0.1 Threshold Moore

Definition 0.1.1. Let $x, y, n \in \mathbb{N}$. We write $x =^{\leq n} y$ if $x = y$ or $x, y > n$.

Let $\mathcal{A} = (Q, \Sigma, q_0, \delta, c)$ be a DPA. For $k \in c(Q)$, we define $\equiv_M^{\leq k} \subseteq Q \times Q$ as a relation, such that $p \equiv_M^{\leq k} q$ if and only if for all $w \in \Sigma^*$, $c(\delta^*(p, w)) =^{\leq k} c(\delta^*(q, w))$. We call $\equiv_M^{\leq k}$ the *k-threshold Moore equivalence*.

Lemma 0.1.1. *Let $\mathcal{A} = (Q, \Sigma, q_0, \delta, c)$ be a DPA and let $\mathcal{A}' = (Q, \Sigma, q_0, \delta, c')$ with $c'(q) = \min\{k + 1, c(q)\}$. Then $\equiv_M^{\leq k}$ of $\mathcal{A}$ is equal to $\equiv_M$ of $\mathcal{A}'$.*

Proof. Follows directly from the definition of $=^{\leq k}$. □

Corollary 0.1.2. *$\equiv_M^{\leq k}$ is a congruence relation.*

Definition 0.1.2. Let $\mathcal{A}$ be a DPA and let R be an equivalence relation on the state space that implies language equivalence. We define a relation $\equiv_{\text{TM}}^R$ such that $p \equiv_{\text{TM}} q$ if and only if all of the following are satisfied:

1. $c(p) = c(q)$
2. $p \equiv_M^{\leq c(p)} q$
3. $(p, q) \in R$

Theorem 0.1.3. *Let $\mathcal{A}$ and R as before and let $\mathcal{A}'$ be a representative merge of $\mathcal{A}$ w.r.t. an equivalence class λ of $\equiv_{TM}^R$. Then $L(\mathcal{A}) = L(\mathcal{A}')$.*

Proof. Let $q \in Q'$ be a state in the representative merge and let $\alpha \in \Sigma^\omega$. Let ρ and ρ' be the runs of $\mathcal{A}$ and $\mathcal{A}'$ on α starting from q. We claim that ρ is accepting iff ρ' is accepting.

We show that for all i, $\rho(i) \equiv_L \rho'(i)$ (in $\mathcal{A}$) and $\rho(i) \equiv_M^{\leq k} \rho'(i)$. If that is true, then there are two cases: if $c(\rho)$ sees infinitely many priorities of at most k, then $c(\rho')$ sees the same priorities at the same positions and thus $\min \text{Inf}(c(\rho)) = \min \text{Inf}(c(\rho'))$. Otherwise there is a position n from which $c(\rho)$ only is greater than k and therefore the same is true for $c(\rho')$. That means reading $\alpha[n, \omega]$ from $\rho'(n)$ in either $\mathcal{A}$ or $\mathcal{A}'$ yields the same run which has the same acceptance as ρ.

Claim For all i, $\rho(i) \equiv_L \rho'(i)$ and $\rho(i) \equiv_M^{\leq k} \rho'(i)$.

First, consider $i = 0$. If $\rho'(0) \notin \lambda$, then $\rho(0) = \rho'(0)$ and we are done. Otherwise, $\rho'(0) = r_\lambda \in \lambda$. Then $\rho(0) \in \lambda$ and thus $\rho(0) \equiv_M^{\leq k} \rho'(0)$ and $(\rho(0), \rho'(0)) \in R$. Since $R \subseteq \equiv_L$, we are done.

Next, consider $i + 1 > 0$. If $\rho'(i + 1) \notin \lambda$, then $\rho'(i + 1) = \delta(\rho'(i), \alpha(i))$. By induction, $\rho(i) \equiv_L \rho'(i)$. Since $\equiv_L$ is a congruence relation this implies $\rho(i + 1) \equiv_L \rho'(i + 1)$. The same argumentation works for $\equiv_M^{\leq k}$.

Otherwise, $\rho'(i + 1) = r_\lambda \in \lambda$. That means $\delta(\rho'(i), \alpha(i)) = p$ for some $p \in \lambda$. Since $\equiv_L$ is a congruence relation, $p \equiv_L \delta(\rho(i), \alpha(i)) = \rho(i + 1)$. Because $p \equiv_{\text{TM}}^R \rho'(i + 1)$, $p \equiv_L \rho'(i + 1)$ and together this gives $\rho'(i + 1) \equiv_L \rho(i + 1)$. The same argumentation works for $\equiv_M^{\leq k}$. □

Lemma 0.1.4. *Let $\mathcal{A}$ be a DPA and let p and q be two states with $p \equiv_M q$. We construct $\mathcal{A}'$ from $\mathcal{A}$ by redirecting all transitions to p to q instead. Then for all states $r \neq p$ and all words w, $c(\delta^*(r, w)) = c'(\delta'^*(r, w))$.*

Proof. Let ρ and ρ' be the runs of $\mathcal{A}$ and $\mathcal{A}'$ on w starting in r. If ρ never visits p, then $\rho = \rho'$ and the proof is done. Otherwise, let n be the last position at which $\rho(n) = p$. Then $\rho'(n) = q$. Since $p \equiv_M q$, $c(\delta^*(p,u)) = c(\delta^*(q,u))$ for all $u \in \Sigma^*$ and especially for $u = w[n, |w|]$. Since n was chosen as the last position where p is visited, $\delta^*(q,u) = \delta'^*(q,u)$ and therefore $c(\delta^*(p,u)) = c'(\delta'^*(q,u))$ which finishes the proof. □

Lemma 0.1.5. *Let $\mathcal{A}$ and R as before and let $\mathcal{A}'$ be a representative merge of $\mathcal{A}$ w.r.t. an equivalence class λ of $\equiv^R_{TM}$. Let k be the priority of the states in λ and let $\equiv^{\leq l}_M$ and $\equiv^{\trianglelefteq l}_M$ be the l-threshold Moore equivalences of $\mathcal{A}$ and $\mathcal{A}'$. If $l \leq k$, then $\equiv^{\leq l}_M$ and $\equiv^{\trianglelefteq l}_M$ are the same.*

Proof. A representative merge w.r.t. λ can be seen as a repeated redirection of transitions, meaning that Lemma 0.1.4 applies. Together with Lemma 0.1.1, that already finishes our proof. □

On the other hand, figures **??** show that if $l > k$, the l-threshold Moore equivalence can both grow or shrink during the merge step.