1 Mean Payoff Games

Definition 1. *A **mean payoff game** is a two player-game with the arena $\mathcal{G} = (V_0, V_1, E, r)$, where (V_0, V_1, E) is a game graph and $r : E \to \mathbb{Z}$.*

The value of a finite play is the average over all edges $r_{fin}(v_0 \dots v_n) = \frac{1}{n} \cdot \sum_{i=0}^{n-1} r(v_i, v_{i+1})$.

The infinite variant has plays $\pi \in V^\omega$. The goal of player 0 is to maximize $r_0(\pi) = \liminf_{n\to\infty} r_{fin}(\pi[0,n])$, whereas the goal of player 1 is to minimize $r_1(\pi) = \limsup_{n\to\infty} r_{fin}(\pi[0,n])$.
The best values the players can reach from $v \in V$ are $val_0(v) := \sup_\sigma \inf_\tau r_0(\pi_{\sigma,\tau,v})$ and $val_0(v) := \inf_\tau \sup_\sigma r_1(\pi_{\sigma,\tau,v})$.

The finite variant $\mathcal{G}_{fin}$ has plays $\pi = v_0 \dots v_n$ for $n \leq |V| + 1$ that terminate as soon as a vertex is reached the second time, i.e. $v_n = v_i$ for some $i < n$. The value is $r_{fin}(v_i \dots v_n)$.

*The finite variant with resets at $u \in V$ ($\mathcal{G}^u_{fin}$) has plays $\pi = v_0 \dots v_n$ for $n \leq 2|V| + 1$. The game behaves like the finite variant, with the addition that the **first** occurrence of u in a π resets the game, i.e. $v_1 v_3 u v_1 v_2 u$ is a valid run with score $r_{fin}(u v_1 v_2 u)$.*

Definition 2. *Let $\mathcal{G} = (V_0, V_1, E, r)$ be a mean payoff game. Let $v \in V$ and let σ and τ be strategies of player 0 and 1 respectively. The unique run of σ and τ from v is $\pi_{\sigma,\tau,v} \in V^\omega$ for the infinite variant or $\beta_{\sigma,\tau,v} \in V^*$ for the finite variant.*

2 Determinacy

Theorem 1. *Let $\mathcal{G} = (V_0, V_1, E, r)$ be a mean payoff game. Then the best value either player can achieve in the finite variant for any $v \in V$ is $\max_\sigma \min_\tau r_{fin}(\beta_{\sigma,\tau,v}) = \min_\tau \max_\sigma r_{fin}(\beta_{\sigma,\tau,v})$. We call this value $val(v, \mathcal{G}_{fin})$.*

Proof. The proof is based on the Minmax theorem bei von Neumann. Given a finite tree of all possible plays in the game, one can derive the value at each node from the leaf upwards. Note that these strategies are not positional in general. $\square$

The same proof can be used to find a unique $\mathrm{val}(v, \mathcal{G}^u_{\mathrm{fin}})$ for the finite variant with resets,

Definition 3. *Let $\mathcal{G} = (V_0, V_1, E, r)$ be a mean payoff game. Let σ be a strategy for player 0/1 in $\mathcal{G}$ or $\mathcal{G}_{fin}$. σ secures $a \in \mathbb{R}$ for that player if every play according to σ has value at least / most a.*

Theorem 2. *Let $\mathcal{G} = (V_0, V_1, E, r)$ be a mean payoff game and let σ be a strategy for one of the players in $\mathcal{G}_{fin}$. We define $\hat{\sigma}(\pi) = \sigma(\pi')$, where π' is obtained by removing all loops from π. If σ secures value a in $\mathcal{G}_{fin}$ for that player, then $\hat{\sigma}$ secures value a in $\mathcal{G}$ for that player.*

Proof. Let $\pi \in V^\omega$ be a play in $\mathcal{G}$ according to $\hat{\sigma}$. After a finite prefix, π consists of loops only (because V is finite). We claim that every one of those loops has a value of at least/most a. If that is true, π also has a value of at least/most a which is what had to be shown.

Let $v_1 \dots v_k v_1$ be a loop in π, so $\rho v_1 \dots v_k v_1 \sqsubseteq \pi$ for some ρ. We can assume that $\rho v_1 \dots v_k$ does not contain any loops. If there would be a loop containing any v_i, then that loop would have

to be the same as $v_1 \dots v_k v_1$ by definition of $\hat{\sigma}$, so we can adapt ρ accordingly. Any other loop in ρ could simply be removed.
Thus, $\rho v_1 \dots v_k v_1$ is a play in $\mathcal{G}_{\text{fin}}$ and moreover it is a play according to σ. Because σ secures value a, the loop $v_1 \dots v_k v_1$ must have value at least/most a. □

Theorem 3. *Let $\mathcal{G} = (V_0, V_1, E, r)$ be a mean payoff game. Then $val_0(v) = val_1(v) = val(v, \mathcal{G}_{fin})$ for all $v \in V$. We call this value $val(v, \mathcal{G})$.*

Proof. Let $a := \text{val}(v, \mathcal{G}_{\text{fin}})$, so there are strategies σ and τ for player 0 and 1 respectively that both secure a in $\mathcal{G}_{\text{fin}}$. By theorem 2, the same is true for $\mathcal{G}$, so $\text{val}_1(v) \leq a \leq \text{val}_0(v)$.

From the definition of val_0 and val_1 we also have $\text{val}_0(v) \leq \text{val}_1(v)$. The two inequations imply our goal. □

3 Positionality

Theorem 4. *For all $u, v \in V$: $val(v, \mathcal{G}_{fin}) = val(v, \mathcal{G}^u_{fin})$.*

Proof. We show that $a := \text{val}(v, \mathcal{G}_{\text{fin}}) \leq \text{val}(v, \mathcal{G}^u_{\text{fin}})$. The other bound and equality can be shown symetrically. Let σ be an optimal strategy for player 0 in $\mathcal{G}_{\text{fin}}$. If there are no plays from v according to σ that visit u, then σ secures a for player 0 in $\mathcal{G}^u_{\text{fin}}$. Otherwise:

Claim 1: $a \leq \text{val}(u, \mathcal{G}^u_{\text{fin}})$.

By theorem 3, we know that $\text{val}(v, \mathcal{G}) = \text{val}(v, \mathcal{G}_{\text{fin}}) = a$. Let τ be an arbitrary strategy for player 1 and let τ' be the strategy that plays from v to u (which is possible by assumption) and then plays according to τ. Note that $\pi_{\hat{\sigma},\tau',v} = v \dots u \pi_{\hat{\sigma},\tau,u}$ and therefore $\text{val}(u, \mathcal{G}_{\text{fin}}) = \text{val}(u, \mathcal{G}) = \text{val}(v, \mathcal{G}) = a$.

It remains to be shown that $\text{val}(u, \mathcal{G}_{\text{fin}}) \leq \text{val}(u, \mathcal{G}^u_{\text{fin}})$. This is clear because the "reset" of every run happens at the initial vertex. □

Theorem 5. *Let $\mathcal{G}$ be a parity game. Both players have positional optimal strategies for $\mathcal{G}_{fin}$, i.e. there are strategies σ^* and τ^* for player 0 and 1 respectively such that $val(v, \mathcal{G}_{fin}) = r_{fin}(\beta_{\sigma^*,\tau^*,v})$.*

Proof. We perform a proof by induction on $|E|$. If $|E| = |V|$, then there is only one choice at every vertex and positional optimal strategies are easy to define.

Otherwise, there is some node $u \in V$ which has at least two outgoing edges. By theorem 4, $\text{val}(v, \mathcal{G}_{\text{fin}}) = \text{val}(v, \mathcal{G}^u_{\text{fin}})$ for all v. Let $\text{val}(v, \mathcal{G}_{\text{fin}}) = a$, then there is a strategy σ which secures value a in $\mathcal{G}^u_{\text{fin}}$.

Let $w \in V$ be arbitrary such that there is a play ρ which leads to $\sigma(u) = w$. Note that we can assume this w to be unique, as any history before a first occurrence of u has no effect on the outcome and a second occurrence terminates the run. We then obtain $\mathcal{G}'$ from $\mathcal{G}$ by removing all outgoing edges from u except for (u, w).

By induction, there is a positional optimal strategy σ' for $\mathcal{G}'_{\text{fin}}$ which secures value $a' = \text{val}(v, \mathcal{G}'_{\text{fin}})$ for the player. Using the same strategy σ' for $\mathcal{G}_{\text{fin}}$ must therefore also secure a'. It remains to be shown that $a = a'$.

By theorem 4, we have $a' = \text{val}(v, \mathcal{G}'^u_{\text{fin}})$ and therefore $a' = \text{val}(v, \mathcal{G}^u_{\text{fin}}) = a$ by choice of σ'. □

Theorem 6. *Let $\mathcal{G}$ be a parity game. Both players have positional optimal strategies for $\mathcal{G}$, i.e. there are strategies σ^* and τ^* for player 0 and 1 respectively such that $val(v, \mathcal{G}) = r_0(\pi_{\sigma^*,\tau^*,v}) = r_1(\pi_{\sigma^*,\tau^*,v})$.*

Proof. Using theorem 2 and 3, we know that $\hat{\sigma^*}$ and $\hat{\tau^*}$ both secure $\mathrm{val}(v, \mathcal{G})$ for their respective players. They are both positional by definition. $\square$

4 Algorithm

Theorem 7. *Let $\mathcal{G}$ be a parity game. For every $v \in V$ the value $val(v, \mathcal{G})$ can be computed in poly$(W + |V|)$ time, where $W = \max\{|x| \mid x \in r(V)\}$.*

Proof. **Claim 1**: For $|V| = n$, the difference between two possible values for a $\mathrm{val}(v)$ is at least $\frac{1}{n(n-1)}$.

Let $x, y \in \mathbb{R}$ be values of a vertex from any mean payoff game with $|V| = n$. Because of the positional determinacy of the game, the value must be defined by the average of some loop in the arena, so $x = \frac{k_1}{l_1}, y = \frac{k_2}{l_2}$ for some $k_1, k_2 \in \mathbb{Z}$ and $l_1, l_2 \in \mathbb{N}$ with $l_1, l_2 \leq n$. Our claim is: $|\frac{k_1}{l_1} - \frac{k_2}{l_2}| \geq \frac{1}{n(n-1)}$.

If $l_1 = l_2$, then $k_1 \neq k_2$ and $|\frac{k_1}{l_1} - \frac{k_2}{l_2}| = |\frac{k_1 - k_2}{l_1}| \geq \frac{1}{l_1} \geq \frac{1}{n} \geq \frac{1}{n(n-1)}$.

If $l_1 \neq l_2$, then $|\frac{k_1}{l_1} - \frac{k_2}{l_2}| = |\frac{k_1 l_2 - k_2 l_1}{l_1 l_2}| \geq |\frac{k_1 l_2 - k_2 l_1}{n(n-1)}| \geq \frac{1}{n(n-1)}$.

Claim 2: Let $\mathrm{val}^k(v, \mathcal{G})$ be the optimal mean value of plays of length k that can be enforced, i.e. the best score if every run is terminated after k steps. Then val^k can be computed in $\mathcal{O}(k \cdot |E|)$.

We have $\mathrm{val}^0(v, \mathcal{G}) = 0$ for all v. Assuming that val^k was already defined, we can compute val^{k+1} in $\mathcal{O}(|E|)$ as follows:

$$\mathrm{val}^{k+1}(v, \mathcal{G}) = \begin{cases} \max\limits_{(u,v) \in E} \frac{k \cdot \mathrm{val}^k(v, \mathcal{G}) + r(u,v)}{k+1} & \text{if } v \in V_0 \\ \min\limits_{(u,v) \in E} \frac{k \cdot \mathrm{val}^k(v, \mathcal{G}) + r(u,v)}{k+1} & \text{if } v \in V_1 \end{cases}$$

Claim 3: For all $v \in V, k \in \mathbb{N}$, we have $|\mathrm{val}^k(v) - \mathrm{val}(v)| \leq \frac{2nW}{k}$.

Claim 4: $\mathrm{val}(v, \mathcal{G})$ can be computed in $\mathrm{poly}(W + |V|)$ time.

Let $k = 4n^3W$. Then $|\mathrm{val}^k(v) - \mathrm{val}(v)| \leq \frac{2nW}{4n^3W} = \frac{1}{2n^2} \leq \frac{1}{2n(n-1)}$, so there is exactly one possible value for $\mathrm{val}(v)$ in the neighborhood of $\mathrm{val}^k(v)$. This can be computed by checking which interval $[(\mathrm{val}^k(v) - \frac{1}{2n(n-1)}) \cdot l, (\mathrm{val}^k(v) + \frac{1}{2n(n-1)}) \cdot l]$ for $0 \leq l \leq n$ contains an integer i. Then $\mathrm{val}(v) = \frac{i}{l}$. $\square$

Theorem 8. *Let $\mathcal{G}$ be a parity game. There is an algorithm to construct the positional optimal strategies σ^* and τ^* in poly$(W + |V|)$ time, where $W = \max\{|x| \mid x \in r(V)\}$.*

Proof. We construct the strategy σ for the player step by step, i.e. for every $v \in V_0$ we define a successor $\sigma(v)$.

If v has exactly one successor, the choice of $\sigma(v)$ is clear. Otherwise, let u be an arbitrary successor of v. We define the game $\mathcal{G}'$ by removing the edge (v, u) from $\mathcal{G}$. Using theorem 7, we can compute $a := \mathrm{val}(v, \mathcal{G})$ and $b := \mathrm{val}(v, \mathcal{G}')$ in polynomial time.

If $a = b$, then the edge (v, u) is not a part of σ^*, so we can continue the algorithm with $\mathcal{G}'$. If $a \neq b$, then define $\sigma(v) = u$. $\square$

5 Connection to Parity Games

Theorem 9. *Let $\mathcal{G}$ be a parity game. We can construct a mean payoff game $\mathcal{G}'$ over the same game graph such that val$(v, \mathcal{G}) \geq 0$ iff player 0 has a winning strategy in $\mathcal{G}'$ from v.*

Proof. Let $\mathcal{G} = (V_0, V_1, E, c)$ with $|V| = n$. We define $\mathcal{G}' = (V_0, V_1, E, r)$ with $r(u, v) = (-n)^{c(u)}$. We claim that $\text{val}(v, \mathcal{G}') \geq 0$ iff player 0 has a winning strategy in $\mathcal{G}$ from v. Due to symmetry, we only show one direction.

Let $\overline{\sigma}$ be a positional winning strategy of player 0 in $\mathcal{G}$ from v. Let τ^* be a positional optimal strategy for player 1 in $\mathcal{G}'$. We have $\text{val}(v, \mathcal{G}') = \text{val}_1(v, \mathcal{G}') = \sup_{\sigma} r_1(\pi_{\sigma, \tau^*, v}) \geq r_1(\pi_{\overline{\sigma}, \tau^*, v})$. It remains to be shown that $r_1(\pi_{\overline{\sigma}, \tau^*, v}) \geq 0$.

Because $\overline{\sigma}$ and τ^* are both positional, $\pi_{\overline{\sigma}, \tau^*, v} = v_0 \ldots v_i (v_{i+1} \ldots v_k v_i)^\omega$ is ultimately periodic with $k - i + 1 \leq n$. Then $r_1(\pi_{\overline{\sigma}, \tau^*, v}) = r_{\text{fin}}(v_{i+1} \ldots v_k v_i)$. Since $\overline{\sigma}$ is a winning strategy for player 0 in $\mathcal{G}$, the maximal priority in that loop must be even. Let that priority be p.

$$\text{val}(v, \mathcal{G}') = r_{\text{fin}}(v_{i+1} \ldots v_k v_i) \geq n^p - k \cdot n^{p-1} \geq n^p - (n-1) \cdot n^{p-1} = n^{p-1} > 0$$

$\square$