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Prove that $g(x)$ is a pdf. (Assume that $F(x_0) < 1$.)

1.53 A certain river floods every year. Suppose that the low-water mark is set at 1 and the high-water mark Y has distribution function

$$F_Y(y) = P(Y \le y) = 1 - \frac{1}{y^2}, \quad 1 \le y < \infty.$$

(a) Verify that $F_Y(y)$ is a cdf.
(b) Find $f_Y(y)$, the pdf of Y.
(c) If the low-water mark is reset at 0 and we use a unit of measurement that is $\frac{1}{10}$ of that given previously, the high-water mark becomes $Z = 10(Y-1)$. Find $F_Z(z)$.

1.53

(a). From Thm 1.5.3, $1 \le y < \infty$, $F_Y(1) = 1-1 = 0$, $\lim_{y \to \infty} F_Y(y) = 1 - 0 = 1$;

when $1 \le y < \infty$, $\frac{1}{y^2}$ is monotonically decreasing, so $F_Y(y)$ is a nondecreasing function of x ;

$\forall y_0$, $\lim_{y \to y_0} F_Y(y) - F_Y(y_0) = \lim_{y \to y_0} \left(\frac{1}{y_0^2} - \frac{1}{y^2} \right) = \lim_{y \to y_0} \frac{y^2 - y_0^2}{y_0^2 y^2} = 0$

Thus $F_Y(y)$ is a cdf

(b). $f_Y(y) = \frac{d}{dy} F_Y(y) = \begin{cases} \frac{2}{y^3} & 1 \le y < \infty. \\ 0 & y < 1 \end{cases}$

(c). $Z = 10(Y-1) \Rightarrow Y = \frac{Z}{10} + 1$, then $F_Z(z) = \begin{cases} 0 & z \le 0 \\ 1 - \frac{1}{(\frac{z}{10}+1)^2}, & z > 0 \end{cases}$

1.54 For each of the following, determine the value of c that makes $f(x)$ a pdf.

(a) $f(x) = c \sin x,\ 0 < x < \pi/2$ (b) $f(x) = ce^{-|x|},\ -\infty < x < \infty$

1.54.

(a). From the hp.s, $x \in (0, \frac{\pi}{2}) \Rightarrow \sin x > 0,\ f(x) \geq 0 \Rightarrow c \geq 0$;

$\int_0^{\frac{\pi}{2}} f(x)dx = \int_0^{\frac{\pi}{2}} c \sin x\, dx = c = 1 \Rightarrow c = 1.$

(b). $e^{-|x|} > 0,\ -\infty < x < \infty$, so $f(x) \geq 0 \Rightarrow c \geq 0.$

Then $\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} c \cdot e^{-|x|} dx = c(\int_{-\infty}^{0} e^{x} dx + \int_0^{\infty} e^{-x} dx)$

$= 2c = 1 \Rightarrow c = \frac{1}{2}$

□

$\lambda > 0,\ \mu > 0.$

2.17 A *median* of a distribution is a value m such that $P(X \le m) \ge \frac{1}{2}$ and $P(X \ge m) \ge \frac{1}{2}$. (If X is continuous, m satisfies $\int_{-\infty}^{m} f(x)\,dx = \int_{m}^{\infty} f(x)\,dx = \frac{1}{2}$.) Find the median of the following distributions.

(a) $f(x) = 3x^2, \quad 0 < x < 1$ (b) $f(x) = \dfrac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$

2.17:

(a). $0 < m < 1$, $\int_0^m f(x)\,dx = \int_m^1 f(x)\,dx$

$\Leftrightarrow \int_0^m 3x^2\,dx = \int_m^1 3x^2\,dx \Leftrightarrow m^3 = 1 - m^3$

$\Rightarrow m = 2^{-\frac{1}{3}}$

(b). $\int_{-\infty}^m f(x)\,dx = \int_m^{\infty} f(x)\,dx \Leftrightarrow \int_{-\infty}^m \frac{1}{\pi(1+x^2)}\,dx = \int_m^{\infty} \frac{1}{\pi(1+x^2)}\,dx$

$\Leftrightarrow \frac{1}{\pi}\left(\arctan m + \frac{\pi}{2}\right) = \frac{1}{\pi}\left(\frac{\pi}{2} - \arctan m\right)$

$\Rightarrow \arctan m = 0 \quad \Rightarrow \quad m = 0.$

$\square$

2.31 Does a distribution exist for which $M_X(t) = t/(1-t), |t| < 1$? If yes, find it. If no, prove it.

2.31: Does not exist,

Pf: If exist:

From Thm 2.3.7, $EX = M_X^{(1)}(0) = \left.\frac{1}{(1-t)^2}\right|_{t=0} = 1$

$EX^2 = M_X^{(2)}(0) = \left.-2(1-t)^{-3}\right|_{t=0} = -2.$

Then we get $Var X = EX^2 - (EX)^2 = -2-1 = -3 < 0$

There's a contradiction, Thus, this distribution does not exist.

$\square$

3.44 For any random variable X for which EX^2 and $E|X|$ exist, show that $P(|X| \geq b)$ does not exceed either EX^2/b^2 or $E|X|/b$, where b is a positive constant. If $f(x) = e^{-x}$ for $x > 0$, show that one bound is better when $b = 3$ and the other when $b = \sqrt{2}$. (Notice Markov's Inequality in Miscellanea 3.8.2.)

3.44. Firstly, we prove $P(|X| \geq b) \leq EX^2/b^2$ and $P(|X| \geq b) \leq \frac{E|X|}{b}$.

$|X|$ is nonnegative, so from Thm 3.6.1 (Chebychev's inequality)

$$P(|X| \geq b) \leq \frac{E|X|}{b}$$

Also, $EX^2 = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx \geq \int_{-\infty}^{-b} x^2 f_X(x)\,dx + \int_b^{+\infty} x^2 f_X(x)\,dx$

$\geq b^2 \left(\int_{-\infty}^{-b} f_X(x)\,dx + \int_b^{+\infty} f_X(x)\,dx\right) = b^2 \cdot P(|X| \geq b)$

i.e $P(|X| \geq b) \leq \frac{EX^2}{b^2}$

Then, let's consider $f(x) = e^{-x}$. $x > 0$

when $b = 3$, $EX^2/b^2 = \frac{1}{b^2}\int_0^{\infty} x^2 \cdot e^{-x}\,dx = \frac{1}{9}\int_0^{\infty} 2x \cdot e^{-x}\,dx$

$= \frac{2}{9}\left(-x \cdot e^{-x}\Big|_0^{\infty} + \int_0^{\infty} e^{-x}\,dx\right) = \frac{2}{9}$

$E|X|/b = \frac{1}{b}\int_0^{\infty} |x| \cdot e^{-x}\,dx = \frac{1}{3}\int_0^{\infty} x \cdot e^{-x}\,dx = \frac{1}{3}\int_0^{\infty} e^{-x}\,dx = \frac{1}{3} > \frac{2}{9}$

So, as a bound, EX^2/b^2 is better.

when $b = \sqrt{2}$, $EX^2/b^2 = 1$, $E|X|/b = \frac{\sqrt{2}}{2} < 1$

So as a bound $E|X|/b$ is better.

$\square$

3.46 Calculate $P(|X - \mu_X| \geq k\sigma_X)$ for $X \sim \text{uniform}(0,1)$ and $X \sim \text{exponential}(\lambda)$, and compare your answers to the bound from Chebychev's Inequality.

3.46. When $X \sim$ uniform $(0,1)$. $\mu_X = \frac{1}{2}$, $\sigma_X^2 = \frac{1}{12}$

$$P(|X-\mu_X| \geq k\sigma_X) = P\left(|X-\tfrac{1}{2}| \geq \tfrac{k}{\sqrt{12}}\right) = 1-P\left(|X-\tfrac{1}{2}| < \tfrac{k}{\sqrt{12}}\right)$$

$$= \begin{cases} 1-\frac{k}{\sqrt{3}} & 0 \leq k \leq \sqrt{3} \\ 0 & k > \sqrt{3} \\ 1 & k<0 \end{cases}$$

We need to let $k>0$

Using Chebychev's Inequality, $P(|X-\mu_X| \geq k\sigma_X) \leq \frac{E|X-\mu_X|}{k\sigma_X}$

$$= \frac{E|X-\frac{1}{2}|}{\frac{k}{\sqrt{12}}} = \sqrt{12}\cdot\frac{\int_0^1 |x-\frac{1}{2}|\,dx}{k} = \frac{\sqrt{3}}{2k}, \text{ Then we compare:}$$

$$\frac{\sqrt{3}}{2k} - \left(1-\frac{k}{\sqrt{3}}\right) = \frac{\sqrt{3}}{2k} + \frac{k}{\sqrt{3}} - 1 \geq 1-1=0,\ 0\leq k \leq \sqrt{3}, \quad \text{when } k>\sqrt{3},\ \frac{\sqrt{3}}{2k}>0.$$

When $X \sim$ exponential(λ). $\mu_X = \frac{1}{\lambda}$, $\sigma_X^2 = \frac{1}{\lambda^2}$

$$P(|X-\mu_X| \geq k\sigma_X) = P\left(|X-\tfrac{1}{\lambda}| \geq \tfrac{k}{\lambda}\right) = 1-P\left(|X-\tfrac{1}{\lambda}| < \tfrac{k}{\lambda}\right)$$

$$= \begin{cases} -e^{k-1} + e^{-k-1} + 1, & 0 \leq k \leq 1 \\ 1 & k<0 \\ e^{-(k+1)} & k>1 \end{cases}$$

Using Chebychev's Inequality, let $k>0$.

$$P(|X-\mu_X| \geq k\sigma_X) \leq \frac{E|X-\mu_X|}{k\sigma_X} = \frac{\lambda E|X-\frac{1}{\lambda}|}{k}$$

$$= \frac{\lambda\int_0^\infty |x-\frac{1}{\lambda}|\cdot\lambda\cdot e^{-\lambda x}\,dx}{k} = \frac{2}{ke}. \text{ Then Let we compare:}$$

$0<k\leq 1$: $\frac{-e^{k-1}+e^{-k-1}+1}{\frac{2}{ke}} = \frac{k}{2}(-e^k+e^{-k}+e) < 1$

$k>1$: $\frac{e^{-(k+1)}}{\frac{2}{ke}} = \frac{k}{2}\cdot e^{-k} < 1$, So chebychev's inequality is correct $\square$

PS: We can also use Thm 3.6.2 and use $\frac{1}{k^2}$ as the bound.

Example 4.1.5 has the pmf given in that example.

4.4 A pdf is defined by

$$f(x,y) = \begin{cases} C(x+2y) & \text{if } 0<y<1 \text{ and } 0<x<2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of C.
(b) Find the marginal distribution of X.
(c) Find the joint cdf of X and Y.
(d) Find the pdf of the random variable $Z = 9/(X+1)^2$.

4.4. (a). $\int_0^1 \int_0^2 C(x+2y)\,dx\,dy = 4C = 1$, thus $C = \frac{1}{4}$

(b). $f_X(x) = \begin{cases} \int_0^1 \frac{1}{4}(x+2y)\,dy = \frac{1}{4}x + \frac{1}{4}, & 0<x<2 \\ 0 & \text{otherwise} \end{cases}$

(c). For $0<x<2$ and $0<y<1$,

$$F_{X,Y}(x,y) = \int_0^x \int_0^y \frac{1}{4}(u+2v)\,dv\,du$$
$$= \int_0^x \frac{1}{4}u\cdot y + \frac{y^2}{4}\,du = \frac{x^2y}{8} + \frac{y^2x}{4}$$

for $0<x<2$, and $1\le y$, we have.

$$F_{XY}(x,y) = \int_0^x \int_0^1 \frac{1}{4}(u+2v)\,dv\,du = \frac{x^2}{8} + \frac{x}{4}$$

Similarly, we get the complete definition.

$$F_{X,Y}(x,y) = \begin{cases} 0 & x\le 0 \text{ or } y\le 0. \\ \frac{x^2y}{8} + \frac{y^2x}{4}, & 0<x<2 \text{ and } 0<y<1 \\ \frac{y}{2} + \frac{y^2}{2} & 2\le x \text{ and } 0<y<1. \\ \frac{x^2}{8} + \frac{x}{4} & 0<x<2 \text{ and } 1\le y \\ 1 & 2\le x \text{ and } 1\le y \end{cases}$$

(d). The function $z = g(x) = \frac{9}{(x+1)^2}$ is monotone on $0<x<2$, so from Thm 2.1.5, we get, $f_Z(z) = f_X\left(\frac{3}{\sqrt{z}}-1\right)\left|\frac{d}{dz}\left(\frac{3}{\sqrt{z}}-1\right)\right|$

$$= \frac{3}{4\sqrt{z}} \cdot \left(\frac{3}{2}\right) \cdot z^{-\frac{3}{2}} = \frac{9}{8z^2}, \quad 1<z<9$$

□

and Jensen's inequality shows that $E\log\left(\frac{\cdot}{a}\right) \le \log\left(E\frac{\cdot}{a}\right)$.)

4.58 For any two random variables X and Y with finite variances, prove that

(a) $\mathrm{Cov}(X,Y) = \mathrm{Cov}(X, \mathrm{E}(Y|X))$.

(b) X and $Y - \mathrm{E}(Y|X)$ are uncorrelated.

(c) $\mathrm{Var}(Y - \mathrm{E}(Y|X)) = \mathrm{E}(\mathrm{Var}(Y|X))$.

4.58. (a). proof: $\mathrm{Cov}(X, E(Y|X)) = E(X\cdot E(Y|X)) - EX\cdot E(E(Y|X))$.

From Thm 4.4.3, $E(E(Y|X)) = EY$

Then, we also have $E(X\cdot E(Y|X)) =$

$$\int_{-\infty}^{\infty}\left[x\cdot\int_{-\infty}^{\infty} y\cdot f(y|x)\,dy\right] f_X(x)\,dx = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} xy\cdot f(y|x) f_X(x)\,dy\right]dx$$

$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} xy\cdot f(x,y)\,dy\right)dx = EXY$. So, $\mathrm{Cov}(X, E(Y|X)) = EXY - EXEY = \mathrm{Cov}(X,Y)$.

(b). $\mathrm{Cov}(X, Y - E(Y|X)) = \mathrm{Cov}(X,Y) - \mathrm{Cov}(X, E(Y|X)) = \mathrm{Cov}(X,Y) - \mathrm{Cov}(X,Y) = 0$. Thus, X and $Y - E(Y|X)$ are uncorrelated.

(c). From Thm 4.4.7, $\mathrm{Var}(Y - E(Y|X)) = \mathrm{Var}(E(Y - E(Y|X)|X)) + E(\mathrm{Var}(Y - E(Y|X)|X))$. We realize that $E(Y - E(Y|X)|X) = E(Y|X) - E(Y|X) = 0$, So, $\mathrm{Var}(Y - E(Y|X)) = E[\mathrm{Var}(Y - E(Y|X)|X)]$

Let, $Z = Y - E(Y|X)$. Then, $\mathrm{Var}(Z|X) = E((Z - E(Z|X))^2|X)$

$E(Z|X) = E(Y - E(Y|X)|X) = 0$. So, $Var(Z|X) = E(Z^2|X)$

$= E((Y - E(Y|X))^2|X) = Var(Y|X)$, according to the definition of conditional variance. Thus, $Var(Y - E(Y|X)) =$

$E(Var(Z|X)) = E(Var(Y|X))$.

$\square$.