State Space Reduction For Parity Automata

Andreas Tollkötter
Supervisor: Dr. Christof Löding

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Overview

Goal: reduce the number of states in a given deterministic parity automaton while keeping the recognized language.

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1. Deterministic Parity Automata
2. Why do we need heuristic reduction?
3. Merger functions as a framework
4. Delayed Simulation
5. Congruence Path Refinement
6. Labeled SCC Filter

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ω-automata

ω-words are words of one-sided infinite length:
Σ^ω = functions from $\mathbb{N}$ to Σ

ω-automata are finite transition structures that describe a language $L \subseteq \Sigma^\omega$

Deterministic parity automata (DPA):

- State set Q
- Alphabet Σ
- Transition function $\delta : Q \times \Sigma \to Q$
- Priority function $c : Q \to \mathbb{N}$

An ω-word α starting in a state $q_0 \in Q$ induces a run $q_0 q_1 q_2 \dots$.
The DPA accepts α iff the **smallest** priority that occurs infinitely often in the sequence $c(q_0)c(q_1)c(q_2)\dots$ is **even**.

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Why do we need heuristic reduction?

Goal: Reduce number of states in the automaton to ease run time of follow up algorithms.

Minimization Problem: Given an automaton $\mathcal{A}$, what is the smallest number of states required to recognize the same language as $\mathcal{A}$?

For DFAs: Minimization is solvable in $\mathcal{O}(n \log n)$. [Hopcroft, 1971]
For DPAs: Minimization is NP-hard. [Schewe, 2010]

Moore Minimization

A DPA can be interpreted as a Moore automaton with c being the output function.

Definition.

$p \equiv_M q$ iff $\forall w \in \Sigma^* : c(\delta^*(p, w)) = c(\delta^*(q, w))$.

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Definition.

$p \equiv_M q$ iff $\forall w \in \Sigma^* : c(\delta^*(p, w)) = c(\delta^*(q, w))$.

Theorem.

Deterministic Moore automata can be minimized in log-linear time.

Idea: Build the quotient automaton w.r.t. $\equiv_M$.

The same algorithm can be used to reduce DPAs but will not give minimal DPAs in general.

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Merger functions

Merger functions μ map from some $D \subseteq 2^Q$ into $2^Q \setminus \{\emptyset\}$.

$M, C \subseteq Q$

$$\mu(\; M \;) = \; C$$

All states from the **merge set** ...

...can be represented by any single one representative from the **candidate set**.

Merger functions generalize quotient automata

Special case: $\mu(M) = M$.
Remove all states from M except for one (arbitrarily chosen) representative.

For a congruence relation $\sim$, let $\mathfrak{C} \subseteq 2^Q$ be the equivalence classes.
The quotient automaton is defined by state set $\mathfrak{C}$.
This is captured by the merger function $\mu_{\div} : \mathfrak{C} \to 2^Q, \kappa \mapsto \kappa$.

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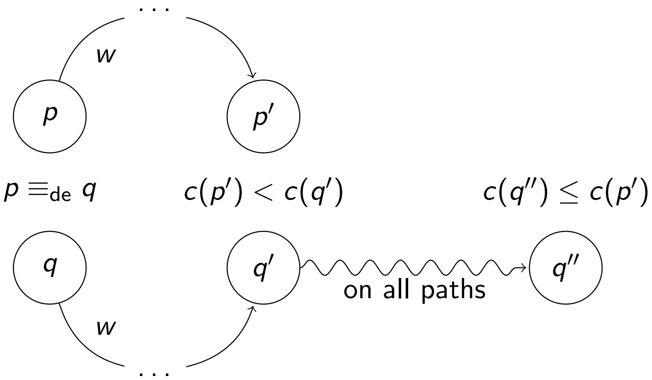
Delayed Simulation

Definition.

$p \equiv_{\mathrm{de}} q$ iff for all $w \in \Sigma^*$, every run that starts in $\delta^*(p, w)$ or $\delta^*(q, w)$ eventually sees a priority of at most $\min\{c(\delta^*(p, w)), c(\delta^*(q, w))\}$.

Delayed Simulation



Delayed Simulation

Definition.

Let $\mathfrak{C}_{\mathsf{de}} = \{[q]_{\equiv_{\mathsf{de}}} \mid q \in Q\}$ be the set of $\equiv_{\mathsf{de}}$-equivalence classes.
Define the **delayed simulation merger** as

$$\mu_{\mathsf{de}} : \mathfrak{C}_{\mathsf{de}} \to 2^Q, \kappa \mapsto \{q \in \kappa \mid c(q) = \min c(\kappa)\}.$$

Theorem.

Merging states according to μ_{de} preserves language.

Computing Delayed Simulation

We define a det. Büchi automaton $\mathcal{G}_{\mathrm{de}}$ with states $q^0_{\mathrm{de}}(p, q)$ such that:
$p \equiv_{\mathrm{de}} q$ iff both $L(\mathcal{G}_{\mathrm{de}}, q^0_{\mathrm{de}}(p, q))$ and $L(\mathcal{G}_{\mathrm{de}}, q^0_{\mathrm{de}}(q, p))$ are universal (Σ^ω).

This automaton uses the state set $Q_{\mathrm{de}} = Q \times Q \times (c(Q) \cup \{\checkmark\})$.
Computing states of universal language in a DBA requires linear time.

Theorem.

μ_{de} can be computed in $\mathcal{O}(n^2 k)$.

Delayed Simulation Automaton

$\mathcal{G}_{\text{de}} = (Q_{\text{de}}, \Sigma, \delta_{\text{de}}, F_{\text{de}})$

- States are $Q_{\text{de}} = Q \times Q \times (c(Q) \cup \{\checkmark\})$.
  The first two components are a "simulation" of the original DPA. The third component are the so called "obligations".
- Transitions δ_{de}.

$$\begin{aligned}
\delta_{\text{de}}((p, q, k), a) = (&\delta(p, a),\\
&\delta(q, a),\\
&\gamma(\quad c(\delta(p, a)), \quad c(\delta(q, a)), \quad k \quad))
\end{aligned}$$

- Accepting states are $F_{\text{de}} = Q \times Q \times \{\checkmark\}$.

Delayed Simulation Automaton: γ

(Actual definition of γ is more complex for some additional properties.)

$\gamma : \mathbb{N} \times \mathbb{N} \times (\mathbb{N} \cup \{\checkmark\}) \to \mathbb{N} \cup \{\checkmark\}$

$$\gamma(i, j, \checkmark) = \begin{cases} \checkmark & \text{if } j \leq i \\ i & \text{else} \end{cases}$$

$$\text{for } k \in \mathbb{N}: \quad \gamma(i, j, k) = \begin{cases} \checkmark & \text{if } j \leq \min\{i, k\} \\ \min\{i, k\} & \text{else} \end{cases}$$

$q^0_{\text{de}}(p, q) = (p, q, \gamma(c(p), c(q), \checkmark))$.

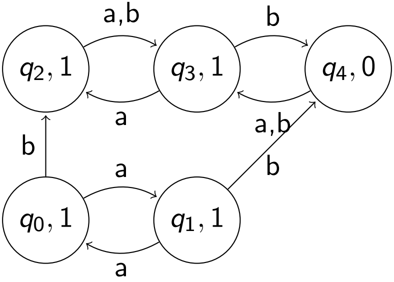
Delayed Simulation Automaton: γ

Let $0 \leq_{\checkmark} 1 \leq_{\checkmark} 2 \leq_{\checkmark} \cdots \leq_{\checkmark} \checkmark$.
For $p, q \in Q$, $k \in c(Q) \cup \{\checkmark\}$, $a \in \Sigma$, set
$\gamma((p, q, k), a) = \gamma'(\delta^*(p, a), \delta^*(q, a), k)$, where γ' is defined as follows:
If any of the following is true, then $\gamma'(i, j, k) = \checkmark$.

- i is odd, j is even, and $i \leq_{\checkmark} k$
- i is odd, j is even, and $j \leq_{\checkmark} k$
- i is odd, j is odd, $j \geq i$, and $i \leq_{\checkmark} k$
- i is even, j is even, $j \leq i$, and $j \leq_{\checkmark} k$

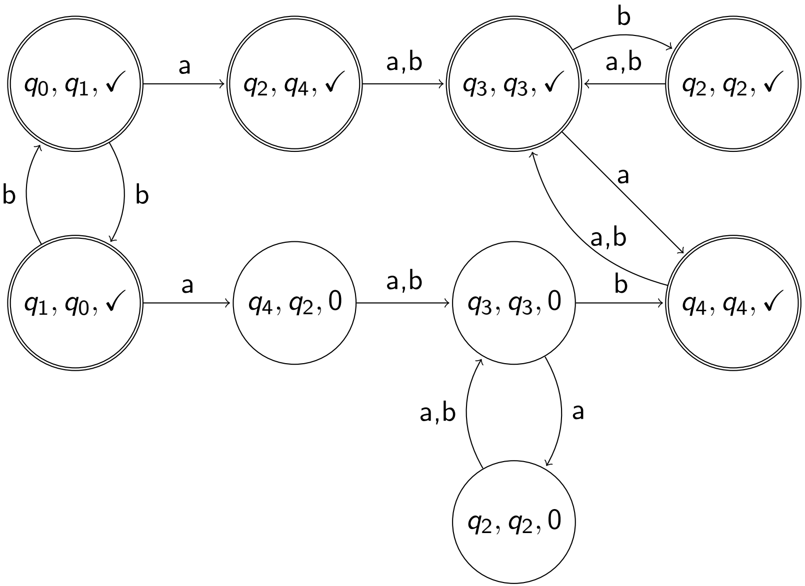
Otherwise, $\gamma'(i, j, k) = min_{\leq_{\checkmark}}\{i, j, k\}$.
$q^0_{\mathrm{de}}(p, q) = (p, q, \gamma'(c(p), c(q), \checkmark))$.

Delayed Simulation Automaton



A DPA with 5 states. We want to check whether $q_0 \equiv_{\mathrm{de}} q_1$ is true.

Delayed Simulation Automaton



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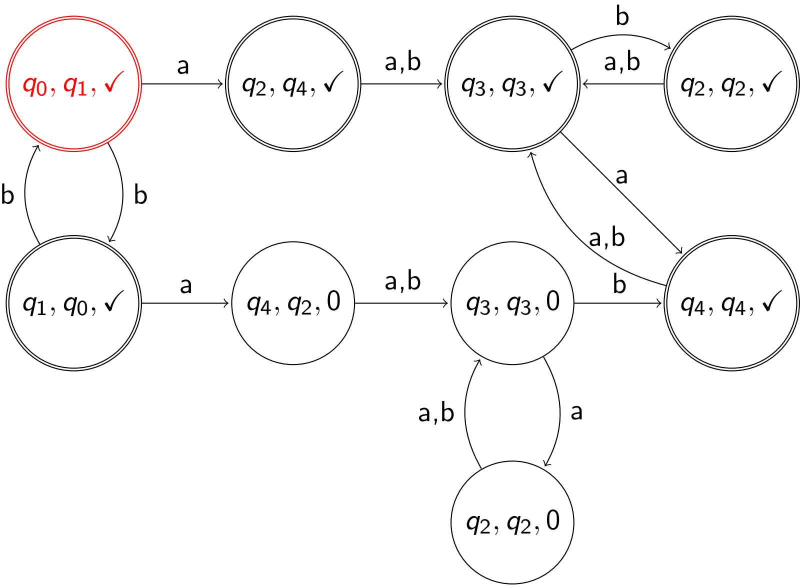

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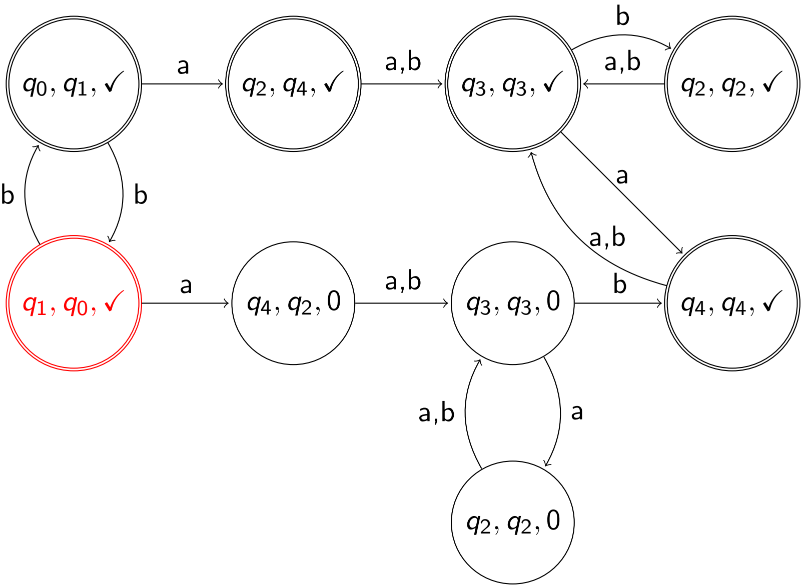

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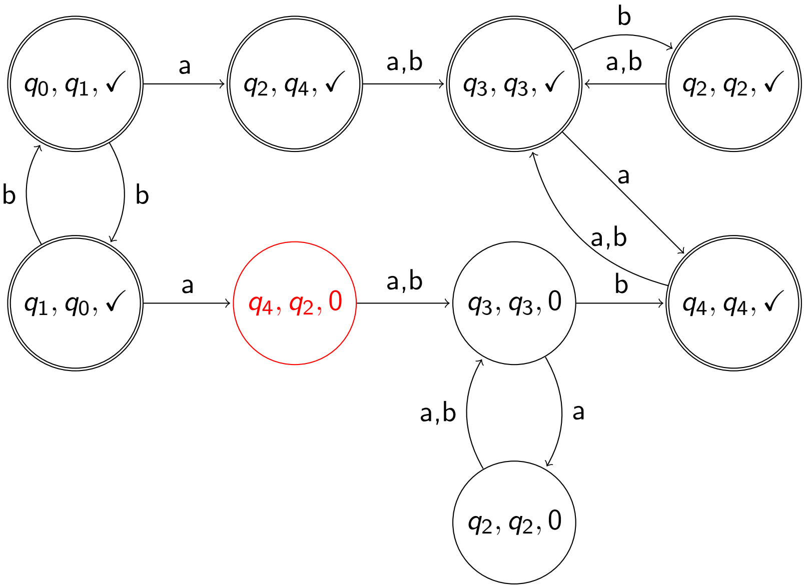

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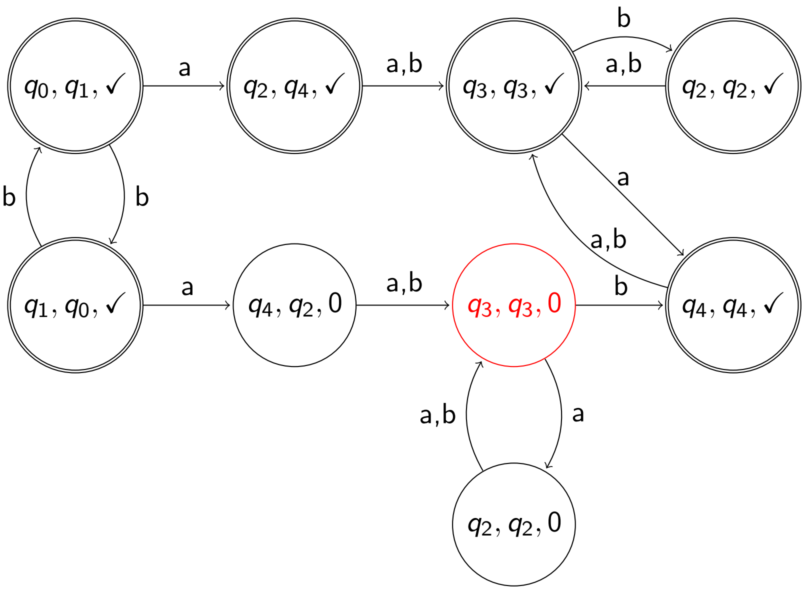

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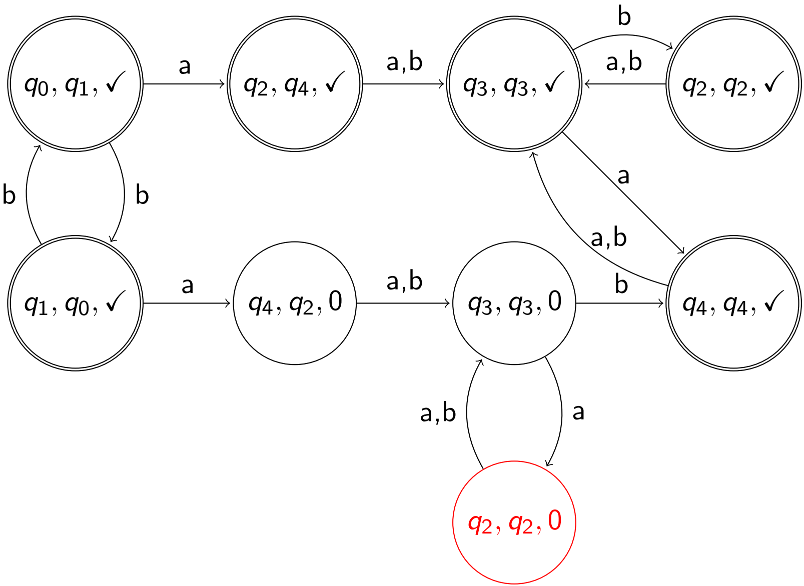
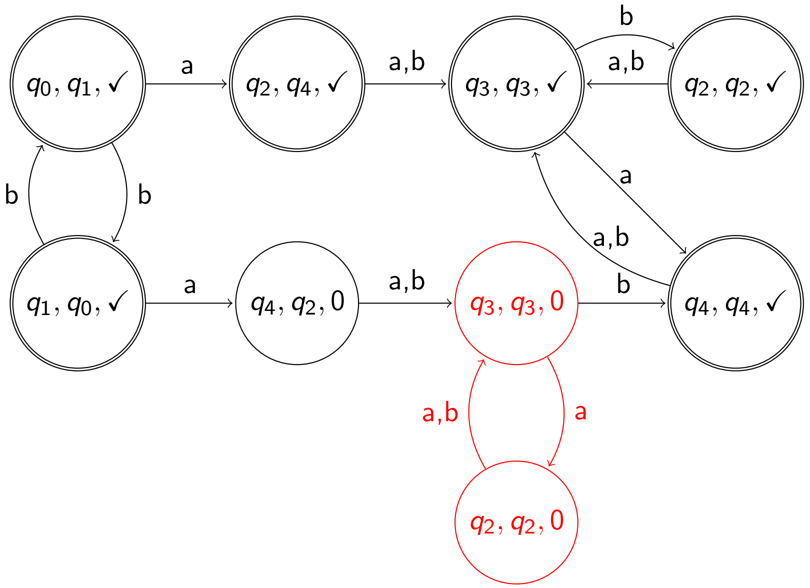
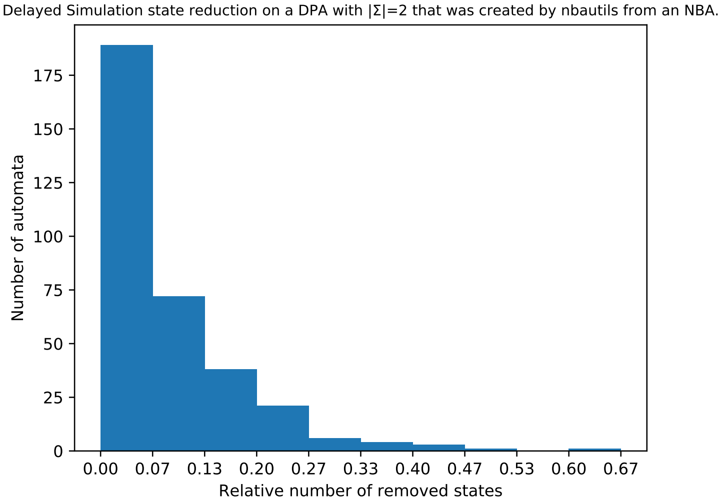


Delayed Simulation Automaton



Efficiency



Delayed Simulation state reduction on a DPA with |Σ|=2 that was created by nbautils from an NBA.
Number of automata
0
25
50
75
100
125
150
175
0.00
0.07
0.13
0.20
0.27
0.33
0.40
0.47
0.53
0.60
0.67
Relative number of removed states

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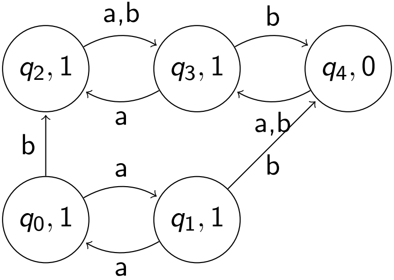
Congruence Path Refinement

Definition.

Let $\sim$ be a congruence relation and let $\lambda \subseteq Q$ be an equivalence class of $\sim$.
We define $L_{\lambda\hookleftarrow} \subseteq \Sigma^*$ as the set of all words such that the induced run from a state in λ moves back to λ exactly once and ends there.
The **path refinement** equivalence $\equiv_{\mathrm{PR}}^{\lambda}$ is the largest relation such that if $p \equiv_{\mathrm{PR}}^{\lambda} q$, then for all $w \in L_{\lambda\hookleftarrow}$, $\delta^*(p, w) \equiv_{\mathrm{PR}}^{\lambda} \delta^*(q, w)$ and the smallest priority seen when reading w is the same from p and from q.

Path Refinement Relation



Potential choices for λ are the equivalence classes of $\equiv_L$:
$\{q_0, q_1\}$, $\{q_2, q_4\}$, or $\{q_3\}$.

Path Refinement Merger

Definition.

Let $\mathfrak{C}^{\lambda}_{\mathsf{PR}} = \{[q]_{\equiv^{\lambda}_{\mathsf{PR}}} \mid q \in Q\}$ be the set of $\equiv^{\lambda}_{\mathsf{PR}}$-equivalence classes. Define the **path refinement merger** as
$\mu^{\lambda}_{\mathsf{PR}} : \mathfrak{C}^{\lambda}_{\mathsf{PR}} \to 2^{Q}, \kappa \mapsto \{q \in \kappa \mid c(q) = \min c(\kappa)\}$.

Theorem.

If all states in λ are pairwise language equivalent, merging states according to μ^{λ}_{PR} preserves language.

Computing Path Refinement

Definition.

Define the **visit graph** DPA $\mathcal{A}^{\lambda}_{\text{visit}} = (Q^{\lambda}_{\text{visit}}, \Sigma, \delta^{\lambda}_{\text{visit}}, c^{\lambda}_{\text{visit}})$.

- $Q^{\lambda}_{\text{visit}} = ((Q \setminus \lambda) \times c(Q) \times \{-1\}) \cup (\lambda \times c(Q) \times c(Q))$.
- $\delta^{\lambda}_{\text{visit}}((q, k, k'), a) = \begin{cases} (q', \min\{k, c(q')\}, -1) & \text{if } q' \notin \lambda \\ (q', c(q'), \min\{k, c(q')\}) & \text{if } q' \in \lambda \end{cases}$, where $q' = \delta(q, a)$.
- $c^{\lambda}_{\text{visit}}((q, k, k')) = k'$.

States consist of three components $q \in Q \times c(Q) \times (c(Q) \cup \{-1\})$.
The first component "simulates" the original automaton $\mathcal{A}$.
The second component tracks the minimal priority seen on one run from λ to λ.
The third component is required to distinguish the different priorities.

Computing Path Refinement

Moore equivalence in $\mathcal{A}^{\lambda}_{\text{visit}}$ corresponds to path refinement equivalence in $\mathcal{A}$.

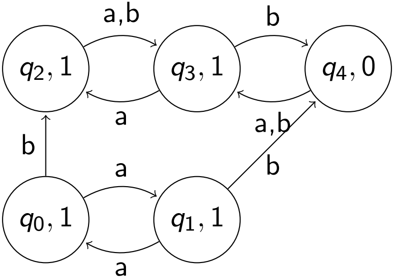
Theorem.

$p \equiv^{\lambda}_{PR} q$ iff $(p, c(p), \max c(Q)) \equiv_M (q, c(q), \max c(Q))$.

Theorem.

$\equiv^{\lambda}_{PR}$ can be computed in $\mathcal{O}(k^2 n \log n)$.

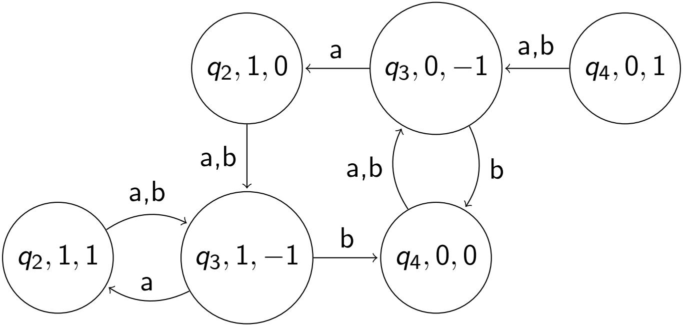
Visit Graph



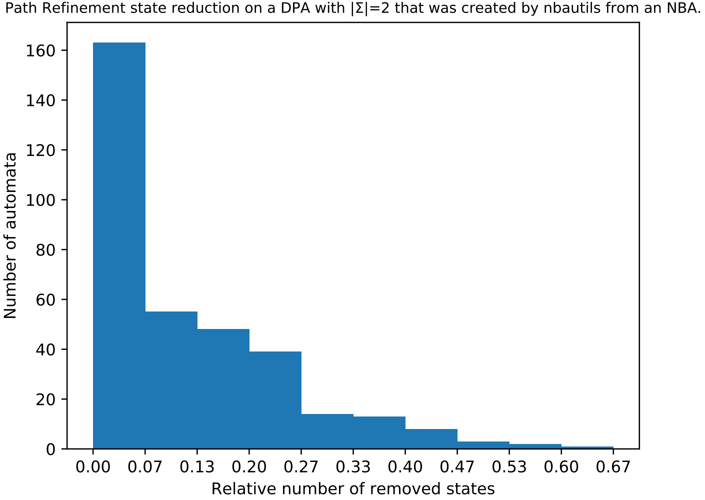
Potential choices for λ are the equivalence classes of $\equiv_L$:
$\{q_0, q_1\}$, $\{q_2, q_4\}$, or $\{q_3\}$.

Visit Graph

$\mathcal{A}_{\text{visit}}^{\{q_2,q_4\}}$



Efficiency


Path Refinement state reduction on a DPA with |Σ|=2 that was created by nbautils from an NBA.
Number of automata
0
20
40
60
80
100
120
140
160
0.00
0.07
0.13
0.20
0.27
0.33
0.40
0.47
0.53
0.60
0.67
Relative number of removed states

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Labeled SCC Filter

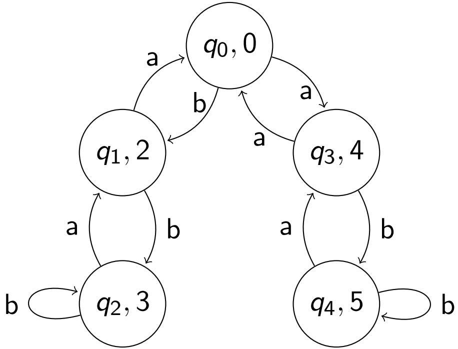
Definition.

Define $\equiv_M^{\leq k}$ as the Moore equivalence that considers all priorities **greater than k** to be equal.
$p \equiv_M^{\leq k} q$ iff for all $w \in \Sigma^*$: $c(\delta^*(p, w)) = c(\delta^*(q, w))$ or $k < c(\delta^*(p, w)), c(\delta^*(q, w))$.

Definition.

Let $\sim$ be an equivalence relation and $k \in \mathbb{N}$. We define the LSF relation $p \equiv_{\mathrm{LSF}}^{k,\sim} q$ iff $p \sim q$ and $p \equiv_M^{\leq k} q$.

LSF Relation



Equivalence classes of $\equiv_{\mathrm{LSF}}^{1,\equiv_L}$: $\{q_0\}$, $\{q_1, q_3\}$, and $\{q_2, q_4\}$.

LSF Merger

From the DPA $\mathcal{A}$, we remove all states which have priority less or equal to k and call the resulting (possibly incomplete) DPA $\mathcal{A}_k$.
We choose some total preorder $\preceq_k$ on the states of $\mathcal{A}_k$ such that p being reachable from q in $\mathcal{A}_k$ implies $q \preceq_k p$, and $p \preceq_k q \preceq_k p$ is only true if p and q are in the same SCC in $\mathcal{A}_k$. ($\preceq_k$ is a total preorder that is a minimal extension of reachability.)

In focus are the set of states that are $\preceq_k$-maximal among a given set $P \subseteq Q$. These are all states in one SCC of $\mathcal{A}_k$ such that no other states in P are reachable.

LSF Merger

Definition.

Let $\mathfrak{C}^{k,\sim}_{\mathsf{LSF}}$ be the set of equivalence classes in $\equiv^{k,\sim}_{\mathsf{LSF}}$ and let κ be such an equivalence class. Define

$$C^k_\kappa = \{r \in \kappa \mid c(r) > k \text{ and } r \text{ is } \preceq_k \text{-maximal among } \kappa\}$$

and $M^k_\kappa = \kappa \setminus C^k_\kappa$.

Definition.

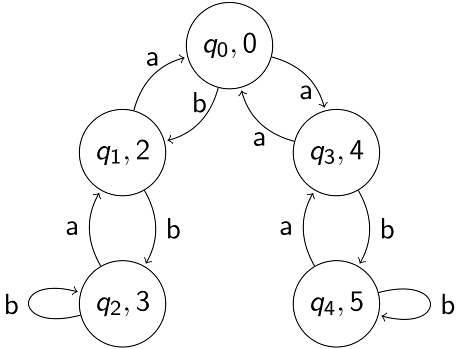
Define the **LSF merger function**

$$\mu^{k,\sim}_{\mathsf{LSF}} : \{M^k_\kappa \mid \kappa \in \mathfrak{C}^{k,\sim}_{\mathsf{LSF}}\} \to 2^Q, M^k_\kappa \mapsto C^k_\kappa$$

Theorem.

If $\sim$ implies language equivalence, merging states according to $\mu^{k,\sim}_{LSF}$ preserves language.

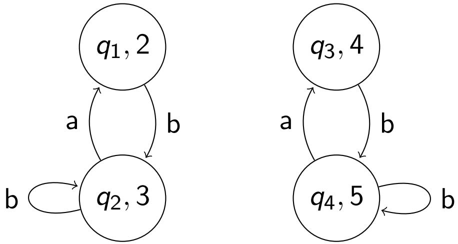
LSF example



Equivalence classes of $\equiv_{\mathsf{LSF}}^{1,\equiv_L}$: $\{q_0\}$, $\{q_1, q_3\}$, and $\{q_2, q_4\}$.

LSF example

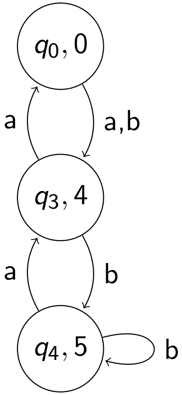
$\mathcal{A}_1$ variant of the automaton.



Possible order: $q_1 \simeq_1 q_2 \prec_1 q_3 \simeq_1 q_4$.
q_3 is the only $\preceq_1$-maximal element in $\{q_1, q_3\}$.
q_4 is the only $\preceq_1$-maximal element in $\{q_2, q_4\}$.
$\mu_{\mathsf{LSF}}^{1,\equiv_L}(\{q_1\}) = \{q_3\}$
$\mu_{\mathsf{LSF}}^{1,\equiv_L}(\{q_2\}) = \{q_4\}$

LSF example

After the merge:



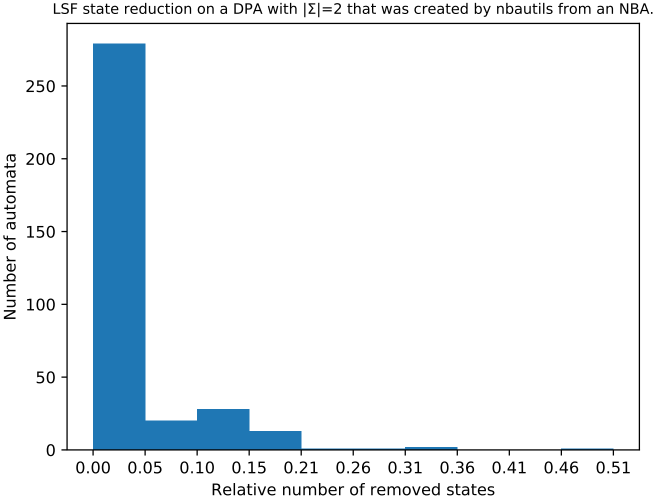
Computing LSF

The definition provides a straight-forward computation: $\equiv_M^{\leq k}$ is only a slight variation of the normal Moore equivalence and $\preceq_k$ can be computed with a topological sorting on the SCCs of $\mathcal{A}_k$.

Theorem.

$\mu_{LSF}^{k,\sim}$ can be computed in $\mathcal{O}(n \log n)$.

Efficiency



LSF state reduction on a DPA with |Σ|=2 that was created by nbautils from an NBA.
Number of automata
0
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150
200
250
0.00
0.05
0.10
0.15
0.21
0.26
0.31
0.36
0.41
0.46
0.51
Relative number of removed states

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