0.1 Introduction

Finite automata are a long established computation model that dates back to sources such as [11] and [14]. A known problem for finite automata is state space reduction, referring to the search of a language-equivalent automaton which uses fewer states than the original object. For deterministic finite automata (DFA), not just reduction but minimization was solved in [8]. Regarding nondeterministic finite automata (NFA), [9] proved the PSPACE-completeness of the minimization problem, which is why reduction algorithms such as [4] and [1] are a popular alternative.

In his prominent work [2], Büchi introduced the model of Büchi automata (BA) as an extension of finite automata to read words of one-sided infinite length. As these ω-automata tend to have higher levels of complexity in comparison to standard finite automata, the potential gain of state space reduction is even greater. Similar to NFAs, exact minimization for deterministic Büchi automata was shown to be NP-complete in [15] and spawned heuristic approaches such as [15], [10], or [5].

As [17] displays, deterministic Büchi automata are a strictly weaker model than nondeterministic Büchi automata. It is therefore interesting to consider different models of ω-automata in which determinism is possible while maintaining enough power to describe all ω-regular languages. Parity automata (PA) are one such model, a mixture of Büchi automata and Moore automata ([12]), that use a parity function rather than the usual acceptance set. [13] showed that deterministic parity automata are in fact sufficient to recognize all ω-regular languages. As for DBAs, the exact minimization problem for DPAs is NP-complete ([15]).

Our goal in this publication is to develop new algorithms for state space reduction of DPAs, partially adapted from existing algorithms for Büchi or Moore automata. We perform theoretical analysis of the algorithms in the form of proofs of correctness and analysis of run time complexity, as well as practical implementation of the algorithms in code to provide empirical data for or against their actual efficiency.

Chapter 1

Theory

1.1 General Results

We first use this section to establish some general results that are used multiple times in the upcoming proofs.

1.1.1 Equivalence Relations

In general, we use the symbol $\equiv$ to denote equivalence relations, mostly between states of an automata. In general, we have automata $\mathcal{A}$ and $\mathcal{B}$ with states p and q from there respective state spaces. Our relations are then defined on $(\mathcal{A}, p) \equiv (\mathcal{B}, q)$.

Definition 1.1.1. Assuming that $\mathcal{A}$ is a fixed automaton that is obvious in context and p and q are both states in $\mathcal{A}$, we shorten $(\mathcal{A}, p) \equiv (\mathcal{A}, q)$ to $p \equiv q$.

Furthermore, we write $\mathcal{A} \equiv \mathcal{B}$ if for every p in $\mathcal{A}$ there is a q in $\mathcal{B}$ such that $(\mathcal{A}, p) \equiv (\mathcal{B}, q)$; and the same holds with $\mathcal{A}$ and $\mathcal{B}$ exchanged.

Definition 1.1.2. Let $\mathcal{A} = (Q_1, \Sigma, \delta_1)$ and $\mathcal{B} = (Q_2, \Sigma, \delta_2)$ be deterministic transition structures and let $\sim$ be an equivalence relation. We call $\sim$ a *congruence relation* if for all $(\mathcal{A}, p) \sim (\mathcal{B}, q)$ and all $a \in \Sigma$, also $(\mathcal{A}, \delta_1(p, a)) \sim (\mathcal{B}, \delta_2(q, a))$.

Definition 1.1.3. Let $\sim$ be an equivalence relation. We define the *congruence refinement* of $\sim$ as $\approx$ with $(\mathcal{A}, p) \approx (\mathcal{B}, q)$ iff for all $w \in \Sigma^*$, $(\mathcal{A}, \delta_1^*(p, w)) \sim (\mathcal{B}, \delta_2(q, w))$.

Lemma 1.1.1. *For a given equivalence relation $\sim$, the congruence refinement of $\sim$ is a congruence relation.*

Proof. □

Lemma 1.1.2. *For a given equivalence relation $\sim$, the congruence refinement on a single automaton $\mathcal{A}$ can be computed in $\mathcal{O}(|\mathcal{A}| \cdot \log |\mathcal{A}|)$.*

Proof. We refer to [8], which describes a special case where $p \sim q$ iff $c(p) = c(q)$. □

The following is a comprehensive list of all relevant equivalence relations that we use.

- Language equivalence, $\equiv_L$. Defined below.
- Moore equivalence, $\equiv_M$. Defined below.
- Priority almost equivalence, $\equiv_\dagger$. Defined below.
- Delayed simulation equivalence, $\equiv_{\mathrm{de}}$. Defined in section **??**.
- Delayed simulation equivalence with resets, $\equiv_{\mathrm{deR}}$. Defined in section **??**.
- Iterated Moore equivalence, $\equiv_{\mathrm{IM}}$. Defined in section **??**.
- Path refinement equivalence, $\equiv_{\mathrm{PR}}$. Defined in section **??**.
- Threshold Moore equivalence, $\equiv_{\mathrm{TM}}^{\sim}$. Defined in section **??**.
- Labeled SCC filter equivalence, $\equiv_{\mathrm{LSF}}^{k,\sim}$. Defined in section **??**.

Immediately we define the three first of these relations and show that they are computable.

Language Equivalence

Definition 1.1.4. Let $\mathcal{A}$ and $\mathcal{B}$ be ω-automata. We define *language equivalence* as $(\mathcal{A}, p) \equiv_L (\mathcal{B}, q)$ if and only if for all words $\alpha \in \Sigma^\omega$, $\mathcal{A}$ accepts α from p iff $\mathcal{B}$ accepts α from q.

Lemma 1.1.3. *$\equiv_L$ is a congruence relation.*

Proof. It is obvious that $\equiv_L$ is an equivalence relation. For two states $(\mathcal{A}, p) \equiv_L (\mathcal{B}, q)$ and some successors $p' = \delta_1(p, a)$ and $q' = \delta_2(q, a)$, it must be true that $(\mathcal{A}, p') \equiv_L (\mathcal{B}, q')$. Otherwise there is a word $\alpha \in \Sigma^\omega$ that is accepted from p' and rejected from q' (or vice-versa). Then $a \cdot \alpha$ is rejected from p and accepted from q and thus $p \not\equiv_L q$. □

Lemma 1.1.4. *Language equivalence of a given DPA can be computed in $\mathcal{O}(|Q|^2 \cdot |c(Q)|^2)$.*

Proof. The algorithm is based partially on [7].

Let $\mathcal{A} = (Q, \Sigma, \delta, c)$ be the DPA that we want to compute $\equiv_L$ on. We construct a labeled deterministic transition structure $\mathcal{B} = (Q \times Q, \Sigma, \delta', d)$ with $\delta'((p_1, p_2), a) = (\delta(p_1, a), \delta(p_2, a))$ and $d((p_1, p_2)) = (c(p_1), c(p_2)) \in \mathbb{N}^2$. Then, for every $i, j \in c(Q)$, let $\mathcal{B}_{i,j} = \mathcal{B} \restriction_{Q_{i,j}}$ with $Q_{i,j} = \{(p_1, p_2) \in Q \times Q \mid c(p_1) \geq i, c(p_2) \geq j\}$, i.e. remove all states which have first priority less than i or second priority less than j.

For each i and j, let $S_{i,j} \subseteq 2^{Q \times Q}$ be the set of all SCCs in $\mathcal{B}_{i,j}$ and let $S = \bigcup_{i,j} S_{i,j}$. From this set S, remove all SCCs $s \subseteq Q \times Q$ in which the parity of the smallest priority in the first component differs from the parity of the smallest priority in the second component. The "filtered" set we call S'. For any two states $p, q \in Q$, $p \not\equiv_L q$ iff there is a pair $(p', q') \in \bigcup S'$ that is reachable from (p, q) in $\mathcal{B}$.

We omit the correctness proof of the algorithm here. Regarding the runtime, observe that $\mathcal{B}$ has size $\mathcal{O}(|Q|^2)$ and we create $\mathcal{O}(|c(Q)|^2)$ copies of it. All other steps like computing the SCCs can then be done in linear time in the size of the automata, which brings the total to $\mathcal{O}(|Q|^2 \cdot |c(Q)|^2)$. □

Priority Almost Equivalence

Definition 1.1.5. Let $\mathcal{A} = (Q_1, \Sigma, \delta_1, c_1)$ and $\mathcal{B} = (Q_2, \Sigma, \delta_2, c_2)$ be DPAs. We define *priority almost equivalence* as $(\mathcal{A}, p) \equiv_{\dagger} (\mathcal{B}, q)$ if and only if for all words $\alpha \in \Sigma^\omega$, $c_1^*(p, \alpha)$ and $c_2^*(q, \alpha)$ differ at only finitely many positions.

Lemma 1.1.5. *Priority almost equivalence is a congruence relation.*

Proof. It is obvious that $\equiv_{\dagger}$ is an equivalence relation. For two states $(\mathcal{A}, p) \equiv_{\dagger} (\mathcal{B}, q)$ and some successors $p' = \delta(p, a)$ and $q' = \delta(q, a)$, it must be true that $(\mathcal{A}, p') \equiv_{\dagger} (\mathcal{B}, q')$. Otherwise there is a word $\alpha \in \Sigma^\omega$ such that $c_1^*(p', \alpha)$ and $c_2^*(q', \alpha)$ differ at infinitely many positions. Then $c_1^*(p, a\alpha)$ and $c_2^*(q, a\alpha)$ also differ at infinitely many positions and thus $(\mathcal{A}, p) \not\equiv_{\dagger} (\mathcal{B}, q)$. $\square$

The following definition is used as an intermediate step on the way to computing $\equiv_{\dagger}$.

Definition 1.1.6. Let $\mathcal{A} = (Q_1, \Sigma, \delta_1, c_1)$ and $\mathcal{B} = (Q_2, \Sigma, \delta_2, c_2)$ be DPAs. We define the deterministic Büchi automaton $\mathcal{A} \intercal \mathcal{B} = (Q_1 \times Q_2, \Sigma, \delta_{\intercal}, F_{\intercal})$ with $\delta_{\intercal}((q_1, q_2), a) = (\delta_1(q_1, a), \delta_2(q_2, a))$. The transition structure is a common product automaton.

The final states are $F_{\intercal} = \{(p, q) \in Q_1 \times Q_2 \mid c_1(p) \neq c_2(q)\}$, i.e. every pair of states at which the priorities differ.

Lemma 1.1.6. *$\mathcal{A} \intercal \mathcal{B}$ can be computed in time $\mathcal{O}(|Q_1| \cdot |Q_2|)$.*

Proof. The definition already provides a rather straightforward description of how to compute $\mathcal{A} \intercal \mathcal{B}$. Each state only requires constant time (assuming that δ and c can be evaluated in such) and has $|\mathcal{A}| \cdot |\mathcal{B}|$ many states. $\square$

Lemma 1.1.7. *Let $\mathcal{A} = (Q_1, \Sigma, \delta_1, c_1)$ and $\mathcal{B} = (Q_2, \Sigma, \delta_2, c_2)$ be DPAs. $(\mathcal{A}, p) \equiv_{\dagger} (\mathcal{B}, q)$ iff $L(\mathcal{A} \intercal \mathcal{B}, (p, q)) = \emptyset$.*

Proof. For the first direction of implication, let $L(\mathcal{A} \intercal \mathcal{B}, (p_0, q_0)) \neq \emptyset$, so there is a word α accepted by that automaton. Let $(p, q)(p_1, q_1)(p_2, q_2) \cdots$ be the accepting run on α. Then $pp_1 \cdots$ and $qq_1 \cdots$ are the runs of $\mathcal{A}$ and $\mathcal{B}$ on α respectively. Whenever $(p_i, q_i) \in F_{\intercal}$, p_i and q_i have different priorities. As the run of the product automaton vists infinitely many accepting states, α is a witness for p and q being not priority almost-equivalent.

For the second direction, let p and q be not priority almost-equivalent, so there is a witness α at which infinitely many positions differ in priority. Analogously to the first direction, this means that the run of $\mathcal{A} \intercal \mathcal{B}$ on the same word is accepting and therefore the language is not empty. $\square$

Corollary 1.1.8. *Priority almost equivalence of a given DPA can be computed in quadratic time.*

Proof. By Lemma 1.1.6, we can compute $\mathcal{A} \intercal \mathcal{A}$ in quadratic time. The emptiness problem for deterministic Büchi automata is solvable in linear time by checking reachability of loops that contain a state in F. $\square$

Moore Equivalence

Definition 1.1.7. Let $\mathcal{A} = (Q_1, \Sigma, \delta_1, c_1)$ and $\mathcal{B} = (Q_2, \Sigma, \delta_2, c_2)$ be DPAs. We define *Moore equivalence* as $(\mathcal{A}, p) \equiv_M (\mathcal{B}, q)$ if and only if for all words $w \in \Sigma^*$, $c_1(\delta^*(p, w)) = c_2(\delta^*(q, w))$.

Lemma 1.1.9. *Moore equivalence is the congruence refinement of $\sim$ with $(\mathcal{A}, p) \sim (\mathcal{B}, q)$ iff $c(p) = c(q)$.*

Corollary 1.1.10. *$\equiv_M$ is a congruence relation.*

Corollary 1.1.11. *Moore equivalence of a given DPA can be computed in log-linear time.*

Theorem 1.1.12. $\equiv_M \subseteq \equiv_\dagger \subseteq \equiv_L$

Proof. Let $\mathcal{A} = (Q_1, \Sigma, \delta_1, c_1)$ and $\mathcal{B} = (Q_2, \Sigma, \delta_2, c_2)$ be DPAs with states $q_1 \in Q_1$ and $q_2 \in Q_2$.

At first, let $(\mathcal{A}, q_1) \equiv_M (\mathcal{B}, q_2)$ and assume towards a contradiction that $(\mathcal{A}, q_1) \not\equiv_\dagger (\mathcal{B}, q_2)$, so there is a word $\alpha \in \Sigma^\omega$ such that $c_1^*(q_1, \alpha)$ and $c_2^*(q_2, \alpha)$ differ at infinitely many positions. In particular, there is some $w \sqsubseteq \alpha$ such that $c_1(\delta_1^*(q_1, w)) \neq c_2(\delta_2^*(q_2, w))$. This would be a contradiction to $(\mathcal{A}, q_1) \equiv_M (\mathcal{B}, q_2)$.

Now assume that $(\mathcal{A}, q_1) \equiv_\dagger (\mathcal{B}, q_2)$ and let $\alpha \in \Sigma^\omega$. Let ρ_1 and ρ_2 be the runs of $\mathcal{A}$ and $\mathcal{B}$ on α starting in q_1 and q_2. Because $(\mathcal{A}, q_1) \equiv_\dagger (\mathcal{B}, q_2)$ is true, there is a position n such that $c_1(\rho_1[n, \omega]) = c_2(\rho_2[n, \omega])$ and therefore $\mathrm{Inf}(c_1(\rho_1[n, \omega])) = \mathrm{Inf}(c_2(\rho_2[n, \omega]))$, which means that the runs have the same acceptance. Thus, $(\mathcal{A}, q_1) \equiv_L (\mathcal{B}, q_2)$. ☐

1.1.2 Representative Merge

Definition 1.1.8. Let $\mathcal{A} = (Q, \Sigma, \delta, c)$ be a DPA and let $\emptyset \neq C \subseteq M \subseteq Q$. Let $\mathcal{A}' = (Q', \Sigma, \delta', c')$ be another DPA. We call $\mathcal{A}'$ a *representative merge of $\mathcal{A}$ w.r.t. M by candidates C* if it satisfies the following:

- There is a state $r_M \in C$ such that $Q' = (Q \setminus M) \cup \{r_M\}$.
- $c' = c \upharpoonright_{Q'}$.
- Let $p \in Q'$ and $\delta(p, a) = q$. If $q \in M$, then $\delta'(p, a) = r_M$. Otherwise, $\delta'(p, a) = q$.

We call r_M the *representative* of M in the merge. We might omit C and implicitly assume $C = M$.

Definition 1.1.9. Let $\mathcal{A} = (Q, \Sigma, \delta, c)$ be a DPA and let $\mu : D \to (2^Q \setminus \emptyset)$ be a function for some $D \subseteq 2^Q$. We call μ a *merger function* if

- all sets in D are pairwise disjoint; and
- for $U = \bigcup D$ and all sets $X \in D$, $\mu(X) \cap (U \setminus X) = \emptyset$

A DPA $\mathcal{A}'$ is a representative merge of $\mathcal{A}$ w.r.t. μ if there is an enumeration $X_1, \ldots, X_{|D|}$ of D and a sequence of automata $\mathcal{A}_0, \ldots, \mathcal{A}_{|D|}$ such that $\mathcal{A}_0 = \mathcal{A}$, $\mathcal{A}_{|D|} = \mathcal{A}'$ and every $\mathcal{A}_{i+1}$ is a representative merge of $\mathcal{A}_i$ w.r.t. X_{i+1} by candidates $\mu(X_{i+1})$.

The following Lemma formally proofs that this definition actually makes sense, as building representative merges is commutative if the merge sets are disjoint.

Lemma 1.1.13. *Let $\mathcal{A} = (Q, \Sigma, \delta, c)$ be a DPA and let $M_1, M_2 \subseteq Q$. Let $\mathcal{A}_1$ be a representative merge of $\mathcal{A}$ w.r.t. M_1 by some candidates C_1. Let $\mathcal{A}_{12}$ be a representative merge of $\mathcal{A}_1$ w.r.t. M_2 by some candidates C_2. If M_1 and M_2 are disjoint, then there is a representative merge $\mathcal{A}_2$ of $\mathcal{A}$ w.r.t. M_2 by candidates C_2 such that $\mathcal{A}_{12}$ is a representative merge of $\mathcal{A}_2$ w.r.t M_1 by candidates C_1.*

Proof. By choosing the same representative r_{M_1} and r_{M_2} in the merges, this is a simple application of the definition. □

The following Lemma, while simple to prove, is interesting and will find use in multiple proofs of correctness later on.

Lemma 1.1.14. *Let $\mathcal{A}$ be a DPA. Let $\sim$ be a congruence relation on Q and let $M \subseteq Q$ such that for all $x, y \in M$, $x \sim y$. Let $\mathcal{A}'$ be a representative merge of $\mathcal{A}$ w.r.t. M by candidates C. Let ρ and ρ' be runs of $\mathcal{A}$ and $\mathcal{A}'$ on some α. Then for all i, $(\mathcal{A}, \rho(i)) \sim (\mathcal{A}, \rho'(i))$.*

Proof. We use a proof by induction. For $i = 0$, we have $\rho(0) = q_0$ for some $q_0 \in Q$ and $\rho'(0) = r_{[q_0]_M}$. By choice of the representative, $q_0 \in M$ and $r_{[q_0]_M} \in M$ and thus $q_0 \sim r_{[q_0]_M}$.

Now consider some $i + 1 > 0$. Then $\rho'(i + 1) = r_{[q]_M}$ for $q = \delta(\rho'(i), \alpha(i))$. By induction we know that $\rho(i) \sim \rho'(i)$ and thus $\delta(\rho(i), \alpha(i)) = \rho(i + 1) \sim q$. Further, we know $q \sim r_{[q]_M}$ by the same argument as before. Together this lets us conclude in $\rho(i + 1) \sim q \sim \rho'(i + 1)$. □

The following is a comprehensive list of all relevant merger functions that we use.

- Quotient merger, $\mu^{\sim}_{\div}$. Defined below.
- Moore merger, μ_M. Defined below.
- Skip merger, $\mu^{\sim}_{\text{skip}}$. Defined in section **??**.
- Delayed simulation merger, μ_{de}. Defined in section **??**.
- Path refinement metger
- Treshold Moore merger, $\mu^{\sim}_{\text{TM}}$. Defined in section **??**.
- Labeled SCC Filter merger, $\mu^{k,\sim}_{\text{LSF}}$. Defined in section **??**.

Quotient merger

Definition 1.1.10. Let $\sim$ be a congruence relation. We define the *quotient merger* $\mu^{\sim}_{\div} : \mathfrak{C}(\sim) \to 2^Q, \kappa \mapsto \kappa$.

Lemma 1.1.15. *Let $\mathcal{A} = (Q, \Sigma, \delta, c)$ be a DPA and let $\sim$ be a congruence relation such that $p \sim q$ implies $c(p) = c(q)$. Then $\mathcal{A}$ is Moore equivalent to every representative merge w.r.t. $\mu^{\sim}_{\div}$.*

Proof. Let $\mathcal{A}' = (Q', \Sigma, \delta', c')$ be a representative merge. For every $q \in Q$, we prove that $(\mathcal{A}, q) \equiv_M (\mathcal{A}', r_{[q]_\sim})$. Since all states in $\mathcal{A}'$ are representatives of that form and every representative exists in $\mathcal{A}$ as well, this suffices to prove Moore equivalence.

Let $\alpha \in \Sigma^\omega$ be a word and let ρ and ρ' be the runs of $\mathcal{A}$ and $\mathcal{A}'$ on α starting in q and $r_{[q]_\sim}$. For every $i \in \mathbb{N}$, we have $\rho'(i) = [\rho(i)]_\sim$ and thus $c'(\rho'(i)) = c(\rho(i))$. Therefore, ρ is accepting iff ρ' is accepting. $\square$

Definition 1.1.11. We define the special *Moore merger* $\mu_M = \mu_{\div}^{\equiv_M}$.

1.1.3 Reachability

Definition 1.1.12. Let $\mathcal{S} = (Q, \Sigma, \delta)$ be a deterministic transition structure. We define the *reachability order* $\preceq_{\text{reach}}^{\mathcal{S}}$ as $p \preceq_{\text{reach}}^{\mathcal{S}} q$ if and only if q is reachable from p.

We want to note here that we always assume for all automata to only have one connected component, i.e. for all states p and q, there is a state r such that p and q are both reachable from r. In practice, most automata have an predefined initial state and a simple depth first search can be used to eliminate all unreachable states.

Lemma 1.1.16. *$\preceq_{\text{reach}}^{\mathcal{S}}$ is a preorder.*

Definition 1.1.13. Let $\mathcal{S} = (Q, \Sigma, \delta)$ be a deterministic transition structure. We call a relation $\preceq$ a *total extension of reachability* if it is a minimal superset of $\preceq_{\text{reach}}^{\mathcal{S}}$ that is also a total preorder.

For $p \preceq q$ and $q \preceq p$, we write $p \simeq q$.

Lemma 1.1.17. *For a given deterministic transition structure $\mathcal{S}$, a total extension of reachability is computable in $\mathcal{O}(|\mathcal{S}|)$.*

Proof. Using e.g. Kosaraju's algorithm [16], the SCCs of $\mathcal{A}$ can be computed in linear time. We can now build a DAG from $\mathcal{A}$ by merging all states in an SCC into a single state; iterate over all transitions (p, a, q) and add an a-transition from the merged representative of p to that of q. Assuming efficient data structures for the computed SCCs, this DAG can be computed in $O(|\mathcal{A}|)$ time.

To finish the computation of $\preceq$, we look for a topological order on that DAG. This is a total preorder on the SCCs that is compatible with reachability. All that is left to be done is to extend that order to all states. $\square$

1.1.4 Changing Priorities

As we mentioned earlier, state reduction of DPAs is difficult and minimization is an NP-hard problem. Priorities of states on the other hand are generally easier to modify. A few of these possibilities are considered in this section.

Lemma 1.1.18. *Let $\mathcal{A} = (Q, \Sigma, \delta, c)$ be a DPA and let $\{s\} \subseteq Q$ be a trivial SCC in $\mathcal{A}$. Let $\mathcal{A}' = (Q, \Sigma, \delta, c')$ be a copy of $\mathcal{A}$ with the only exception that $c'(s) = k$ for some arbitrary k. Then $\mathcal{A} \equiv_L \mathcal{A}'$.*

Proof. Let $q \in Q$ be any state. We show that $L(\mathcal{A}, q) = L(\mathcal{A}', q)$. Let ρ and ρ' be the runs of $\mathcal{A}$ and $\mathcal{A}'$ starting in q on some $\alpha \in \Sigma^\omega$. As s lies in a trivial SCC, the runs visit that state at most once. Therefore, $\mathrm{Inf}c(\rho) = \mathrm{Inf}c'(\rho')$ and the runs have the same acceptance. □

An interesting property that parity automata can have is being *normalized*. Even more important is that a normalized version of a DPA can be computed rather easily.

Definition 1.1.14. Let $\mathcal{A} = (Q, \Sigma, \delta, c)$ be a DPA. We call $\mathcal{A}$ *c-normalized* if for every state $q \in Q$ that does not lie in a trivial SCC and all priorities $k \leq c(q)$, there is a path from q to q such that the lowest priority visited is k.

Algorithm 1 shows how an equivalent normalized priority function can be computed in $\mathcal{O}(|Q| \cdot |c(Q)|)$. The algorithm is a slight adaption of that presented in [3], which is why we will not go into further details here and just refer to the original source.

Algorithm 1 Normalizing the priority function of a DPA.

function NORMALIZE($\mathcal{A}$)
    $c' : Q \to \mathbb{N}, q \mapsto c(q)$
    $\mathrm{M}(\mathcal{A}, c')$
    return c'
end function

function $\mathrm{M}(\mathcal{A} \restriction_P, c')$
    if $P = \emptyset$ **then**
        return 0
    end if
    $min \leftarrow 0$
    for SCC S in $\mathcal{A} \restriction_P$ **do**
        $m := \min c(S) \mod 2$
        $X := c^{-1}(m)$
        for $q \in X$ **do**
            $c'(q) \leftarrow m$
        end for
        $S' := S \setminus X$
        $m' \leftarrow \mathrm{M}(\mathcal{A} \restriction_{S'}, c')$
        if m' even **then**
            if m even **then**
                $\delta := m$
            else
                $\delta := m - 2$
            end if
        else
            $\delta := m - 1$
        end if
        for $q \in S'$ **do**
            $c'(q) \leftarrow c'(q) - \delta$
        end for
        $min \leftarrow \min\{min, m\}$
    end for
    return min
end function

1.2 Schewe

This section is based heavily on [15] and partially adapts their results from Büchi to parity automata.

Definition 1.2.1. Let $\mathcal{A} = (Q, \Sigma, \delta, c)$ be a DPA and let $\emptyset \neq C \subseteq M \subseteq Q$. Let $\mathcal{A}' = (Q', \Sigma, \delta', c')$ be another DPA. We call $\mathcal{A}'$ a *Schewe merge of $\mathcal{A}$ w.r.t. M by candidates C* if it satisfies the following:

- There is a state $r_M \in C$ such that $Q' = (Q \setminus M) \cup \{r_M\}$.

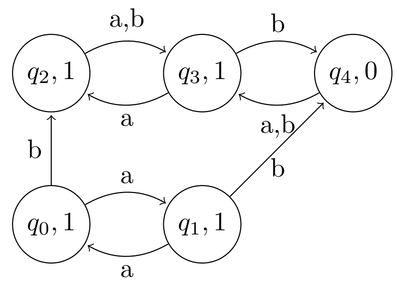

Figure 1.1: Example to show the effect of a Schewe merge.

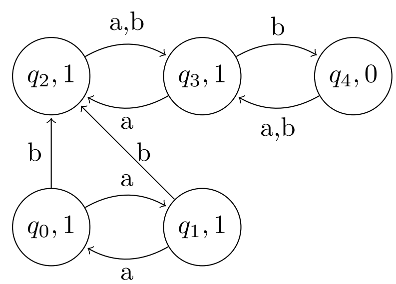

Figure 1.2: Automaton from figure 1.1 after Schewe merge.

- $c' = c\restriction_{Q'}$.
- Let $p \in Q'$ and $\delta(p, a) = q$. If $q \in M$ or if ($q \in C$ and p is not reachable from q), then $\delta'(p, a) = r_M$. Otherwise, $\delta'(p, a) = q$.

The definition of a Schewe merge is almost identical to that of a representative merge. The only difference lies therein that some additional transitions are redirected to the representative: when a transition leads to a candidate that is not in M while also moving to a different SCC.

Using the Schewe merge instead of the representative merge does not actually remove any additional states from the automaton, it only provides a better "framework" for following algorithms such as the Moore reduction. Example figure 1.1 shows that using a Schewe merge followed by another merge of μ_M creates a better end result.

The automaton has six states. Regarding Moore equivalence, every state builds a singleton class. For priority almost equivalence, the classes are $\{q_0, q_1\}$, $\{q_2, q_4\}$, and $\{q_3\}$. As all states within one equivalence class lie in the same SCC, a representative merge w.r.t. $\mu_{\text{skip}}^{\equiv_\dagger}$ will not change the number of states. However, a Schewe merge w.r.t. $\mu_{\text{skip}}^{\equiv_\dagger}$ can result in figure 1.2. Now, q_0 and q_1 are actually Moore equivalent and could be merged with μ_M.

Definition 1.2.2. Let $\mathcal{A}$ be DPA with a merger function $\mu : D \to 2^Q$. For a representative merge $\mathcal{A}'$, we define the *candidate relation* $\sim_C^\mu$ as $p \sim_C^\mu q$ iff there is a $C \in \mu(D)$ with $p, q \in C$.

We say that μ is *Schewe suitable* if for all representative merges $\mathcal{A}'$, $\sim_C^\mu$ is a congruence relation, it implies language equivalence, and the reachability order restricted to any $\kappa \in \mathfrak{C}(\sim_C^\mu)$ is an equivalence relation.

Lemma 1.2.1. *Let $\mathcal{A}$ be a DPA and $\mu : D \to 2^Q$ be a merger function that is Schewe suitable. Let $\mathcal{A}'$ be a representative merge of $\mathcal{A}$ w.r.t. μ and let $\mathcal{A}''$ be the Schewe merge that uses the same choice of representatives. For all $p \sim_{\mathcal{C}}^{\mu} q$, $\delta'(p, a) \sim_{\mathcal{C}}^{\mu} \delta''(q, a)$.*

Proof. If $\delta''(q, a) = \delta'(q, a)$, then $\delta'(p, a) \sim_{\mathcal{C}}^{\mu} \delta'(q, a)$ because μ is Schewe suitable and the claim is true.

Otherwise $\delta'(q, a) = q' \in \mu(M)$ for some M and q is not reachable from q'. Then $\delta''(q, a) = r_M$. By definition, $r_M \in \mu(M)$ as well. By definition of $\sim_{\mathcal{C}}^{\mu}$, $r_m \sim_{\mathcal{C}}^{\mu} q'$ and thus $\delta'(p, a) \sim_{\mathcal{C}}^{\mu} \delta'(q, a) \sim_{\mathcal{C}}^{\mu} \delta''(q, a)$. □

Lemma 1.2.2. *Let $\mathcal{A}$ be a DPA and $\mu : D \to 2^Q$ be a merger function that is Schewe suitable. Let $\mathcal{A}'$ be a representative merge of $\mathcal{A}$ w.r.t. μ and let $\mathcal{A}''$ be the Schewe merge that uses the same choice of representatives. Every run of $\mathcal{A}''$ has a suffix that is a run of $\mathcal{A}'$.*

Proof. Let $K \subseteq \mathbb{N}$ be the set of positions at which $\mathcal{A}''$ uses a transition that is not in $\mathcal{A}'$. That is, given a run ρ on α, $\rho(k + 1) \neq \delta'(\rho(k)\alpha(k))$ for all $k \in K$. If we can prove that K is finite, then that means $\rho[\max K + 1, \omega]$ is a run in $\mathcal{A}'$.

More precisely, we show that for every $\kappa \in \mathfrak{C}(\sim_{\mathcal{C}}^{\mu})$, there is at most one $k \in K$ such that $\rho(k + 1) \in \kappa$. Towards a contradiction assume the opposite, so there are $k < k'$ which both have this property. We label the positions in K in ascending order and have $k = k_l$ and $k' = k_{l'}$ for some $l < l'$.

Consider k_{l+1}. By definition of the Schewe merge, $\delta'(\rho(k_{l+1}), \alpha(k_{l+1})) \not\preceq_{\text{reach}}^{\mathcal{A}} \rho(k_{l+1})$. By Lemma 1.2.1, we know that $(\delta')^*(\rho(k_{l+1}), \alpha[k_{l+1}, k_{l'} + 1]) \sim_{\mathcal{C}}^{\mu} (\delta'')^*(\rho(k_{l+1}), \alpha[k_{l+1}, k_{l'} + 1]) = \rho(k_{l'} + 1)$. That means there is a state $r \in \kappa$ that is reachable from $\delta'(\rho(k_{l+1}), \alpha(k_{l+1}))$ in $\mathcal{A}$ and therefore from $\rho(k_l + 1)$ as well.

Reachability order restricted to κ is an equivalence relation, so r and $\rho(k_l + 1)$ must lie in the same SCC in $\mathcal{A}'$. That however contradicts the fact that at position k_{l+1} a redirected transition was taken. □

Lemma 1.2.3. *Let $\mathcal{A}$ be a DPA and $\mu : D \to 2^Q$ be a merger function that is Schewe suitable. Let $\mathcal{A}'$ be a representative merge of $\mathcal{A}$ w.r.t. μ that was built with representatives $R \subseteq Q$. If a Schewe merge $\mathcal{A}''$ is built with the same representatives, then $(\mathcal{A}', p) \equiv_L (\mathcal{A}'', q)$ for all $q \sim_{\mathcal{C}}^{\mu} q$.*

Proof. Let $\alpha \in \Sigma^\omega$ be some word. Let ρ' be the run of $\mathcal{A}'$ on α starting in p and let ρ'' be the run of $\mathcal{A}''$ on α starting in q. Let $k_1, \ldots, k_n$ be the positions at which ρ'' uses a transition that is not present in $\mathcal{A}'$, i.e. $\rho''(k_i + 1) \neq \delta'(\rho(k_i), \alpha(k_i))$. By Lemma 1.2.2, this list must be finite.

Our goal now is to prove $\rho'(k_i + 1) \sim_{\mathcal{C}}^{\mu} \rho''(k_i + 1)$ for all i. If that is true, then $\rho'(k_n + 1) \sim_{\mathcal{C}}^{\mu} \rho''(k_n+1)$ in particular is true and therefore $\rho'(k_n+1) \equiv_L \rho''(k_n+1)$. By choice of k_n, $\rho''[k_n+1, \omega]$ is also a run in $\mathcal{A}'$ which has the same acceptance as ρ''. Since the two states are language equivalent, this is also the same acceptance as ρ'.

Assume that for some i, $\rho'(k_j + 1) \sim_{\mathcal{C}}^{\mu} \rho''(k_j + 1)$ is true for all $j < i$. As $\sim_{\mathcal{C}}^{\mu}$ is a congruence relation in $\mathcal{A}'$, $\rho'(k_i) \sim_{\mathcal{C}}^{\mu} \rho''(k_i)$ and $\rho'(k_i + 1) = \delta'(\rho'(k_i), \alpha(k_i)) \sim_{\mathcal{C}}^{\mu} \delta'(\rho''(k_i), \alpha(k_i))$. By Lemma 1.2.1, $\delta'(\rho''(k_i), \alpha(k_i)) \sim \delta''(\rho''(k_i), \alpha(k_i)) = \rho''(k_i + 1)$. □

Lemma 1.2.4. *Let $\sim$ be a congruence relation that implies language equivalence. Then $\mu_{skip}^{\sim}$ is Schewe suitable.*

Proof. Let $\mathcal{A}'$ be a representative merge of a DPA $\mathcal{A}$ w.r.t. $\mu^{\sim}_{\text{skip}}$. We prove that $p \sim q$ iff $p \sim_{\mathcal{C}}^{\mu^{\sim}_{\text{skip}}} q$. The required properties then follow from the assumptions in the statement and from Lemma **??**.

We have $\text{dom}(\mu^{\sim}_{\text{skip}}) = D = \{M_\kappa \mid \kappa \in \mathfrak{C}(\sim)\}$ and $\mu^{\sim}_{\text{skip}}(M_\kappa) = C_\kappa \subseteq \kappa$. Thus, $p \sim_{\mathcal{C}}^{\mu^{\sim}_{\text{skip}}} q$ implies $p \sim q$.

On the other hand, $\kappa = C_\kappa \cup M_\kappa$, so all states of κ that remain in $\mathcal{A}'$ are C_κ and thus lie in the same equivalence class of $\sim_{\mathcal{C}}^{\mu^{\sim}_{\text{skip}}}$. □

Corollary 1.2.5. *Let $\mathcal{A}$ be a DPA and let $\sim$ be a congruence relation that implies language equivalence. For each Schewe merge $\mathcal{A}'$ of $\mathcal{A}$ w.r.t. $\mu^{\sim}_{skip}$, $\mathcal{A} \equiv_L \mathcal{A}'$.*

Proof. Follows from Lemma 1.2.3, Lemma 1.2.4, and Theorem **??**. □

For our final proof in this section we want to adapt a result from [15] to show a special relation between Schewe merges and priority almost equivalence.

Lemma 1.2.6. *Let $\mathcal{A}$ be a DPA, $\mathcal{S}$ be a Schewe merge of $\mathcal{A}$ w.r.t. $\mu^{\equiv_\dagger}_{skip}$, and $\mathcal{S}'$ be a representative merge of $\mathcal{S}$ w.r.t. μ_M. There is no smaller DPA than $\mathcal{S}'$ that is priority almost equivalent to $\mathcal{A}$.*

Proof. Let $\mathcal{B}$ be a DPA that is smaller than $\mathcal{S}'$. Our goal is to show that $\mathcal{A} \not\equiv_\dagger \mathcal{B}$.

At first observe that $\mathcal{A} \equiv_\dagger \mathcal{S}'$: $\mathcal{A}$ and $\mathcal{S}$ are priority almost equivalent as for every state in $\mathcal{A}$, there is an equivalent representative in $\mathcal{S}$. $\mathcal{S}$ and $\mathcal{S}'$ are Moore equivalent by Lemma 1.1.15 and thus they are also priority almost equivalent by Theorem 1.1.12.

Assume that $\mathcal{S}' \equiv_\dagger \mathcal{B}$ holds, so for all states in $\mathcal{S}'$, there is a a priority almost equivalent state in $\mathcal{B}$. We define a function f that maps to each equivalence class of $\equiv_\dagger$ in $\mathcal{S}'$ all states in $\mathcal{B}$ that are equivalent to it, i.e. for $\kappa \in \mathfrak{C}(\equiv_\dagger, \mathcal{S}')$ then $f(\kappa) = \{q \in Q_\mathcal{B} \mid \exists p \in \kappa : (\mathcal{S}', p) \equiv_\dagger (\mathcal{B}, q)\}$. Note that $f(\kappa)$ can never be empty by the assumption of $\mathcal{S}' \equiv_\dagger \mathcal{B}$.

As $\mathcal{B}$ is smaller than $\mathcal{S}'$, the pigeonhole principle applies and we can fix κ to be one equivalence class such that $|f(\kappa)| < |\kappa|$.

There is an SCC C in $\mathcal{S}'$ that contains all states in κ (argumentation is done similar to Lemma **??**). Without loss of generality we can assume that likewise there is an SCC D in $\mathcal{B}$ that contains $f(\kappa)$. If no such SCC would exist, we could simply apply the Schewe merger to $\mathcal{B}$ to find an automaton that is smaller than $\mathcal{S}'$ and does have this property.

C and D must be non-trivial SCCs: if C would be trivial, κ would contain only one element and $f(\kappa)$ would be empty. If D would be trivial, $f(\kappa) = \{q\}$ would contain of only one state. Since $\mathcal{S}' \equiv_\dagger \mathcal{B}$, there is a state $p \in \kappa \subseteq C$ in $\mathcal{S}'$ with $(\mathcal{S}', p) \equiv_\dagger (\mathcal{B}, q)$. Since C is not trivial, there is a word w from which $\mathcal{S}'$ moves from p back to p. $\mathcal{B}$ however, leaves $f(\kappa)$ with that word, as D is trivial, so q cannot reach itself again. This is a contradiction, as $\equiv_\dagger$ is a congruence relation.

We claim that there is a state $p \in \kappa$ such that there is a family of words $(w_q)_{q \in f(\kappa)}$ such that $\mathcal{S}'$ does not leave C reading these words from p and $c_{\mathcal{S}'}(\delta^*_{\mathcal{S}'}(p, w)) \neq c_\mathcal{B}(\delta^*_\mathcal{B}(q, w))$. (in other words, w_q is a witness for p and q being not Moore-equivalent.)

Towards a contradiction, assume that the claim is false and that for every $p \in \kappa$, there is a state $q_p \in f(\kappa)$ that does not satisfy the property. As $|\kappa| > |f(\kappa)|$ we can again use the pigeonhole principle and obtain two states $p_1, p_2 \in \kappa$ such that $q_{p_1} = q_{p_2}$. We call this state $q := q_{p_1}$.

For each word $w \in \Sigma^*$, $c_{\mathcal{S}'}(\delta^*_{\mathcal{S}'}(p_1, w)) = c_\mathcal{B}(\delta^*_\mathcal{B}(q, w))$ or $\mathcal{S}'$ leaves C while reading w from p_1. The same holds for p_2. As p_1 and p_2 are distinct states in $\mathcal{S}'$, they cannot be Moore equivalent. That means there has to be a word u with $c_{\mathcal{S}'}(\delta^*_{\mathcal{S}'}(p_1, u)) \neq c_{\mathcal{S}'}(\delta^*_{\mathcal{S}'}(p_2, u))$. However, $\mathcal{S}'$ cannot leave C while reading u from p_1; if it did, it would reach a class λ and the transition would lead to the representative r_λ. As $\equiv_\dagger$ is a congruence relation, reading u from p_2 would lead to r_λ at the same position, so from that point on the two runs would be exactly the same.

Hence, $p_1' = \delta^*_{\mathcal{S}'}(p_1, u)$ and $p_2' = \delta^*_{\mathcal{S}'}(p_2, u)$ are still in C and $c_{\mathcal{S}'}(p_1') \neq c_{\mathcal{S}'}(p_2')$. That means at least one of these values must be different to $c_{\mathcal{B}}(\delta^*_{\mathcal{B}}(q, u))$, which contradicts our assumption.

We can use the claim that he have just shown to finish our overall proof. Fix an arbitrary $q_0 \in f(\kappa)$. We define a sequence of finite words $(\alpha_n)_{n\in\mathbb{N}}$ such that every α_n is a prefix of α_{n+1} and the runs of $\mathcal{S}'$ and $\mathcal{B}$ from p and q_0 respectively differ in priority at least n times. Then $\alpha := \bigcup_n \alpha_n$ is an ω-word that is a witness for $(\mathcal{S}', p)$ and $(\mathcal{B}, q_0)$ not being priority almost equivalent.

We make sure that after reading any α_n from p, $\mathcal{S}'$ moves back to p. Let $\alpha_0 := \varepsilon$ and assume for induction that α_n has already been defined. After reading α_0, $\mathcal{S}'$ reaches p and $\mathcal{B}$ reaches some q. If $q \notin f(\kappa)$, i.e. $(\mathcal{S}', p) \not\equiv_\dagger (\mathcal{B}, q)$, there is a witness β and we can simply set $\alpha := \alpha_n\beta$. Otherwise, $q \in f(\kappa)$. With the claim we have proven, we can find a word w_q such that reading the word from p and q leads to states p' and q' that have different priorities but p' is still in C. Thus, there is a word u that leads back from p' to p. We then set $\alpha_{n+1} := \alpha_n w_q u$. This finishes our proof. $\square$

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