1 Structures

1.1 Words

- alphabet Σ
- Σ^*, Σ^ω
- ω-word $(\alpha : \mathbb{N} \to \Sigma) \in \Sigma^\omega$
- $\mathrm{Occ}(\alpha) = \{\alpha(n) \mid n \in \mathbb{N}\}$
- $\mathrm{Inf}(\alpha) = \bigcap_{i \in \mathbb{N}} \{\alpha(n) \mid n > i\}$

1.2 Finite Ranked Trees

- ranked alphabet $\Sigma_0, \ldots, \Sigma_m$
- trees T_Σ
- trees (inductive definition) $a \in \Sigma_i$, $t_1, \ldots, t_i \in T_\Sigma \Rightarrow a(t_1, \ldots, t_i) \in T_\Sigma$
- trees (labeled definition)
  - tree $t = (\mathrm{dom}_t, \mathrm{val}_t)$
  - $\mathrm{dom}_t \subseteq (\mathbb{N}^*_{>0}$
  - $\mathrm{val}_t : \mathrm{dom}_t \to \Sigma$
  - if $\mathrm{val}_t(w) \in \Sigma_i$, then $wi \in \mathrm{dom}_t$ and $w(i+1) \notin \mathrm{dom}_t$
  - if $w = uv \in \mathrm{dom}_t$, then $u \in \mathrm{dom}_t$
  - if $w(i+1) \in \mathrm{dom}_t$, then $wi \in \mathrm{dom}_t$

1.3 Finite Special Trees

- $S_\Sigma = \{t \in T_{\Sigma \cup \{\circ\}} \mid \circ$ occurs exaxctly once in $t\}$
- Notation, $s \in S_\Sigma$, $t \in T_\Sigma$, $u \in \mathrm{dom}(s)$:
  - $s \cdot t$: replace $\circ$ in s by t
  - $s|_u$: subtree of s with root u
  - $s[\circ/u]$: replace u and its subtree by $\circ$

1.4 Finite Unranked Trees

- alphabet Σ
- trees T_Σ
- trees (inductive definition)
  - $h \in (T_\Sigma)^*$ is a **hedge**
  - $a \in \Sigma$, h hedge $\Rightarrow a(h) \in T_\Sigma$
- trees (labeled definition)
  - tree $t = (\mathrm{dom}_t, \mathrm{val}_t)$
  - $\mathrm{dom}_t \subseteq (\mathbb{N}^*_{>0}$
  - $\mathrm{val}_t : \mathrm{dom}_t \to \Sigma$
  - if $w = uv \in \mathrm{dom}_t$, then $u \in \mathrm{dom}_t$
  - if $w(i+1) \in \mathrm{dom}_t$, then $wi \in \mathrm{dom}_t$

1.5 Infinite Trees

- alphabet Σ
- trees T_Σ
- tree $t : \{0,1\}^* \to \Sigma$
- path $\pi \in \{0,1\}^\omega$
- $t|_\pi \in \Sigma^\omega : n \mapsto t(\pi(0) \cdots \pi(n))$

1.6 Game Arenas

- arena / game graph $G = (V_0, V_1, E, c)$
  - V_0, V_1 player vertices, $V = V_0, V_1$
  - $E \subseteq V \times V$
  - colors C finite
  - colors $c : V \to C$
- winning condition $\mathrm{Win} \subseteq C^\omega$
- game $\mathcal{G} = (G, \mathrm{Win})$
- play $\alpha \in V^\omega$
- winner of $\alpha = \begin{cases} 0 & \text{if } c(\alpha) \in \mathrm{Win} \\ 1 & \text{else} \end{cases}$

2 Acceptance Conditions

Q set of states/colors
$\rho \in Q^\omega$

E $F \subseteq Q$. ρ accepted iff $F \cap \mathrm{Occ}(\rho) \neq \emptyset$

A $F \subseteq Q$. ρ accepted iff $\mathrm{Occ}(\rho) \subseteq F$

Staiger-Wagner $\mathcal{F} \subseteq 2^Q$. ρ accepted iff $\mathrm{Occ}(\rho) \in \mathcal{F}$

weak Parity $c : Q \to C$. ρ accepted iff $\max \mathrm{Occ}(c(\rho))$ is even

Büchi $F \subseteq Q$. ρ accepted iff $F \cap \mathrm{Inf}(\rho) \neq \emptyset$

Generalized Büchi $F_1, \dots, F_k \subseteq Q$. ρ accepted iff for all $1 \leq i \leq k$: $F_i \cap \mathrm{Inf}(\rho) \neq \emptyset$

coBüchi $F \subseteq Q$. ρ accepted iff $\mathrm{Inf}(\rho) \subseteq F$

Rabin $\Omega \subseteq 2^Q \times 2^Q$. ρ accepted iff there is $(E, F) \in \Omega$ s.t. $\mathrm{Inf}(\rho) \cap E = \emptyset$ and $\mathrm{Inf} \cap F \neq \emptyset$

- Rabin Chain: $\Omega = \{(E_i, F_i) \mid 1 \leq i \leq k\}$ with $E_k \subseteq F_k \subseteq E_{k-1} \subseteq \cdots \subseteq E_1 \subseteq F_1$

Streett $\Omega \subseteq 2^Q \times 2^Q$. ρ accepted iff for all $(E, F) \in \Omega$: $\mathrm{Inf}(\rho) \cap E \neq \emptyset$ or $\mathrm{Inf} \cap F = \emptyset$

Parity $c : Q \to C$. ρ accepted iff $\max \mathrm{Inf}(c(\rho))$ is even

Muller $\mathcal{F} \subseteq 2^Q$. ρ accepted iff $\mathrm{Occ}(\rho) \in \mathcal{F}$

3 Word Automata

$(Q, \Sigma, q_0, \Delta, \mathrm{Acc})$
$L(A) =$ accepted words

Deterministic $\delta : Q \times \Sigma \to Q$

Non-Deterministic $\Delta \subseteq Q \times \Sigma \times Q$

Alternating $\delta : Q \times \Sigma \to B_+(Q)$ where $B_+(Q)$ are boolean formulas over $Q \cup \{0, 1\}$ without negation

3.1 Deterministic Word Automata

- DBA
- weak DBA
- coDBA
- E-automaton
- A-automaton
- Staiger-Wagner automaton
- DMA
- DRA
- DPA

3.2 Nondeterministic & Alternating Word Automata

- NBA
- ABA

4 Finite Tree Automata

4.1 Ranked Top-Down Automata

(Q, Σ, Δ, F) where $\Delta \subseteq \bigcup_{i=0}^{m} Q^i \times \Sigma_i \times Q$ or deterministic δ

- DTA
- NTA

4.2 Ranked Bottom-Up Automata

(Q, Σ, Q_0, Δ) where $Q_0 \subseteq Q$ initial states, $\Delta \subseteq \bigcup_{i=0}^{m} Q \times \Sigma_i \times Q^i$ or deterministic δ

- $\downarrow$DTA
- $\downarrow$NTA

4.3 Unranked Bottom-Up Automata

(Q, Σ, Δ, F) where $\Delta \subseteq \mathrm{Reg}(Q) \times \Sigma \times Q$

Normalized For all $(L_1, a_1, q_1), (L_2, a_2, q_2)$: If $a_1 = a_2$ and $q_1 = q_2$, then $L_1 = L_2$

Deterministic For all $(L_1, a_1, q_1), (L_2, a_2, q_2)$: $L_1 \cap L_2 = \emptyset$ or $a_1 \neq a_2$ or $q_1 = q_2$

- DUTA
- NUTA

4.4 Ranked Tree Walking Automata

$\mathrm{Types} = \{\mathrm{root}, 1, \ldots, m\}$
$\mathrm{Dir} = \{\uparrow, 0, 1, \ldots m\}$
$$\mathrm{type}_t(u) = \begin{cases} \mathrm{root} & \text{if } u = \epsilon \\ i & \text{if } u = vi \end{cases}$$

$(Q, \Sigma, q_0, \Delta, F)$ where $\Delta \subseteq Q \times \mathrm{Types} \times \Sigma \times Q \times \mathrm{Dir}$
$(\text{current state}, \text{current type}, \text{current symbol}, \text{new state}, \text{movement}) \in \Delta$

- TWA
- DTWA

4.5 Tree Transducers

- Bottom-up Tree Transducer (BTT)
- Top Down Transducer (TDT)

BTT

$(Q, \Sigma, \Gamma, \Delta, F)$
Σ, Γ input / output alphabet
Δ: tree transition & ε transition
$F \subseteq Q$

Tree with variables X at leafs: $T_\Sigma(X)$

Tree transitions: $f(q_1(x_1), \ldots, q_n(x_n)) \to q(u)$
for some $f \in \Sigma_n$, $q, q_1, \ldots, q_n \in Q$, $x_1, \ldots, x_n \in X$, $u \in T_\Gamma(\{x_1, \ldots, x_n\})$

ε transitions: $q(x) \to q'(u)$
for some $q, q' \in Q$, $x \in X$, $u \in T_\Gamma(\{x_1\})$

Configurations are trees over $\Sigma \cup \Gamma \cup Q$.
BTTs define relations $R(\mathcal{A}) = \{(t, t') \mid \exists q \in F : t \to_{\mathcal{A}}^* q(t')\}$.

Transition relation: $s \cdot f(q_1(t_1), \ldots, q_n(t_n)) \to_{\mathcal{A}} s \cdot q(u[x_1 \leftarrow t_1, \ldots, x_n \leftarrow t_n])$

TDT

Similar properties as BTT but incomparable class of relations

5 Infinite Tree Automata

$(Q, \Sigma, q_0, \Delta, \mathrm{Acc})$
$\Delta \subseteq Q \times \Sigma \times Q \times Q$
$\mathrm{Acc} \subseteq Q^\omega$

Run $\rho : \{0,1\}^* \to Q$ accepting iff $\rho|_\pi \in \mathrm{Acc}$ for all paths π
$T(A)$ = accepted trees

5.1 Nondeterministic

- BTA
- MTA
- PTA

5.2 Deterministic

$|\{(q, a, q_1, q_2) \mid q_1, q_2\}| = 1$ for all q, a

- DTBA

5.3 Regular Tree Automata

$(Q_{\mathcal{B}}, \{0,1\}, q_0^{\mathcal{B}}, \delta_{\mathcal{B}}, f_{\mathcal{B}})$ reads $\{0,1\}$-words
$\delta_{\mathcal{B}} : Q \times \{0,1\} \to Q$
$f_{\mathcal{B}} : Q_{\mathcal{B}} \to \Sigma$

Defines tree $t_{\mathcal{B}} : u \mapsto f_{\mathcal{B}}(\delta_{\mathcal{B}}^*(u))$

6 Infinite Games

- Reachability (E-condition)
- Safety (A-condition)
- Staiger-Wagner
- weak Parity (assuming $C \subset \mathbb{N}$)
- Büchi
- Parity (assuming $C \subset \mathbb{N}$)
- Muller
- Rabin
- Streett

6.1 Strategy Automata

$(M, C, m_0, \sigma^u, \sigma^n)$ strategy for player 0
Memory states M
Input colors C
Transition / memory update function $\sigma^u : M \times C \to M$
Next move function $\sigma^n : M \times V_0 \to V$
Strategy $\sigma_{\mathcal{A}}(v_0 \dots v_n) = \sigma^n((\sigma^u)^*(v_0 \dots v_{n-1}), v_n)$

7 Logics

7.1 LTL

$\varphi ::= p_i \mid \neg\varphi \mid \varphi \vee \varphi \mid X\varphi \mid F\varphi \mid G\varphi \mid \varphi U \varphi$
$(\alpha, i) \models \varphi$

- ω-languages
- Game winning conditions

7.2 FO

$\varphi ::= t_1 = t_2 \mid R\bar{t} \mid \neg\varphi \mid \varphi \vee \varphi \mid \exists x\varphi \mid \forall x\varphi$

- Finite tree languages

7.3 MSO

$\varphi ::= \mathrm{FO} + (\exists X \mid \forall X)$
set quantification

- Finite tree languages
- S1S, S2S

7.4 wMSO

MSO with only finite set quanitification

- ω-tree languages

7.5 S1S

MSO over $(\mathbb{N}, +1, <, 0)$

- ω-languages
- Game winning conditions

$\mathbf{S1S_0}$

S1S - element variables + $(X \subseteq Y \mid \mathrm{Sing}(X) \mid \mathrm{Succ}(X))$

7.6 S2S

MSO over $\underline{T}_2 = (\{0,1\}^*, \varepsilon, S_0, S_1)$

- ω-tree languages

$\mathbf{S2S_0}$

S2S - element variables + $(X \subseteq Y \mid \mathrm{Sing}(X) \mid \mathrm{Succ}_0(X) \mid \mathrm{Succ}_1(X))$