1 MSO interpretations

Definition 1. *Let $\mathfrak{A}$ be a structure. An* ***MSO interpretation*** *is* $\mathcal{I} = (\varphi_{dom}(x), (\varphi_{R_i}(x_1, \ldots, x_{k_i}))_{1 \leq i \leq n})$.

$\mathcal{I}$ defines the structure $\mathcal{I}(\mathfrak{A}) = (D^{\mathcal{I}(\mathfrak{A})}, (R_i^{\mathcal{I}(\mathfrak{A})})_{1 \leq i \leq n})$, where $D^{\mathcal{I}(\mathfrak{A})} = \{a \in A \mid \mathfrak{A} \models \varphi_{dom}(a)\}$ and $R_i^{\mathcal{I}(\mathfrak{A})} = \{(x_1, \ldots, x_{k_i} \mid \mathfrak{A} \models \varphi_{R_i}(x_1, \ldots, x_{k_i})\}$.

Definition 2. *Let $\mathfrak{A}$ be a structure. We define $MTh_2(\mathfrak{A}) = \{\varphi \in MSO \mid \mathfrak{A} \models \varphi\}$ as the monadic second-order theory of $\mathfrak{A}$.*

Theorem 1. *Let $\mathfrak{A}$ be a structure and let $\mathcal{I}$ be an MSO interpretation of $\mathfrak{A}$. If $MTh_2(\mathfrak{A})$ is decidable, then $MTh_2(\mathcal{I}(\mathfrak{A}))$ is decidable.*

Proof. Define a computable transformation $\psi \mapsto \hat{\psi}$ such that $\mathcal{I}(\mathfrak{A}) \models \psi$ iff $\mathfrak{A} \models \hat{\psi}$.

- $R_i\overline{x} \mapsto \varphi_{R_i}(\overline{x})$
- $\wedge$, $\vee$, $\neg$ are kept.
- $\exists x \psi(x) \mapsto \exists x(\varphi_{\mathrm{dom}}(x) \wedge \hat{\psi}(x))$

□

2 Transferring (Un)Decidability

Theorem 2. *$MTh_2(\underline{T_2})$ is decidable.*

Proof: see S2S to PTA transformation

Theorem 3. *$MTh_2((\mathbb{Q}, \leq))$ is decidable.*

Proof. We define an MSO interpretation $\mathcal{I} = (\varphi_{\mathrm{dom}}, \varphi_{\leq})$.

$$\varphi_{\mathrm{dom}}(x) = \exists y x = S_1 y$$
$$\varphi_{\leq}(x, y) = (x = y \vee x <_{\mathrm{lexi}} y)$$

□

Theorem 4. *Let $G_2 = (\mathbb{N} \times \mathbb{N}, (0,0), s_h, s_v)$ with $s_h(x, y) = (x+1, y)$ and $s_v(x, y) = (x, y+1)$. Then $MTh_2(G_2)$ is undecidable.*

Proof. Encode Turing tapes by rows in G_2 and encode succeeding Turing configurations by different rows. Then the Halting problem can be reduced to the decidability of $\mathrm{MTh}_2(G_2)$. □

Theorem 5. *Let $(\underline{T_2}, el)$ be the extension of $\underline{T_2}$ with $el = \{(u, v) \in \{0,1\}^* \times \{0,1\}^* \mid |u| = |v|\}$. Then $MTh_2((\underline{T_2}, el))$ is undecidable.*

Proof. We provide an interpretation $\mathcal{I} = (\varphi_{\mathrm{dom}}(x), \varphi_{(0,0)}(x), \varphi_{s_h}(x,y), \varphi_{s_v}(x,y))$ for $(\underline{T_2}, \mathrm{el})$ such that $\mathcal{I}((\underline{T_2}, \mathrm{el})) = G_2$. Then the statement follows from theorem 1.

$$\varphi_{\mathrm{dom}}(x) = \exists y (y \sqsubseteq x \wedge \forall z (z \sqsubseteq x \to (y \sqsubseteq z \leftrightarrow (\exists u\, z = S_1 u))) \mathrel{\hat{=}} 0^* 1^*$$

$$\varphi_{(0,0)}(x) = (x = \varepsilon)$$

$$\varphi_{s_h}(x,y) = \mathrm{el}(x,y) \wedge y < x \wedge \forall z (\mathrm{el}(x,z) \wedge z < x \wedge y < z \to \neg \varphi_{\mathrm{dom}}(z))$$

$$\varphi_{s_v}(x,y)) = (y = S_1 x)$$

□