Definition 0.0.1. Let $\mathcal{A} = (Q, \Sigma, q_0, \delta, c)$ be a deterministic parity automaton. For $w \in \Sigma^* \cup \Sigma^\omega$ and $q \in Q$, we define $\lambda_{\mathcal{A}}(q, w) \in \mathbb{N}^{1+|w|}$ as follows: Let $q_0 q_1 \cdots \in Q^{1+|w|}$ be the unique run of $\mathcal{A}$ on w. Then $\lambda_{\mathcal{A}}(q, w)(n) = c(q_n)$.

Two DPAs $\mathcal{A}$ and $\mathcal{B}$ are **priority almost-equivalent**, if for all words $\alpha \in \Sigma^\omega$, $\lambda_{\mathcal{A}}(q_0^{\mathcal{A}}, \alpha)$ and $\lambda_{\mathcal{B}}(q_0^{\mathcal{B}}, \alpha)$ differ in only finitely many positions. We call two states $p, q \in Q$ of $\mathcal{A}$ priority almost-equivalent, $\mathcal{A}_q$ and $\mathcal{A}_p$ are priority almost-equivalent, where $\mathcal{A}_q$ behaves like $\mathcal{A}$ with initial state q.

We define the **reachability order** $\preceq^{\mathcal{A}}_{\text{reach}} \subseteq Q \times Q$ as $p \preceq^{\mathcal{A}}_{\text{reach}} q$ iff q is reachable from p. ("p is closer to q_0 than q"). Note that $p \preceq^{\mathcal{A}}_{\text{reach}} q$ and $q \preceq^{\mathcal{A}}_{\text{reach}} p$ together mean that p and q reside in the same SCC.

Lemma 0.0.1. *Priority almost-equivalence is a congruence relation.*

Definition 0.0.2. Let $\mathcal{A} = (Q, \Sigma, q_0, \delta, c)$ be a DPA and let $\sim \subseteq Q \times Q$ be a congruence relation on $\mathcal{A}$. We define the **Schewe** automaton $\mathcal{S}$ as follows:

For each state q, let $[q]_\sim = \{p \in Q \mid q \sim p\}$ be its equivalence class of $\sim$ and let ${}^{Q}/_{\sim} = \{[q]_\sim \mid q \in Q\}$ be the set of equivalence classes. For each such class $\mathfrak{c}$ we fix a representative $r_{\mathfrak{c}} \in \mathfrak{c}$ which is $\preceq^{\mathcal{A}}_{\text{reach}}$-maximal in its class, meaning that all states in $\mathfrak{c}$ that are reachable from $r_{\mathfrak{c}}$ are also in its SCC.

The automaton is then almost the same as the original DPA, with only a few modifications. Namely, $\mathcal{S} = (Q, \Sigma, r_{[q_0]_\sim}, \delta_{\mathcal{S}}, c)$.

For each transition $\delta_{\mathcal{S}}(q, a)$, let $\delta(q, a) = p$. If $q \prec^{\mathcal{A}}_{\text{reach}} r_{[p]_\sim}$ (i.e. q is not reachable from the representative of $[p]_\sim$), then $\delta_{\mathcal{S}}(q, a) = r_{[p]_\sim}$. Otherwise, we keep $\delta_{\mathcal{S}}(q, a) = p$. In other words, every time a transition moves to a different quotient class, it skips to the representative which lies as "deep" inside the automaton as possible.

Lemma 0.0.2. *For a given $\mathcal{A}$ and $\sim$, the Schewe automaton $\mathcal{S}$ can be computed in $\mathcal{O}(|\mathcal{A}|)$.*

Proof. Using e.g. Kosaraju's algorithm **??**, the SCCs of $\mathcal{A}$ can be computed in $\mathcal{O}(|\mathcal{A}|)$ if we assume an adjacency list or similar as the underlying data structure. A topological sorting of the states and therefore the SCCs can then be computed in $\mathcal{O}(|\mathcal{A}|)$ again, e.g. by DFS. ☐

We focus on a specialized version of the Schewe automaton. Let $\sim$ be the priority equivalence and let $\mathcal{S}$ be the according automaton. We define $\mathcal{S}_m$ as the Moore-minimization of $\mathcal{S}$.

Lemma 0.0.3. *Priority almost-equivalence implies language equivalence.*

Proof. Let $\mathcal{A} = (Q_{\mathcal{A}}, \Sigma, q_0^{\mathcal{A}}, \delta_{\mathcal{A}}, c_{\mathcal{A}})$ and $\mathcal{B} = (Q_{\mathcal{B}}, \Sigma, q_0^{\mathcal{B}}, \delta_{\mathcal{B}}, c_{\mathcal{B}})$ be two DPA that are priority almost-equivalent and assume towards a contradiction that they are not language equivalent. Due to symmetry we can assume that there is a $w \in L(\mathcal{A}) \setminus L(\mathcal{B})$.

Consider $\alpha = \lambda_{\mathcal{A}}(q_0^{\mathcal{A}}, w)$ and $\beta = \lambda_{\mathcal{B}}(q_0^{\mathcal{B}}, w)$, the priority outputs of the automata on w. By choice of w, we know that $a := \max \mathrm{Inf}(\alpha)$ is even and $b := \max \mathrm{Inf}(\beta)$ is odd. Without loss of generality, assume $a > b$. That means a is seen only finitely often in β but infinitely often in a. Hence, α and β differ at infinitely many positions where a occurs in α. That would mean w is a witness that the two automata are not priority almost-equivalent, contradicting our assumption. ☐

Lemma 0.0.4. *Let $\mathcal{A}$ a DPA, $\sim$ the relation of priority almost-equivalence, and $\mathcal{S}$ be the Schewe automaton. Then $\mathcal{A}$ and $\mathcal{S}$ are priority almost-equivalent.*

Proof. Let $\mathcal{A} = (Q, \Sigma, q_0^{\mathcal{A}}, \delta_{\mathcal{A}}, c)$ and $\mathcal{S} = (Q, \Sigma, q_0^{\mathcal{S}}, \delta_{\mathcal{S}}, c)$. Let $\alpha \in \Sigma^\omega$. We have to show that $\lambda_{\mathcal{A}}(q_0^{\mathcal{A}}, \alpha)$ and $\lambda_{\mathcal{S}}(q_0^{\mathcal{S}}, \alpha)$ differ in only finitely many positions. For that, let $a_0 a_1 \cdots \in Q^\omega$ and $s_0 s_1 \cdots \in Q^\omega$ be the respective runs of the automata on α.

Claim 1 For all i, $\mathcal{A}_{a_i}$ and $\mathcal{A}_{s_i}$ are priority almost-equivalent. ($a_i \sim s_i$)

For $i = 0$, we have $a_i = q_0^{\mathcal{A}}$ and $s_i = q_0^{\mathcal{S}} = r_{[q_0^{\mathcal{A}}]_\sim}$. By definition, $s_i = r_{[q_0^{\mathcal{A}}]_\sim} \in [q_0^{\mathcal{A}}]_\sim$, so $s_i \sim a_i$.

Using induction, assume $a_i \sim s_i$ and consider $i+1$. We seperate two cases: If $\delta_{\mathcal{S}}(s_i, \alpha(i)) = \delta_{\mathcal{A}}(s_i, \alpha(i))$, then $a_{i+1} \sim s_{i+1}$ follows from $\sim$ being a congruence relation.

For the second case, consider $\delta_{\mathcal{S}}(s_i, \alpha(i)) = r_{[p]_\sim}$, where $\delta_{\mathcal{A}}(s_i, \alpha(i)) = p$. Again we have $s_{i+1} = r_{[p]_\sim} \in [p]_\sim$, so $s_{i+1} \sim p$. Since $\sim$ is a congruence relation, we have $\delta_{\mathcal{A}}(a_i, \alpha(i)) \sim \delta_{\mathcal{A}}(s_i, \alpha(i))$, which is $a_{i+1} \sim p$ and therefore $a_{i+1} \sim s_{i+1}$ by transitivity.

Claim 2 $\lambda_{\mathcal{A}}(q_0^{\mathcal{A}}, \alpha)$ and $\lambda_{\mathcal{S}}(q_0^{\mathcal{S}}, \alpha)$ differ in only finitely many positions.

We can see from the definition of $\mathcal{S}$ that the "new type of transition" is taken only when the target state of $\delta_{\mathcal{A}}$ is strictly bigger w.r.t. $\preceq^{\mathcal{A}}_{\text{reach}}$. Since this is a partial order on Q, this means in particular that from some point k onwards, only the transition type $\delta_{\mathcal{S}}(q, a) = \delta_{\mathcal{A}}(q, a)$ is taken. Hence, $s_k s_{k+1} \cdots$ is the valid run of $\mathcal{A}_{s_k}$ on some suffix β of α. If now $\lambda_{\mathcal{A}}(q_0^{\mathcal{A}}, \alpha)$ and $\lambda_{\mathcal{S}}(q_0^{\mathcal{S}}, \alpha)$ would differ at infinitely many positions, then also $\lambda_{\mathcal{A}}(a_k, \beta)$ and $\lambda_{\mathcal{A}}(s_k, \beta)$ would (as these are suffixes of the former). However, we have shown in claim 1 that $\mathcal{A}_{a_k}$ and $\mathcal{A}_{s_k}$ are priority almost-equivalent. □

Lemma 0.0.5. *Let $\mathcal{A}$ a DPA, $\sim$ the relation of priority almost-equivalence, and $\mathcal{S}$ be the Schewe automaton. If p and q are priority almost-equivalent states in $\mathcal{S}_m$, then they lie in the same SCC.*

Proof. Let $\sim$ the relation of priority almost-equivalence.

Claim 1 If p and q are reachable in $\mathcal{S}$ and $\mathcal{A}_p$ and $\mathcal{A}_q$ are priority almost-equivalent, then $\mathcal{S}_p$ and $\mathcal{S}_q$ are priority almost-equivalent.

Assume towards a contradiction that p, q form a counterexample with $\lambda_{\mathcal{S}}(p, \alpha)$ and $\lambda_{\mathcal{S}}(q, \alpha)$ differing at infinitely many positions. Let ρ_p, ρ_q, π_p, and π_q be the runs of $\mathcal{A}_p$, $\mathcal{A}_q$, $\mathcal{S}_p$, and $\mathcal{S}_q$ on α respectively. Similar to the proof of **??**, we can

Claim 2 Let p and q be reachable states in $\mathcal{S}$ that are priority almost-equivalent and in $\mathcal{A}$ there is a path ρ from p to q. Then ρ is also a valid path in $\mathcal{S}$.

Let $\rho = p_0 \cdots p_m$ with $p_0 = p$ and $p_m = q$. Assume towards a contradiction that k is the first position at which this path is not valid in $\mathcal{S}$, i.e. $\delta_{\mathcal{S}}(p_k, a) \neq p_{k+1}$. That means that the transition was redirected to $\delta_{\mathcal{S}}(p_k, a) = r_{[p_{k+1}]_\sim}$. Therefore, $r_{[p_k]_\sim} \prec^{\mathcal{A}}_{\text{reach}} r_{[p_{k+1}]_\sim}$ by definition of the Schewe automaton. Let $u, v \in \Sigma^*$ with $p_0 \underset{\mathcal{A}}{\overset{u}{\to}} p_k \underset{\mathcal{A}}{\overset{a}{\to}} p_{k+1} \underset{\mathcal{A}}{\overset{v}{\to}} p_m$.

We define $\delta^*_{\mathcal{A}}(p_{k+1}, vu) = s$ and $\delta^*_{\mathcal{A}}(r_{[p_{k+1}]_\sim}, vu) = t$. Since $\mathcal{A}_{p_{k+1}}$ and $\mathcal{A}_{r_{[p_{k+1}]_\sim}}$ are priority almost-equivalent and $\sim$ is a congruence relation, also $\mathcal{A}_s$ and $\mathcal{A}_t$ are priority almost-equivalent. The former is priority-almost equivalent to $\mathcal{A}_{p_k}$. That means, in $\mathcal{A}$, we have $t \sim s \sim p_k \sim r_{[p_k]_\sim}$.

On the other hand we have shown that via vu we can reach t from $r_{[p_{k+1}]_\sim}$, so $r_{[p_{k+1}]_\sim} \preceq^{\mathcal{A}}_{\text{reach}} t$ and therefore $r_{[p_k]_\sim} \preceq^{\mathcal{A}}_{\text{reach}} t$. This together with the fact that $t \sim r_{[p_k]_\sim}$ contradicts the choice of $r_{[p_k]_\sim}$ as a $\preceq^{\mathcal{A}}_{\text{reach}}$-maximal element in its equivalence class. Hence, k cannot exist and ρ is a valid path in $\mathcal{S}$.

Claim 3 In the (not-minimized) Schewe automaton $\mathcal{S}$, for all reachable states q we have $q \preceq^{\mathcal{S}}_{\text{reach}} r_{[q]_\sim}$.

Let q be a reachable state, so there is a run $q_0 \cdots q_m$ in $\mathcal{S}$ on w with $q_m = q$. By definition of the automaton, there is a position k on this run at which a $\sim$-equivalence class $\mathfrak{c}$ is entered with $r_{[q]_\sim} \preceq^{\mathcal{A}}_{\text{reach}} r_{\mathfrak{c}}$. At latest this happens when $[q]_\sim$ is reached for the first time, as then the transition is directed to $r_{[q]_\sim}$ directly. Let $k' \geq k$ be the position in this run at which $[q]_\sim$ is entered for the first time.

Let $p_k \cdots p_{k'}$ be the run of $\mathcal{A}$ on $w(k) \cdots w(k'-1)$ with $p_k = q_k$. Using induction we can show that $\mathcal{A}_{p_i}$ and $\mathcal{A}_{q_i}$ are priority almost-equivalent for all $k \leq i \leq k'$. For $i = k$ this is obvious since $p_k = q_k$. Otherwise consider index $i+1$. Note that because $\sim$ is a congruence relation, $\mathcal{A}_{\delta_\mathcal{A}(p_i, w(i))}$ and $\mathcal{A}_{\delta_\mathcal{A}(q_i, w(i))}$ are priority almost-equivalent. The definition of the Schewe automaton sets $\delta_\mathcal{S}(q_i, w(i))$ either to the state $\delta_\mathcal{A}(q_i, w(i))$ or to its representative; in both cases, $\mathcal{A}_{\delta_\mathcal{S}(q_i, w(i))} = \mathcal{A}_{q_{i+1}}$ is priority almost-equivalent to $\mathcal{A}_{\delta_\mathcal{A}(q_i, w(i))}$ and therefore to $\mathcal{A}_{\delta_\mathcal{A}(p_i, w(i))} = \mathcal{A}_{q_i}$.

This observation together with claim 1 implies that all $\mathcal{S}_{p_i}$ and $\mathcal{S}_{q_i}$ are priority almost-equivalent as well. In particular, $p_{k'} \sim q$ (in $\mathcal{S}$).

Claim 4 If p and q are priority almost-equivalent in $\mathcal{S}$ and reachable, then they lie in the same SCC.

$\square$

Lemma 0.0.6. *There is no DPA priority almost-equivalent to $\mathcal{A}$ that is smaller than $\mathcal{S}_m$.*

Lemma 0.0.7. *The priority almost-equivalence of a DPA $\mathcal{A}$ can be computed in $\mathcal{O}(|\mathcal{A}|^2)$.*

Theorem 0.0.8. *For a given DPA $\mathcal{A}$, a minimal almost priority-equivalent automaton can be computed in $\mathcal{O}(|\mathcal{A}|^2)$.*

Definition 0.0.3. Let $\mathcal{A} = (Q, \Sigma, q_0, \delta, c)$ be a DPA. We define the **Moore-minimization** $\mathcal{B}$ as the parity automaton corresponding to the minimal Moore automaton of $\mathcal{A}$. That means it is the minimal automaton such that $\lambda_\mathcal{A}(\alpha) = \lambda_\mathcal{B}(\alpha)$ for all $\alpha \in \Sigma^\omega$.

More specifically, we define the congruence relation $\sim \subseteq Q \times Q$ by $p \sim q$ iff $\forall w \in \Sigma^* : \lambda_\mathcal{A}(p, w) = \lambda_\mathcal{A}(q, w)$. Then $\mathcal{A}'$ is constructed from $\mathcal{A}$ by removing unreachable states (from q_0). $\mathcal{B} = (Q/_\sim, \Sigma, [q_0]_\sim, \delta_\mathcal{B}, c_\mathcal{B})$ is the quotient automaton of $\mathcal{A}'/_\sim$ with $\delta_\mathcal{B}([q]_\sim, a) = [\delta(q, a)]_\sim$ and $c_\mathcal{B}([q]_\sim) = c(q)$.

Lemma 0.0.9. *For a given DPA $\mathcal{A}$, the Moore-minimization can be computed in $\mathcal{O}$.*