

Probabilistic Artificial Intelligence

Problem Set 1

Sep 23, 2019

$$\begin{aligned} P(H) &= 10^{-4} \\ P(P|H) &= 99\% \\ P(\bar{P}|\bar{H}) &= 99\% \end{aligned}$$

$$\begin{aligned} P(H|P) &= \frac{P(P|H)P(H)}{P(P)} \\ &= \frac{P(P|H)P(H)}{P(P|H)P(H) + P(P|\bar{H})P(\bar{H})} \\ &= \frac{0.99 \times 10^{-4}}{0.010098} \\ &\approx 0.98\% \end{aligned}$$

1. Bayes rule

As a result of a medical screening, one of the tests revealed a serious disease in a person. The test has a high accuracy of 99% (the probability of a positive response in the presence of a disease is 99% and the probability of a negative response in the absence of a disease is also 99%). However, the disease is quite rare and occurs only in one person per 10,000. Calculate the probability of the examined person having the identified disease.

2. Conditional Probabilities

For each statement below, either prove that it is true, or give a counterexample showing that it is false. Let a, b, c be events in some probability space.

(a) If $P(a|b, c) = P(b|a, c)$, then $P(a|c) = P(b|c)$ ✓

(b) If $P(a|b, c) = P(a)$, then $P(b|c) = P(b)$ ✗

(c) If $P(a|b) = P(a)$, then $P(a|b, c) = P(a|c)$ ✗

$$\begin{aligned} P(a|b, c) &= \frac{P(a, b, c)}{P(b, c)} = \frac{P(a, b|c)}{P(b|c)} \Rightarrow P(a|c) = P(b|c) \\ P(b|a, c) &= \frac{P(b, a, c)}{P(a, c)} = \frac{P(b, a|c)}{P(a|c)} \end{aligned}$$

3. Random Walk on Graphs

$$p_{x,v} = P\{Z_{x,v}\} =$$

Let G be a simple connected finite graph. We start at a vertex of G . At every second, we move to one of the neighbors of the current vertex uniformly at random, e.g., if the vertex has 3 neighbors, we move to one of them, each with probability $1/3$. Let u and v be two vertices of G . Starting from u , what is the probability that the walk visits v eventually? Does the same result hold for infinite graphs?

4. Multivariate Gaussian Distribution

A vector-valued random variable $x \in \mathbb{R}^n$ is said to have a multivariate normal distribution with mean $\mu \in \mathbb{R}^n$ and covariance matrix $\Sigma \in \mathbf{S}_{++}^n$ if its pdf is:

$$p(x; \mu, \Sigma) = \frac{1}{(2\pi)^{n/2} \det(\Sigma)^{1/2}} \exp\left(-\frac{1}{2}(x - \mu)^T \Sigma^{-1}(x - \mu)\right)$$

In this exercise you will need to show that:

1. Marginal of a joint Gaussian is Gaussian,
2. Conditional of a joint Gaussian is Gaussian.

Specifically, consider $x = \begin{bmatrix} x_A \\ x_B \end{bmatrix}$, $\mu = \begin{bmatrix} \mu_A \\ \mu_B \end{bmatrix}$, and $\Sigma = \begin{bmatrix} \Sigma_{AA} & \Sigma_{AB} \\ \Sigma_{BA} & \Sigma_{BB} \end{bmatrix}$, the joint distribution:

$$p(x) = p(x_A, x_B) = \frac{1}{Z} \exp \left(-\frac{1}{2} \begin{bmatrix} x_A - \mu_A \\ x_B - \mu_B \end{bmatrix}^T \begin{bmatrix} \Sigma_{AA} & \Sigma_{AB} \\ \Sigma_{BA} & \Sigma_{BB} \end{bmatrix}^{-1} \begin{bmatrix} x_A - \mu_A \\ x_B - \mu_B \end{bmatrix} \right).$$

Compute the marginal of joint Gaussians::

$$p(x_A) = \frac{1}{Z} \int_{x_B} p(x_A, x_B) dx_B.$$

Compute the conditional of joint Gaussians:

$$p(x_B|x_A) = \frac{p(x_A, x_B)}{p(x_A)}.$$

The following notations can ease the computation:

$$V = \begin{bmatrix} V_{AA} & V_{AB} \\ V_{BA} & V_{BB} \end{bmatrix} = \begin{bmatrix} \Sigma_{AA} & \Sigma_{AB} \\ \Sigma_{BA} & \Sigma_{BB} \end{bmatrix}^{-1}, \quad \begin{bmatrix} x_A - \mu_A \\ x_B - \mu_B \end{bmatrix} = \begin{bmatrix} \Delta_A \\ \Delta_B \end{bmatrix}$$