Integration by Parts

Trial Lecture: A-Level Edexcel Math C4

Wusheng Zhang

Learning Outcome:

Recognize when to use integration by parts; Use the integration-by-parts formula to solve integration problems

About Using Software to help you solve exercises

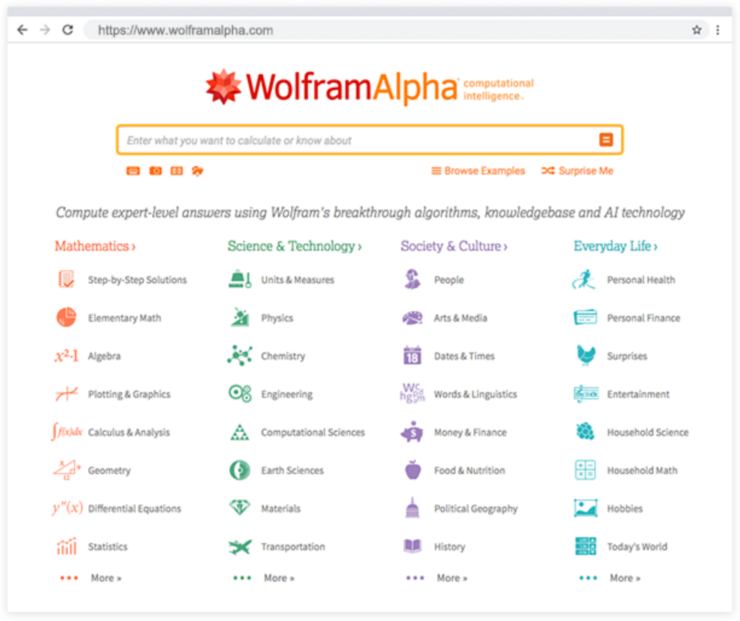
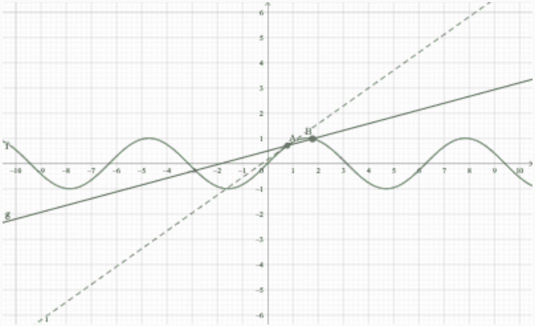


Table of Contents

Derivative

Given a function $f(x)$, and there is a point A has the coordinate $(x, f(x))$ in the function and another point B near A with $(x+h, f(x+h))$, and the **derivative** is defined as the tangent line AB as h becomes small and the chord becomes close to the tangent, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$



Basic Functions and their derivatives

Basic functions

$$\text{Polynomial}: \; f(x) = \sum_{k=0}^{n} a_k x^k$$
$$\text{Constant}: \; f(x) = a$$
$$\text{Expotential}: \; f(x) = e^x$$
$$f(x) = a^x$$
$$\text{Logarithmic}: \; f(x) = \ln(x)$$
$$\text{Trigonometric}: \; f(x) = \sin(x)$$
$$f(x) = \cos(x)$$

Derivatives

$$f'(x) = \sum_{k=1}^{n} a_k k \cdot x^{(k-1)}$$
$$f'(x) = 0$$
$$f'(x) = e^x$$
$$f'(x) = a^x \ln(x)$$
$$f'(x) = \frac{1}{x}$$
$$f'(x) = \cos(x)$$
$$f'(x) = -\sin(x)$$

Properties of Derivatives

Derivation Rules for all function $f(x)$ and $g(x)$ and all real numbers α and β

Sum Rule:

$$(\alpha f(x) + \beta g(x))' = \alpha f'(x) + \beta g'(x)$$

Product Rule:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x) \quad \star \tag{1}$$

Quotient Rule:

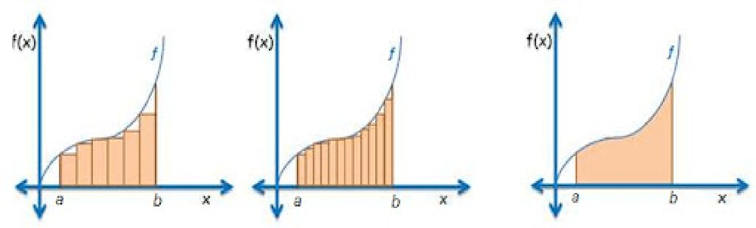
$$(\frac{f(x)}{g(x)})' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}, \quad \text{where } g(x) \neq 0$$

Chain Rule:

$$(f \circ g)'(x) = f'(g(x)) = f'(g(x)) \cdot g'(x).$$

What is Integration

Integration can be seen as the inverse of derivation.
Alternatively it is the calculation of the **area under a function**.(Thus all rules for sums apply here as well.).



Basic Function and Their Integral

Basic Function

$$\text{Polynomial}: \; f(x) = x^n$$
$$f(x) = \frac{1}{x}$$
$$\text{Constant}: \; f(x) = a$$
$$f(x) = 0$$
$$\text{Expotential}: \; f(x) = e^x$$
$$f(x) = a^x$$
$$\text{Trigonometric}: \; f(x) = \sin(x)$$
$$f(x) = \cos(x)$$

Integral

$$F(x) = \frac{x^{n+1}}{n+1} + C$$
$$F(x) = \ln(|x|) + C$$
$$F(x) = ax + C$$
$$F(x) = a$$
$$F(x) = e^x + C$$
$$F(x) = x\ln(x) - 1$$
$$F(x) = -\cos(x)$$
$$F(x) = \sin(x)$$

Properties of Integration

Integration Rules

Sum Rule:

$$\int (f \pm g)\,\mathrm{d}x = \int f\,\mathrm{d}x \pm \int g\,\mathrm{d}x$$

Properties of Integration

Integration Rules

Sum Rule:

$$\int (f \pm g)\,\mathrm{d}x = \int f\,\mathrm{d}x \pm \int g\,\mathrm{d}x$$

Definite Integral:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

Properties of Integration

Integration Rules

Sum Rule:

$$\int (f \pm g)\,\mathrm{d}x = \int f\,\mathrm{d}x \pm \int g\,\mathrm{d}x$$

Definite Integral:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

Substitution Rule:

$$\int_a^b f(\phi(x))\phi'(x)\,\mathrm{d}x = \int_{\phi(a)}^{\phi(b)} f(u)\,\mathrm{d}x$$

where $u = \phi(x)$.

Exercise from Past Paper

Consider the following integral

$$\int x\sqrt{x+4}\,\mathrm{d}x.$$

$$\int (e^x + 1)^3\,\mathrm{d}x.$$

$$\int \cos^3(x) \cdot \sin(x)\,\mathrm{d}x$$

Exercise from Past Paper

Consider the following integral

$$\int x\sqrt{x+4}\,\mathrm{d}x.$$

$$\int (e^x+1)^3\,\mathrm{d}x.$$

$$\int \cos^3(x)\cdot\sin(x)\,\mathrm{d}x$$

Answer

$$\int x\sqrt{x+4}\,\mathrm{d}x = \frac{2}{15}(x+4)^{3/2}(3x-8)+C$$

$$\int (e^x+1)^3\,\mathrm{d}x = x+3e^x+\frac{3e^{2x}}{2}+\frac{e^{3x}}{3}+C$$

$$\int \cos^3(x)\cdot\sin(x)\,\mathrm{d}x = -\frac{1}{4}\cos^4(x)+C$$

Table of Contents

An Exercise

Example

Consider the following integral

$$\int \ln x \, \mathrm{d}x.$$

Verbose mode

Example

Consider the following integral

$$\int \ln x \, \mathrm{d}x.$$

▸ Verbose mode ♠

▸▸ See Solution in ??

Product of two Functions

Given two *continuous differentiable* function $f(x)$ and $g(x)$, and product rule states:

$$(fg)' = f'g + fg'.$$

integrating both sides with respect to x,

$$\begin{aligned}
\int (fg)'(x)\,\mathrm{d}x &= \int f'(x)\cdot g(x)\,\mathrm{d}x + \int f(x)\cdot g'(x)\,\mathrm{d}x \\
\implies \quad f(x)\cdot g(x) &= \int f'(x)\cdot g(x)\,\mathrm{d}x + \int f(x)\cdot g'(x)\,\mathrm{d}x \\
\implies \quad \int f'(x)\cdot g(x)\,\mathrm{d}x &= f(x)\cdot g(x) - \int f(x)\cdot g'(x)\,\mathrm{d}x
\end{aligned}$$

This yields the formula for definte integral:

$$\int_a^b f'(x)\cdot g(x)\,\mathrm{d}x = f(x)\cdot g(x)\Big|_a^b - \int_a^b f(x)\cdot g'(x)\,\mathrm{d}x. \tag{2}$$

Answer of Question in ??

We choose $f'(x) = 1$ and $g(x) = \ln(x)$ in **??**, so we get $f(x) = x$ and $g'(x) = \frac{1}{x}$:

$$\int \underbrace{1 \cdot \ln(x)}_{f'(x) \cdot g(x)} \, \mathrm{d}x = \underbrace{x \cdot \ln(x)}_{f(x) \cdot g(x)} - \int \underbrace{x \cdot \frac{1}{x}}_{f(x) \cdot g'(x)} \, \mathrm{d}x$$

$$= x \ln(x) - x + C.$$

Choosing $g(x)$ that comes first in the following list

1. **L**: logarithmic function
2. **I**: inverse trigonometric function
3. **A**: algebraic function (e.g. $x^3, 3x^{50}$)
4. **T**: trigonometric function
5. **E**: expotential function

Calculation of Integrals through Trick:LIATE Rule

Exercises

1. $$\int_0^{\pi} \sin(x) \cdot x \, dx \tag{3}$$

Exercises

1.
$$\int_0^\pi \sin(x) \cdot x \, \mathrm{d}x \tag{3}$$

See Solution in ??

Exercises

1.
$$\int_0^\pi \sin(x) \cdot x \, \mathrm{d}x \tag{3}$$

See Solution in ??

2.
$$\int e^x \cdot (2 - x^2) \, \mathrm{d}x \tag{4}$$

Exercises

1.
$$\int_0^\pi \sin(x) \cdot x \, \mathrm{d}x \tag{3}$$

▸ See Solution in ??

2.
$$\int e^x \cdot (2 - x^2) \, \mathrm{d}x \tag{4}$$

▸ See Solution in ??

Exercises

1.
$$\int_0^{\pi} \sin(x) \cdot x \, \mathrm{d}x \tag{3}$$

See Solution in ??

2.
$$\int e^x \cdot (2 - x^2) \, \mathrm{d}x \tag{4}$$

See Solution in ??

3.
$$\int x^3 e^{x^2} \, \mathrm{d}x \tag{5}$$

Exercises

1.
$$\int_0^{\pi} \sin(x) \cdot x \, \mathrm{d}x \tag{3}$$

See Solution in ??

2.
$$\int e^x \cdot (2 - x^2) \, \mathrm{d}x \tag{4}$$

See Solution in ??

3.
$$\int x^3 e^{x^2} \, \mathrm{d}x \tag{5}$$

See Solution in ??

Answer of ??

We choose $f'(x) = \sin(x)$ and $g(x) = x$ in **??**, so that we get $f(x) = -\cos(x)$ and $g'(x) = 1$:

$$\begin{aligned}
\int_0^\pi \underbrace{\sin(x) \cdot x}_{f'(x)\cdot g(x)} \,\mathrm{d}x &= \underbrace{-\cos(x) \cdot x}_{f'(x)\cdot g(x)}\Big|_0^\pi - \int_0^\pi \underbrace{(-\cos(x)) \cdot 1}_{f'(x)\cdot g(x)} \,\mathrm{d}x \\
&= -\cos(x) \cdot x\Big|_0^\pi + \int_0^\pi \cos(x)\,\mathrm{d}x \\
&= -\cos(x) \cdot x\Big|_0^\pi + \sin(x)\Big|_0^\pi \\
&= -\pi\cos(\pi) + 0 \cdot \cos(0) + \sin(\pi) - \sin(0) \\
&= -\pi \cdot (-1) \\
&= \pi
\end{aligned}$$

Return

Answer of ??

We choose $f'(x) = e^x$ and $g(x)$ as the remaining polynomial term in **??**, so that we get:

$$\begin{aligned}
\int \underbrace{e^x \cdot (2 - x^2)}_{f'(x) \cdot g(x)} \, \mathrm{d}x &= \underbrace{e^x \cdot (2 - x^2)}_{f(x) \cdot g(x)} - \int \underbrace{e^x \cdot (-2x)}_{f(x) \cdot g'(x)} \, \mathrm{d}x \\
&= e^x \cdot (2 - x^2) + \int \underbrace{e^x \cdot 2x}_{f'(x) \cdot g(x)} \, \mathrm{d}x \\
&= e^x \cdot (2 - x^2) + \left(\underbrace{e^x \cdot (2x)}_{f'(x) \cdot g(x)} - \int \underbrace{e^x \cdot 2}_{f(x) \cdot g'(x)} \, \mathrm{d}x \right) \\
&= e^x \cdot (2 - x^2) + e^x \cdot 2x - 2 \cdot e^x + C \\
&= e^x (2x - x^2) + C
\end{aligned}$$

◂ Return ♣

Answer of ??

We choose $f'(x) = x \cdot e^{x^2}$ and $g(x) = x^2$ in **??**, so that we get $f(x) = \frac{e^{x^2}}{2}$ and $g'(x) = 2x$:

$$\begin{aligned}
\int \underbrace{(xe^{x^2}) \cdot (x^2)}_{f'(x) \cdot g(x)} \, \mathrm{d}x &= \underbrace{\frac{e^{x^2}}{2} \cdot x^2}_{f(x) \cdot g(x)} - \int \underbrace{\frac{e^{x^2}}{2} \cdot 2x}_{f(x) \cdot g'(x)} \, \mathrm{d}x \\
&= \frac{x^2 e^{x^2}}{2} - \int x \cdot e^{x^2} \, \mathrm{d}x \\
&= \frac{x^2 e^{x^2}}{2} - \frac{e^{x^2}}{2} + C \\
&= \frac{e^{x^2}(x^2 - 1)}{2} + C
\end{aligned}$$

Return

Indirect Calculation of Integrals by ??

Exercise

Consider the following integral:

$$\int \sin(x) \cdot \cos(x) \, \mathrm{d}x \tag{6}$$

Exercise

Consider the following integral:

$$\int \sin(x) \cdot \cos(x)\, \mathrm{d}x \tag{6}$$

See Solution in ??

We choose $f'(x) = \sin(x)$ and $g(x) = \cos(x)$ in **??**, so that we get $f(x) = -\cos(x)$ and $g'(x) = -\sin(x)$:

$$\int \underbrace{\sin(x) \cdot \cos(x)}_{f'(x) \cdot g(x)} \, \mathrm{d}x = \underbrace{-\cos(x) \cdot \cos(x)}_{f(x) \cdot g(x)} - \int \underbrace{(-\cos(x)) \cdot (-\sin(x))}_{f(x) \cdot g'(x)} \, \mathrm{d}x$$

$$= -\cos^2(x) - \int \cos(x) \cdot \sin(x) \, \mathrm{d}x$$

We add up both side of equation the integral $\int \cos(x) \cdot \sin(x) \, \mathrm{d}x$, so we get

$$2 \int \cos(x) \cdot \sin(x) \, \mathrm{d}x = -\cos^2(x)$$

$$\Longrightarrow \int \cos(x) \cdot \sin(x) \, \mathrm{d}x = -\frac{1}{2} \cos^2(x) + C.$$

Return

Calculation of Recursion Formula by ??

Example

Given the following integral,

$$\int \sin^n(x)\,\mathrm{d}x,$$

where n is integer, prove the following recursion formula:

$$\int \sin^n(x)\,\mathrm{d}x = -\frac{1}{n}\cos(x)\cdot\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,\mathrm{d}x \tag{7}$$

Example

Given the following integral,

$$\int \sin^n(x)\,\mathrm{d}x,$$

where n is integer, prove the following recursion formula:

$$\int \sin^n(x)\,\mathrm{d}x = -\frac{1}{n}\cos(x)\cdot\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,\mathrm{d}x \tag{7}$$

See Solution in ??

$$\int \sin^n(x)\,\mathrm{d}x = \int \sin(x)\cdot\sin^{n-1}(x)\,\mathrm{d}x$$

$$\downarrow \text{Choose } f'(x) = \sin(x) \text{ and } g(x) = \sin^{n-1}(x)$$

$$\implies f(x) = -\cos(x) \text{and } g'(x) = (n-1)\sin^{n-2}(x)\cos(x)$$

$$= \underbrace{-\cos(x)\cdot\sin^{n-1}(x)}_{f(x)\cdot g(x)} - \int \underbrace{(-\cos(x))\cdot(n-1)\sin^{n-2}(x)\cos(x)}_{f(x)\cdot g(x)}\,\mathrm{d}x$$

$$= -\cos(x)\cdot\sin^{n-1}(x) + (n-1)\int \cos^2(x)\sin^{n-2}(x)\,\mathrm{d}x$$

We would like to make the integral in the right side in the form $\int \sin^m(x)\,\mathrm{d}x$ with $m \leq n$, so that we use Pythagorean trigonometric identity,

$$\sin^2(x) + \cos^2(x) = 1 \quad \implies \cos^{(x)} = 1 - \sin^2(x)$$

and we get

Next

Cont. Answer of ??

$$\int \sin^n(x)\,\mathrm{d}x = -\cos(x)\cdot\sin^{n-1}(x) + (n-1)\int \cos^2(x)\sin^{n-2}(x)\,\mathrm{d}x$$

$\downarrow$ Pythagorean identity

$$= -\cos(x)\cdot\sin^{n-1}(x) + (n-1)\int (1-\sin^2(x))\sin^{n-2}(x)\,\mathrm{d}x$$

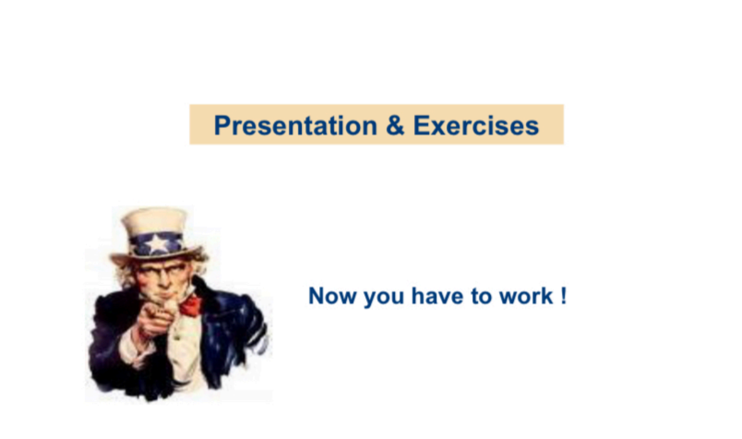
$$= -\cos(x)\cdot\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\,\mathrm{d}x - (n-1)\int \sin^n(x)\,\mathrm{d}x$$

We add up both side of equation the integral $(n-1)\int \sin^n(x)\,\mathrm{d}x$, so we get

$$n\int \sin^n(x)\,\mathrm{d}x = -\cos(x)\cdot\sin^{n-1}(x) + (n-1)\int \sin^{n-2}(x)\,\mathrm{d}x.$$

We get the recursion formula through dividing the both side of equation by n

$$\int \sin^n(x)\,\mathrm{d}x = -\frac{1}{n}\cos(x)\cdot\sin^{n-1}(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,\mathrm{d}x$$


Presentation & Exercises
Now you have to work !

Basic Function and Their Integral

Basic Function

Return

$$\text{Polynomial}: f(x) = x^n$$

$$f(x) = \frac{1}{x}$$

$$\text{Constant}: f(x) = a$$

$$f(x) = 0$$

$$\text{Expotential}: f(x) = e^x$$

$$f(x) = a^x$$

$$\text{Trigonometric}: f(x) = \sin(x)$$

$$f(x) = \cos(x)$$

Integral

Next

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

$$F(x) = \ln(|x|) + C$$

$$F(x) = ax + C$$

$$F(x) = a$$

$$F(x) = e^x + C$$

$$F(x) = x\ln(x) - 1$$

$$F(x) = -\cos(x)$$

$$F(x) = \sin(x)$$

Properties of Integration

Integration Rules

Sum Rule:

$$\int (f \pm g)\,\mathrm{d}x = \int f\,\mathrm{d}x \pm \int g\,\mathrm{d}x$$

Definite Integral:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

Substitution Rule:

$$\int_a^b f(\psi(x))\psi'(x)\,\mathrm{d}x = \int_{\psi(a)}^{\psi(b)} f(u)\,\mathrm{d}x$$

where $u = \psi(x)$.

◂ Return ■ ◂ Return ♣

Temporary page!

LaTeX was unable to guess the total number of pages correctly. As there was some unprocessed data that should have been added to the final page this extra page has been added to receive it.

If you rerun the document (without altering it) this surplus page will go away, because LaTeX now knows how many pages to expect for this document.