Environnement et Changement climatique Canada
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Next Generation of the Dynamical Core for the Model of the Atmosphere : Theoretical Foundations

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Collaborators

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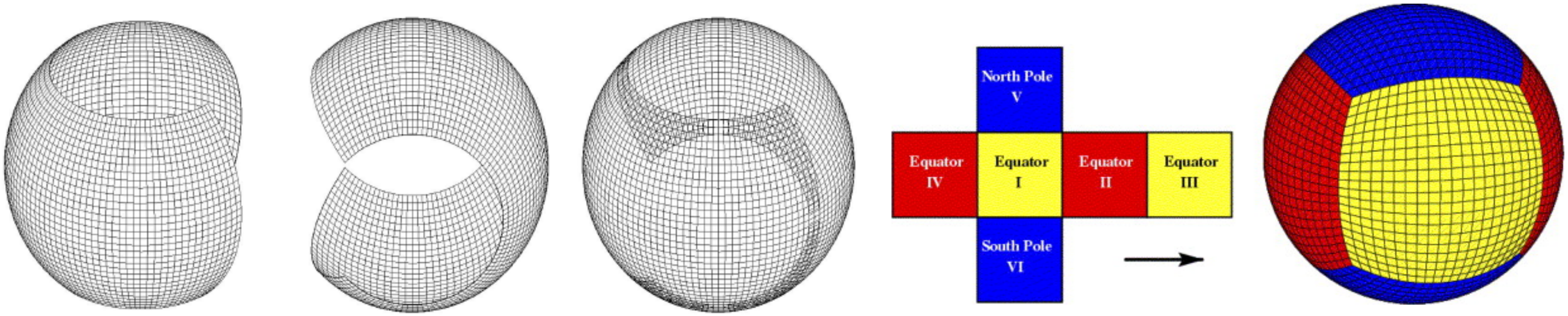
This is the first of three seminars. Stéphane and Christopher will present discretization approaches in space and time as well as applications in 2D and 3D with mountains (May 12 and June 2).

Talk Outline

- What are the objectives?
- What are the basic tools?
  - Curvilinear coordinates
  - Tensors
  - Equations governing the atmosphere in flux form
  - Cubed-sphere coordinates
- Is this useful?

Main Objectives

- We want the new atmospheric model's dynamical core to optimize compute intensity on the HPC and be very scalable
  - The overlapping region of the Yin-Yang grid as well as the semi-Lagrangian approach may become limiting factors



- We want mass to be conserved to machine precision
  - Adequately cover weather, air quality and climate applications
  - But time steps should ideally remain relatively large
  - Use equations governing the atmosphere in flux form

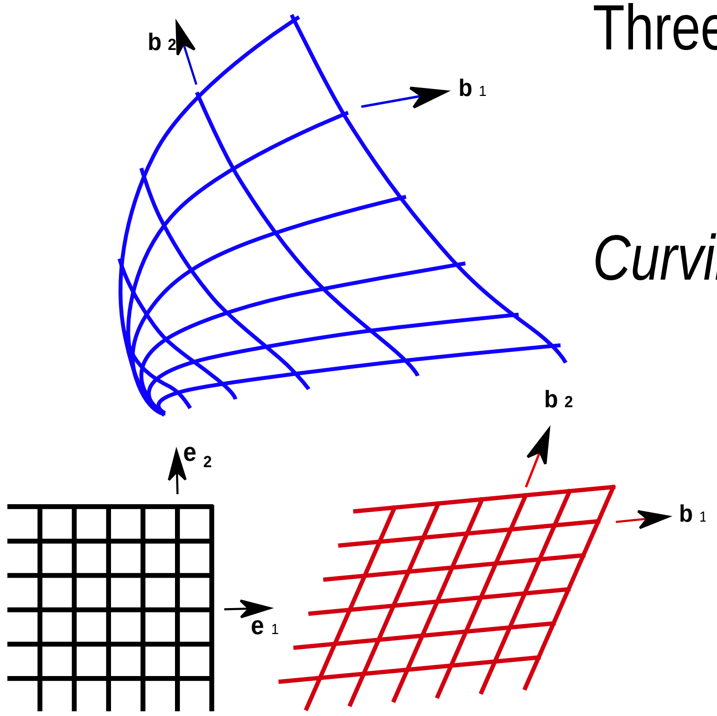
The Basic Tools

- For models of the atmosphere, a coordinate system is required. For instance, GEM uses rotated longitude, latitude and a terrain-following vertical coordinate
- These coordinates are not straight lines in three dimensions but are curvilinear
- *Tensors* are the most natural mathematical objects to deal with curvilinear coordinates
- Tensors in *4-dimensional* space-time are most adequate to describe a coordinate system that moves with time (e.g. the rotation of the Earth, a vertical coordinate based on hydrostatic pressure, the artifice of growing orography in a terrain-following coordinate, etc)

Curvilinear Coordinates (1)

- Any coordinate system that can be used to refer to space

Three examples in two-dimensional space :



Curvilinear coordinates

Rectilinear coordinates

Curvilinear Coordinates (2)

- In three spatial dimensions : x^1, x^2, x^3
- Time is also a dimension : $x^0 = 1 \text{ m/s } \times t$
- Our coordinates will always be interpreted as representing the 4-dimensional space-time : x^0, x^1, x^2, x^3
- We write x^μ to refer to any of the four coordinates ($\mu =$0, 1, 2 or 3) and x^i to refer to any of the three spatial coordinates ($i =$1, 2 or 3)
- We may decide to change the coordinates and prefer $\tilde{x}^\mu$:

$$
\begin{aligned}
\tilde{x}^0 &= x^0 + \tau \\
\tilde{x}^1 &= \tilde{x}^1(x^0, x^1, x^2, x^3) \\
\tilde{x}^2 &= \tilde{x}^2(x^0, x^1, x^2, x^3) \\
\tilde{x}^3 &= \tilde{x}^3(x^0, x^1, x^2, x^3)
\end{aligned}
$$

What Is a Tensor? (1)

- A zeroth-rank tensor is called a scalar
  - A quantity having, at a given point in space and time, the same value in all coordinate systems. Examples : density, pressure, gravitational potential, temperature, the number 3.14159265358979...
  - It bears no indices
  - In a given coordinate system x^μ , a scalar $f(x^0, x^1, x^2, x^3)$ keeps the same values as in the coordinates $\tilde{x}^\mu$:

$$\tilde{f}(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = f(x^0, x^1, x^2, x^3)$$

$$\boxed{\tilde{f}(\tilde{x}) = f(x)}$$

What Is a Tensor? (2)

- A first-rank tensor (4-vector) transforms as the coordinate increments

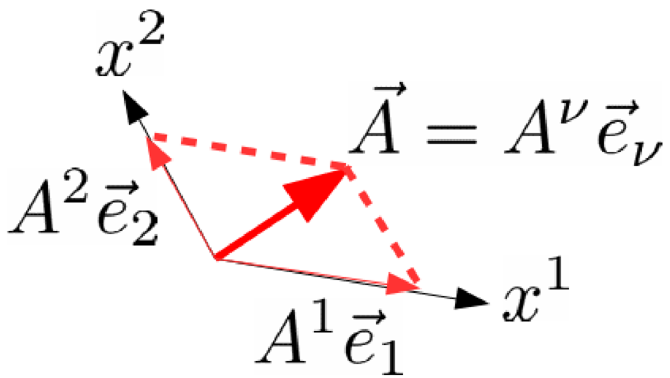
$$d\tilde{x}^\mu = \sum_{\nu=0}^{3} \frac{\partial \tilde{x}^\mu}{\partial x^\nu} dx^\nu \equiv \frac{\partial \tilde{x}^\mu}{\partial x^\nu} dx^\nu$$

  – Notation : Repeated Greek indices are summed from 0 to 3

$$\boxed{\tilde{A}^\mu(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} A^\nu(x)}$$

What Is a Tensor? (3)

- The tangent basis $\vec{e}_1, \ \vec{e}_2$



- The normal basis $\vec{e}^{\,1}, \ \vec{e}^{\,2}$

$$\vec{e}^{\,1} \cdot \vec{e}_1 = 1$$
$$\vec{e}^{\,1} \cdot \vec{e}_2 = 0$$
$$\vec{e}^{\,2} \cdot \vec{e}_1 = 0$$
$$\vec{e}^{\,2} \cdot \vec{e}_2 = 1$$

$$\vec{A} = A_\nu \vec{e}^{\,\nu} = A^\nu \vec{e}_\nu$$

What Is a Tensor? (4)

- Because time intervals are absolute in Newtonian mechanics,

$$d\tilde{x}^0 = dx^0 = dt$$
$$\frac{d\tilde{x}^\mu}{dt} = \frac{\partial \tilde{x}^\mu}{\partial x^\nu}\frac{dx^\nu}{dt}$$

- We define the 4-velocity as

$$\frac{dx^\mu}{dt} \equiv u^\mu(x) \text{ and } \frac{d\tilde{x}^\mu}{dt} \equiv \tilde{u}^\mu(\tilde{x})$$
$$u^0 = \tilde{u}^0 = 1$$

What Is a Tensor? (5)

- A second-rank tensor transforms as the product of coordinate increments

$$d\tilde{x}^{\mu} d\tilde{x}^{\nu} = \frac{\partial \tilde{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \tilde{x}^{\nu}}{\partial x^{\beta}} dx^{\alpha} dx^{\beta}$$

- Therefore, if $B^{\mu\nu}$ is a second-rank tensor :

$$\boxed{\tilde{B}^{\mu\nu}(\tilde{x}) = \frac{\partial \tilde{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \tilde{x}^{\nu}}{\partial x^{\beta}} B^{\alpha\beta}(x)}$$

The Metric Tensor (1)

- Consider two vectors $\vec{A} = A^\mu \vec{e}_\mu$ and $\vec{B} = B^\nu \vec{e}_\nu$
- Their scalar product $\vec{A} \cdot \vec{B}$ is

$$\vec{A} \cdot \vec{B} = (\vec{e}_\mu \cdot \vec{e}_\nu) A^\mu B^\nu$$

- We define the metric tensor $g_{\mu\nu} \equiv \vec{e}_\mu \cdot \vec{e}_\nu$
- A scalar product between the vectors $\vec{A}$ and $\vec{B}$ is

$$\boxed{\vec{A} \cdot \vec{B} = g_{\mu\nu} A^\mu B^\nu}$$

The Metric Tensor (2)

- In Newtonian mechanics
  - Time intervals dt are absolute
  - Spatial distances dl are absolute
- It is convenient to define the sum of their squared values

$$(ds)^2 \equiv (dt)^2 + (dl)^2$$

$$d\vec{s} = dx^\mu \vec{e}_\mu \Longrightarrow \boxed{(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu}$$

$$(dt)^2 = dx^0 dx^0 = \delta^0_\mu \delta^0_\nu dx^\mu dx^\nu$$

$$\Longrightarrow \boxed{(dl)^2 = (g_{\mu\nu} - \delta^0_\mu \delta^0_\nu) dx^\mu dx^\nu \equiv h_{\mu\nu} dx^\mu dx^\nu}$$

The Metric Tensor (3)

- In tensor notation, the position of indices has a specific meaning :

$$\vec{A} \cdot \vec{B} = g_{\mu\nu} A^\mu B^\nu$$

- Define $\boxed{B_\mu \equiv g_{\mu\nu} B^\nu} \Longrightarrow \vec{A} \cdot \vec{B} = A^\mu B_\mu$

$$\vec{B} = B^\mu \vec{e}_\mu = B_\mu \vec{e}^{\,\mu}$$

$$\begin{aligned} \vec{B} \cdot \vec{e}_\nu &= B^\mu e_\mu \cdot \vec{e}_\nu = B_\mu \vec{e}^{\,\mu} \cdot \vec{e}_\nu \\ &= g_{\mu\nu} B^\mu = B_\nu \\ &\Longrightarrow \vec{e}^{\,\mu} \cdot \vec{e}_\nu = \delta^\mu_\nu \end{aligned}$$

The Metric Tensor (4)

$$\vec{B} = B^{\mu}\vec{e}_{\mu} = B_{\mu}\vec{e}^{\,\mu}$$

$$\vec{B}\cdot\vec{e}^{\,\nu} = B^{\mu}\vec{e}_{\mu}\cdot\vec{e}^{\,\nu} = B_{\mu}\vec{e}^{\,\mu}\cdot\vec{e}^{\,\nu}$$
$$= \delta^{\nu}_{\mu}B^{\mu} = B^{\nu}$$

- Define $g^{\mu\nu} \equiv \vec{e}^{\,\mu}\cdot\vec{e}^{\,\nu} \Longrightarrow \boxed{B^{\nu} = g^{\mu\nu}B_{\mu}}$
- For a second-rank tensor, this relation generalizes to

$$\boxed{T^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}T_{\alpha\beta}}$$

The Metric Tensor (5)

- For the space metric tensor :

$$
\begin{aligned}
h^{\mu\nu} &= g^{\mu\alpha} g^{\nu\beta} h_{\alpha\beta} \\
&= g^{\mu\alpha} g^{\nu\beta} \left(g_{\alpha\beta} - \delta^0_\alpha \delta^0_\beta \right)
\end{aligned}
$$

$$
\Longrightarrow \boxed{h^{\mu\nu} = g^{\mu\nu} - g^{\mu 0} g^{\nu 0}}
$$

What Is a Derivative? (1)

- Let us verify if the derivative of a first-rank tensor is still a tensor :

$$\frac{\partial \tilde{A}^\mu}{\partial \tilde{x}^\nu} = \frac{\partial}{\partial \tilde{x}^\nu}\left(\frac{\partial \tilde{x}^\mu}{\partial x^\alpha} A^\alpha\right) = \frac{\partial x^\beta}{\partial \tilde{x}^\nu}\frac{\partial}{\partial x^\beta}\left(\frac{\partial \tilde{x}^\mu}{\partial x^\alpha} A^\alpha\right)$$

$$= \frac{\partial x^\beta}{\partial \tilde{x}^\nu}\frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial A^\alpha}{\partial x^\beta} + \frac{\partial x^\beta}{\partial \tilde{x}^\nu}\frac{\partial^2 \tilde{x}^\mu}{\partial x^\alpha \partial x^\beta} A^\alpha$$

- We need to redefine what a derivative is in order to have a tensor

What Is a Derivative? (2)

- The partial derivative of a vector is

$$\frac{\partial \vec{A}}{\partial x^\nu} = \frac{\partial}{\partial x^\nu}\left(A^\alpha \vec{e}_\alpha\right) = \vec{e}_\alpha \frac{\partial A^\alpha}{\partial x^\nu} + A^\alpha \frac{\partial \vec{e}_\alpha}{\partial x^\nu}$$

- We project this vector derivative along the normal basis

$$\begin{aligned}\vec{e}^\mu \cdot \frac{\partial \vec{A}}{\partial x^\nu} &= \vec{e}^\mu \cdot \vec{e}_\alpha \frac{\partial A^\alpha}{\partial x^\nu} + A^\alpha \vec{e}^\mu \cdot \frac{\partial \vec{e}_\alpha}{\partial x^\nu} = \delta^\mu_\alpha \frac{\partial A^\alpha}{\partial x^\nu} + A^\alpha \vec{e}^\mu \cdot \frac{\partial \vec{e}_\alpha}{\partial x^\nu} \\ &= \frac{\partial A^\mu}{\partial x^\nu} + A^\alpha \Gamma^\mu_{\alpha\nu} \text{ where } \Gamma^\mu_{\alpha\nu} \equiv \vec{e}^\mu \cdot \frac{\partial \vec{e}_\alpha}{\partial x^\nu}\end{aligned}$$

- It may be shown that this quantity transforms as a tensor
- Each time a vector in the governing equations needs to be differentiated, a symbol $\Gamma^\mu_{\alpha\nu}$ must appear

The Equations of Motion

- In arbitrary coordinates, the inviscid momentum equations are

$$\frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} + \Gamma^i_{00} + 2\Gamma^i_{j0} u^j + \Gamma^i_{jk} u^j u^k = -\frac{1}{\rho} h^{ij} \frac{\partial p}{\partial x^j} - h^{ij} \frac{\partial \Phi}{\partial x^j}$$

Eulerian tendency

Nonlinear advection

Centrifugal acceleration

Coriolis acceleration

Nonlinear metric terms

Pressure force / unit mass

Gravitational force / unit mass

The Equations of Motion in Flux Form (1)

- The three momentum equations :

$$\frac{\partial}{\partial t}\left(\sqrt{g}\rho u^1\right) + \frac{\partial}{\partial x^j}\left(\sqrt{g}\left[\rho u^1 u^j + h^{1j}p + \sigma^{1j}\right]\right) = -2\sqrt{g}\rho\,\Gamma^1_{0j}u^j - \sqrt{g}\,\Gamma^1_{jk}\left(\rho u^j u^k + h^{jk}p + \sigma^{jk}\right)$$

$$\frac{\partial}{\partial t}\left(\sqrt{g}\rho u^2\right) + \frac{\partial}{\partial x^j}\left(\sqrt{g}\left[\rho u^2 u^j + h^{2j}p + \sigma^{2j}\right]\right) = -2\sqrt{g}\rho\,\Gamma^2_{0j}u^j - \sqrt{g}\,\Gamma^2_{jk}\left(\rho u^j u^k + h^{jk}p + \sigma^{jk}\right)$$

$$\frac{\partial}{\partial t}\left(\sqrt{g}\rho u^3\right) + \frac{\partial}{\partial x^j}\left(\sqrt{g}\left[\rho u^3 u^j + h^{3j}p + \sigma^{3j}\right]\right) = -2\sqrt{g}\rho\,\Gamma^3_{0j}u^j - \sqrt{g}\,\Gamma^3_{jk}\left(\rho u^j u^k + h^{jk}p + \sigma^{jk}\right) - \sqrt{g}\rho h^{3j}\frac{\partial \varphi}{\partial x^j}$$

- The mass continuity equation :

$$\frac{\partial}{\partial t}\left(\sqrt{g}\rho\right) + \frac{\partial}{\partial x^j}\left(\sqrt{g}\rho u^j\right) = 0$$

The Equations of Motion in Flux Form (2)

- The thermodynamic equation :

$$\frac{\partial}{\partial t}\left(\sqrt{g}\rho\theta\right) + \frac{\partial}{\partial x^j}\left(\sqrt{g}\rho u^j\theta\right) = \frac{\sqrt{g}\rho}{c_p}\left(\frac{p_r}{p}\right)^{\kappa}\dot{Q} - \frac{\sqrt{g}}{c_p}\left(\frac{p_r}{p}\right)^{\kappa} u_{i:j}\sigma^{ij}$$

- The ideal gas law :

$$p = p_r\left(\frac{\rho R\theta}{p_r}\right)^{\frac{c_p}{c_v}}$$

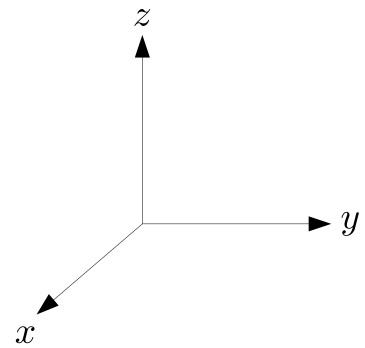
Example : Inertial, Cartesian Coordinates

- The coordinates are

$$x^0 = t \qquad x^1 = x \qquad x^2 = y \qquad x^3 = z$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad h^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

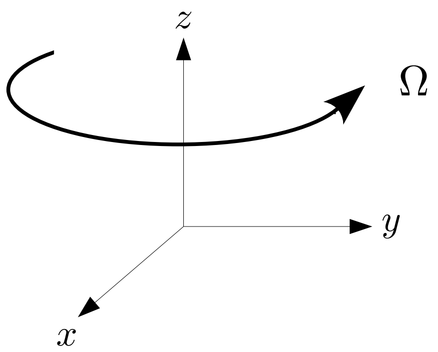
$$\Gamma^{\alpha}_{\mu\nu} = 0$$



Example : Rotating, Cartesian Coordinates

- The coordinates are

$$
\begin{aligned}
x^0 &= t \\
x^1 &= x\cos(\Omega t) + y\sin(\Omega t) \\
x^2 &= -x\sin(\Omega t) + y\cos(\Omega t) \\
x^3 &= z
\end{aligned}
$$



$$
g_{\mu\nu} = \begin{pmatrix} 1 + \Omega^2\left[(x^1)^2 + (x^2)^2\right] & -\Omega x^2 & \Omega x^1 & 0 \\ -\Omega x^2 & 1 & 0 & 0 \\ \Omega x^1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\qquad
h^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
\Gamma^1_{\mu\nu} = \begin{pmatrix} -x^1\Omega^2 & 0 & -\Omega & 0 \\ 0 & 0 & 0 & 0 \\ -\Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\qquad
\Gamma^2_{\mu\nu} = \begin{pmatrix} -x^2\Omega^2 & \Omega & 0 & 0 \\ \Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
\Gamma^3_{\mu\nu} = 0
$$

Example : Rotating, Rotated, Terrain-Following Cubed-Sphere Coordinates (1)

$$
\begin{aligned}
h^{11} &= \frac{\delta^2}{r^2(1+X^2)},\\
h^{12} &= \frac{XY\delta^2}{r^2(1+X^2)(1+Y^2)} = h^{21},\\
h^{13} &= -\left(\frac{\partial r}{\partial \eta}\right)^{-1}\left(\frac{\partial r}{\partial x^1}\right)\frac{\delta^2}{r^2(1+X^2)} - \left(\frac{\partial r}{\partial \eta}\right)^{-1}\left(\frac{\partial r}{\partial x^2}\right)\frac{XY\delta^2}{r^2(1+X^2)(1+Y^2)} = h^{31},\\
h^{22} &= \frac{\delta^2}{r^2(1+Y^2)},\\
h^{23} &= -\left(\frac{\partial r}{\partial \eta}\right)^{-1}\left(\frac{\partial r}{\partial x^1}\right)\frac{XY\delta^2}{r^2(1+X^2)(1+Y^2)} - \left(\frac{\partial r}{\partial \eta}\right)^{-1}\left(\frac{\partial r}{\partial x^2}\right)\frac{\delta^2}{r^2(1+Y^2)} = h^{32},\\
h^{33} &= \left(\frac{\partial r}{\partial \eta}\right)^{-2}\left[1+\left(\frac{\partial r}{\partial x^1}\right)^2\frac{\delta^2}{r^2(1+X^2)} + 2\left(\frac{\partial r}{\partial x^1}\right)\left(\frac{\partial r}{\partial x^2}\right)\frac{XY\delta^2}{r^2(1+X^2)(1+Y^2)} + \left(\frac{\partial r}{\partial x^2}\right)^2\frac{\delta^2}{r^2(1+Y^2)}\right].
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{g} &= \frac{r^2(1+X^2)(1+Y^2)}{\delta^3}\left|\frac{\partial r}{\partial \eta}\right|\\
X &= \tan x^1\\
Y &= \tan x^2\\
r &= x^3\\
\delta^2 &= 1+X^2+Y^2
\end{aligned}
$$

Example : Rotating, Rotated, Terrain-Following Cubed-Sphere Coordinates (2)

$$
\begin{aligned}
\Gamma^1_{01} &= \frac{\Omega XY}{\delta^2}\left(\sin\phi_p - X\cos\phi_p\sin\alpha_p + Y\cos\phi_p\cos\alpha_p\right) + \\
&\quad \left(\frac{\partial r}{\partial x^1}\right)\frac{\Omega}{r(1+X^2)}\left((1+X^2)\cos\phi_p\cos\alpha_p - Y\sin\phi_p + XY\cos\phi_p\sin\alpha_p\right) = \Gamma^1_{10}, \\
\Gamma^1_{02} &= -\frac{\Omega(1+Y^2)}{\delta^2}\left(\sin\phi_p - X\cos\phi_p\sin\alpha_p + Y\cos\phi_p\cos\alpha_p\right) + \\
&\quad \left(\frac{\partial r}{\partial x^2}\right)\frac{\Omega}{r(1+X^2)}\left((1+X^2)\cos\phi_p\cos\alpha_p - Y\sin\phi_p + XY\cos\phi_p\sin\alpha_p\right) = \Gamma^1_{20}, \\
\Gamma^1_{03} &= \left(\frac{\partial r}{\partial \eta}\right)\frac{\Omega}{r(1+X^2)}\left((1+X^2)\cos\phi_p\cos\alpha_p - Y\sin\phi_p + XY\cos\phi_p\sin\alpha_p\right) = \Gamma^1_{30}, \\
\Gamma^1_{11} &= \frac{2XY^2}{\delta^2} + \left(\frac{\partial r}{\partial x^1}\right)\frac{2}{r}, \\
\Gamma^1_{12} &= -\frac{Y(1+Y^2)}{\delta^2} + \left(\frac{\partial r}{\partial x^2}\right)\frac{1}{r} = \Gamma^1_{21}, \\
\Gamma^1_{13} &= \left(\frac{\partial r}{\partial \eta}\right)\frac{1}{r} = \Gamma^1_{31}, \\
\Gamma^1_{22} &= 0, \\
\Gamma^1_{23} &= 0 = \Gamma^1_{32}, \\
\Gamma^1_{33} &= 0,
\end{aligned}
$$

Example : Rotating, Rotated, Terrain-Following Cubed-Sphere Coordinates (3)

$$
\begin{aligned}
\Gamma^2_{01} &= \frac{\Omega(1+X^2)}{\delta^2}(\sin\phi_p - X\cos\phi_p\sin\alpha_p + Y\cos\phi_p\cos\alpha_p) + \\
&\quad \left(\frac{\partial r}{\partial x^1}\right)\frac{\Omega}{r(1+Y^2)}\left((1+Y^2)\cos\phi_p\sin\alpha_p + X\sin\phi_p + XY\cos\phi_p\cos\alpha_p\right) = \Gamma^2_{10}, \\
\Gamma^2_{02} &= -\frac{\Omega XY}{\delta^2}(\sin\phi_p - X\cos\phi_p\sin\alpha_p + Y\cos\phi_p\cos\alpha_p) + \\
&\quad \left(\frac{\partial r}{\partial x^2}\right)\frac{\Omega}{r(1+Y^2)}\left((1+Y^2)\cos\phi_p\sin\alpha_p + X\sin\phi_p + XY\cos\phi_p\cos\alpha_p\right) = \Gamma^2_{20}, \\
\Gamma^2_{03} &= \left(\frac{\partial r}{\partial \eta}\right)\frac{\Omega}{r(1+Y^2)}\left((1+Y^2)\cos\phi_p\sin\alpha_p + X\sin\phi_p + XY\cos\phi_p\cos\alpha_p\right) = \Gamma^2_{30}, \\
\Gamma^2_{11} &= 0, \\
\Gamma^2_{12} &= -\frac{X(1+X^2)}{\delta^2} + \left(\frac{\partial r}{\partial x^1}\right)\frac{1}{r} = \Gamma^2_{21}, \\
\Gamma^2_{13} &= 0 = \Gamma^2_{31}, \\
\Gamma^2_{22} &= \frac{2X^2Y}{\delta^2} + \left(\frac{\partial r}{\partial x^2}\right)\frac{2}{r}, \\
\Gamma^2_{23} &= \left(\frac{\partial r}{\partial \eta}\right)\frac{1}{r} = \Gamma^2_{32}, \\
\Gamma^2_{33} &= 0,
\end{aligned}
$$

Example : Rotating, Rotated, Terrain-Following Cubed-Sphere Coordinates (4)

$$\Gamma^3_{01} = -\left(\frac{\partial r}{\partial \eta}\right)^{-1}\left[\left(\frac{\partial r}{\partial x^1}\right)\Gamma^1_{01} + \left(\frac{\partial r}{\partial x^2}\right)\Gamma^2_{01} + \frac{r}{\delta^2}\Omega(1+X^2)(\cos\phi_p\cos\alpha_p - Y\sin\phi_p)\right] = \Gamma^3_{10},$$

$$\Gamma^3_{02} = -\left(\frac{\partial r}{\partial \eta}\right)^{-1}\left[\left(\frac{\partial r}{\partial x^1}\right)\Gamma^1_{02} + \left(\frac{\partial r}{\partial x^2}\right)\Gamma^2_{02} + \frac{r}{\delta^2}\Omega(1+Y^2)\left(\cos\phi_p\sin\alpha_p + X\sin\phi_p\right)\right] = \Gamma^3_{20},$$

$$\Gamma^3_{03} = -\left(\frac{\partial r}{\partial x^1}\right)\frac{\Omega}{r(1+X^2)}\left((1+X^2)\cos\phi_p\cos\alpha_p - Y\sin\phi_p + XY\cos\phi_p\sin\alpha_p\right)$$
$$-\left(\frac{\partial r}{\partial x^2}\right)\frac{\Omega}{r(1+Y^2)}\left((1+Y^2)\cos\phi_p\sin\alpha_p + X\sin\phi_p + XY\cos\phi_p\cos\alpha_p\right) = \Gamma^3_{30},$$

$$\Gamma^3_{11} = \left(\frac{\partial r}{\partial \eta}\right)^{-1}\left[\frac{\partial^2 r}{\partial x^1 \partial x^1} - \left(\frac{\partial r}{\partial x^1}\right)\Gamma^1_{11} - \frac{r}{\delta^4}(1+X^2)^2(1+Y^2)\right],$$

$$\Gamma^3_{12} = \left(\frac{\partial r}{\partial \eta}\right)^{-1}\left[\frac{\partial^2 r}{\partial x^1 \partial x^2} - \left(\frac{\partial r}{\partial x^1}\right)\Gamma^1_{12} - \left(\frac{\partial r}{\partial x^2}\right)\Gamma^2_{12} + \frac{r}{\delta^4}XY(1+X^2)(1+Y^2)\right] = \Gamma^3_{21},$$

$$\Gamma^3_{13} = \left(\frac{\partial r}{\partial \eta}\right)^{-1}\frac{\partial^2 r}{\partial x^1 \partial \eta} - \left(\frac{\partial r}{\partial x^1}\right)\frac{1}{r} = \Gamma^3_{31},$$

$$\Gamma^3_{22} = \left(\frac{\partial r}{\partial \eta}\right)^{-1}\left[\frac{\partial^2 r}{\partial x^2 \partial x^2} - \left(\frac{\partial r}{\partial x^2}\right)\Gamma^2_{22} - \frac{r}{\delta^4}(1+X^2)(1+Y^2)^2\right],$$

$$\Gamma^3_{23} = \left(\frac{\partial r}{\partial \eta}\right)^{-1}\frac{\partial^2 r}{\partial x^2 \partial \eta} - \left(\frac{\partial r}{\partial x^2}\right)\frac{1}{r} = \Gamma^3_{32},$$

$$\Gamma^3_{33} = \left(\frac{\partial r}{\partial \eta}\right)^{-1}\frac{\partial^2 r}{\partial \eta \partial \eta}.$$

Why This Is Useful

- It is a systematic method : nothing needs to be guessed
- The equations of motion always take the same form in all coordinates, only $\sqrt{g},\ h^{ij}$ and $\Gamma^{i}_{\mu\nu}$ need to be calculated
- When adopted, if we decide to change the coordinates (horizontal and / or vertical), no need to change the governing equations
- It systematizes the search for approximated equations : they remain consistent with the laws of physics to a chosen order
- We implement this approach for the next-generation atmospheric dynamical core