1 List Of Models

1.1 Infinite Words

- DBA
- NBA
- GBA
- Rabin automaton
- Muller automaton
- Parity automaton
- det. E automaton
- det. A automaton
- det. coBA
- weak BA
- Staiger-Wagner automaton
- ABA
- LTL
- S1S
- ∃S1S
- $S1S_0$

1.2 Finite Trees

- DTA
- NTA
- ↓DTA
- ↓NTA
- DUTA
- NUTA
- deterministic DTD
- DTD
- deterministic EDTD

- single-type EDTD
- EDTD
- Relax NG
- FO
- MSO
- Regular expressions
- DTWA
- TWA

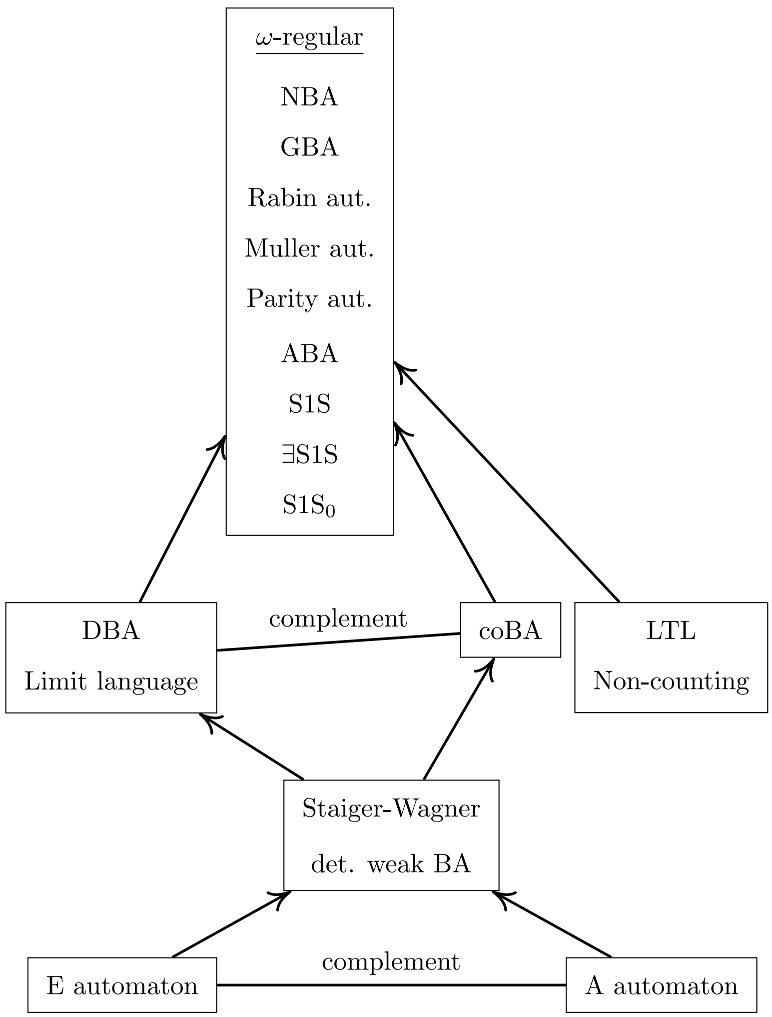
1.3 Infinite Trees

- BTA
- Muller TA
- Parity TA
- DMTA
- S2S (MSO / WMSO)
- $S2S_0$ (MSO / WMSO)

2 List Of Games

- Büchi
- Staiger-Wagner
- weak Parity
- Reachability
- Safety
- Muller
- Parity
- Rabin
- Streett
- Gale-Stewart
- Wadge

3 Infinite Word Models



3.1 Class Inclusions

- E aut. $\subseteq$ Staiger-Wagner
  Proof: SWA with $\mathcal{F} = \{Q' \subseteq Q \mid F \cap Q' \neq \emptyset\}$.

- A. aut. $\subseteq$ Staiger-Wagner
  Proof: SW closed under complement,

- Staiger-Wagner $\subseteq$ DBA / coBA
  Proof: $\mathcal{A}$ SWA $\Rightarrow \mathcal{A}' = (Q \times 2^Q, \Sigma, (q_0, \emptyset), \delta', F')$
  Collect all visited states and accept if that set stays in $\mathcal{F}$.

- DBA $\subseteq$ NBA
  trivial

- coBA $\subseteq$ NBA
  Proof: NBA closed under complement.

- LTL $\subseteq$ GBA (exponential size transformation)
  Proof: Theorem 2

- LTL $\subseteq$ ABA (linear size transformation)
  Proof: Similar to theorem **??**. $\mathcal{A}_\varphi = (\mathrm{cl}(\varphi), \mathbb{B}^n, \varphi, \delta, F)$ with $F = \{\psi \in \mathrm{cl}(\varphi) \mid \psi = G\vartheta\}$ and δ is defined as follows:
  - $\delta(p_i, a)$ and $\delta(\neg p_i, a)$ are tt or ff depending on the i-th component of a.
  - $\delta(\psi_1 \oplus \psi_2, a) = \delta(\psi_1, a) \oplus \delta(\psi_2, a)$ for $\oplus \in \{\wedge, \vee\}$.
  - $\delta(X\psi, a) = \psi$.
  - $\delta(G\psi, a) = \delta(\psi, a) \wedge G\psi$.
  - $\delta(\psi_1 U \psi_2, a) = (\delta(\psi_1, a) \wedge \psi_1 U \psi_2) \vee \delta(\psi_2, a)$.

3.2 Class Exclusions

- E aut. $\not\subseteq$ A aut.
  Example: $b^*a(a+b)^\omega$
  Proof: Assume the A automaton $\mathcal{A} = (Q, \Sigma, q_0, \delta, F)$ recognizes L, so for every $b^n ab^\omega$, the run ρ_n is accepting. Let $\rho^*(n) = \rho_{n+1}(n)$. This is an accepting run on b^ω, which is a contradiction.

- A aut. $\not\subseteq$ E aut.
  Example: $\{a^\omega\}$
  Proof: Assume the E automaton $\mathcal{A} = (Q, \Sigma, q_0, \delta, F)$ recognizes L, so the run ρ of $\mathcal{A}$ on a^ω is accepting. That means there is an n s.t. $\rho(n) \in F$. Therefore, $\mathcal{A}$ accepts the word $a^n b^\omega$, which is a contradiction.

- DBA $\not\subseteq$ coBA
  Example: $(a^*b)^\omega$
  Proof: Assume $\mathcal{A} = (Q, \Sigma, q_0, \delta, F)$ is a coBA recognizing L. Let $n = |Q|$, $w = (a^n b)^\omega$, and ρ the run of $\mathcal{A}$ on w. From some m on, ρ only visits states in F. Consider $(a^n b)^m a^n$. By choice of n, there must be $i < j$ such that after $(a^n b)^m a^i$ and $(a^n b)^m a^j$, the automaton is in the same state $q \in F$. Therefore, the run on $(a^n b)^m a^\omega$ is accepting, which is a contradiction.

- coBA $\not\subseteq$ DBA
  Example: $(a+b)^*a^\omega$
  Proof: Assume $\mathcal{A} = (Q, \Sigma, q_0, \delta, F)$ is a DBA recognizing L. We inductively define words w_n with runs ρ_n s.t. $w_n \sqsubseteq w_{n+1}$ and ρ_n visits F at least n times and $|w_n|_b = n$. Then the "limit" of those words contains infinitely many b but is accepted by $\mathcal{A}$, which is a contradiction.
  Let $w_0 = \varepsilon$ and $\rho_0 = q_0$. For $n+1$, consider $w_n a^\omega$ with the run $\rho_n \pi$. Since $w_n a^\omega \in L$, there is a k s.t. $\pi(k) \in F$. Let $w_{n+1} = w_n a^k b$ and ρ_{n+1} accordingly.

- LTL $\not\subseteq$ NBA
  Example: $((a+b)a)^\omega$
  Proof: Show that L is counting. Then it follows that it is not LTL-definable. Assume that L is non-counting, so there is an n_0 according to the definition. Let $n = n_0 + 1$, $u = \varepsilon$, $v = a$, and $\beta = ba^\omega$. Due to symmetry, we can assume that n_0 is even, so $uv^n\beta \notin L$ but $uv^{n+1}\beta \in L$.

3.3 Class Equalities

3.3.1 NBA

- NBA $\Leftrightarrow$ ω-regular
  Proof: $\Leftarrow$ All three operations used in regular expressions $(\cup, \cdot, ^\omega)$ correspond to easy NBA constructions.

  $\Rightarrow$ Let $\mathcal{A} = (Q, \Sigma, q_0, \Delta, F)$ be an NBA. For every final state $q \in F$, we consider the finite words U_q which lead from q_0 to q, and the finite words V_q which lead from q back to q itself. Then $L(\mathcal{A})$ is the union of $U_q \cdot V_q^\omega$ for all final states q.

- GBA $\Rightarrow$ NBA
  Proof: Let $\mathcal{A} = (Q, \Sigma, q_0, \Delta, (F_1, \ldots, F_k))$ be a GBA. An equivalent NBA is $\mathcal{A}' = (Q \times \{1, \ldots, k\}, \Sigma, (q_0, 1), \Delta', F_k \times \{k\})$. The transitions Δ' contain all those from Δ and allow to switch from (q, i) to $(q, i+1)$ if $q \in F_i$.

- NBA $\Leftrightarrow$ S1S
  Proof: Theorem 4.

- Det. Muller $\Leftrightarrow$ NBA
  Proof: Theorem 5.

- (det.) Muller $\Rightarrow$ (det.) Parity
  Proof: Theorem 6.

- ABA $\Rightarrow$ NBA
  Proof: Theorem 7.

3.3.2 LTL

LTL $\Leftrightarrow$ Non-counting
No proof. Remarks in F8.

A language L is called **non-counting** if there is an n_0 such that for all $n > n_0$, all $u, v \in \Sigma^*$, and all $\beta \in \Sigma^\omega$: $uv^n\beta \in L \leftrightarrow uv^{n+1}\beta \in L$.

3.3.3 SW

Staiger-Wagner $\Leftrightarrow$ det. weak BA
Proof: **??**

3.4 Closures

3.4.1 NBA

- Closed under union
  Proof: Product automaton with $F = (F_1 \times Q_2) \cup (Q_1 \times F_2)$.

- Closed under intersection
  Proof: GBA $\mathcal{A} = (Q_1 \times Q_2, \Sigma, (q_0^1, q_0^2), \Delta, (F_1 \times Q_2, Q_1 \times F_2))$ where
  $\Delta = \{((p_1, p_2), a, (q_1, q_2)) \mid (p_1, a, q_1) \in \Delta_1, (p_2, a, q_2) \in \Delta_2\}$.

- Closed under complement
  Proof: Theorem 1.

3.4.2 DBA

- Closed under union and intersection
  Proof: Same construction as for NBA.

- Not closed under complement (inf. many $a \leftrightarrow$ fin. many a)

3.4.3 SW

- Closed under union and intersection
  Proof: Product automaton with $\mathcal{F}_\cap = \{F \subseteq Q_1 \times Q_2 \mid \pi_1(F) \in \mathcal{F}_1, \pi_2(F) \in \mathcal{F}_2\}$ where $\pi_i((x_1, x_2)) = x_1$.

- Closed under complement
  Proof: $\overline{\mathcal{F}} = 2^Q \setminus \mathcal{F}$.

3.5 Characterizations

- Parity conditions are directly convertible to Rabin chain conditions and vice-versa.
  Proof: Assign priorities in ascending order; $E_k \to 0$, $F_k \setminus E_k \to 1$, $E_{k-1} \setminus F_k \to 2 \ldots$

- U is ω-regular iff U is a Boolean combination of DBA-languages
  Proof: $\Leftarrow$ NBAs are closed under Boolean operations.

  $\Rightarrow$ Let $\mathcal{A}$ be a DRA for U. It suffices to consider $\Omega = \{(E, F)\}$ as any other language is a finite union of these conditions. Let $\mathcal{A}_E$ and $\mathcal{A}_F$ be the modifications of $\mathcal{A}$ with conditions $\{(E, Q)\}$ and $\{\emptyset, F\}$ respectively. Then $L(\mathcal{A}_E)^\complement$ and $L(\mathcal{A}_F)$ are DBA-recognizable.

- U is DBA-recog. iff $U = \lim(L)$ for some regular $L \subseteq \Sigma^*$.
  Proof: Use the DBA as a DFA or vice-versa.

- U is E-recog. iff $U = L \cdot \Sigma^*$ for some regular $L \subseteq \Sigma^*$.
  Proof: $\Rightarrow$ Let $\mathcal{A}$ be an E-automaton for U. For every $q \in F, a \in \Sigma$, add a transition (q, a, q). The resulting automaton as an NFA accepts L with $U = L \cdot \Sigma^*$.

  $\Leftarrow$ Let $\mathcal{A}$ be a DFA for L. The same automaton as an E-automaton recognizes $L \cdot \Sigma^*$.

- Landweber's theorem
  Proof: Theorem 3

- DBA $\cap$ coBA $\Rightarrow$ SW
  Proof: **??**

- SW $\Leftrightarrow$ Boolean combination of E-recognizable languages.
  Proof: $\Leftarrow$ SWAs are closed under Boolean operations.

  $\Rightarrow$ Let $\mathcal{A} = (Q, \Sigma, q_0, \delta, \mathcal{F})$ be the SWA and let $\mathcal{A}_q = (Q, \Sigma, q_0, \delta, \{q\})$. For every $q \in Q$, the language $L(\mathcal{A}_q)$ is E-recognizable. For every $F \subseteq Q$, let $L_F = \bigcap\limits_{q \in F} L(A_q) \cap \bigcap\limits_{q \notin F} L(A_q)^{\complement}$, which is a Boolean combination of E-recognizable languages. Then $L(\mathcal{A}) = \bigcup\{L_F \mid F \in \mathcal{F}\}$.

3.6 Duality

- U is A-recog. iff $\Sigma^\omega \setminus U$ is E-recog.
  Proof: same automaton, $\overline{F} = Q \setminus F$

- U is coBA-recog. iff $\Sigma^\omega \setminus U$ is DBA-recog.
  Proof: same automaton, $\overline{F} = Q \setminus F$

3.7 Problems / Complexity

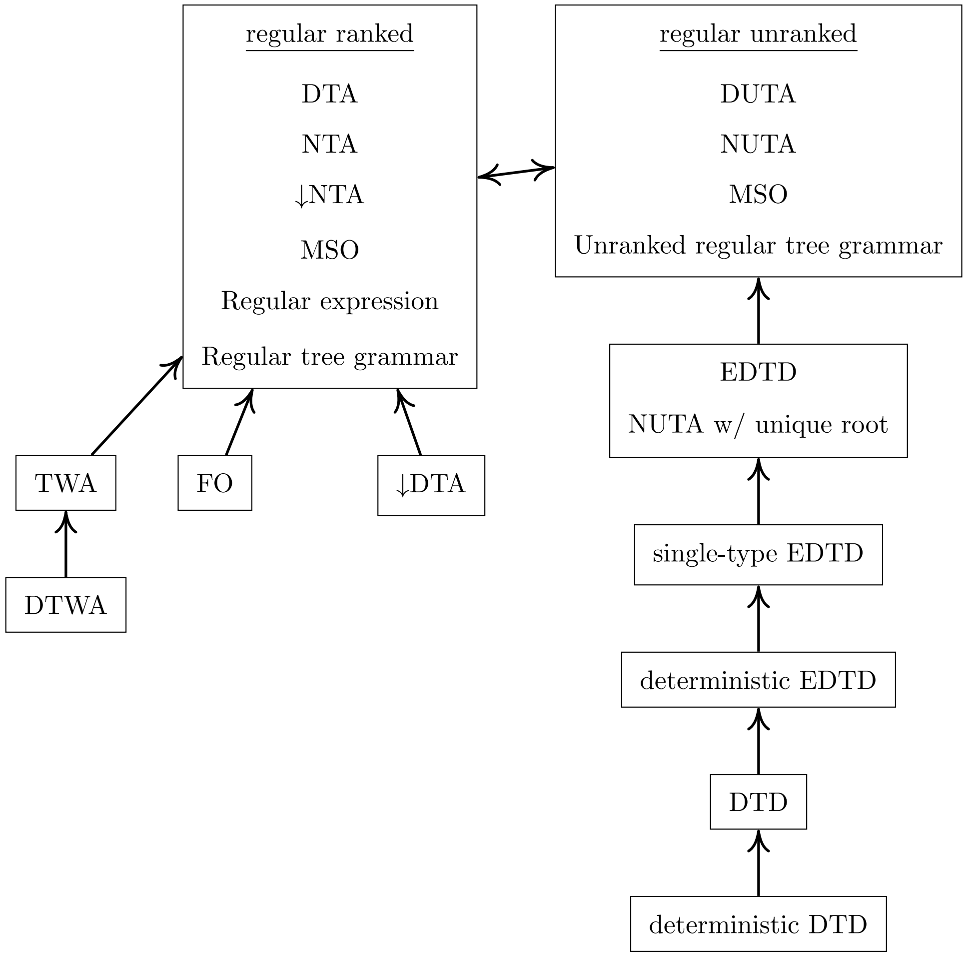
- Emptiness problem for NBAs is decidable in poly. time.
  Algorithm:

  1. Compute all states that are reachable from q_0. ($\mathcal{O}(|Q| + |\Delta|)$).
  2. For each of these reachable states that is also accepting, check if there is a path to itself. ($\mathcal{O}(|F| \cdot (|Q| + |\Delta|))$).
  3. If any state like that was found, the language is not empty.

- Emptiness problem for ABAs is PSPACE-complete.

- Membership problem for ABAs is decidable in poly. time.
  Proof: Model the problem as a Büchi game. In each turn, starting at a node $v \in Q$, player 1 chooses an $a \in \Sigma$. Then, according to the Boolean operators of $\delta(v, a)$, the players choose a next state. If player 0 wins, the choices of player 1 model an element of the ABA language.

4 Finite Tree Models



4.1 Class Inclusions

- Regular Ranked $\subseteq$ Regular Unranked
  FCNS: Let Σ be an unranked alphabet. We define $\Gamma_0 = \{\#\}$ and $\Gamma_2 = \Sigma$. Let $\bar{t} = t_1 \dots t_n \in T_\Sigma^*$ with $t_1 = a(t'_1 \dots t'_m)$. We define

  $$\mathrm{fcns}(\bar{t}) = \begin{cases} \# & \text{if } n = 0 \\ a(\mathrm{fcns}(t'_1, \dots, t'_m), \mathrm{fcns}(t_2 \dots t_n)) & \text{else} \end{cases}$$

  See Theorem 12.

- det. DTD $\subseteq$ DTD $\subseteq$ det. EDTD $\subseteq$ single-type EDTD $\subseteq$ EDTD
  trivial

- EDTD $\subseteq$ Regular tree grammar
  Proof: $N = \Sigma'$, $P_{\text{gram}} = P \cup \{a^{(n)} \to a \mid a, n\}$

- TWA $\subseteq$ NTA
  Proof: Theorem 8

4.2 Class Exclusions

- $\downarrow$DTA $\not\subseteq$ NTA
  Example: $T = \{f(a,b), f(b,a)\}$
  Proof: Assume the $\downarrow$DTA $\mathcal{A} = (Q, \Sigma, q_0, \Delta)$ accepts T. Let $(q_0, f, (q_1, q_2)) \in \Delta$, so also $(q_1, a), (q_1, b), (q_2, a), (q_2, b) \in \Delta$. However, that means the tree $f(a,a)$ is accepted by the run $q_0(q_1, q_2)$ which is a contradiction. On the other hand, NTAs can clearly recognize this property.

- DTD $\not\subseteq$ single-type EDTD
  Example: $T = \{t \in T_{\{a,b\}} \mid$ there is a path in t on which a occurs exactly twice

- NUTA w/ unique root $\not\subseteq$ NUTA
  Example: $T = \{a, b\}$

- FO $\not\subseteq$ MSO
  Example: $T =$ positive boolean terms that evaluate to true

- DTWA $\not\subseteq$ TWA
  Example: $T_{N \setminus D}$
  $\Sigma_0 = \{a, b\}$, $\Sigma_2 = \{f\}$
  $t \in T_{N \setminus D}$ iff $|t|_a = 3 \wedge \text{lca}(u, v) \sqsubseteq \text{lca}(v, w)$ **Proof**: DTWAs cannot recognize this tree language (no proof). TWAs can:
  - Check whether there are exactly three a and move to the right-most one (DFS).
  - While going up, guess a node and go the the left-most ancestor.
  - The tree is in $T_{N \setminus D}$ iff there are exactly two leafs labeled a right of that node. (DFS)

- TWA $\not\subseteq$ NTA
  Example: all paths in the skeleton have even length
  Proof: $\Sigma_0 = \{a, c\}$, $\Sigma_2 = \{b\}$
  Skeleton of a tree t: replace all subtrees that contain exactly one a.
  TWAs cannot recognize this tree language (no proof). NTAs can (find "merge" points and count the parity)

4.3 Class Equalities

4.3.1 Regular Ranked

- NTA $\Rightarrow$ DTA
  Proof: Subset construction.

- NTA $\Leftrightarrow$ $\downarrow$NTA
  Proof: Reverse the transitions and initial states $\leftrightarrow$ final states.

- $\downarrow$NTA $\Leftrightarrow$ Regular Tree Grammar
  Proof: Non-terminals correspond to states and transitions $(q, a, q_1, \ldots, q_n)$ correspond to rules $A_q \to a(A_{q_1}, \ldots, A_{q_n})$

- MSO $\Leftrightarrow$ NTA
  Proof: same as S2S

- Reg. exp. $\Leftrightarrow$ NTA
  Proof: Theorem 9

4.3.2 Regular Unranked

- NUTA $\Rightarrow$ DUTA
  Proof: Specialized subset construction.
  $\mathcal{A} = (Q, \Sigma, \Delta, F) \Rightarrow \mathcal{A}' = (2^Q, \Sigma, \Delta', \{P \subseteq Q \mid P \cap F \neq \emptyset\})$
  $\Delta' = \{(L_{a,P}, a, P) \mid a \in \Sigma, P \subseteq Q\}$ with $L_{a,P} = \bigcap\limits_{p \in P} K_{a,p} \cap \bigcap\limits_{p \notin P} K^{\complement}_{a,p}$
  $K_{a,p} = \{P_1 \ldots P_n \in (2^Q)^* \mid \exists p_1 \in P_1, \ldots, p_n \in P_n : p_1 \ldots p_n \in L_{a,p}\}$.

- MSO $\Leftrightarrow$ NUTA

4.3.3 EDTD

- NUTA with unique root $\Leftrightarrow$ EDTD (in poly. time)
  Proof: $\Rightarrow$ Let $q_0, \ldots, q_n$ be an enumeration of Q. We then set $D = \{\{a^{(i)} \mid a \in \Sigma, 0 \leq i \leq n\}, P, a^{q_0})$ where a is the unique root symbol of the automaton. Rules in P guess arbitrary symbols for fitting states.

  $\Leftarrow$ Analogously, we use $Q = \Sigma \times \{1, \ldots, n\}$ where n is the maximal type in Σ' and rules mimic the transitions.

4.4 Closures

4.4.1 Regular Ranked

- Regular (ranked) trees are closed under complement.
  Proof: Make Δ total (add sink state) and set $F' := Q \setminus F$.

- Regular (ranked) trees are closed under union and intersection.
  Proof: Product construction ($F_\cap = F_1 \times F_2$, $F_\cup = (Q_1 \times F_2) \cup (F_1 \times Q_2)$)

4.4.2 Regular Unranked

- Regular unranked trees are closed under complement, union, and intersection.
  Proof: via FCNS, because ranked trees are closed under these operations

4.4.3 TWA

- TWAs are closed under union.
  Proof: Non-deterministically choose at the start whether to execute $\mathcal{A}_1$ or $\mathcal{A}_2$.
- TWAs and DTWAs are closed under intersection.
  Proof: Execute $\mathcal{A}_1$. If it accepts, move to the root and execute $\mathcal{A}_2$.
- Open Problem: Are TWAs closed under complement?
- DTWAs are closed under complement.
  Proof: Theorem 11

4.5 Problems / Complexity

4.5.1 Regular Ranked

- Membership problem for NTAs is decidable.
- Reachable states of NTAs can be computed in linear time in $|\mathcal{A}|$. **Algorithm**:
  1. Initial reachable states $R = \{q \in Q \mid \exists a \in \Sigma_0 : (a, q) \in \Delta\}$
  2. For each transition $\tau = \{\overline{q}, a, p\} \in \Delta$, count $\text{in}(\tau) = |\overline{q}|$ and remember $\forall q \in \overline{q} : \tau \in \text{tr}(q)$.
  3. Grow R by processing each reachable state once: Decrement $\text{in}(\tau)$ by 1; if that value reaches 0, every ingoing state of τ is reachable and therefore, the outgoing state is.
- Emptiness of an NTA can be decided in linear time in $|\mathcal{A}|$.
  Algorithm: $T(\mathcal{A}) = \emptyset$ iff $\text{Reachable}(\mathcal{A}) \cap F = \emptyset$
- Given a DTA $\mathcal{A}$, $\sim_{T(\mathcal{A})}$ can be computed in time $\text{poly}(|Q^m \times \Sigma \times Q|)$ where m is the maximal arity in Σ.
  Algorithm:
  1. Mark all (q, q') with $(q, q') \in (F \times Q \setminus F) \cup (Q \setminus F \times F)$.
  2. As long as there is still change in the marks, execute step 3:
  3. If (p, p') is already marked and there are $q_1, \dots, q_{i-1}, q, q', q_{i+1}, \dots, q_n \in Q$ and $a \in \Sigma_n$ such that $p = \delta(q_1, \dots, q, \dots, q_n, a)$ and $p' = \delta(q_1, \dots, q', \dots, q_n, a)$, then mark the pair (q, q').
  4. $p \sim_{T(\mathcal{A})} q$ iff (p, q) is not marked.

4.5.2 Regular Unranked

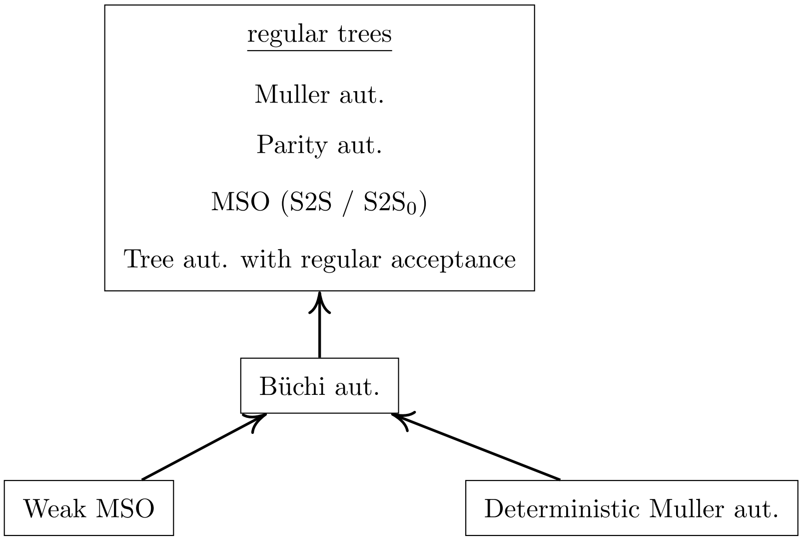
- Emptiness / membership / inclusion for NUTAs is decidable in polynomial / polynomial / exponential time.
  Algorithm (Emptiness): Use FCNS transformation.
  Algorithm (Membership): Construct $\mathcal{A}_t$ with $T(\mathcal{A}_t) = \{t\}$ and check $\mathcal{A} \cap \mathcal{A}_t \stackrel{?}{=} \emptyset$.
  Algorithm (Inclusion): Check $T(\mathcal{A}_1) \cap T(\mathcal{A}_2)^{\complement} \stackrel{?}{=} \emptyset$. Complementation is exponential.
- Inclusion for complete DUTAs is decidable in polynomial time.
  Algorithm: Same algorithm as for general NUTAs but complementation can be done in polynomial time.

4.5.3 Grammars

- Emptiness / membership for EDTDs is decidable in polynomial time.
  Proof: EDTD can be converted to NUTA in polynomial time.

- Inclusion for deterministic EDTDs is decidable in polynomial time.
  Proof: Let D_1, D_2 be deterministic EDTDs. Using the previous results and theorem 10, we can construct NUTAs $\mathcal{A}, \mathcal{B}$ with $T(\mathcal{A}) = T(D_1)$ and $T(\mathcal{B}) = T(D_2)^{\complement}$ in polynomial time. Then $T(D_1) \subseteq T(D_2)$ iff $T(\mathcal{A}) \cap T(\mathcal{B}) = \emptyset$.

5 Infinite Tree Models



5.1 Class Exclusions

- BTA $\not\subseteq$ Regular tree
  Example: $T_{\text{fin}} = \{t \in T_{\{a,b\}} \mid \text{every infinite path has only fin. many } b\}$
  Proof:

- DMTA $\not\subseteq$ BTA
  Example: $T_{\text{fin}}^{\complement} = \{t \in T_{\{a,b\}} \mid \text{there is a path with inf. many } b\}$
  Proof: BTA-recognizable (see below) but not a path tree language.

- WMSO $\not\subseteq$ BTA
  Example: $\{t \in T_{\{a,b\}} \mid \text{every infinite path has infinitely many } b\}$
  Proof:

5.2 Class Equalities

5.2.1 Regular Trees

- PTA, MTA $\Rightarrow$ TA with regular acceptance
  Proof: By definition and PA/MA regularity.

- TA with reg. acc. $\Rightarrow$ PTA
  Proof: TA $\mathcal{A}$, DPA $\mathcal{A}'$ over alphabet Q that defines Acc. Define PTA with state space $Q \times P$.
  $(q, a, q_1, q_2) \in \Delta \Rightarrow ((q, p), a, (q_1, \delta(p, q)), (q_2, \delta(p, q))) \in \Delta'$

- PTA $\Leftrightarrow$ MSO (S2S) **Proof**: **??**

5.3 Closures

5.3.1 Regular Trees

- The class of regular tree languages is closed under union and intersection.
  Proof: **??**

- The class of regular tree languages is closed under projection.
  Proof: **??**

- PTAs are closed under complement ($2^{\mathcal{O}(kn\cdot\log(kn))}$ states for $|Q| = n$, $|\text{img}(c)| = k$) **Proof**: **??**

5.3.2 BTA

- BTAs are not closed under complement.
  Proof: $T_{\text{fin}}^{\complement} = \{t \in T_{\{a,b\}} \mid \text{there is a path with inf. many } b\}$
  $T_{\text{fin}}^{\complement}$ is BTA-recognizable but its complement is not.

5.3.3 DMTA

- DMTAs are not closed under union or complement.

- DMTAs are closed under intersection.
  Proof: product automaton

5.4 Characterizations

- $T \subseteq T_\Sigma$ is DMTA-recognizable iff T is a path tree language.
  $T : 2^{(\{0,1\}\times\Sigma)^\omega} \to 2^{T_\Sigma}, L \mapsto \{t \in T_\Sigma \mid \forall \pi \in \{0,1\}^\omega : \pi^\frown(t|_\pi) \in L\}$
  Proof:

- $T \subseteq T_\Sigma$ is WMSO-definable iff T and $T^{\complement}$ are BTA-recognizable.

5.5 Problems

5.5.1 Regular Trees

- The membership problem for regular tree languages is decidable.
  Proof: via membership game as in complementation

- The emptiness problem for regular tree languages is decidable.
  Proof: **??**