

Integration by Parts

Trial Lecture: A-Level Edexcel Math C4

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Learning Outcome:

Recognize when to use integration by parts; Use the integration-by-parts formula to solve integration problems

About Using Software to help you solve exercises

The screenshot shows the WolframAlpha website in a web browser. The address bar displays `https://www.wolframalpha.com`. The page features the WolframAlpha logo with the tagline "computational intelligence." Below the logo is a search bar with the placeholder text "Enter what you want to calculate or know about". To the right of the search bar are links for "Browse Examples" and "Surprise Me". A descriptive sentence reads: "Compute expert-level answers using Wolfram's breakthrough algorithms, knowledgebase and AI technology". The page is organized into four columns of category links, each with an icon and a "More >" link at the bottom.

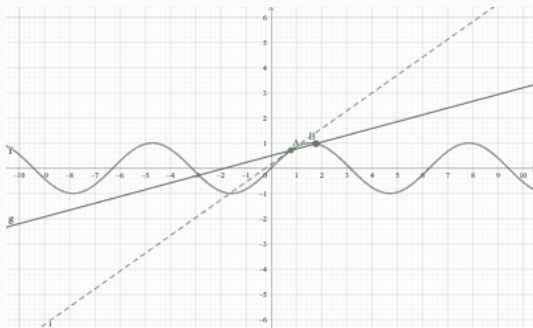
Mathematics >	Science & Technology >	Society & Culture >	Everyday Life >
Step-by-Step Solutions	Units & Measures	People	Personal Health
Elementary Math	Physics	Arts & Media	Personal Finance
Algebra	Chemistry	Dates & Times	Surprises
Plotting & Graphics	Engineering	Words & Linguistics	Entertainment
Calculus & Analysis	Computational Sciences	Money & Finance	Household Science
Geometry	Earth Sciences	Food & Nutrition	Household Math
Differential Equations	Materials	Political Geography	Hobbies
Statistics	Transportation	History	Today's World
More >	More >	More >	More >

Review: Derivative

Derivative

Given a function $f(x)$, and there is a point A has the coordinate $(x, f(x))$ in the function and another point B near A with $(x + h, f(x + h))$, and the **derivative** is defined as the tangent line AB as h becomes small and the chord becomes close to the tangent, i.e.

$$\frac{d}{dx}f(x) = f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}.$$



Basic Functions and their derivatives

Basic functions

Polynomial : $f(x) = \sum_{k=0}^n a_k x^k$

Constant : $f(x) = a$

Exponential : $f(x) = e^x$

$$f(x) = a^x$$

Logarithmic : $f(x) = \ln(x)$

Trigonometric : $f(x) = \sin(x)$

$$f(x) = \cos(x)$$

Derivatives

$$f'(x) = \sum_{k=1}^n a_k k \cdot x^{(k-1)}$$

$$f'(x) = 0$$

$$f'(x) = e^x$$

$$f'(x) = a^x \ln(a)$$

$$f'(x) = \frac{1}{x}$$

$$f'(x) = \cos(x)$$

$$f'(x) = -\sin(x)$$

Properties of Derivatives

Derivation Rules for all function $f(x)$ and $g(x)$ and all real numbers α and β

Sum Rule:

$$(\alpha f(x) + \beta g(x))' = \alpha f'(x) + \beta g'(x)$$

Product Rule:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x) \quad \star \quad (1)$$

Quotient Rule:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}, \quad \text{where } g(x) \neq 0$$

Chain Rule:

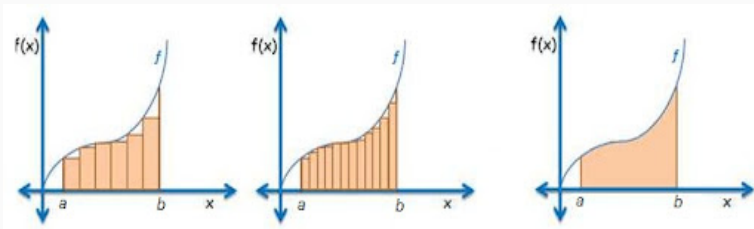
$$(f \circ g)'(x) = f'(g(x)) = f'(g(x)) \cdot g'(x).$$

Integration

What is Integration

Integration can be seen as the inverse of derivation.

Alternatively it is the calculation of the **area under a function**. (Thus all rules for sums apply here as well.).



Basic Function and Their Integral

Basic Function

Polynomial : $f(x) = x^n$

$$f(x) = \frac{1}{x}$$

Constant : $f(x) = a$

$$f(x) = 0$$

Exponential : $f(x) = e^x$

$$f(x) = a^x$$

Trigonometric : $f(x) = \sin(x)$

$$f(x) = \cos(x)$$

Integral

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

$$F(x) = \ln(|x|) + C$$

$$F(x) = ax + C$$

$$F(x) = a$$

$$F(x) = e^x + C$$

$$F(x) = x \ln(x) - 1$$

$$F(x) = -\cos(x)$$

$$F(x) = \sin(x)$$

Integration Rules

Sum Rule:

$$\int (f \pm g) \, dx = \int f \, dx \pm \int g \, dx$$

Properties of Integration

Integration Rules

Sum Rule:

$$\int (f \pm g) \, dx = \int f \, dx \pm \int g \, dx$$

Definite Integral:

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

Properties of Integration

Integration Rules

Sum Rule:

$$\int (f \pm g) \, dx = \int f \, dx \pm \int g \, dx$$

Definite Integral:

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

Substitution Rule:

$$\int_a^b f(\phi(x))\phi'(x) \, dx = \int_{\phi(a)}^{\phi(b)} f(u) \, dx$$

where $u = \phi(x)$.

Exercise from Past Paper

Consider the following integral

$$\int x\sqrt{x+4} \, dx.$$

$$\int (e^x + 1)^3 \, dx.$$

$$\int \cos^3(x) \cdot \sin(x) \, dx$$

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$$\int \cos^3(x) \cdot \sin(x) \, dx$$

Answer

$$\int x\sqrt{x+4} \, dx = \frac{2}{15}(x+4)^{3/2}(3x-8) + C$$

$$\int (e^x + 1)^3 \, dx = x + 3e^x + \frac{3e^{2x}}{2} + \frac{e^{3x}}{3} + C$$

$$\int \cos^3(x) \cdot \sin(x) \, dx = -\frac{1}{4}\cos^4(x) + C$$

Example

Consider the following integral

$$\int \ln x \, dx.$$

► Verbose mode 

Example

Consider the following integral

$$\int \ln x \, dx.$$

► Verbose mode ♠

►► See Solution in ??

Product of two Functions

Given two *continuous differentiable* function $f(x)$ and $g(x)$, and product rule states:

$$(fg)' = f'g + fg'.$$

integrating both sides with respect to x ,

$$\begin{aligned}\int (fg)'(x) \, dx &= \int f'(x) \cdot g(x) \, dx + \int f(x) \cdot g'(x) \, dx \\ \implies f(x) \cdot g(x) &= \int f'(x) \cdot g(x) \, dx + \int f(x) \cdot g'(x) \, dx \\ \implies \int f'(x) \cdot g(x) \, dx &= f(x) \cdot g(x) - \int f(x) \cdot g'(x) \, dx\end{aligned}$$

This yields the formula for definite integral:

$$\int_a^b f'(x) \cdot g(x) \, dx = f(x) \cdot g(x) \Big|_a^b - \int_a^b f(x) \cdot g'(x) \, dx. \quad (2)$$

Answer of Question in ??

We choose $f'(x) = 1$ and $g(x) = \ln(x)$ in ??, so we get $f(x) = x$ and $g'(x) = \frac{1}{x}$:

$$\begin{aligned}\int \underbrace{1 \cdot \ln(x)}_{f'(x) \cdot g(x)} dx &= \underbrace{x \cdot \ln(x)}_{f(x) \cdot g(x)} - \int \underbrace{x \cdot \frac{1}{x}}_{f(x) \cdot g'(x)} dx \\ &= x \ln(x) - x + C.\end{aligned}$$

Choosing $g(x)$ that comes first in the following list

1. L: logarithmic function
2. I: inverse trigonometric function
3. A: algebraic function (e.g. x^3 , $3x^{50}$)
4. T: trigonometric function
5. E: exponential function

Calculation of Integrals through Trick:LIATE Rule

Exercises

1.

$$\int_0^{\pi} \sin(x) \cdot x \, dx \quad (3)$$

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2.

$$\int e^x \cdot (2 - x^2) \, dx \quad (4)$$

Calculation of Integrals through Trick:LIATE Rule

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Calculation of Integrals through Trick:LIATE Rule

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$$\int_0^{\pi} \sin(x) \cdot x \, dx \quad (3)$$

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2.

$$\int e^x \cdot (2 - x^2) \, dx \quad (4)$$

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3.

$$\int x^3 e^{x^2} \, dx \quad (5)$$

Calculation of Integrals through Trick:LIATE Rule

Exercises

1.

$$\int_0^{\pi} \sin(x) \cdot x \, dx \quad (3)$$

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2.

$$\int e^x \cdot (2 - x^2) \, dx \quad (4)$$

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3.

$$\int x^3 e^{x^2} \, dx \quad (5)$$

► See Solution in ??

We choose $f'(x) = \sin(x)$ and $g(x) = x$ in ??, so that we get $f(x) = -\cos(x)$ and $g'(x) = 1$:

$$\begin{aligned}\int_0^{\pi} \underbrace{\sin(x) \cdot x}_{f'(x) \cdot g(x)} dx &= \underbrace{-\cos(x) \cdot x}_{f'(x) \cdot g(x)} \Big|_0^{\pi} - \int_0^{\pi} \underbrace{(-\cos(x)) \cdot 1}_{f'(x) \cdot g(x)} dx \\&= -\cos(x) \cdot x \Big|_0^{\pi} + \int_0^{\pi} \cos(x) dx \\&= -\cos(x) \cdot x \Big|_0^{\pi} + \sin(x) \Big|_0^{\pi} \\&= -\pi \cos(\pi) + 0 \cdot \cos(0) + \sin(\pi) - \sin(0) \\&= -\pi \cdot (-1) \\&= \pi\end{aligned}$$

We choose $f'(x) = e^x$ and $g(x)$ as the remaining polynomial term in ??, so that we get:

$$\begin{aligned}
 \int \underbrace{e^x \cdot (2 - x^2)}_{f'(x) \cdot g(x)} dx &= \underbrace{e^x \cdot (2 - x^2)}_{f'(x) \cdot g(x)} - \int \underbrace{e^x \cdot (-2x)}_{f'(x) \cdot g'(x)} dx \\
 &= e^x \cdot (2 - x^2) + \int \underbrace{e^x \cdot 2x}_{f'(x) \cdot g'(x)} dx \\
 &= e^x \cdot (2 - x^2) + \left(\underbrace{e^x \cdot (2x)}_{f'(x) \cdot g(x)} - \int \underbrace{e^x \cdot 2}_{f'(x) \cdot g'(x)} dx \right) \\
 &= e^x \cdot (2 - x^2) + e^x \cdot 2x - 2 \cdot e^x + C \\
 &= e^x(2x - x^2) + C
 \end{aligned}$$

We choose $f'(x) = x \cdot e^{x^2}$ and $g(x) = x^2$ in ??, so that we get $f(x) = \frac{e^{x^2}}{2}$ and $g'(x) = 2x$:

$$\begin{aligned}\int \underbrace{(xe^{x^2}) \cdot (x^2)}_{f'(x) \cdot g(x)} dx &= \underbrace{\frac{e^{x^2}}{2} \cdot x^2}_{f(x) \cdot g(x)} - \int \underbrace{\frac{e^{x^2}}{2} \cdot 2x}_{f(x) \cdot g'(x)} dx \\&= \frac{x^2 e^{x^2}}{2} - \int x \cdot e^{x^2} dx \\&= \frac{x^2 e^{x^2}}{2} - \frac{e^{x^2}}{2} + C \\&= \frac{e^{x^2}(x^2 - 1)}{2} + C\end{aligned}$$

Indirect Calculation of Integrals by ??

Exercise

Consider the following integral:

$$\int \sin(x) \cdot \cos(x) dx \quad (6)$$

Exercise

Consider the following integral:

$$\int \sin(x) \cdot \cos(x) dx \quad (6)$$

► See Solution in ??

Answer of ??

We choose $f'(x) = \sin(x)$ and $g(x) = \cos(x)$ in ??, so that we get $f(x) = -\cos(x)$ and $g'(x) = -\sin(x)$:

$$\begin{aligned}\int \underbrace{\sin(x) \cdot \cos(x)}_{f'(x) \cdot g(x)} dx &= \underbrace{-\cos(x) \cdot \cos(x)}_{f(x) \cdot g(x)} - \int \underbrace{(-\cos(x)) \cdot (-\sin(x))}_{f(x) \cdot g'(x)} dx \\ &= -\cos^2(x) - \int \cos(x) \cdot \sin(x) dx\end{aligned}$$

We add up both side of equation the integral $\int \cos(x) \cdot \sin(x) dx$, so we get

$$\begin{aligned}2 \int \cos(x) \cdot \sin(x) dx &= -\cos^2(x) \\ \Rightarrow \int \cos(x) \cdot \sin(x) dx &= -\frac{1}{2} \cos^2(x) + C.\end{aligned}$$

Calculation of Recursion Formula by ??

Example

Given the following integral,

$$\int \sin^n(x) dx,$$

where n is integer, prove the following recursion formula:

$$\int \sin^n(x) dx = -\frac{1}{n} \cos(x) \cdot \sin^{n-1}(x) + \frac{n-1}{n} \int \sin^{n-2}(x) dx \quad (7)$$

Example

Given the following integral,

$$\int \sin^n(x) \, dx,$$

where n is integer, prove the following recursion formula:

$$\int \sin^n(x) \, dx = -\frac{1}{n} \cos(x) \cdot \sin^{n-1}(x) + \frac{n-1}{n} \int \sin^{n-2}(x) \, dx \quad (7)$$

► See Solution in ??

$$\int \sin^n(x) dx = \int \sin(x) \cdot \sin^{n-1}(x) dx$$

↓ Choose $f'(x) = \sin(x)$ and $g(x) = \sin^{n-1}(x)$

$$\implies f(x) = -\cos(x) \text{ and } g'(x) = (n-1) \sin^{n-2}(x) \cos(x)$$

$$= \underbrace{-\cos(x) \cdot \sin^{n-1}(x)}_{f(x) \cdot g(x)} - \int \underbrace{(-\cos(x)) \cdot (n-1) \sin^{n-2}(x) \cos(x)}_{f(x) \cdot g(x)} dx$$

$$= -\cos(x) \cdot \sin^{n-1}(x) + (n-1) \int \cos^2(x) \sin^{n-2}(x) dx$$

We would like to make the integral in the right side in the form $\int \sin^m(x) dx$ with $m \leq n$, so that we use Pythagorean trigonometric identity,

$$\sin^2(x) + \cos^2(x) = 1 \implies \cos^2(x) = 1 - \sin^2(x)$$

and we get

$$\int \sin^n(x) dx = -\cos(x) \cdot \sin^{n-1}(x) + (n-1) \int \cos^2(x) \sin^{n-2}(x) dx$$

↓ Pythagorean identity

$$= -\cos(x) \cdot \sin^{n-1}(x) + (n-1) \int (1 - \sin^2(x)) \sin^{n-2}(x) dx$$

$$= -\cos(x) \cdot \sin^{n-1}(x) + (n-1) \int \sin^{n-2}(x) dx - (n-1) \int \sin^n(x) dx$$

We add up both side of equation the integral $(n-1) \int \sin^n(x) dx$, so we get

$$n \int \sin^n(x) dx = -\cos(x) \cdot \sin^{n-1}(x) + (n-1) \int \sin^{n-2}(x) dx.$$

We get the recursion formula through dividing the both side of equation by n

$$\int \sin^n(x) dx = -\frac{1}{n} \cos(x) \cdot \sin^{n-1}(x) + \frac{n-1}{n} \int \sin^{n-2}(x) dx$$

Presentation & Exercises



Now you have to work !

Basic Function and Their Integral

Basic Function

► Return

Polynomial : $f(x) = x^n$

$$f(x) = \frac{1}{x}$$

Constant : $f(x) = a$

$$f(x) = 0$$

Exponential : $f(x) = e^x$

$$f(x) = a^x$$

Trigonometric : $f(x) = \sin(x)$

$$f(x) = \cos(x)$$

Integral

► Next

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

$$F(x) = \ln(|x|) + C$$

$$F(x) = ax + C$$

$$F(x) = a$$

$$F(x) = e^x + C$$

$$F(x) = x \ln(x) - 1$$

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Properties of Integration

Integration Rules

Sum Rule:

$$\int (f \pm g) \, dx = \int f \, dx \pm \int g \, dx$$

Definite Integral:

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

Substitution Rule:

$$\int_a^b f(\psi(x))\psi'(x) \, dx = \int_{\psi(a)}^{\psi(b)} f(u) \, dx$$

where $u = \psi(x)$.

◀ Return ■

◀ Return ♣

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