0.1 Threshold Moore

Definition 0.1.1. Let $x, y, n \in \mathbb{N}$. We write $x =^{\leq n} y$ if $x = y$ or $x, y > n$.

Let $\mathcal{A} = (Q, \Sigma, q_0, \delta, c)$ be a DPA. For $k \in c(Q)$, we define $\equiv_M^{\leq k} \subseteq Q \times Q$ as a relation, such that $p \equiv_M^{\leq k} q$ if and only if for all $w \in \Sigma^*$, $c(\delta^*(p, w)) =^{\leq k} c(\delta^*(q, w))$. We call $\equiv_M^{\leq k}$ the *k-threshold Moore equivalence.*

Lemma 0.1.1. *Let $\mathcal{A} = (Q, \Sigma, q_0, \delta, c)$ be a DPA and let $\mathcal{A}' = (Q, \Sigma, q_0, \delta, c')$ with $c'(q) = \min\{k + 1, c(q)\}$. Then $\equiv_M^{\leq k}$ of $\mathcal{A}$ is equal to $\equiv_M$ of $\mathcal{A}'$.*

Proof. Follows directly from the definition of $=^{\leq k}$. $\square$

Corollary 0.1.2. *$\equiv_M^{\leq k}$ is a congruence relation.*

Definition 0.1.2. Let $\mathcal{A}$ be a DPA and let R be an equivalence relation on the state space that implies language equivalence. We define a relation $\equiv_{\text{TM}}^R$ such that $p \equiv_{\text{TM}} q$ if and only if all of the following are satisfied:

1. $c(p) = c(q)$
2. $p \equiv_M^{\leq c(p)} q$
3. $(p, q) \in R$

Lemma 0.1.3. *Let $\mathcal{A}$ be a DPA. Let $\equiv$ be a congruence relation on Q and let R be a equivalence relation on Q such that $R \subseteq \equiv$. Let $\mathcal{A}'$ be a representative merge of $\mathcal{A}$ w.r.t. R. Let ρ and ρ' be the runs of $\mathcal{A}$ and $\mathcal{A}'$ on some α. Then for all i, $\rho(i) \equiv \rho'(i)$.*

Proof. We use a proof by induction. For $i = 0$, we have $\rho(0) = q_0$ and $\rho'(0) = r_{[q_0]_R}$. By choice of the representative, $(q_0, r_{[q_0]_R}) \in R$ and thus $q_0 \equiv r_{[q_0]_R}$.

Now consider some $i + 1 > 0$. Then $\rho'(i + 1) = r_{[q]_R}$ for $q = \delta(\rho'(i), \alpha(i))$. By induction we know that $\rho(i) \equiv \rho'(i)$ and thus $\delta(\rho(i), \alpha(i)) = \rho(i + 1) \equiv q$. Further, we know $q \equiv r_{[q]_R}$ by the same argument as before. Together this lets us conclude in $\rho(i + 1) \equiv q \equiv \rho'(i + 1)$. $\square$

Theorem 0.1.4. *Let $\mathcal{A}$ and R as before and let $\mathcal{A}'$ be a representative merge of $\mathcal{A}$ w.r.t. an equivalence class λ of $\equiv_{\text{TM}}^R$. Then $L(\mathcal{A}) = L(\mathcal{A}')$.*

Proof. Let $q \in Q'$ be a state in the representative merge and let $\alpha \in \Sigma^\omega$. Let ρ and ρ' be the runs of $\mathcal{A}$ and $\mathcal{A}'$ on α starting from q. We claim that ρ is accepting iff ρ' is accepting.

By Lemma 0.1.3, $\rho(i) \equiv_L \rho'(i)$ and $\rho(i) \equiv_M^{\leq k} \rho'(i)$ for all i. Now there are two cases: if $c(\rho)$ sees infinitely many priorities of at most k, then $c(\rho')$ sees the same priorities at the same positions and thus $\min \text{Inf}(c(\rho)) = \min \text{Inf}(c(\rho'))$. Otherwise there is a position n from which $c(\rho)$ only is greater than k and therefore the same is true for $c(\rho')$. That means reading $\alpha[n, \omega]$ from $\rho'(n)$ in either $\mathcal{A}$ or $\mathcal{A}'$ yields the same run which has the same acceptance as ρ. $\square$

Lemma 0.1.5. *Let $\mathcal{A}$ be a DPA and let p and q be two states with $p \equiv_M q$. We construct $\mathcal{A}'$ from $\mathcal{A}$ by redirecting all transitions to p to q instead. Then for all states $r \neq p$ and all words w, $c(\delta^*(r, w)) = c'(\delta'^*(r, w))$.*

Proof. Let ρ and ρ' be the runs of $\mathcal{A}$ and $\mathcal{A}'$ on w starting in r. If ρ never visits p, then $\rho = \rho'$ and the proof is done. Otherwise, let n be the last position at which $\rho(n) = p$. Then $\rho'(n) = q$. Since $p \equiv_M q$, $c(\delta^*(p, u)) = c(\delta^*(q, u))$ for all $u \in \Sigma^*$ and especially for $u = w[n, |w|]$. Since n was chosen as the last position where p is visited, $\delta^*(q, u) = \delta'^*(q, u)$ and therefore $c(\delta^*(p, u)) = c'(\delta'^*(q, u))$ which finishes the proof. □

Lemma 0.1.6. *Let $\mathcal{A}$ and R as before and let $\mathcal{A}'$ be a representative merge of $\mathcal{A}$ w.r.t. an equivalence class λ of $\equiv^R_{\mathrm{TM}}$. Let k be the priority of the states in λ and let $\equiv^{\leq l}_M$ and $\equiv^{\trianglelefteq l}_M$ be the l-threshold Moore equivalences of $\mathcal{A}$ and $\mathcal{A}'$. If $l \leq k$, then $\equiv^{\leq l}_M$ and $\equiv^{\trianglelefteq l}_M$ are the same.*

Proof. A representative merge w.r.t. λ can be seen as a repeated redirection of transitions, meaning that Lemma 0.1.5 applies. Together with Lemma 0.1.1, that already finishes our proof. □

On the other hand, figures **??** show that if $l > k$, the l-threshold Moore equivalence can both grow or shrink during the merge step.

0.2 Labeled SCC Filter

Definition 0.2.1. Let $\mathcal{A} = (Q, \Sigma, q_0, \delta, c)$ be a DPA. We define $\mathcal{A} \upharpoonright^c_{=k} := \mathcal{A} \upharpoonright_P$ with $P = \{q \in Q \mid c(q) = k\}$. Analogously, we define $\mathcal{A} \upharpoonright^c_{>k}$.

We define a relation $R_k \subseteq Q \times Q$ such that $(p, q) \in R_k$ if and only if all of the following are true:

1. $\min\{c(p), c(q)\} > k$
2. $p \equiv_L q$
3. $p \equiv^{\leq k}_M q$
4. In $\mathcal{A} \upharpoonright^c_{>k}$, p and q lie in different SCCs.

We define $\equiv^k_{\mathrm{LSF}} \subseteq Q \times Q$ to be the reflexive and transitive closure of R_k.

Lemma 0.2.1. *$\equiv^k_{LSF}$ is an equivalence relation.*

Definition 0.2.2. Let $\mathcal{A}$ be a DPA and $k \in \mathbb{N}$. We define $\preceq_k \subseteq Q \times Q$ to be a total extension of the reachability preorder in $\mathcal{A} \upharpoonright^c_{\geq k}$.

Let λ be an equivalence class of $\equiv^k_{\mathrm{LSF}}$. Let $r \in \lambda$ be a representative of λ that is $\preceq_k$-maximal. We set $\lambda' := \{q \in \lambda \mid q \prec_k r\} \cup \{r\}$. We call an automaton $\mathcal{A}'$ a *LSF^k_λ-merge* of $\mathcal{A}$ if it is a representative merge of $\mathcal{A}$ w.r.t. λ' that uses the representative $r_{\lambda'} = r$.

Theorem 0.2.2. *Let $\mathcal{A}$ be a DPA and let $\mathcal{A}'$ be a LSF^k_λ-merge of $\mathcal{A}$. Then $L(\mathcal{A}) = L(\mathcal{A}')$.*

Proof. Let r_λ be the representative that is used in the construction of $\mathcal{A}'$. Let $q \in Q'$ be a state in the representative merge and let $\alpha \in \Sigma^\omega$. Let ρ and ρ' be the runs of $\mathcal{A}$ and $\mathcal{A}'$ on α starting from q. We claim that ρ is accepting iff ρ' is accepting.

By Lemma 0.1.3, we know that $\rho(i) \equiv_L \rho'(i)$ and $\rho(i) \equiv^{\leq k}_M \rho'(i)$ for all i. If there is a position n from which on $\rho'[n, \omega]$ is both a valid run in $\mathcal{A}$ and $\mathcal{A}'$, then we know that ρ is accepting if and only if ρ' is accepting since $\rho(n) \equiv_L \rho'(n)$.

If ρ' visits infinitely many states with priority equal to or less than k, then ρ and ρ' share the same minimal priority that is visited infinitely often and thus have the same acceptance.

For the last case, assume that ρ' uses infinitely many redirected edges but from some point n_1 on stays in $\mathcal{A}\restriction^c_{>k}$. Let $n_3 > n_2 > n_1$ be the next two positions at which ρ' uses a redirected edge, i.e. $\delta(\rho'(n_2), \alpha(n_2)) \neq \delta'(\rho'(n_2), \alpha(n_2))$ and analogous for n_3. Note that $\delta'(\rho'(n_2), \alpha(n_2)) = \delta'(\rho'(n_3), \alpha(n_3)) = r_\lambda$, since all redirected transition target the representative state. Let we call $\delta(\rho'(n_3), \alpha(n_3)) = q$. Since between n_2 and n_3 no redirected transition is taken, $\rho'[n_2, n_3]$ is a valid path in $\mathcal{A}$, so we have $r_\lambda \preceq_k q$ by choice of n_1. The fact that transitions to q are redirected to r_λ however requires that $q \prec_k r_\lambda$, which would be a contradiction. □

Lemma 0.2.3. *Let $\mathcal{A}$ be a DPA and let $\mathcal{A}'$ be a LSF$^k_\lambda$-merge of $\mathcal{A}$. Let $\equiv^l_{LSF}$ be the LSF-relation in $\mathcal{A}$ and let $\equiv^l_{LSF'}$ be the LSF-relation in $\mathcal{A}'$. If $l \leq k$, then $\equiv^l_{LSF}\restriction_{Q' \times Q'} \supseteq \equiv^l_{LSF'}$.*

Proof. Let R_l and R'_l be the relations used in the definition of $\equiv^l_{\mathrm{LSF}}$ and $\equiv^l_{\mathrm{LSF'}}$. We prove $R'_l \subseteq R'_l\restriction_{Q' \times Q'}$. If that is true, then so is the statement of our Lemma. We do so by considering the four properties of R_l individually.

The first point is clear; $c' = c\restriction_{Q'}$, so $c'(p) = c(p)$ and $c'(q) = c(q)$.

For the second point, consider Theorem 0.2.2. By making p or q the initial state of our DPA, we observe that neither state has its language changed by the construction, so they must still be equal.

For the third point, let $\equiv^{\trianglelefteq l}_M$ be the l-threshold Moore equivalence in $\mathcal{A}'$. Let $w \in \Sigma^*$ be an arbitrary word, $p \equiv^{\leq l}_M$, $p' := (\delta')^*(p, w)$, and $q' := (\delta')^*(q, w)$. Using Lemma 0.1.3, we know that, since $p \equiv^{\leq l}_M q$, also $p' \equiv^{\leq l}_M q'$. In particular, this means $c(p') =^{\leq l} c(q')$. As w was chosen to be arbitrary, that means $p \equiv^{\trianglelefteq l}_M q$.

Lastly, for the fourth point, assume that there are states p, q which lie in different SCCs in $\mathcal{A}\restriction^c_{>l}$ but not in $\mathcal{A}'\restriction^c_{>l}$. Without loss of generality, we assume that in $\mathcal{A}\restriction^c_{>l}$, p is not reachable from q. In $\mathcal{A}'\restriction^c_{>l}$ however, this is possible, so let ρ' be a path from q to p. We can assume ρ' to pay exactly one visit to λ; there has to be at least one visit, as otherwise the path would also be available in $\mathcal{A}$; if there would be multiple visits, all of them would end at r_λ, so we could cut those parts from the run. Let uv be words that induce that run, i.e. $\delta^*(q, u) \in \lambda$ and $\delta^*(r_\lambda, v) = p$.

We distinguish two cases. In the first case, q is reachable from p in $\mathcal{A}$ by some word w. Here, consider reading the word vwu from r_λ in $\mathcal{A}$. The run moves to p by v, then to q by w, then to $\delta^*(q, u) \in \lambda$. $\delta^*(q, u)$ was the state from which the redirected transition was taken in ρ', so it cannot be reachable from r_λ by definition of the merge. This is a contradiction.

For the second case, q is not reachable from p in $\mathcal{A}$. Since the two states lie in a common SCC in $\mathcal{A}'$ however, there is a path π' from p to q. With the same argument as before, we can assume that π' leads to r_λ via some word u' and from there to q via some v'. As in the first case, the word $v'u$ gives us a path from r_λ to $\delta^*(q, u)$ which is a contradiction. □

The two previous statements provide us with a possible algorithm to perform state space reduction with the LSF method. Starting at $k = \min c(Q) - 1$ and iterating up to $\max c(Q)$, compute $\equiv^k_{\mathrm{LSF}}$ and build representative merges of each equivalence class. By Lemma 0.2.3, strictly iterating once through all k in ascending order gives us all possible merges.

The final question that remains is how to compute $\equiv^k_{\mathrm{LSF}}$ itself. This can be done rather easily