1 Infinite Words

Theorem 1.1. *Every non-empty ω-regular language contains an ultimately periodic word.*

Theorem 1.2. *For a Kripke structure $\mathcal{K}$ with initial state s and $\varphi \in$ LTL, the model checking problem $L(\mathcal{K}, s) \subseteq L(\varphi)$? is PSPACE-complete.*

Proof. **PSPACE** Compute the intersection automaton for $L(\mathcal{K}, s) \cap L(\neg\varphi)$ and test it for emptiness.

PSPACE-hard Encode a poly.-length Turing tape as a Kripke structure and its correct behavior in LTL. □

Theorem 1.3 (Büchi)**.** *The MSO theory of $(\mathbb{N}, +1, <, 0)$ is decidable.*

Proof. Corresponds to S1S formula. Can be checked with NBA emptiness test. □

Theorem 1.4. *The FO theory of $(\mathbb{R}, +, <, 0)$ is decidable.*

Proof. Encode real numbers x by triples of sets (X_s, X_i, X_f) with the number's sign ($X_s = \emptyset$ or $\{0\}$), the positive decimal digits in binary encoding, and the positive fractional digits in binary encoding. Then an FO sentence can be transformed to an equi-satisfiable MSO sentence over $(\mathbb{N}, +1, <, 0)$. □

Theorem 1.5. *Subset-construction does not suffice to determinize NBAs.*

Theorem 1.6. *For every n, there is $L_n \subseteq \Sigma^\omega$ s.t. there is an NBA that recognizes L_n with $n+2$ states, but every det. Rabin automaton that recognizes L_n has at least $n!$ states.*

Theorem 1.7. *There is a DBA-recog. language which does not have a unique minimal DBA. DBAs minimized with the DFA minimization algorithm can be arbitrarily bad compared to a minimal DBA.*

Theorem 1.8. *Weak DBAs can be minimized uniquely in polynomial time.*

Theorem 1.9. *Given an ABA $\mathcal{A}$, the dual $\tilde{\mathcal{A}}$ is an alternating co-Büchi automaton which accepts $\overline{L(\mathcal{A})}$, with $\tilde{F} = Q \setminus F$ and $\tilde{\delta}$ exchanging true/false and $\wedge/\vee$.*

2 Finite Trees

Theorem 2.1 (Pumping Principle)**.** *Let $T \subseteq T_\Sigma$ be a regular ranked tree language. There is a $n \in \mathbb{N}$ such that for all trees $t \in T$, all $m > n$, and all paths $\pi_1 \dots \pi_m$, there are $1 \leq i < j \leq m$ such that for all $k \in \mathbb{N}$:*

$$t[\circ/u] \cdot (t[\circ/v]|_u)^k \cdot t|_v \in T$$

where $u = \pi_1 \dots \pi_i$ and $v = \pi_1 \dots \pi_j$.

Definition 1. *Let $T \subseteq T_\Sigma$. The **Myhill-Nerode equivalence** is $\sim_T \subseteq T_\Sigma \times T_\Sigma$ with*

$$t_1 \sim_T t_2 \Leftrightarrow \forall s \in S_\Sigma : s \cdot t_1 \in T \leftrightarrow s \cdot t_2 \in T$$

The index of T is Index$(\sim_T) := |T/\sim_T|$.

Theorem 2.2. *Let $T \subseteq T_\Sigma$. T is regular iff Index$(\sim_T)$ is finite. If T is regular, $\mathcal{A}_T$ is the minimal DTA.*

Theorem 2.3. *The emptiness problem for NTAs can be reduced to HORN-SAT in linear time.*

2.1 BTTs

Theorem 2.4. *The equivalence problem for BTTs is undecidable.*

Theorem 2.5. *The emptiness problem for BTTs is decidable in polynomial time.*

Theorem 2.6. *The type-checking problem (given regular T, T', is $\mathcal{A}(T) \subseteq T'$?) is decidable.*

Theorem 2.7. *If T is regular, then $\mathcal{A}^{-1}(T)$ is regular.*
If $\mathcal{A}$ is linear, then $\mathcal{A}(T_\Sigma)$ is regular.

Theorem 2.8. *There are BTT-definable relations R_1, R_2 such that $R_1 \circ R_2$ is not BTT-definable.*

Theorem 2.9. *If $\mathcal{A}_1$ is linear* ***or*** *$\mathcal{A}_2$ is deterministic Then $R(\mathcal{A}_1) \circ R(\mathcal{A}_2)$ is BTT-definable.*

3 Infinite Trees

Theorem 3.1 (BTA Pumping)**.** *For $t \in T_\Sigma, x \in \{0,1\}^*, y \in \{0,1\}^+$, let*

$$t^*_{[x,y]} : \{0,1\}^* \to \Sigma, z \mapsto \begin{cases} t(z) & \text{if } xy \not\sqsubseteq z \\ xz' & \text{if } \exists n > 0 : z = xy^n z' \text{ with } y \not\sqsubseteq z' \end{cases}.$$

Let $\mathcal{A}$ be a BTA, $t \in T(\mathcal{A})$, ρ an accepting run of $\mathcal{A}$ on t, and $x, y, y' \in \{0,1\}^*$ s.t. $\rho(x) = \rho(xy)$, $y' \sqsubset y$, and $\rho(xy') \in F$. Then $t^*_{[x,y]} \in T(\mathcal{A})$.

Theorem 3.2. *Every non-empty regular tree language contains a regular tree.*

Theorem 3.3 (Rabin's Tree Theorem)**.** *The MSO theory of $\underline{T_2}$ is decidable for formulas $\varphi(X_1, \ldots, X_n)$ and a model $X_1, \ldots X_n \subseteq \{0,1\}^*$ is computable.*

Proof. Transform φ into an equivalent PTA. A model can be found by solving the emptiness game. $\square$