State Space Reduction For Parity Automata

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Overview

Reduce the number of states in a given deterministic parity automaton while keeping the recognized language.

1. Deterministic Parity Automata
2. Why do we need heuristic reduction?
3. Merger functions as a framework
4. Delayed Simulation
5. Congruence Path Refinement
6. Efficiency

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ω-automata

ω-words are words of one-sided infinite length:
Σ^ω = functions from $\mathbb{N}$ to Σ

ω-automata are finite transition structures that describe a language $L \subseteq \Sigma^\omega$

Deterministic parity automata (DPA):

- State set Q
- Alphabet Σ
- Transition function $\delta : Q \times \Sigma \to Q$
- Priority function $c : Q \to \mathbb{N}$

An ω-word α starting in a state $q_0 \in Q$ induces a run $q_0 q_1 q_2 \ldots$.
The DPA accepts α iff the **smallest** priority that occurs infinitely often in the sequence $c(q_0)c(q_1)c(q_2)\ldots$ is **even**.

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Why do we need heuristic reduction?

Goal: Reduce number of states in the automaton to ease run time of follow up algorithms.

Minimization Problem: Given an automaton $\mathcal{A}$, what is the smallest number of states required to recognize the same language as $\mathcal{A}$?

For DFAs: Minimization is solvable in $\mathcal{O}(n \log n)$. [Hopcroft, 1971]
For DPAs: Minimization is NP-hard. [Schewe, 2010]

Moore Minimization

A DPA can be interpreted as a Moore automaton with c being the output function.

Definition.

$p \equiv_M q$ iff $\forall w \in \Sigma^* : c(\delta^*(p, w)) = c(\delta^*(q, w))$.

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Theorem.

Deterministic Moore automata can be minimized in log-linear time.

Idea: Build the quotient automaton w.r.t. $\equiv_M$.

The same algorithm can be used to reduce DPAs but will not give minimal DPAs in general.

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Merger functions

Merger functions μ map from some $D \subseteq 2^Q$ into $2^Q \setminus \{\emptyset\}$.

$M, C \subseteq Q$

$$\mu(\ M\ ) =\ C$$

All states from the **merge set** . . .

. . . can be represented by any single one representative from the **candidate set**.

Merger functions generalize quotient automata

Special case: $\mu(M) = M$.
Remove all states from M except for one (arbitrarily chosen) representative.

For a congruence relation $\sim$, let $\mathfrak{C} \subseteq 2^Q$ be the equivalence classes.
The quotient automaton is defined by state set $\mathfrak{C}$.
This is captured by the merger function $\mu_{\div} : \mathfrak{C} \to 2^Q, \kappa \mapsto \kappa$.

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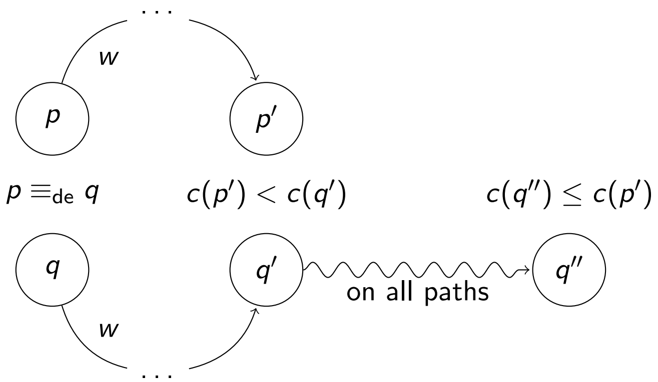
Delayed Simulation

Definition.

$p \equiv_{\mathrm{de}} q$ iff for all $w \in \Sigma^*$, every run that starts in $\delta^*(p, w)$ or $\delta^*(q, w)$ eventually sees a priority of at most $\min\{c(\delta^*(p, w)), c(\delta^*(q, w))\}$.

Delayed Simulation



Delayed Simulation

Definition.

Let $\mathfrak{C}_{\text{de}} = \{[q]_{\equiv_{\text{de}}} \mid q \in Q\}$ be the set of $\equiv_{\text{de}}$-equivalence classes.
Define the **delayed simulation merger** as

$$\mu_{\text{de}} : \mathfrak{C}_{\text{de}} \to 2^Q, \kappa \mapsto \{q \in \kappa \mid c(q) = \min c(\kappa)\}.$$

Theorem.

Merging states according to μ_{de} preserves language.

Computing Delayed Simulation

We define a det. Büchi automaton $\mathcal{G}_{\mathrm{de}}$ with states $q^0_{\mathrm{de}}(p,q)$ such that:
$p \equiv_{\mathrm{de}} q$ iff both $L(\mathcal{G}_{\mathrm{de}}, q^0_{\mathrm{de}}(p,q))$ and $L(\mathcal{G}_{\mathrm{de}}, q^0_{\mathrm{de}}(q,p))$ are universal (Σ^ω).

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$\mathcal{G}_{\text{de}} = (Q_{\text{de}}, \Sigma, \delta_{\text{de}}, F_{\text{de}})$

- States are $Q_{\text{de}} = Q \times Q \times (c(Q) \cup \{\checkmark\})$.
  The first two components are a "simulation" of the original DPA.
  The third component are the so called "obligations".
- Accepting states are $F_{\text{de}} = Q \times Q \times \{\checkmark\}$.
- Transitions δ_{de}.

$$\begin{aligned}\delta_{\text{de}}((p, q, k), a) = (&\delta(p, a),\\ &\delta(q, a),\\ &\gamma(\quad c(\delta(p, a)), \quad c(\delta(q, a)), \quad k \quad))\end{aligned}$$

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Delayed Simulation Automaton: γ

(Actual definition of γ is more complex for some additional properties.)

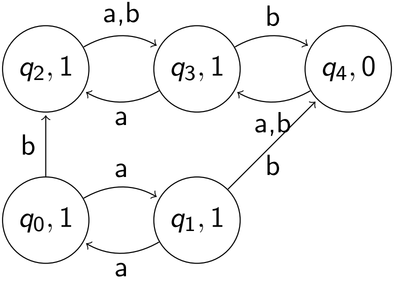
$\gamma : \mathbb{N} \times \mathbb{N} \times (\mathbb{N} \cup \{\checkmark\}) \to \mathbb{N} \cup \{\checkmark\}$

$$\gamma(i, j, \checkmark) = \begin{cases} \checkmark & \text{if } j \leq i \\ i & \text{else} \end{cases}$$

$$\text{for } k \in \mathbb{N}: \quad \gamma(i, j, k) = \begin{cases} \checkmark & \text{if } j \leq \min\{i, k\} \\ \min\{i, k\} & \text{else} \end{cases}$$

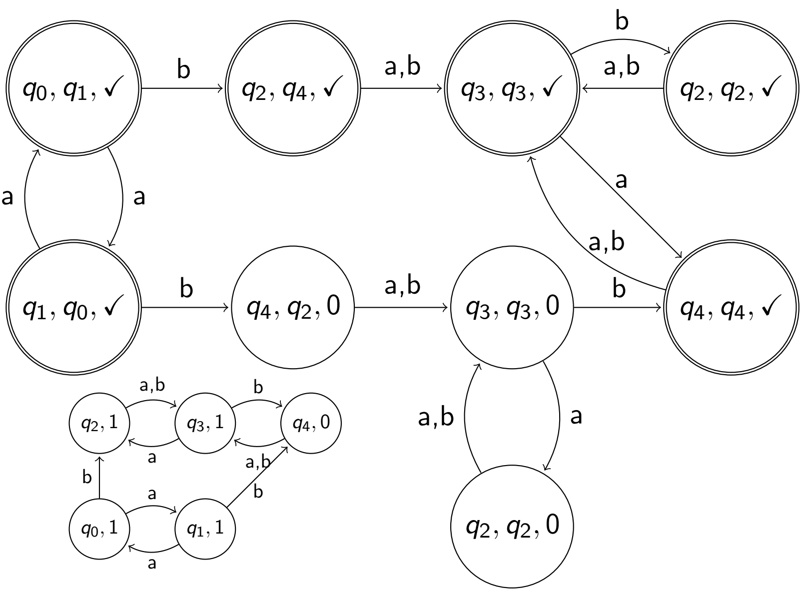
$q^0_{\mathrm{de}}(p, q) = (p, q, \gamma(c(p), c(q), \checkmark))$.

Delayed Simulation Automaton



A DPA with 5 states. We want to check whether $q_0 \equiv_{\text{de}} q_1$ is true.

Delayed Simulation Automaton



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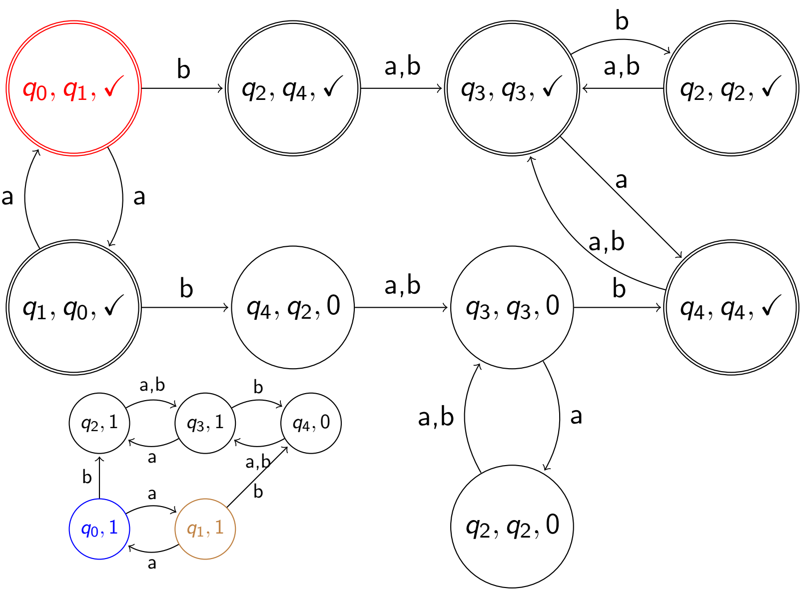

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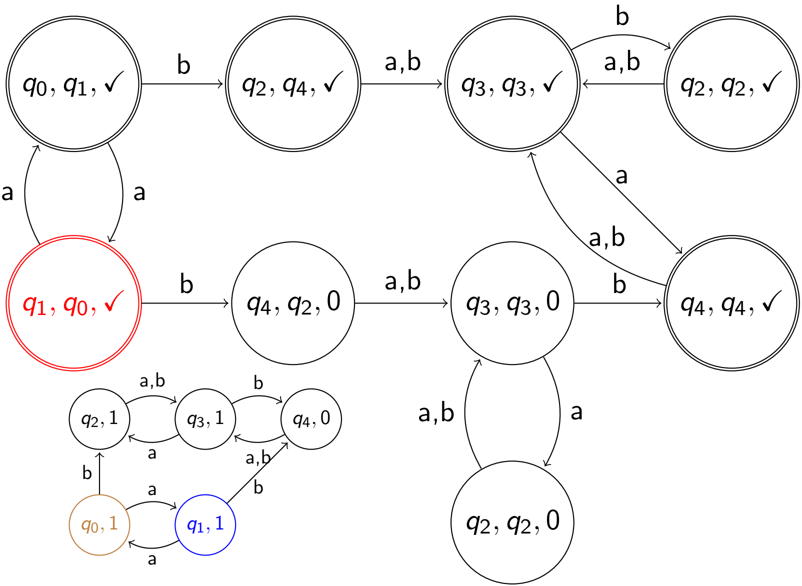

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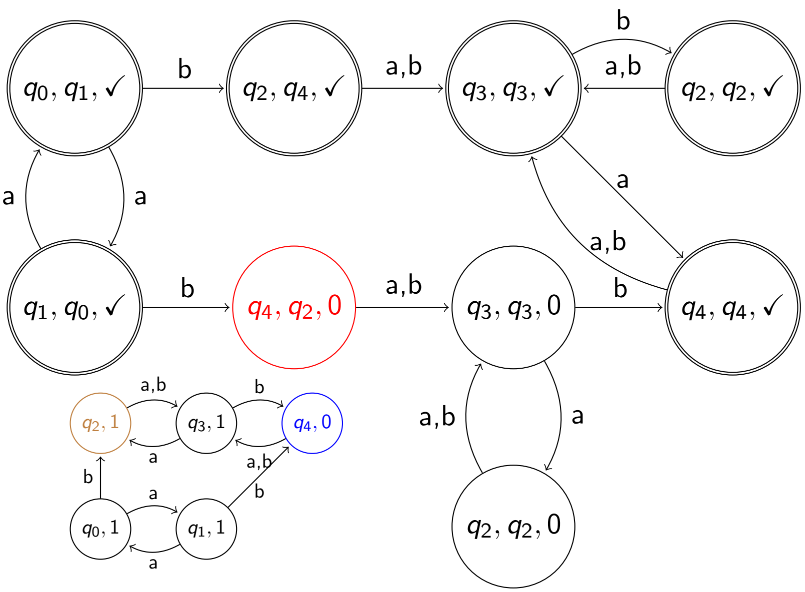

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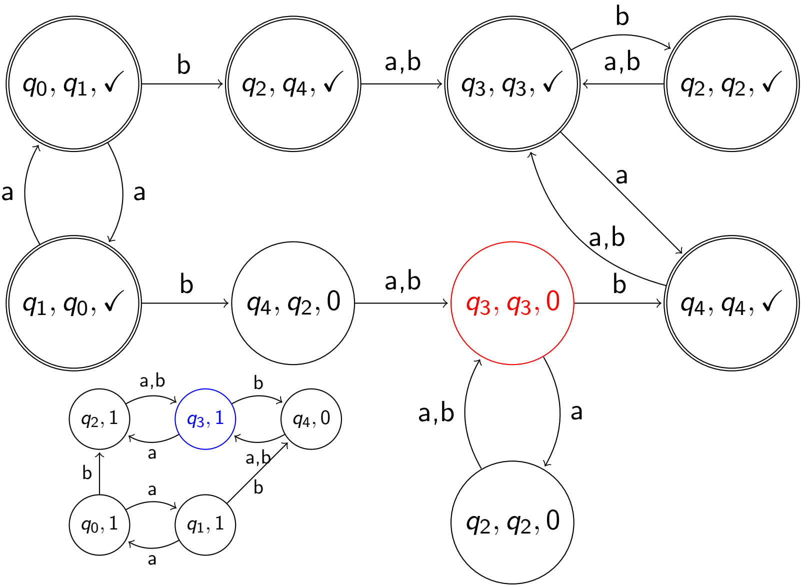

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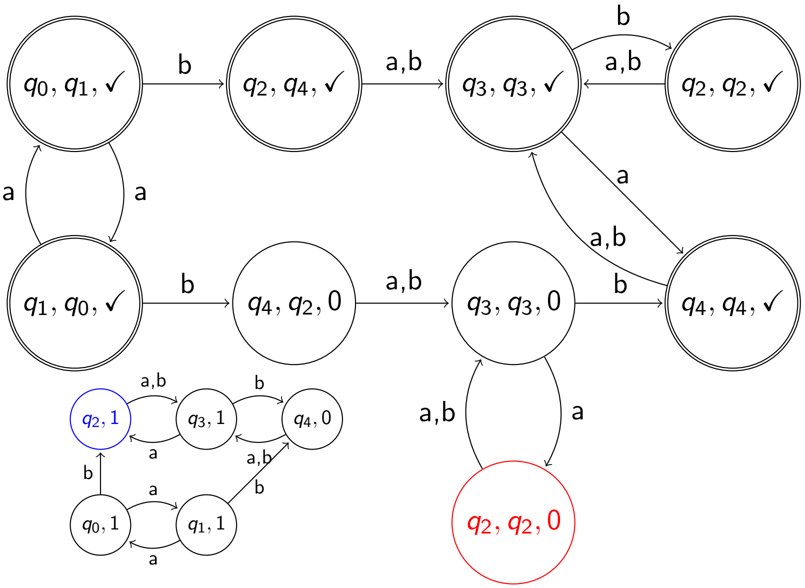

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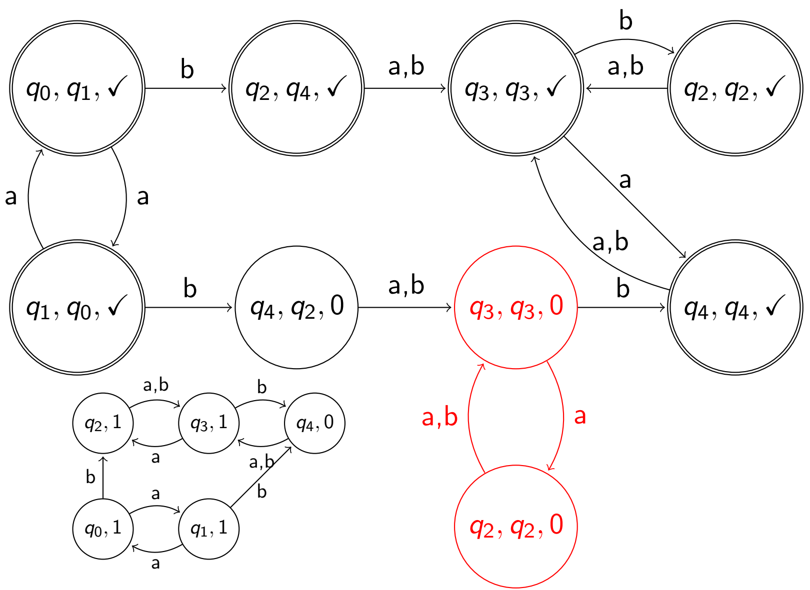

Complexity

$\mathcal{G}_{\text{de}}$ uses the state set $Q_{\text{de}} = Q \times Q \times (c(Q) \cup \{\checkmark\})$.
Computing states of universal language in a DBA requires linear time.

Theorem.

μ_{de} can be computed in $\mathcal{O}(n^2 k)$.

$n = |Q|$, $k = |c(Q)|$

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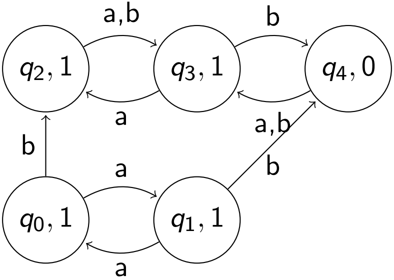
Congruence Path Refinement

Idea: take a given relation and refine it until states can be merged.

Definition.

Let $\sim$ be a congruence relation and let $\lambda \subseteq Q$ be an equiv. class of $\sim$. We define $L_{\lambda\hookleftarrow} \subseteq \Sigma^*$ as the set of all words such that the induced run from a state in λ moves back to λ exactly once and ends there.

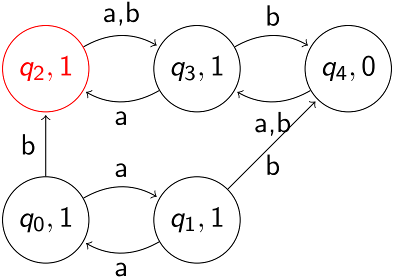
Congruence Path Refinement



$\lambda = \{q_2, q_4\}$
Because $\sim$ is a congruence relation, we only need to consider one state.

$L_{\lambda \hookleftarrow} = \{$

Congruence Path Refinement



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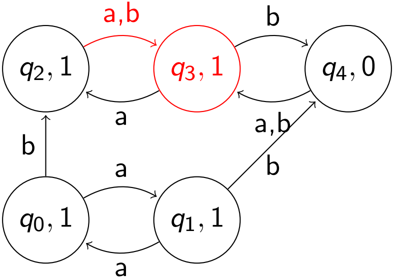

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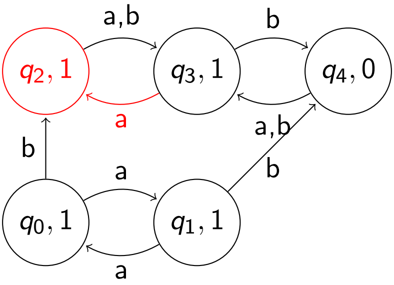
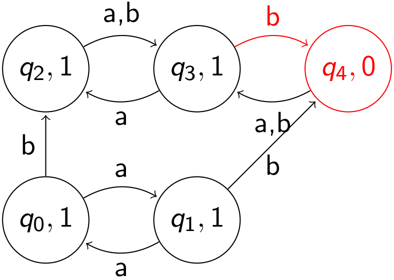


$\lambda = \{q_2, q_4\}$

$L_{\lambda\hookleftarrow} = \{aa, ba$

Congruence Path Refinement



$\lambda = \{q_2, q_4\}$

$L_{\lambda\hookleftarrow} = \{aa, ba, ab, bb\}$

Congruence Path Refinement

Definition.

The **path refinement** equivalence $\equiv_{\mathrm{PR}}^{\lambda}$ is the **largest relation** s.t.:
For $p, q \in \lambda$, $p \equiv_{\mathrm{PR}}^{\lambda} q$ if and only if

- $\forall w \in L_{\lambda\hookleftarrow}$: the smallest priority seen when reading w is the same from p and from q.
- $\forall w \in L_{\lambda\hookleftarrow} : \delta^*(p, w) \equiv_{\mathrm{PR}}^{\lambda} \delta^*(q, w)$

First point: Makes sure path segments have the same acceptance.
Second point: Makes sure the argument can be applied repeatedly.

Path Refinement Merger

Definition.

Let $\mathfrak{C}^{\lambda}_{\mathsf{PR}} = \{[q]_{\equiv^{\lambda}_{\mathsf{PR}}} \mid q \in Q\}$ be the set of $\equiv^{\lambda}_{\mathsf{PR}}$-equivalence classes. Define the **path refinement merger** as
$\mu^{\lambda}_{\mathsf{PR}} : \mathfrak{C}^{\lambda}_{\mathsf{PR}} \to 2^Q, \kappa \mapsto \{q \in \kappa \mid c(q) = \min c(\kappa)\}$.

Theorem.

If all states in λ are pairwise language equivalent, merging states according to $\mu^{\lambda}_{\mathsf{PR}}$ preserves language.

Computing Path Refinement

Define a DPA that relates Moore equivalence to $\equiv_{\mathsf{PR}}$.

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Definition.

Define the **visit graph** DPA $\mathcal{A}^{\lambda}_{\text{visit}} = (Q^{\lambda}_{\text{visit}}, \Sigma, \delta^{\lambda}_{\text{visit}}, c^{\lambda}_{\text{visit}})$.

- $Q^{\lambda}_{\text{visit}} = Q \times c(Q) \times (c(Q) \cup \{-1\})$
- $\delta^{\lambda}_{\text{visit}}((q,k,k'),a) = \begin{cases} (q', \min\{k, c(q')\}, -1) & \text{if } q' \notin \lambda \\ (q', c(q'), \min\{k, c(q')\}) & \text{if } q' \in \lambda \end{cases}$,
  where $q' = \delta(q,a)$.
- $c^{\lambda}_{\text{visit}}((q,k,k')) = k'$.

The first component "simulates" the original automaton $\mathcal{A}$.
The second component tracks the minimal priority seen on one run from λ to λ.
The third component is required to distinguish the different priorities.

Computing Path Refinement

Define a DPA that relates Moore equivalence to $\equiv_{\mathrm{PR}}$.

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Computing Path Refinement

Definition.

For $q \in Q$, we set $\iota_q := (q, c(q), \max c(Q)) \in Q^{\lambda}_{\text{visit}}$.

Theorem.

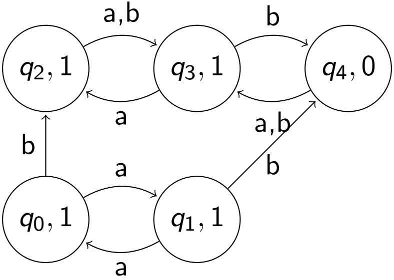
$p \equiv^{\lambda}_{PR} q$ iff $\iota_p \equiv_M \iota_q$.

Theorem.

$\equiv^{\lambda}_{PR}$ can be computed in $\mathcal{O}(k^2 n \log n)$.

$n = |Q|$, $k = |c(Q)|$

Visit Graph



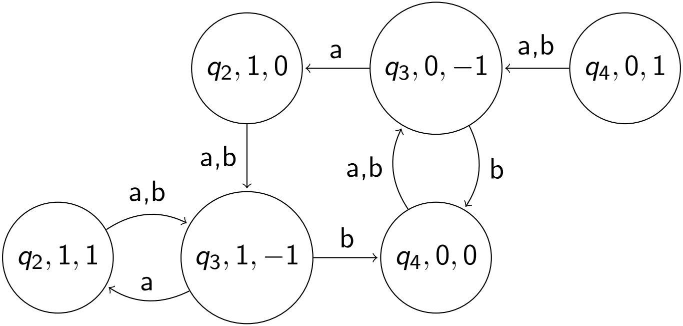
Potential choices for λ are the equivalence classes of $\equiv_L$:
$\{q_0, q_1\}$, $\{q_2, q_4\}$, or $\{q_3\}$.

We take $\lambda = \{q_2, q_4\}$ and ask if $q_2 \equiv^{\lambda}_{\mathrm{PR}} q_4$ is true.

Visit Graph

$\mathcal{A}_{\text{visit}}^{\{q_2,q_4\}}$

$\iota_{q_2} = (q_2, 1, 1)$ $\iota_{q_4} = (q_4, 0, 1)$ Question: $\iota_{q_2} \equiv_M \iota_{q_4}$?



(Reminder: the third component defines the color of a state)

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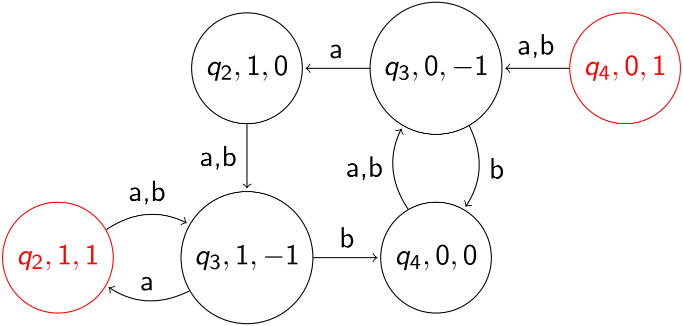

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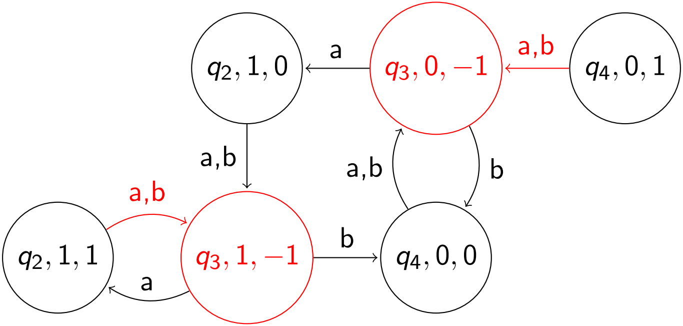

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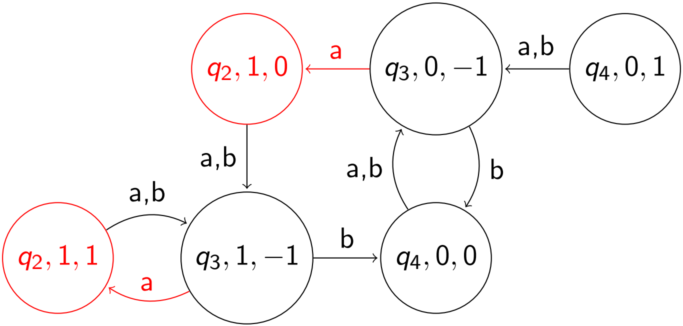

q_2 and q_4 are not PR-equivalent.
From q_4 we can move to q_2 with min. priority 0. From q_2 we can move to q_2 with min. priority 1.

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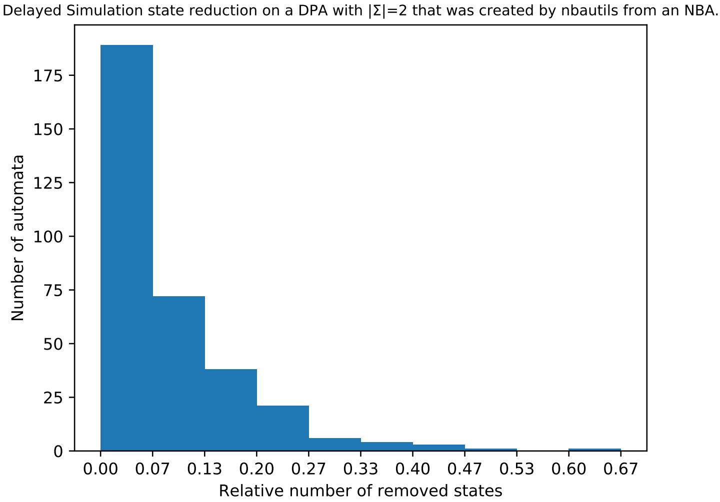
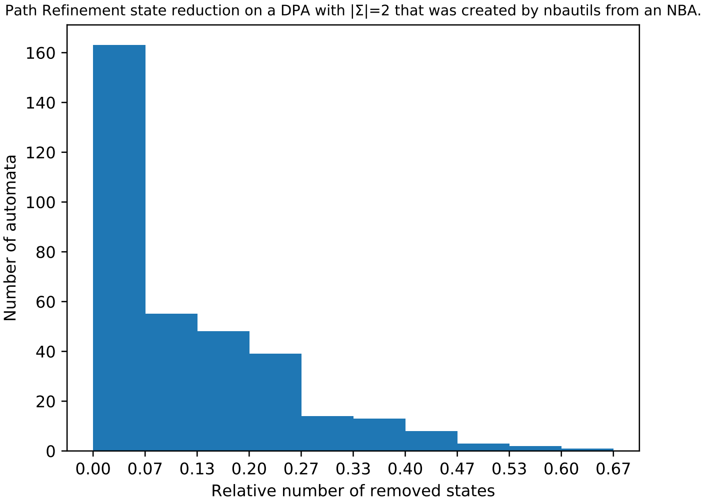
Delayed Simulation



Path Refinement



Summary

- Moore
- Skip
- Delayed Simulation
- Iterated Moore
- Path Refinement
- Threshold Moore
- LSF

$\Rightarrow$ Merger function $\Rightarrow$

- Representative merge
- Schewe merge

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