q_1, \dots q_n \in Q$ such that $\forall i : q_i \in \Delta^*(t_i)$ and $q_1 \dots q_n \in L(\mathcal{B}_{a,q}$ init $p)$.

$\Leftarrow$ Let $\mathcal{A} = (Q, \Sigma, \Delta, F)$ be an NTA with $T(\mathcal{A}) = \mathrm{fcns}(T)$. We define □