1 Infinite Words

Theorem 1.1. *Every non-empty ω-regular language contains an ultimately periodic word.*

Proof. Let U be ω-regular, so there is a regular expression $r = \bigcup_{i=1}^{n} U_i \cdot V_i^{\omega}$ for it. Since $U \neq \emptyset$, n must be larger than 0. Let $u \in U_1$ and $v \in V_1$ be arbitrary. Then $uv^{\omega} \in U$. $\square$

Theorem 1.2. *For a Kripke structure $\mathcal{K}$ with initial state s and $\varphi \in$ LTL, the model checking problem $L(\mathcal{K}, s) \subseteq L(\varphi)$? is PSPACE-complete.*

Proof. **PSPACE** Compute the intersection automaton for $L(\mathcal{K}, s) \cap L(\neg\varphi)$ and test it for emptiness.

PSPACE-hard Encode a poly.-length Turing tape as a Kripke structure and its correct behavior in LTL. $\square$

Theorem 1.3 (Büchi)**.** *The MSO theory of* $(\mathbb{N}, +1, <, 0)$ *is decidable.*

Proof. Corresponds to S1S formula. Can be checked with NBA emptiness test. $\square$

Theorem 1.4. *The FO theory of* $(\mathbb{R}, +, <, 0)$ *is decidable.*

Proof. Encode real numbers x by triples of sets (X_s, X_i, X_f) with the number's sign ($X_s = \emptyset$ or $\{0\}$), the positive decimal digits in binary encoding, and the positive fractional digits in binary encoding. Then an FO sentence can be transformed to an equi-satisfiable MSO sentence over $(\mathbb{N}, +1, <, 0)$. $\square$

Theorem 1.5. *Subset-construction does not suffice to determinize NBAs.*

Proof. Example: Let $\mathcal{A} = (Q, \{a\}, q_0, \Delta, F)$ be an NBA with $Q = \{q_0, q_1\}$, $F = \{q_1\}$, and $\Delta = \{(q_0, a, q_0), (q_0, a, q_1)\}$. Then $L(\mathcal{A}) = \emptyset$.

The subset construction yields $\mathcal{A}' = \{\{\{q_0\}, \{q_0, q_1\}\}, \{a\}, \{q_0\}, \Delta, \{P \subseteq Q \mid P \cap F \neq \emptyset\})$ with $\Delta = \{(\{q_0\}, a\{q_0, q_1\}), (\{q_0, q_1\}, a, \{q_0, q_1\})\}$. Therefore, the word a^{ω} has an accepting run $\{q_0\}\{q_0, q_1\}^{\omega}$. $\square$

Theorem 1.6. *Let $\mathcal{A}$ be a Rabin automaton and let ρ_1, ρ_2, ρ be runs of $\mathcal{A}$. If $Inf(\rho_1) \cup Inf(\rho_2) = Inf(\rho)$ and ρ_1 and ρ_2 are both non-accepting, then ρ is also non-accepting.*

Proof. Let $\Omega = \{(E_i, F_i) \mid 1 \leq i \leq m\}$. Let $I_1 = \mathrm{Inf}(\rho_1)$, $I_2 = \mathrm{Inf}(\rho_2)$, and $I = \mathrm{Inf}(\rho)$. Assume ρ is accepting, so there is an $1 \leq i \leq m$ s.t. $I \cap E_i = \emptyset$ and $I \cap F_i \neq \emptyset$. That means $I \cap E_i = (I_1 \cup I_2) \cap E_i = \emptyset$, so $I_1 \cap E_i = I_2 \cap E_i = \emptyset$. Since ρ_1 and ρ_2 are non-accepting, that means $I_1 \cap F_i = I_2 \cap F_i = \emptyset$ and therefore $I \cap F_i = \emptyset$, which is a contradiction. $\square$

Theorem 1.7. *For every n, there is $L_n \subseteq \Sigma^\omega$ s.t. there is an NBA that recognizes L_n with $n+2$ states, but every det. Rabin automaton that recognizes L_n has at least $n!$ states.*

Proof. We define the NBA $\mathcal{A}_n = (Q, \Sigma_n, q_0, \Delta, F)$ with

- $\Sigma_n = \{\#, 1, \ldots, n\}$
- $Q = \{q_0, q_f, q_1, \ldots, q_n\}$
- $F = \{q_f\}$
- Δ:
  - For all $1 \leq i \leq n$, for all $a \in \Sigma_n$: $q_i \xrightarrow{a} q_i$ and $q_0 \xrightarrow{a} q_i$
  - For all $1 \leq i \leq n$: $q_i \xrightarrow{i} q_f$ and $q_f \xrightarrow{i} q_i$

Then $\alpha \in L(\mathcal{A}_n)$ iff there are $a_1, \ldots, a_k \in \{1, \ldots, n\}$ such that the words $a_1a_2, a_2a_3, \ldots, a_ka_1$ occur infinitely often in α.

Let $\mathcal{A}' = (Q', \Sigma_n, q_0', \delta, \Omega)$ be a DRA for $L(\mathcal{A}_n)$. We want to show that $\mathcal{B}$ has at least $n!$ states. A permutation word is a word $(i_1 \ldots i_n\#)^\omega$ with $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$; let PW be the set of permutation words. Let $\alpha, \beta \in \mathrm{PM}$ with $\alpha \neq \beta$ and let ρ_α, ρ_β the respective runs of $\mathcal{B}$. We claim that $\mathrm{Inf}(\rho_\alpha) \cap \mathrm{Inf}(\rho_\beta) = \emptyset$. If that is true, then $|Q'| \geq |\mathrm{PM}| = n!$.

Claim : $\mathrm{Inf}(\rho_\alpha) \cap \mathrm{Inf}(\rho_\beta) = \emptyset$

Let $\alpha = (\bar{i}\#)^\omega$ and $\beta = (\bar{j}\#)^\omega$ with $\bar{i} = i_1 \ldots i_n$ and $\bar{j} = j_1 \ldots j_n$. Assume there is a $q \in \mathrm{Inf}(\rho_\alpha) \cap \mathrm{Inf}(\rho_\beta)$. We extract the following segments from the two runs:

- ρ_1 from ρ_α with $\rho_1 : q_0 \xrightarrow{w} q$ for some $w \in \Sigma_n^*$.
- ρ_2 from ρ_α with $\rho_2 : q \xrightarrow{w} q$ for some $w \in \Sigma_n^* \bar{i} \Sigma_n^*$.
- ρ_3 from ρ_β with $\rho_3 : q \xrightarrow{w} q$ for some $w \in \Sigma_n^* \bar{j} \Sigma_n^*$.
- $\pi := \rho_1(\rho_2\rho_3)^\omega$

Then π is a run of $\mathcal{A}'$ on a word γ with $\mathrm{Inf}(\pi) = \mathrm{Inf}(\rho_\alpha) \cup \mathrm{Inf}(\rho_\beta)$. Since $\bar{i} \neq \bar{j}$, there is a minimal k with $i_k \neq j_k$. By this choice, there must also be l, l' such that $i_k = j_l$ and $j_k = i_{l'}$. Therefore, the following pairs all occur infinitely often in γ:

$$i_k i_{k+1}, i_{k+1} i_{k+2}, \ldots, i_{l'-1} \underbrace{i_{l'}}_{=j_k}, j_k j_{k+1}, \ldots, j_{l-1} \underbrace{j_l}_{=i_k}$$

As shown before, that means $\gamma \in L(\mathcal{A}_n)$, so π is an accepting run. This contradicts the Union Lemma (theorem 1.6). □

Theorem 1.8. *Given an ABA $\mathcal{A}$, the dual $\tilde{\mathcal{A}}$ is an alternating co-Büchi automaton which accepts $L(\mathcal{A})^\complement$, with $\tilde{F} = Q \setminus F$ and $\tilde{\delta}$ exchanging true/false and $\wedge$/$\vee$.*

Proof. We claim that $L(\mathcal{A})^\complement = L(\tilde{\mathcal{A}})$.

$\subseteq$ Let $\alpha \notin L(\mathcal{A})$, so for every run tree t of $\mathcal{A}$ on α, there is a path π_t which is rejecting. One can show that the combination of all these paths is an accepting run of $\tilde{\mathcal{A}}$.

$\supseteq$ Let $\alpha \in L(\mathcal{A})$, so there is an accepting run tree t. Let $\tilde{t}$ be a run tree of $\tilde{\mathcal{A}}$ on α. One can show (by induction on the path length) that there is a path in t that is also a path in $\tilde{t}$. Since that path is accepting in t, it must be rejecting in $\tilde{t}$. $\square$

1.1 Minimization

Theorem 1.9. *There is a DBA-recog. language which does not have a unique minimal DBA. DBAs minimized with the DFA minimization algorithm can be arbitrarily bad compared to a minimal DBA.*

Proof. The language $(a^*b)^\omega$ has two non-isomorphic DBAs.

For every $n > 0$, consider $L_n = \{a^n\}$. The minimal NFA for L_n has $n+1$ states but $\lim(L_n) = \{a^\omega\}$ can be recognized by a DBA with a single state. $\square$

Theorem 1.10. *Let $U \subseteq \Sigma^\omega$ be regular and $\sim_U$ be the Myhill-Nerode equivalence relation. Every DBA for U has at least index($\sim_U$) many states.*

Proof. Assume $\mathcal{A} = (Q, \Sigma, q_0, \delta, F)$ is a DBA for U with $|Q| < \text{index}(\sim_U)$. That means there are $u, v \in \Sigma^*$ with $u \not\sim_U v$ but $\delta^*(q_0, u) = \delta^*(q_0, v)$. By definition of the automaton, that means $u \sim_U v$, which is a contradiction. $\square$

Theorem 1.11. *For every n, there is a $U_n \subseteq \Sigma^\omega$ such that index($\sim_{U_n}$) $= 1$ but every DBA for U_n has at least n states.*

Proof. Let $U_n = \{\alpha \mid a^n b \text{ appears infinitely often as a substring in } \alpha\}$. The language is prefix independent, so $\text{index}(\sim_{U_n}) = 1$. However, a DBA for U_n clearly requires $n+1$ states. $\square$

Theorem 1.12. *The problem* $\text{DBAMin} = \{(\mathcal{A}, k) \mid \mathcal{A} \textit{ DBA}, k \in \mathbb{N}, \textit{There is a } k\textit{-state DBA for } L(\mathcal{A})\}$ *is NP-complete.*

Definition 1. *For $U \subseteq \Sigma^\omega$, we define the canonical automaton $\mathcal{A}_U = (Q_U, \Sigma, q_0^U, \delta_U, F_U)$ with*

- $Q_U = \{[u]_{\sim_U} \mid u \in \Sigma^*\}$
- $q_0^U = [\varepsilon]_{\sim_U}$
- $\delta_U([u]_{\sim_U}, a) = [ua]_{\sim_U}$
- $F_U = \{[u]_{\sim_U} \mid \exists v \in \Sigma^+ : u \sim_U uv \text{ and } uv^\omega \in U\}$

Theorem 1.13. *Let U be WDBA-recognizable. Then $\mathcal{A}$ is a WDBA and $L(\mathcal{A}_U) = U$.*

Proof. We show the statement in two steps.

Claim : $\mathcal{A}_U$ is a WDBA.

Let $[u]_{\sim_U}, [v]_{\sim_U} \in Q_U$ be two states in the same SCC and $[u]_{\sim_U} \in F_U$. We need to show that $[v]_{\sim_U} \in F_U$ as well. Since the two states are in the same SCC, let $\mathcal{A}_U : [u]_{\sim_U} \xrightarrow{y} [v]_{\sim_U} \xrightarrow{z} [u]_{\sim_U}$. We have $u(yz)^\omega \in U$.

Hence, $uy(zy)^\omega \in U$. By definition we know $[uy]_{\sim_U} = [u']_{\sim_U}$, so $u'(zy)^\omega inU$ which means that $[u']_{\sim_U} \in F_U$.

Claim : $L(\mathcal{A}_U) = U$.

Let uv^ω be an ultimately periodic word. By the pigeon hole principle, there are $m, n > 0$ such that $uv^m \sim_U uv^m v^n$.

$$\begin{aligned}
& uv^\omega \in U \\
\Leftrightarrow & uv^m (v^n)^\omega \in U \\
\Leftrightarrow & [uv^m]_{\sim_U} \in F_U \\
\overset{uv^m \sim_U uv^m v^n}{\Leftrightarrow} & uv^m (v^n)^\omega \in L(\mathcal{A}_U) \\
\Leftrightarrow & uv^\omega \in L(\mathcal{A}_U)
\end{aligned}$$

Now consider $X = (U \cup L(\mathcal{A}_U)) \setminus (U \cap L(\mathcal{A}_U)$. If the two languages are equal, then X must be empty. As we have shown above, U and $L(\mathcal{A}_U)$ contain the same ultimately periodic words, so X does not contain any. It is also constructed by Boolean operations on regular languages, so X must be regular. That means that $X = \emptyset$. □

Definition 2. *Let $\mathcal{A} = (Q, \Sigma, q_0, \Delta, F)$ be an NBA. A **looping state** is a state in Q from which there is a non-empty path that ends in that same state.*

*We define the **acceptance height** $ah : Q \to \mathbb{N}$ as follows: Let $q \in Q$. If all looping states that are reachable from q are accepting, then $ah(q) := 0$. Otherwise, let n be the maximal acceptance height of a state that is reachable from q but not in the same SCC. If q is non-loopinig, set $ah(q) := n$. Otherwise,*

$$ah(q) := \begin{cases} 2 \cdot \lfloor \frac{n-1}{2} \rfloor + 2 & \text{if } q \in F \\ 2 \cdot \lfloor \frac{n}{2} \rfloor + 1 & \text{if } q \notin F \end{cases}$$

*$\mathcal{A}$ is in **normal form** if $F = \{q \in Q \mid ah(q) \text{ is even}\}$.*

Theorem 1.14. *Let $\mathcal{A}$ be a weak DBA. One can compute an equivalent weak DBA in normal form in linear time.*

Proof. SCCs can be computed in linear time. Given the SCCs, the inductive definition of ah can be computed in linear time. □

Theorem 1.15. *Let $\mathcal{A}$ be a weak DBA in normal form. Minimizing $\mathcal{A}$ as a DFA also results in a minimal weak DBA.*

1.2 Simulation Game

Definition 3. *Let $\mathcal{A} = (Q, \Sigma, q_0, \Delta, F)$ be an NBA. We define the delayed simulation game $\mathcal{G}_{\mathcal{A}}(G_{\mathcal{A}}, Win)$ as follows*

- $G_{\mathcal{A}} = (V_0, V_1, E, c)$
- $V_0 = \Sigma \times Q \times Q$
- $V_1 = Q \times Q$
- *Player 0 moves from (a, p, q) to a (p, p') with $(q, a, p') \in \Delta$*
- *Player 1 moves from (q, q') to a (a, p, q') with $(q, a, p) \in \Delta$*
- $c : V \to \{-1, 0, 1\}$ *with* $c(v) = \begin{cases} -1 & \text{if } v \in F \times (Q \setminus F) \\ 1 & \text{if } v \in Q \times F \\ 0 & \text{otherwise} \end{cases}$
- *$\alpha \in Win$ iff after every -1 in α, there is a 1 later on.*

Write $q \preceq_{de} q'$ if player 0 has a winning strategy from (q, q').

Idea: Player 1 chooses symbols in Σ and transitions on the second state. Player 0 has to answer with transition on the first state which lead to a run of the same acceptance.

Theorem 1.16. *If $q \preceq_{de} q'$ and $q' \preceq_{de} q$, then q and q' can be merged in $\mathcal{A}$ without changing the language of the automaton.*

Theorem 1.17. *The delayed simulation game can be reduced to a Büchi game in linear time.*

2 Finite Trees

Theorem 2.1 (Pumping Principle). *Let $T \subseteq T_\Sigma$ be a regular ranked tree language. There is a $n \in \mathbb{N}$ such that for all trees $t \in T$, all $m > n$, and all paths $\pi_1 \dots \pi_m$, there are $1 \leq i < j \leq m$ such that for all $k \in \mathbb{N}$:*

$$t[\circ/u] \cdot (t[\circ/v]|_u)^k \cdot t|_v \in T$$

where $u = \pi_1 \dots \pi_i$ and $v = \pi_1 \dots \pi_j$.

Definition 4. *Let $T \subseteq T_\Sigma$. The **Myhill-Nerode equivalence** is $\sim_T \subseteq T_\Sigma \times T_\Sigma$ with*

$$t_1 \sim_T t_2 \Leftrightarrow \forall s \in S_\Sigma : s \cdot t_1 \in T \leftrightarrow s \cdot t_2 \in T$$

The index of T is $Index(\sim_T) := |T/\sim_T|$.

Definition 5. *Let $T \subseteq T_\Sigma$. We define the canonical DTA $\mathcal{A}_T = (Q_T, \Sigma, \delta_T, F_T)$ as*

- *$Q_T = \{[t]_{\sim_T} \mid t \in T_\Sigma\}$.*
- *For all $a \in \Sigma_i$: $\delta_T([t_1]_{\sim_T}, \dots, [t_i]_{\sim_T}, a) = [a(t_1, \dots, t_i)]_{\sim_T}$.*
- *$F_T = \{[t]_{\sim_T} \mid t \in T\}$.*

Theorem 2.2. *Let $T \subseteq T_\Sigma$. T is regular iff $Index(\sim_T)$ is finite. If T is regular, $\mathcal{A}_T$ is the minimal DTA.*

Proof. via induction on t, prove $\delta_T^*(t) = [t]$ □

Theorem 2.3. *The emptiness problem for NTAs can be reduced to HORN-SAT in linear time.*

Proof. Let $\mathcal{A} = (Q, \Sigma, \Delta, F)$ be an NTA. For every $\tau = (q_1, \dots, q_i, a, p) \in \Delta$, let $\psi_\tau = (X_{q_1} \wedge \dots \wedge X_{q_i} \rightarrow X_q)$. Then we define $\varphi = \bigwedge_{\tau \in \Delta} \psi_\tau \wedge \bigwedge_{q \in F} X_q \rightarrow 0$. φ is satisfiable iff $L(\mathcal{A}) \neq \emptyset$. □

2.1 BTTs

Theorem 2.4. *The equivalence problem for BTTs is undecidable.*

Proof. Follows from the equivalence problem of rational word transducer. □

Theorem 2.5. *The emptiness problem for BTTs is decidable in polynomial time.*

Proof. One can construct in polynomial time a ↑NTA that recognizes the domain of a given BTT. The emptiness problem of BTTs then is equivalent to the emptiness problem of ↑NTAs. □

Theorem 2.6. *The type-checking problem (given regular T, T', is $\mathcal{A}(T) \subseteq T'$?) is decidable.*

Proof. $\mathcal{A}(T) \subseteq T'$ iff $\mathcal{A}^{-1}((T')^{\complement}) \cap T = \emptyset$. If T' is regular then so is $\mathcal{A}^{-1}((T')^{\complement}) \cap T$ and this can be decided. □

Theorem 2.7. *If T is regular, then $\mathcal{A}^{-1}(T)$ is regular.*
If $\mathcal{A}$ is linear, then $\mathcal{A}(T_\Sigma)$ is regular.

Theorem 2.8. *There are BTT-definable relations R_1, R_2 such that $R_1 \circ R_2$ is not BTT-definable.*

Proof. Let $\Sigma = \{f, g, h, c\}$ and for both $i \in \{1, 2\}$: $\mathcal{A}_i = (\{q, q_f\}, \Sigma, \Sigma, \Delta_i, \{q_f\})$ where

- Δ_1:
  - $c \to q(c)$
  - $g(q(x_1)) \to q(g(x_1))$
  - $q(x_1) \to q_f(f(x_1))$
- Δ_2:
  - $c \to q(c)$
  - $g(q(x_1)) \to q(g(x_1))$
  - $g(q(x_1)) \to q(h(x_1))$
  - $f(q(x_1), q(x_2)) \to q_f(f(x_1, x_2))$

$\mathcal{A}_1$ converts a tree $t \in T_{\{}c, g\}$ to $f(t, t)$. $\mathcal{A}_2$ replaces an arbitrary number of gs by hs. $R_1 \circ R_2$ defines the set of trees $f(t_1, t_2)$ where t_1 and t_2 are of equal height. □

Theorem 2.9. *If $\mathcal{A}_1$ is linear **or** $\mathcal{A}_2$ is deterministic Then $R(\mathcal{A}_1) \circ R(\mathcal{A}_2)$ is BTT-definable.*

3 Infinite Trees

Theorem 3.1 (BTA Pumping)**.** *For $t \in T_\Sigma, x \in \{0,1\}^*, y \in \{0,1\}^+$, let*

$$t^*_{[x,y]} : \{0,1\}^* \to \Sigma, z \mapsto \begin{cases} t(z) & \text{if } xy \not\sqsubseteq z \\ xz' & \text{if } \exists n > 0 : z = xy^n z' \text{ with } y \not\sqsubseteq z' \end{cases}.$$

Let $\mathcal{A}$ be a BTA, $t \in T(\mathcal{A})$, ρ an accepting run of $\mathcal{A}$ on t, and $x, y, y' \in \{0,1\}^*$ s.t. $\rho(x) = \rho(xy)$, $y' \sqsubset y$, and $\rho(xy') \in F$. Then $t^*_{[x,y]} \in T(\mathcal{A})$.

Proof. $\rho^*_{[x,y]}$ is an accepting run of $\mathcal{A}$ on $t^*_{[x,y]}$. □

Theorem 3.2. *Every non-empty regular tree language contains a regular tree.*

Proof. Let $\mathcal{A}$ be a PTA for $T = T(\mathcal{A}) \neq \emptyset$ and let $\mathcal{G}_\mathcal{A}$ be the emptiness game for $\mathcal{A}$. Since T is not empty, player Automaton has a positional winning strategy $\sigma : Q \to \Sigma$. For $\sigma(q) = (q, a, q_0, q_1)$, we define $\delta_\sigma(q, 0) = q_0$, $\delta_\sigma(q, 1) = q_1$, and $f_\sigma(q) = a$. Then $t \in T$ where t is the regular tree defined by $\mathcal{B} = (Q, \{0,1\}, q_0, \delta_\sigma, f_\sigma)$. □

Theorem 3.3 (Rabin's Tree Theorem)**.** *The MSO theory of $\underline{T_2}$ is decidable for formulas $\varphi(X_1, \dots, X_n)$ and a model $X_1, \dots X_n \subseteq \{0,1\}^*$ is computable.*

Proof. Transform φ into an equivalent PTA. A model can be found by solving the emptiness game. □