0.1 Fritz & Wilke

0.1.1 Delayed Simulation Game

In this section we consider delayed simulation games and variants thereof on DPAs. This approach is based on the paper [] which considered the games for alternating parity automata. The DPAs we use are a special case of these APAs and therefore worth examining.

Definition 0.1.1. We define $\leq_{\checkmark} \subseteq (\mathbb{N} \cup \{\checkmark\}) \times (\mathbb{N} \cup \{\checkmark\})$ as follows:

- For $i, j \in \mathbb{N}$, we set $i \leq_{\checkmark} j$ iff $i \leq j$.
- For all $i \in \mathbb{N}$, we have $i \leq_{\checkmark} \checkmark$ and $\checkmark \not\leq_{\checkmark} i$.
- $\checkmark \leq_{\checkmark} \checkmark$.

Further, we define an order of "goodness" on parity priorities as $\preceq_{\mathrm{p}} \subseteq \mathbb{N} \times \mathbb{N}$ as $0 \preceq_{\mathrm{p}} 2 \preceq_{\mathrm{p}} 4 \preceq_{\mathrm{p}} \cdots \preceq_{\mathrm{p}} 5 \preceq_{\mathrm{p}} 3 \preceq_{\mathrm{p}} 1$.

Definition 0.1.2. Let $\mathcal{A} = (Q, \Sigma, q_0, \delta, c)$ be a DPA. We define the *delayed simulation automaton* $\mathcal{A}_{\mathrm{de}}(p, q) = (Q_{\mathrm{de}}, \Sigma, (p, q, \gamma(c(p), c(q), \checkmark)), \delta_{\mathrm{de}}, F_{\mathrm{de}})$, which is a deterministic Büchi automaton, as follows.

- $Q_{\mathrm{de}} = Q \times Q \times (\mathrm{img}(c) \cup \{\checkmark\})$, i.e. the states are given as triples in which the first two components are states from $\mathcal{A}$ and the third component is either a priority from $\mathcal{A}$ or $\checkmark$.
- The alphabet remains Σ.
- The starting state is a triple $(p, q, \gamma(c(p), c(q), \checkmark))$, where $p, q \in Q$ are parameters given to the automaton, and γ is defined below.
- $\delta_{\mathrm{de}}((p, q, i), a) = (p', q', \gamma(i, c(p'), c(q')))$, where $p' = \delta(p, a)$, $q' = \delta(q, a)$, and γ is the same function as used in the initial state. The first two components behave like a regular product automaton.
- $F_{\mathrm{de}} = Q \times Q \times \{\checkmark\}$.

$\gamma : \mathbb{N} \times \mathbb{N} \times (\mathbb{N} \cup \{\checkmark\}) \to \mathbb{N} \cup \{\checkmark\}$ is the update function of the third component and defines the "obligations" as they are called in []. It is defined as

$$\gamma(i, j, k) = \begin{cases} \checkmark & \text{if } i \text{ is odd and } i \leq_{\checkmark} k \text{ and } j \preceq_{\mathrm{p}} i \\ \checkmark & \text{if } j \text{ is even and } j \leq_{\checkmark} k \text{ and } j \preceq_{\mathrm{p}} i \\ \min_{\leq_{\checkmark}} \{i, j, k\} & \text{else} \end{cases}$$

Definition 0.1.3. Let $\mathcal{A}$ be a DPA and let $\mathcal{A}_{\mathrm{de}}$ be the delayed simulation automaton of $\mathcal{A}$. We say that a state p *de*-simulates a state q if $L(\mathcal{A}_{\mathrm{de}}(p, q)) = \Sigma^{\omega}$. In that case we write $p \leq_{\mathrm{de}} q$. If also $q \leq_{\mathrm{de}} p$ holds, we write $p \equiv_{\mathrm{de}} q$.

$\equiv_{\text{de}}$ is a congruence relation.

Our overall goal is to use $\equiv_{\text{de}}$ to build a quotient automaton of our original DPA. The first step towards this goal is to show that the result is actually a well-defined DPA, by proving that the relation is a congruence.

Lemma 0.1.1. *γ is monotonous in the third component, i.e. if $k \leq_{\checkmark} k'$, then $\gamma(i,j,k) \leq_{\checkmark} \gamma(i,j,k')$ for all $i,j \in \mathbb{N}$.*

Proof. We consider each case in the definition of γ. If i is odd, $i \leq_{\checkmark} k$ and $j \preceq_{\text{p}} i$, then also $i \leq_{\checkmark} k'$ and $\gamma(i,j,k) = \gamma(i,j,k') = \checkmark$.

If j is even, $j \leq_{\checkmark} k$ and $j \preceq_{\text{p}} i$, then also $j \leq_{\checkmark} k'$ and $\gamma(i,j,k) = \gamma(i,j,k') = \checkmark$.

Otherwise, $\gamma(i,j,k) = \min\{i,j,k\}$ and $\gamma(i,j,k') = \min\{i,j,k'\}$. Since $k \leq_{\checkmark} k'$, $\gamma(i,j,k) \leq_{\checkmark} \gamma(i,j,k')$. □

Lemma 0.1.2. *Let $\mathcal{A}$ be a DPA and let $p,q \in Q$, $k \in \mathbb{N} \cup \{\checkmark\}$. If the run of $\mathcal{A}_{\text{de}}$ starting at (p,q,k) on some $\alpha \in \Sigma^\omega$ is accepting, then for all $k \leq_{\checkmark} k'$ also the run of $\mathcal{A}_{\text{de}}$ starting at (p,q,k') on α is accepting.*

Proof. Let ρ be the run starting at (p,q,k) and let ρ' be the run starting at (p,q,k'). Further, let p_i, q_i, k_i, and k'_i be the components of the states of those runs in the i-th step. Via induction we show that $k_i \leq_{\checkmark} k'_i$ for all i. Since k_i is $\checkmark$ infinitely often, the same must be true for k'_i and ρ' is accepting.

For $i = 0$, we have $k_0 = k \leq_{\checkmark} k' = k'_0$. Otherwise, we have $k_{i+1} = \gamma(c(p_{i+1}), c(q_{i+1}), k_i)$ and k'_{i+1} analogously. The rest follows from Lemma **??**. □

Lemma 0.1.3. *Let $\mathcal{A}$ be a DPA and $\rho \in Q_{\text{de}}^\omega$ be a run of $\mathcal{A}_{\text{de}}$ on some word, where the third component is $k \in (\mathbb{N} \cup \{\checkmark\})^\omega$. For all i, $k(i+1) \leq_{\checkmark} k(i)$ or $k(i+1) = \checkmark$.*

Proof. Follows directly from the definition of γ. □

Lemma 0.1.4. *Let $\mathcal{A}$ be a DPA and ρ be a run of $\mathcal{A}_{\text{de}}$ on some word α with $\rho(i) = (p(i), q(i), k(i))$. Let $m < n$ be two distinct positions such that $k(m) = k(n) = \checkmark$ and at no positition inbetween does k become $\checkmark$. Then for all $m < i \leq n$, $c(q(n)) \preceq_p c(p(i))$ and $c(q(n)) \preceq_p c(q(i))$.*

Proof. □

Lemma 0.1.5. *Let $\mathcal{A}$ be a DPA. Then $\leq_{\text{de}}$ is reflexive and transitive.*

Proof. For reflexivitiy, we need to show that $q \leq_{\text{de}} q$ for all states q. This is rather easy to see. For a word $\alpha \in \Sigma^\omega$, the third component of states in the run of $\mathcal{A}_{\text{de}}(q,q)$ on α is always $\checkmark$, as $\gamma(i,i,\checkmark) = \checkmark$.

For transitivity, let $q_1 \leq_{\text{de}} q_2$ and $q_2 \leq_{\text{de}} q_3$. Assume towards a contradiction that $q_1 \not\leq_{\text{de}} q_3$, so there is a word $\alpha \notin L(\mathcal{A}_{\text{de}}(q_1,q_3))$. We consider the three runs ρ_{12}, ρ_{23}, and ρ_{13} of $\mathcal{A}_{\text{de}}(q_1,q_2)$, $\mathcal{A}_{\text{de}}(q_2,q_3)$, anbd $\mathcal{A}_{\text{de}}(q_1,q_3)$ respectively on α. Then ρ_{12} and ρ_{23} are accepting, whereas ρ_{13} is not.

Moreover, we use the notation $q_1(i), q_2(i), q_3(i)$ for the states of the run and $k_{12}(i), k_{23}(i), k_{13}(i)$ for the obligations. More specifically for a run ρ_{ij}, it is true that $\rho_{ij}(n) = (q_i(n), q_j(n), k_{ij}(n))$.

As ρ_{13} is not accepting, k_{13} becomes $\checkmark$ only finitely often. By Lemma **??**, that means k_{13} only grows smaller from some point on and reaches a minimum eventually. Let $n_0 \in \mathbb{N}$ be such a position from which on k_{13} does not change anymore. We split the rest of the proof into two cases, depending on which priority from the following two states caused the change to $k_{13}(n_0)$.

Case 1: $k_{13}(n_0) = c(q_1(n_0))$**.** We know $c(q_1(n_0)) < c(q_3(n_0))$ and $c(q_1(n_0)) \prec_p c(q_3(n_0))$, so $c(q_1(n_0))$ must be even. Let $n_1 \geq n_0$ be the smallest position such that $k_{12}(n_1) = \checkmark$. By Lemma **??**, $c(q_2(n_1)) \preceq_p c(q_1(n_0))$.

Let $n_2 \geq n_1$ be the smallest position such that $k_{23} = \checkmark$. Again by lemma **??**, $c(q_3(n_2)) \preceq_p c(q_2(n_1)) \preceq_p c(q_1(n_0))$. As $c(q_1(n_0))$ is even, so must be $c(q_3(n_2))$. From these facts and the definition of γ we can deduce

$$\begin{aligned}
&k_{13}(n_2) \\
&= \gamma(c(q_1(n_2), c(q_3(n_2)), k_{13}(n_0)) \\
&= \begin{cases} \checkmark & \text{if } c(q_3(n_2)) \leq_\checkmark k_{13}(n_0) \\ \min_{\leq_\checkmark}\{c(q_1(n_2), c(q_3(n_2)), k_{13}(n_0)\} & \text{else} \end{cases}
\end{aligned}$$

Finally, $k_{13}(n_0)$ is the same as $c(q_1(n_0))$, so $c(q_3(n_2)) \leq_\checkmark k_{13}(n_0)$ as both values are even. Thus, $k_{13}(n_2) = \checkmark$ and $n_2 \geq n_0$ which contradicts our choice of n_0.

Case 2: $k_{13}(n_0) = c(q_3(n_0))$**.** We know $c(q_3(n_0)) < c(q_1(n_0))$ and $c(q_1(n_0)) \prec_p c(q_3(n_0))$, so $c(q_3(n_0))$ is odd. □

Lemma 0.1.6. *Let $\mathcal{A}$ be a DPA. Then $\equiv_{\text{de}}$ is a congruence relation.*

Proof. The three properties that are required for $\equiv_{\text{de}}$ to be a equivalence relation are rather easy to see. Reflexivity and transitivity have been shown for $\leq_{\text{de}}$ already and symmetry follows from the definition. Congruence requires more elaboration.

Let $p \equiv_{\text{de}} q$ be two equivalent states. Let $a \in \Sigma$ and $p' = \delta(p, a)$ and $q' = \delta(q, a)$. We have to show that also $p' \equiv_{\text{de}} q'$. Towards a contradiction, assume that $p' \not\leq_{\text{de}} q'$, so there is a word $\alpha \notin L(\mathcal{A}_{\text{de}}(p', q'))$. Let $(p', q', k) = \delta_{\text{de}}((p, q, \checkmark), a)$. By Lemma **??**, the run of $\mathcal{A}_{\text{de}}$ on α from (p', q', k) cannot be accepting; otherwise, the run of $\mathcal{A}_{\text{de}}$ from $(p', q', \checkmark)$ would be accepting and $\alpha \in L(\mathcal{A}_{\text{de}}(p', q'))$. Hence, $a\alpha \notin L(\mathcal{A}_{\text{de}}(p, q))$, which means that $p \not\equiv_{\text{de}} q$. □

Corollary 0.1.7. *Let $\mathcal{A}$ be a DPA and $\equiv_{\text{de}}$ the corresponding delayed simulation-relation. The quotient automaton $\mathcal{A}/_{\equiv_{\text{de}}}$ is well-defined and deterministic.*