

1 Structures

1.1 Words

- alphabet Σ
- Σ^*, Σ^ω
- ω -word $(\alpha : \mathbb{N} \rightarrow \Sigma) \in \Sigma^\omega$
- $\text{Occ}(\alpha) = \{\alpha(n) \mid n \in \mathbb{N}\}$
- $\text{Inf}(\alpha) = \bigcap_{i \in \mathbb{N}} \{\alpha(n) \mid n > i\}$

1.2 Finite Ranked Trees

- ranked alphabet $\Sigma_0, \dots, \Sigma_m$
- trees T_Σ
- trees (inductive definition) $a \in \Sigma_i, t_1, \dots, t_i \in T_\Sigma \Rightarrow a(t_1, \dots, t_i) \in T_\Sigma$
- trees (labeled definition)
 - tree $t = (\text{dom}_t, \text{val}_t)$
 - $\text{dom}_t \subseteq (\mathbb{N}_{>0}^*)$
 - $\text{val}_t : \text{dom}_t \rightarrow \Sigma$
 - if $\text{val}_t(w) \in \Sigma_i$, then $wi \in \text{dom}_t$ and $w(i+1) \notin \text{dom}_t$
 - if $w = uv \in \text{dom}_t$, then $u \in \text{dom}_t$
 - if $w(i+1) \in \text{dom}_t$, then $wi \in \text{dom}_t$

1.3 Finite Special Trees

- $S_\Sigma = \{t \in T_{\Sigma \cup \{\circ\}} \mid \circ \text{ occurs exactly once in } t\}$
- Notation, $s \in S_\Sigma, t \in T_\Sigma, u \in \text{dom}(s)$:
 - $s \cdot t$: replace \circ in s by t
 - $s|_u$: subtree of s with root u
 - $s[\circ/u]$: replace u and its subtree by \circ

1.4 Finite Unranked Trees

- alphabet Σ
- trees T_Σ
- trees (inductive definition)
 - $h \in (T_\Sigma)^*$ is a **hedge**
 - $a \in \Sigma, h \text{ hedge} \Rightarrow a(h) \in T_\Sigma$
- trees (labeled definition)
 - tree $t = (\text{dom}_t, \text{val}_t)$
 - $\text{dom}_t \subseteq (\mathbb{N}_{>0}^*)$
 - $\text{val}_t : \text{dom}_t \rightarrow \Sigma$
 - if $w = uv \in \text{dom}_t$, then $u \in \text{dom}_t$
 - if $w(i+1) \in \text{dom}_t$, then $wi \in \text{dom}_t$

1.5 Infinite Trees

- alphabet Σ
- trees T_Σ
- tree $t : \{0, 1\}^* \rightarrow \Sigma$
- path $\pi \in \{0, 1\}^\omega$
- $t|_\pi \in \Sigma^\omega : n \mapsto t(\pi(0) \cdots \pi(n))$

1.6 Game Arenas

- arena / game graph $G = (V_0, V_1, E, c)$
 - V_0, V_1 player vertices, $V = V_0, V_1$
 - $E \subseteq V \times V$
 - colors C finite
 - colors $c : V \rightarrow C$
- winning condition $\text{Win} \subseteq C^\omega$
- game $\mathcal{G} = (G, \text{Win})$
- play $\alpha \in V^\omega$
- winner of $\alpha = \begin{cases} 0 & \text{if } c(\alpha) \in \text{Win} \\ 1 & \text{else} \end{cases}$

2 Acceptance Conditions

Q set of states/colors

$\rho \in Q^\omega$

E $F \subseteq Q$. ρ accepted iff $F \cap \text{Occ}(\rho) \neq \emptyset$

A $F \subseteq Q$. ρ accepted iff $\text{Occ}(\rho) \subseteq F$

Staiger-Wagner $\mathcal{F} \subseteq 2^Q$. ρ accepted iff $\text{Occ}(\rho) \in \mathcal{F}$

weak Parity $c : Q \rightarrow C$. ρ accepted iff $\max \text{Occ}(c(\rho))$ is even

Büchi $F \subseteq Q$. ρ accepted iff $F \cap \text{Inf}(\rho) \neq \emptyset$

Generalized Büchi $F_1, \dots, F_k \subseteq Q$. ρ accepted iff for all $1 \leq i \leq k$: $F_i \cap \text{Inf}(\rho) \neq \emptyset$

coBüchi $F \subseteq Q$. ρ accepted iff $\text{Inf}(\rho) \subseteq F$

Rabin $\Omega \subseteq 2^Q \times 2^Q$. ρ accepted iff there is $(E, F) \in \Omega$ s.t. $\text{Inf}(\rho) \cap E = \emptyset$ and $\text{Inf} \cap F \neq \emptyset$

- Rabin Chain: $\Omega = \{(E_i, F_i) \mid 1 \leq i \leq k\}$ with $E_k \subseteq F_k \subseteq E_{k-1} \subseteq \dots \subseteq E_1 \subseteq F_1$

Streett $\Omega \subseteq 2^Q \times 2^Q$. ρ accepted iff for all $(E, F) \in \Omega$: $\text{Inf}(\rho) \cap E \neq \emptyset$ or $\text{Inf} \cap F = \emptyset$

Parity $c : Q \rightarrow C$. ρ accepted iff $\max \text{Inf}(c(\rho))$ is even

Muller $\mathcal{F} \subseteq 2^Q$. ρ accepted iff $\text{Occ}(\rho) \in \mathcal{F}$

3 Word Automata

$(Q, \Sigma, q_0, \Delta, \text{Acc})$

$L(A) = \text{accepted words}$

Deterministic $\delta : Q \times \Sigma \rightarrow Q$

Non-Deterministic $\Delta \subseteq Q \times \Sigma \times Q$

Alternating $\delta : Q \times \Sigma \rightarrow B_+(Q)$ where $B_+(Q)$ are boolean formulas over $Q \cup \{0, 1\}$ without negation

3.1 Deterministic Word Automata

- DBA
- weak DBA
- coDBA
- E-automaton
- A-automaton
- Staiger-Wagner automaton
- DMA
- DRA
- DPA

3.2 Nondeterministic & Alternating Word Automata

- NBA
- ABA

4 Finite Tree Automata

4.1 Ranked Top-Down Automata

(Q, Σ, Δ, F) where $\Delta \subseteq \bigcup_{i=0}^m Q^i \times \Sigma_i \times Q$ or deterministic δ

- DTA
- NTA

$$\text{Size } |\mathcal{A}| = |Q| + \sum_{\tau \in \delta} |\tau|$$

4.2 Ranked Bottom-Up Automata

(Q, Σ, Q_0, Δ) where $Q_0 \subseteq Q$ initial states, $\Delta \subseteq \bigcup_{i=0}^m Q \times \Sigma_i \times Q^i$ or deterministic δ

- \downarrow DTA ($|Q_0| = 1$)
- \downarrow NTA

4.3 Unranked Bottom-Up Automata

(Q, Σ, Δ, F) where $\Delta \subseteq \text{Reg}(Q) \times \Sigma \times Q$

Normalized For all $(L_1, a_1, q_1), (L_2, a_2, q_2)$: If $a_1 = a_2$ and $q_1 = q_2$, then $L_1 = L_2$

Deterministic For all $(L_1, a_1, q_1), (L_2, a_2, q_2)$: $L_1 \cap L_2 = \emptyset$ or $a_1 \neq a_2$ or $q_1 \neq q_2$

- DUTA
- NUTA

4.4 Ranked Tree Walking Automata

Types = $\{\text{root}, 1, \dots, m\}$

Dir = $\{\uparrow, 0, 1, \dots, m\}$

$$\text{type}_t(u) = \begin{cases} \text{root} & \text{if } u = \epsilon \\ i & \text{if } u = vi \end{cases}$$

$(Q, \Sigma, q_0, \Delta, F)$ where $\Delta \subseteq Q \times \text{Types} \times \Sigma \times Q \times \text{Dir}$

(current state, current type, current symbol, new state, movement) $\in \Delta$

- TWA
- DTWA

4.5 Tree Transducers

- Bottom-up Tree Transducer (BTT)
- Top Down Transducer (TDT)

BTT

$(Q, \Sigma, \Gamma, \Delta, F)$

Σ, Γ input / output alphabet

Δ : tree transition & ε transition

$F \subseteq Q$

Tree with variables X at leafs: $T_\Sigma(X)$

Tree transitions: $f(q_1(x_1), \dots, q_n(x_n)) \rightarrow q(u)$

for some $f \in \Sigma_n$, $q, q_1, \dots, q_n \in Q$, $x_1, \dots, x_n \in X$, $u \in T_\Gamma(\{x_1, \dots, x_n\})$

ε transitions: $q(x) \rightarrow q'(u)$

for some $q, q' \in Q$, $x \in X$, $u \in T_\Gamma(\{x_1\})$

Configurations are trees over $\Sigma \cup \Gamma \cup Q$.

BTTs define relations $R(\mathcal{A}) = \{(t, t') \mid \exists q \in F : t \rightarrow_{\mathcal{A}}^* q(t')\}$.

Transition relation: $s \cdot f(q_1(t_1), \dots, q_n(t_n)) \rightarrow_{\mathcal{A}} s \cdot q(u[x_1 \leftarrow t_1, \dots, x_n \leftarrow t_n])$

TDT

Similar properties as BTT but incomparable class of relations

5 Infinite Tree Automata

$(Q, \Sigma, q_0, \Delta, \text{Acc})$

$\Delta \subseteq Q \times \Sigma \times Q \times Q$

$\text{Acc} \subseteq Q^\omega$

Run $\rho : \{0, 1\}^* \rightarrow Q$ accepting iff $\rho|_\pi \in \text{Acc}$ for all paths π

$T(A) = \text{accepted trees}$

5.1 Nondeterministic

- BTA
- MTA
- PTA

5.2 Deterministic

$|\{(q, a, q_1, q_2) \mid q_1, q_2\}| = 1$ for all q, a

- DTBA

5.3 Regular Tree Automata

$(Q_{\mathcal{B}}, \{0, 1\}, q_0^{\mathcal{B}}, \delta_{\mathcal{B}}, f_{\mathcal{B}})$ reads $\{0, 1\}$ -words

$\delta_{\mathcal{B}} : Q \times \{0, 1\} \rightarrow Q$

$f_{\mathcal{B}} : Q_{\mathcal{B}} \rightarrow \Sigma$

Defines tree $t_{\mathcal{B}} : u \mapsto f_{\mathcal{B}}(\delta_{\mathcal{B}}^*(u))$

6 Infinite Games

- Reachability (E-condition)
- Safety (A-condition)
- Staiger-Wagner
- weak Parity (assuming $C \subset \mathbb{N}$)
- Büchi
- Parity (assuming $C \subset \mathbb{N}$)
- Muller
- Rabin
- Streett

6.1 Strategy Automata

$(M, C, m_0, \sigma^u, \sigma^n)$ strategy for player 0

Memory states M

Input colors C

Transition / memory update function $\sigma^u : M \times C \rightarrow M$

Next move function $\sigma^n : M \times V_0 \rightarrow V$

Strategy $\sigma_{\mathcal{A}}(v_0 \dots v_n) = \sigma^n((\sigma^u)^*(v_0 \dots v_{n-1}), v_n)$

7 Logics

7.1 LTL

$\varphi ::= p_i \mid \neg\varphi \mid \varphi \vee \varphi \mid X\varphi \mid F\varphi \mid G\varphi \mid \varphi U\varphi$
 $(\alpha, i) \models \varphi$

- ω -languages
- Game winning conditions

7.2 FO

$\varphi ::= t_1 = t_2 \mid R\bar{t} \mid \neg\varphi \mid \varphi \vee \varphi \mid \exists x\varphi \mid \forall x\varphi$

- Finite tree languages

Finite Tree Structure

Ranked $\underline{t} = (\text{dom}_t, (S_i^t)_{1 \leq i \leq m}, \sqsubseteq^t, (P_a^t)_{a \in \Sigma})$
Unranked $\underline{t} = (\text{dom}_t, S^t, \sqsubseteq^t, <^t, (P_a^t)_{a \in \Sigma})$
 \sqsubseteq prefix order; $<$ sibling order

7.3 MSO

$\varphi ::= \text{FO} + (\exists X \mid \forall X)$
set quantification

- Finite tree languages
- S1S, S2S

7.4 wMSO

MSO with only finite set quantification

- ω -tree languages

7.5 S1S

MSO over $(\mathbb{N}, +1, <, 0)$

- ω -languages
- Game winning conditions

S1S₀

S1S - element variables + $(X \subseteq Y \mid \text{Sing}(X) \mid \text{Succ}(X))$

7.6 S2S

MSO over $\underline{T}_2 = (\{0, 1\}^*, \varepsilon, S_0, S_1)$

- ω -tree languages

S2S₀

S2S - element variables + $(X \subseteq Y \mid \text{Sing}(X) \mid \text{Succ}_0(X) \mid \text{Succ}_1(X))$