Theorem 1. *Let $\mathcal{A} = (Q, \Sigma, q_0, \Delta, F)$ be a TWA. There is an NTA $\mathcal{A}'$ of size exponential in $|Q|$ that recognizes $T(\mathcal{A})$.*

Proof. Let $\sim_{T(\mathcal{A})}$ be the usual equivalence relation, i.e. $t_1 \sim_{T(\mathcal{A})} t_2$ iff $\forall s \in S_\Sigma : s \cdot t_1 \in T(\mathcal{A}) \leftrightarrow s \cdot t_2 \in T(\mathcal{A})$. We define a relation $\sim \subseteq T_\Sigma \times T_\Sigma$ such that $\text{index}(\sim_{T(\mathcal{A})}) \leq \text{index}(\sim) \leq 2^{|Q|^2 \cdot m+1}$, where m is the maximal rank in Σ.

Let $t_0 \in T_\Sigma$ and $a_m \in \Sigma_m$ be arbitrary. For every $t \in T_\Sigma$ and $1 \leq i \leq m$, we define $t^{(i)} = a_m(t_0, t_0, \ldots, t, \ldots, t_0)$, meaning the i-th subtree below the root is t. Further, we define a relation $B_t^i \subseteq Q \times Q$ with $(p, q) \in B_t^i$ iff there is a run segment ρ of $\mathcal{A}$ on $t^{(i)}$, such that the run begins at the root of t, never leaves that subtree until the end. Meaning, $\rho = (p, i)(q_1, iu_1) \ldots (q_n, iu_n)(q, \varepsilon)$.

Finally, let $t_1 \sim t_2$ iff $t_1 \in T(\mathcal{A}) \leftrightarrow t_2 \in T(\mathcal{A})$ and $\forall i : B_{t_1}^i = B_{t_2}^i$.

Idea: $(p, q) \in B_t^i$ if $\mathcal{A}$ can enter t as i-th child with state p and after some while leaves it again with state q.

Claim : Let $t_1 \sim t_2$. Then $t_1 \sim_{T(\mathcal{A})} t_2$.

Let $s \in S_\Sigma$. Due to the symmetric definition of $\sim$, it suffices to show that $t_1 \in T(\mathcal{A})$ implies $t_2 \in T(\mathcal{A})$, so let $t_1 \in T(\mathcal{A})$. If $s = \circ$, then $s \cdot t_1 = t_1 \in T(\mathcal{A})$. By definition of $\sim$, this implies $s \cdot t_2 = t_2 \in T(\mathcal{A})$.

Otherwise $s \neq \circ$. Let $\rho_1\rho_2\rho_3$ be an accepting run of $\mathcal{A}$ on $s \cdot t_1$ such that ρ_1 only stays outside of t_1 and ρ_2 only stays inside of t_1. Since $B_{t_1}^i = B_{t_2}^i$, there is a run segment of $\mathcal{A}$ on t_2 which enters and exits the tree with the same states as ρ_2 does, meaning it can replace ρ_2 in the accepting run. Repeating this procedure gives an accepting run of $\mathcal{A}$ on $s \cdot t_2$, so $t_2 \in T(\mathcal{A})$.

Notes on the construction: each state in the NTA corresponds to a list of Q-states that $\mathcal{A}$ had when visiting this node, together with the direction from which it was coming. That can be used to check the correctness of a run. □

Theorem 2. *A language of finite trees $T \subseteq T_\Sigma$ can be recognized by an NTA iff it can be described by a regular tree expression.*

Proof. □

Theorem 3. *Let D be an EDTD. We can construct a NUTA $\mathcal{A}$ with $T(D) = T(\mathcal{A})$ in polynomial time in $|D|$.*

Proof. □

Theorem 4. *The class of DTWA-recognizable languages is closed under complement.*

Proof. □

Theorem 5. *Let $T \subseteq T_\Sigma$. T is regular iff fcns(T) is regular.*

Proof. □