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Four-dimensional tensor equations for a classical fluid in an external gravitational field

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A four-dimensional tensor formalism suitable for the equations of motion of a classical fluid in the presence of a given external gravitational field is presented. The formalism allows for arbitrary time-dependent transformations of spatial coordinates. Some well-known conservation laws are derived in covariant form. The metric tensor and the associated Christoffel symbols are calculated for coordinate systems useful in meteorology. The vertical momentum equation employed in the Canadian operational weather forecasting model is obtained using the proposed tensor formalism.

Key Words: 4D tensors; time-varying curvilinear coordinate systems; classical fluids

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1. Introduction

Tensor equations have the appealing property of preserving their form in all coordinate systems, a property known as *covariance*. If the equations of motion of a given system are available in tensor form, one may transform those equations from one coordinate system to another by systematically following a set of well-defined transformation rules. In this article, a set of tensor equations describing the motion of a classical (in the sense of non-relativistic), viscous fluid in an external gravitational field is presented. Tensor quantities will be defined with respect to arbitrary spatial coordinate transformations involving both space and time, while keeping time unchanged. The only other approximation used will be that the fluid is light compared with the external mass generating the gravitational field, so that the influence of the fluid mass on the gravitational field may be neglected.

Numerous authors have proposed tensor formulations of the dynamical equations for fluids (e.g. Vinokur, 1974; Avis, 1976; Aris, 1989; Carlson *et al.*, 1995; Dutton, 2002; Luo and Bewley, 2004). However, Aris (1989), Carlson *et al.* (1995), Dutton (2002) and Luo and Bewley (2004) do not address time-dependent transformations of spatial coordinates in a four-dimensional fashion. Instead, spatial coordinate transformations involving time are often avoided (Aris, 1989) or treated in an *ad hoc* (Dutton, 2002; Luo and Bewley, 2004) and at times unsatisfactory (Carlson *et al.*, 1995) manner. In the case of Carlson *et al.* (1995), Luo and Bewley (2004) showed that they omitted some terms of the equations of motion in the case of time-varying coordinates. A four-dimensional formalism has been described in various forms by Vinokur (1974), specializing in heat transfer, and by Avis (1976). However, these technical reports are not widely known and therefore the potential usefulness of the four-dimensional tensor formalism in meteorology has not been fully considered. In the present article, spatial coordinate transformations involving time are also incorporated in a four-dimensional formalism in which inertial acceleration terms (such as the Coriolis and centrifugal accelerations) appear explicitly via Christoffel symbols. The formalism presented here was derived independently of Vinokur (1974) and Avis (1976) and extends their work for different practical applications in meteorology. One of the main objectives of this work is to present a clear, compact set of tensor equations and indeed demonstrate their practical usefulness for rigorously establishing governing equations of geophysical fluids in time-varying curvilinear coordinates.

The article is organized as follows. Section 2 introduces the notation and summarizes known properties of tensors in general four-dimensional space–time; section 3 defines a subset of coordinate transformations and the associated properties that are appropriate for classical fluids; section 4 provides the dynamical equations in tensor form; section 5 deals with conservation properties; sections 6 and 7 illustrate the methodology for two coordinate systems useful in meteorology; section 8 describes a general method for treating geometric approximations (of which the thin-shell approximation is an example) consistently with fundamental conservation laws. A discussion is presented in section 9. Appendix A treats more general coordinate transformations in detail, in particular two types of spheroidal coordinates.

2. Curvilinear coordinates and tensors in four dimensions

This section serves mainly to introduce the notation and to present general results of tensor analysis using curvilinear coordinates in four-dimensional space–time. Its content can be found in standard textbooks on curvilinear coordinates such as those of Dirac (1996) and Landau and Lifshitz (1975). The notation of Dirac (1996) is mostly used here, with some exceptions.

A general change of coordinate system S, with coordinates x^0, x^1, x^2, x^3, to $\tilde{S}$, with coordinates $\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3$, can be written as

$$\tilde{x}^0 = \tilde{x}^0(x^0, x^1, x^2, x^3), \tag{1}$$

$$\tilde{x}^1 = \tilde{x}^1(x^0, x^1, x^2, x^3), \tag{2}$$

$$\tilde{x}^2 = \tilde{x}^2(x^0, x^1, x^2, x^3), \tag{3}$$

$$\tilde{x}^3 = \tilde{x}^3(x^0, x^1, x^2, x^3). \tag{4}$$

In the tensor formalism, a scalar is nothing but a zeroth-rank tensor. Therefore, its value at a given point in space–time is the same in all coordinate systems.

Regarding first-rank tensors (i.e. vectors), consider the transformation law for infinitesimal displacements:

$$\mathrm{d}\tilde{x}^\mu = \sum_{\nu=0}^{3} \frac{\partial \tilde{x}^\mu}{\partial x^\nu} \mathrm{d}x^\nu \equiv \tilde{x}^\mu_{,\nu} \mathrm{d}x^\nu. \tag{5}$$

The convention of implicit summation from 0 to 3 over repeated lower and upper Greek indices and from 1 to 3 over repeated lower and upper Latin indices will be used, unless otherwise stated. A comma followed by a lower index ν indicates ordinary partial differentiation with respect to x^ν. By definition, the components of a contravariant first-rank tensor A^ν in S are transformed to $\tilde{S}$ according to (5):

$$\tilde{A}^\mu = \tilde{x}^\mu_{,\nu} A^\nu. \tag{6}$$

The quantity A^ν is the contravariant component of a 4-vector (in short, A^ν is said to be a contravariant 4-vector), the basis vectors of which are tangent to coordinate axes.

Similarly, a second-rank contravariant tensor $B^{\alpha\beta}$ is defined to follow the transformation law

$$\tilde{B}^{\mu\nu} = \tilde{x}^\mu_{,\alpha} \tilde{x}^\nu_{,\beta} B^{\alpha\beta}, \tag{7}$$

and so on for higher rank tensors.

An infinitesimal distance-like scalar quantity $\mathrm{d}s$ serves to define the fundamental metric properties at each point of the four-dimensional space–time and can be written as

$$\mathrm{d}s^2 \equiv g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{8}$$

where $g_{\mu\nu}$ is a symmetric object ($g_{\mu\nu} = g_{\nu\mu}$) called the four-dimensional covariant metric tensor. In space–time geometry, a 4-volume element is defined as $\sqrt{g}\mathrm{d}^4x$, where g is the determinant of $g_{\mu\nu}$ and $\mathrm{d}^4x = \mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3$. The factor $\sqrt{g}$ ensures that 4-volume elements are scalars.

The covariant metric tensor transforms contravariant components (upper index) to covariant components (lower index) of the dual (sometimes called normal) basis:

$$A_\mu = g_{\mu\nu} A^\nu. \tag{9}$$

The quantity A_μ is the covariant component of a 4-vector (in short, A_μ is said to be a covariant 4-vector), the basis vectors of which are normal to the coordinate axes.

The ordinary partial derivative of a scalar f with respect to coordinate x^μ is written

$$\frac{\partial f}{\partial x^\mu} \equiv f_{,\mu}. \tag{10}$$

The quantity $f_{,\mu}$ is a covariant 4-vector. Similarly, the quantity $\partial f/\partial x_\mu \equiv f^{,\mu}$ is a contravariant 4-vector.

The contravariant metric tensor $g^{\mu\nu}$ is the inverse of the covariant metric tensor $g_{\mu\nu}$, thus $A^\mu = g^{\mu\nu}A_\nu$ (in particular, $f^{,\mu} = g^{\mu\nu}f_{,\nu}$) and similarly $B^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}B_{\alpha\beta}$. The metric tensors obey the relation

$$g^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}g_{\alpha\beta}. \tag{11}$$

Covariant tensors transform as

$$\tilde{A}_\mu = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} A_\nu, \tag{12}$$

$$\tilde{B}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} B_{\alpha\beta}. \tag{13}$$

When a quantity is written as a product of mixed covariant and contravariant tensors, it is a scalar when all indices balance. For example, the quantity $A^\mu B^\nu{}_\mu C_\nu$ is a scalar if A^μ, $B^\nu{}_\mu$ and C_ν are tensors. It is a 4-vector when one index is unbalanced (e.g. $A^\mu B^\nu{}_\mu$) and so on. An example of mixed tensor transformation is

$$\tilde{C}^\mu{}_\nu = \tilde{x}^\mu_{,\alpha} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} C^\alpha{}_\beta. \tag{14}$$

Differentiation of tensors in curvilinear coordinates is performed through covariant derivatives. The covariant derivative in the ν direction of a contravariant 4-vector is a tensor and is written

$$A^\mu{}_{:\nu} = A^\mu{}_{,\nu} + \Gamma^\mu_{\alpha\nu} A^\alpha. \tag{15}$$

A colon followed by a lower index ν indicates covariant differentiation in the ν direction and similarly for contravariant derivatives: $A^{\mu:\nu} = g^{\nu\alpha}A^\mu{}_{:\alpha}$. The $\Gamma^\mu_{\alpha\nu}$ are Christoffel symbols of the second kind. They are given by

$$\Gamma^\mu_{\alpha\nu} = \frac{1}{2} g^{\mu\beta}(g_{\beta\alpha,\nu} + g_{\beta\nu,\alpha} - g_{\nu\alpha,\beta}). \tag{16}$$

Christoffel symbols are not tensors. Their transformation law can be written

$$\tilde{\Gamma}^\beta_{\tau\kappa} = \tilde{x}^\beta_{,\alpha} \frac{\partial^2 x^\alpha}{\partial \tilde{x}^\tau \partial \tilde{x}^\kappa} + \frac{\partial x^\mu}{\partial \tilde{x}^\tau} \frac{\partial x^\nu}{\partial \tilde{x}^\kappa} \tilde{x}^\beta_{,\alpha} \Gamma^\alpha_{\mu\nu}. \tag{17}$$

The covariant derivative in the ν direction of a second-rank contravariant tensor is a tensor and is written

$$B^{\mu\nu}{}_{:\alpha} = B^{\mu\nu}{}_{,\alpha} + \Gamma^\mu_{\alpha\beta}B^{\beta\nu} + \Gamma^\nu_{\alpha\beta}B^{\mu\beta}. \tag{18}$$

One can verify that the product rule with covariant differentiation is valid: $(A^\mu B^\nu)_{:\alpha} = A^\mu{}_{:\alpha}B^\nu + A^\mu B^\nu{}_{:\alpha}$. The covariant derivative of a scalar $f_{:\mu}$ is equal to its ordinary partial derivative $f_{,\mu}$. The following relations will be useful in the next sections. They are demonstrated in standard textbooks:

$$g_{\mu\nu:\alpha} = 0, \tag{19}$$

$$g^{\mu\nu}{}_{:\alpha} = 0, \tag{20}$$

$$g_{:\mu} = 0, \tag{21}$$

$$(\sqrt{g})^{-1}(\sqrt{g})_{,\alpha} = \Gamma^\nu_{\alpha\nu}. \tag{22}$$

Note that (15) and (22) imply that the (covariant) divergence of a 4-vector A^μ may be expressed in terms of ordinary partial derivatives:

$$A^\mu{}_{:\mu} = \left(\sqrt{g}\right)^{-1}\left(\sqrt{g}A^\mu\right)_{,\mu} \tag{23}$$

The Levi–Cevita symbol $\varepsilon^{\alpha\beta\mu\nu}$ is useful to define antisymmetric quantities. The symbol is equal to $+1$ for an even number of permutations of the indices 0123, to -1 for an odd number of permutations of the indices 0123 and to zero when two or more indices are the same. The symbol $\varepsilon_{\alpha\beta\mu\nu}$ is defined similarly. The Levi–Cevita symbol is not a tensor; however, it can be shown that $(\sqrt{g})^{-1}\varepsilon^{\alpha\beta\mu\nu}$ and $(\sqrt{g})\varepsilon_{\alpha\beta\mu\nu}$ are tensors.

3. Synchronous coordinate systems

For the rest of this article, a constraint on coordinate transformations is imposed: time is chosen to be the same in all coordinate systems. In other words, all admissible coordinate systems are *synchronous*. This constraint means that a change from a coordinate system S to $\tilde{S}$ is written as

$$\tilde{x}^0 = x^0, \tag{24}$$

$$\tilde{x}^1 = \tilde{x}^1(x^0, x^1, x^2, x^3), \tag{25}$$

$$\tilde{x}^2 = \tilde{x}^2(x^0, x^1, x^2, x^3), \tag{26}$$

$$\tilde{x}^3 = \tilde{x}^3(x^0, x^1, x^2, x^3). \tag{27}$$

Here, $x^0 \equiv u^0 t$ is time multiplied by a non-zero constant u^0. The chosen value for u^0 is irrelevant. Without loss of generality, the time unit can be chosen such that $u^0 = 1$. Subsequently, tensors will be defined with respect to this subset of transformations from synchronous coordinate systems.

The 4-velocity is defined as $u^\mu \equiv \mathrm{d}x^\mu/\mathrm{d}t$. By construction, it transforms as a 4-vector, since $\mathrm{d}x^\mu$ itself is a 4-vector (see (5)) and the time interval $\mathrm{d}t$ is a scalar.

Consider an observer moving with an inertial, rectilinear coordinate system, and two simultaneous events located at points x^i and $x^i + \mathrm{d}x^i$, where i takes the values 1, 2, and 3. The infinitesimal spatial distance $\mathrm{d}l$ between these two simultaneous events can be written in terms of $\mathrm{d}x^i$ and does not depend on $\mathrm{d}x^0$. If one now considers a time-varying coordinate system, this is not true any more. Coordinate transformations such as (25)–(27) imply that $\mathrm{d}l$ generally depends on $\mathrm{d}x^\mu$, where μ takes the values 0 to 3. This shows that, in the tensor formalism, an invariant infinitesimal spatial distance must be formulated in four dimensions. This is written

$$\mathrm{d}l^2 \equiv h_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{28}$$

Here, $h_{\mu\nu}$ is a symmetric tensor, distinct from the metric tensor $g_{\mu\nu}$ of the space–time geometry.

Cartesian coordinates are rectilinear and orthogonal. For an inertial, Cartesian coordinate system (up to a Galilean transformation), the metric tensor is here defined as $g_{\mu\nu} = 1$ for $\mu = \nu$ and zero otherwise. Note, however, that $g_{00} = 1$ for an inertial, Cartesian coordinate system is an arbitrary choice and one could choose other non-zero constants without loss of generality. In that coordinate system, $\mathrm{d}l$ and $\mathrm{d}s$ are related by

$$\mathrm{d}l^2 = \mathrm{d}s^2 - (\mathrm{d}x^0)^2. \tag{29}$$

Noting that $\mathrm{d}l$, $\mathrm{d}s$ and $\mathrm{d}x^0$ are all invariant scalars, (29) is necessarily valid in all synchronous coordinate systems. It follows that

$$\mathrm{d}l^2 = h_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu - \delta^0_\mu \delta^0_\nu \mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{30}$$

where δ^0_μ is the Kronecker symbol. This is valid for any $\mathrm{d}x^\mu$ and $\mathrm{d}x^\nu$. Thus, the relation between $h_{\mu\nu}$ and the metric tensor is

$$h_{\mu\nu} = g_{\mu\nu} - \delta^0_\mu \delta^0_\nu, \tag{31}$$

provided that $h_{\mu\nu}$ and $g_{\mu\nu}$ are symmetric.

The distinction between $\mathrm{d}s$ and $\mathrm{d}l$ may be understood in the following way. Recall that $\mathrm{d}s$ is the infinitesimal 'distance' between two events in space–time. By definition, the quantities $\mathrm{d}s$ and $\mathrm{d}l$ are equal only when the events are simultaneous ($\mathrm{d}x^0 = 0$) and in that case they both represent the spatial distance between those events. Consider now two events that are not necessarily simultaneous. In the inertial, Cartesian coordinate system where $g_{\mu\nu}$ is diagonal with entries unity, $\mathrm{d}l$ is the spatial distance between those events. The definitions (28) and equation (31) show how to obtain that same distance $\mathrm{d}l$ in any other synchronous coordinate system.

Equation (24) implies that $\tilde{x}^0_{,\nu} = \delta^0_\nu$ and that the zeroth component A^0 of a 4-vector A^μ is a scalar. The component B^{00} of a second-rank tensor is also a scalar, implying that g^{00} is unity in all synchronous coordinate systems. The components $B^{0\mu}$ transform as 4-vectors. Also, since $\mathrm{d}x^0$ is a scalar, a spatial volume element $\sqrt{g}\mathrm{d}^3x$ is a scalar since $\sqrt{g}\mathrm{d}^4x$ is a scalar.

The contravariant tensor $h^{\mu\nu}$ is

$$\begin{aligned} h^{\mu\nu} &= g^{\mu\alpha}g^{\nu\beta}h_{\alpha\beta} \\ &= g^{\mu\alpha}g^{\nu\beta}(g_{\alpha\beta} - \delta^0_\alpha \delta^0_\beta) \\ &= g^{\mu\nu} - g^{\mu 0}g^{0\nu}. \end{aligned} \tag{32}$$

The components $h^{\mu 0} = h^{0\mu}$ are zero in all synchronous coordinate systems, i.e. only components with spatial indices are non-zero. It will be seen that $h^{\mu\nu}$ is essential to formulating four-dimensional tensor dynamical equations for classical fluids.

One can verify from (17) that (24)–(27) imply that $\Gamma^0_{\mu\nu} = 0$ in all synchronous coordinate systems. Consider an inertial, rectilinear coordinate system S with axes x^μ and a curvilinear coordinate system $\tilde{S}$ synchronous to S with axes $\tilde{x}^\mu$. In S, the Christoffel symbols are all zero, since all metric tensor components are constants (see (16)). From (17), the Christoffel symbols in $\tilde{S}$ with upper index zero are thus

$$\tilde{\Gamma}^0_{\mu\nu} = \tilde{x}^0_{,\alpha}\frac{\partial^2 x^\alpha}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu} = \delta^0_\alpha \frac{\partial^2 x^\alpha}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu} = \frac{\partial^2 x^0}{\partial \tilde{x}^\mu \partial \tilde{x}^\nu} = \frac{\partial \delta^0_\nu}{\partial \tilde{x}^\mu} = 0.$$

The intrinsic derivative of a tensor of any rank $C^{\alpha\beta\nu\dots}$ is defined as

$$\frac{\mathrm{D}C^{\alpha\beta\nu\dots}}{\mathrm{D}t} \equiv u^\mu C^{\alpha\beta\nu\dots}{}_{;\mu} \tag{33}$$

and is a tensor. However, the material derivative of a tensor

$$\frac{\mathrm{d}C^{\alpha\beta\nu\dots}}{\mathrm{d}t} \equiv u^\mu C^{\alpha\beta\nu\dots}{}_{,\mu} \tag{34}$$

is not a tensor, except when the material derivative is applied on a scalar. For a 4-vector, one thus has

$$\frac{\mathrm{D}C^\alpha}{\mathrm{D}t} = \frac{\mathrm{d}C^\alpha}{\mathrm{d}t} + \Gamma^\alpha_{\mu\nu}u^\mu C^\nu. \tag{35}$$

Both the material and intrinsic derivatives describe the change in time of a quantity along a trajectory following the motion of a fluid element. However, the intrinsic derivative is the only one that accounts for changes of basis vectors along that trajectory.

Note that the meaning of 'synchronous coordinate systems' and the definition found in relativity books such as Landau and Lifshitz (1975) are not the same. In relativity theory, a coordinate system is synchronous if clocks can be synchronized at all points in space, which, for finite light speed, translates into the condition $g_{0\mu} = 0$. In the framework considered in this article, which is non-relativistic, clocks are assumed to be already synchronized at each point in the original inertial, Cartesian coordinate system, there is no such constraint on $g_{0\mu}$ and the only condition imposed is that any coordinate transformation must maintain the original time/clocks intact.

4. The mass–momentum tensor and the equations of motion

The quantity

$$T^{\mu\nu} = \rho u^\mu u^\nu + h^{\mu\nu}p + \sigma^{\mu\nu} \tag{36}$$

is a second-rank tensor that plays a key role in the equations of motion. The field ρ is the fluid density (a scalar) given by $\mathrm{d}m/\mathrm{d}V$, where $\mathrm{d}m$ is the mass and $\mathrm{d}V=\sqrt{g}\mathrm{d}^3x$ is the invariant three-dimensional volume of a fluid element, the vector u^μ is the 4-velocity described previously, p is pressure (also a scalar) and $\sigma^{\mu\nu}$ is a stress tensor describing viscous terms only. One possible representation for $\sigma^{\mu\nu}$ is given by

$$\sigma^{\mu\nu}=-K_\mathrm{d}\left(h^{\alpha\nu}u^\mu{}_{:\alpha}+h^{\alpha\mu}u^\nu{}_{:\alpha}\right)-\left(K_\mathrm{e}-\frac{2}{3}K_\mathrm{d}\right)h^{\mu\nu}u^\alpha{}_{:\alpha}, \tag{37}$$

where K_d and K_e are scalar viscosity coefficients (Landau and Lifshitz, 1987). Viscous terms for incompressible flows are obtained by keeping $-K_\mathrm{d}\,(h^{\alpha\nu}u^\mu{}_{:\alpha}+h^{\alpha\mu}u^\nu{}_{:\alpha})$ only.

The component $T^{00}=\rho$ is the fluid mass density, while $T^{0i}=T^{i0}=\rho u^i$ is the momentum (and mass-flux) density in the ith direction. The terms $T^{ij}=T^{ji}$ are stress and momentum-flux densities. Thus, the symmetric quantity $T^{\mu\nu}$ can be called the mass–momentum-stress tensor density, or in short the mass–momentum tensor.

In the presence of an external gravitational potential Φ (also a scalar), it will be shown that the tensor equation

$$T^{\mu\nu}{}_{:\nu}=-\rho h^{\mu\nu}\Phi_{,\nu} \tag{38}$$

represents the continuity equation when $\mu=0$ and the momentum equation when $\mu=1,2$ or 3.

If an inertial, Cartesian coordinate system is used (up to a Galilean transformation), $g^{\mu\nu}=1$ when $\mu=\nu$ and $g^{\mu\nu}=0$ when $\mu\neq\nu$. In that coordinate system, covariant derivatives reduce to ordinary partial derivatives, and contravariant and covariant components coincide. For the component $\mu=0$ of (38), one can write

$$T^{0\nu}{}_{,\nu}=-\rho h^{0\nu}\Phi_{,\nu}=0$$

since $h^{0\nu}=0$. Separating the summation for time and space and noting that $u^0{}_{:\nu}=0$, one obtains

$$T^{00}{}_{,0}+T^{0j}{}_{,j}=\frac{\partial\rho}{\partial t}+(\rho u^j)_{,j}=0,$$

i.e. the continuity equation. Recall that repeated upper and lower Latin indices are summed from 1 to 3. For the spatial components, one puts $\mu=i$ in (38) and obtains

$$\begin{aligned}T^{i0}{}_{,0}+T^{ij}{}_{,j}=&\frac{\partial(\rho u^i)}{\partial t}+(\rho u^iu^j)_{,j}+p^{,i}-(K_\mathrm{d}u^{i,j})_{,j}\\&-(K_\mathrm{d}u^{j,i})_{,j}-\left(\left[K_\mathrm{e}-\frac{2}{3}K_\mathrm{d}\right]u^k{}_{,k}\right)^{,i}\\=&-\rho\Phi^{,i},\end{aligned}$$

which is the momentum equation in inertial, Cartesian coordinates in the presence of viscosity and a gravitational potential.

Since the tensor equation (38) correctly represents the continuity and momentum equations in inertial, Cartesian coordinates, it must be valid in all synchronous coordinate systems. To derive the continuity and momentum equations in any arbitrarily complicated synchronous coordinate system, one may apply the transformation rules given in sections 2 and 3 to terms found in (38). Terms like the left-hand side of (38) are ubiquitous in theoretical physics. Here, a particular characteristic of (36), (37) and (38) is the presence of $h^{\mu\nu}$, which allows for truly four-dimensional spatial coordinate transformations of the equations of motion in tensor form.

A thermodynamic equation can also be written as a tensor equation. If θ is potential temperature (considered a scalar) and Q some scalar diabatic forcing term, one can write

$$(\rho u^\mu\theta)_{:\mu}=\rho Q, \tag{39}$$

or equivalently $u^\mu\theta_{,\mu}=Q$, after using the continuity equation $(\rho u^\mu)_{:\mu}=0$. The equation of state relating pressure, density and potential temperature is a scalar equation.

The tensorial momentum equation (38) can be rewritten for $\mu=i$ in a more traditional non-tensorial form:

$$\begin{aligned}\frac{\mathrm{D}u^i}{\mathrm{D}t}=&\frac{\mathrm{d}u^i}{\mathrm{d}t}+\Gamma^i_{00}+2\Gamma^i_{j0}u^j+\Gamma^i_{jk}u^ju^k\\=&-h^{ij}(\rho^{-1}p_{,j}+\Phi_{,j})-\rho^{-1}\sigma^{ij}{}_{:j},\end{aligned} \tag{40}$$

after using the continuity equation $(\rho u^\nu)_{:\nu}=0$. From the shape of the previous equation, it is clear that all inertial accelerations and nonlinear metric terms must appear through Christoffel symbols. From (16), (31) and (32), the term Γ^i_{00} can be written as $-\frac{1}{2}h^{ij}g_{00,j}+h^{ij}g_{j0,0}+g^{0\mu}g^{i0}(g_{\mu0,0}-\frac{1}{2}g_{00,\mu})$. The first term of this sum can always be combined with the gravitational potential to define an effective gravitational potential:

$$\Phi_e\equiv\Phi-\frac{1}{2}g_{00}. \tag{41}$$

Note that if one had kept u^0 unspecified then it would have appeared explicitly in some of the metric components and Christoffel symbols, but not in the actual equations of motion.

5. Conservation laws

An equation for the conservation of total energy is obtained by contracting (38) with the covariant vector u_μ and putting viscous terms to zero. After some manipulation, this yields

$$\left(\rho u^\nu\left[\frac{1}{2}h^{\mu\alpha}u_\mu u_\alpha+\Phi\right]+u_\mu h^{\mu\nu}p\right)_{:\nu}=pu^\nu{}_{:\nu}+\rho\Phi^{,0}. \tag{42}$$

An expression for the term $pu^\nu{}_{:\nu}$ is obtained from the following considerations. The laws of thermodynamics stipulate that the change of specific internal energy I of a fluid depends on the change of specific entropy s_e and the change of specific volume ρ^{-1} as $\mathrm{d}I=T\mathrm{d}s_\mathrm{e}-p\,\mathrm{d}(\rho^{-1})$ or, equivalently, as $\rho\mathrm{d}I/\mathrm{d}t=\rho T\mathrm{d}s_\mathrm{e}/\mathrm{d}t+p\,\rho^{-1}\mathrm{d}\rho/\mathrm{d}t$, where T is temperature. Changes of specific entropy and potential temperature are related by $\mathrm{d}s_\mathrm{e}/\mathrm{d}t=c_\mathrm{p}\theta^{-1}\mathrm{d}\theta/\mathrm{d}t$, where $c_\mathrm{p}=c_\mathrm{p}(p_\mathrm{r},\theta)$ is the specific heat capacity at constant pressure evaluated at a reference pressure p_r. From the continuity equation, the material derivative of density is related to the wind 4-divergence by $\rho^{-1}\mathrm{d}\rho/\mathrm{d}t=-u^\nu{}_{:\nu}$. From (39) and $\rho\mathrm{d}I/\mathrm{d}t=(\rho u^\nu I)_{:\nu}$, one obtains

$$pu^\nu{}_{:\nu}=\frac{\rho c_\mathrm{p}T}{\theta}Q-(\rho u^\nu I)_{:\nu}. \tag{43}$$

In adiabatic conditions, the total energy equation is then

$$E^\nu{}_{:\nu}=\rho\Phi^{,0}, \tag{44}$$

with

$$E^\nu=\rho u^\nu\left(\frac{1}{2}h^{\mu\alpha}u_\mu u_\alpha+\Phi+I\right)+u_\mu h^{\mu\nu}p. \tag{45}$$

The right-hand side, $\rho\Phi^{,0}$, of the equation for total energy conservation is the energy source contributed by the gravitational field and is an invariant scalar. It is zero if the gravitational potential is static in physical space. This represents the conservation of total energy when the external forcing is independent of time.

Vorticity can be defined as

$$\omega^\alpha = (\sqrt{g})^{-1}\varepsilon^{0\alpha\mu\nu}g_{\nu\beta}u^\beta{}_{:\mu}. \tag{46}$$

It is a 4-vector since, due to synchronous condition (24), $(\sqrt{g})^{-1}\varepsilon^{0\alpha\mu\nu}$ is a tensor. Ertel's potential vorticity, q, is defined as

$$q \equiv \frac{\omega^\alpha\theta_{,\alpha}}{\rho} \tag{47}$$

and is a scalar. Using (36) with $K_\mathrm{d} = 0 = K_\mathrm{e}$, (38) and (39) with $Q = 0$, an equation of state $\theta = \theta(\rho, p)$ and the identity $\varepsilon_{0\alpha\mu\nu}\varepsilon^{0\alpha\beta\gamma} = \delta^\beta_\mu\delta^\gamma_\nu - \delta^\gamma_\mu\delta^\beta_\nu$, one obtains

$$(\rho u^\mu q)_{:\mu} = 0, \tag{48}$$

which is the conservation equation of Ertel's potential vorticity for an inviscid fluid in adiabatic conditions. The basic steps of the derivation in vector notation are outlined in Pedlosky (1987) and one may re-derive this conservation equation in tensor notation following similar steps.

Dynamical fields can be written as the sum of a basic state (denoted below with overbars) and a term representing the departure from the basic state (denoted below with primes). A quantity called pseudo-energy is conserved when the basic state is independent of time, the flow is inviscid and adiabatic and the gravitational potential is static in physical space (Haynes, 1988). The pseudo-energy 4-vector $\mathcal{A}^\nu$ obeying $\mathcal{A}^\nu{}_{:\nu} = 0$ can be expressed as

$$\mathcal{A}^\nu = E^\nu + \rho u^\nu\mathcal{C} - (\overline{E}^\nu + \overline{\rho}\,\overline{u}^\nu\overline{\mathcal{C}}), \tag{49}$$

where

$$\overline{E}^\nu = \overline{\rho}\,\overline{u}^\nu\left(\frac{1}{2}h^{\mu\alpha}\overline{u}_\mu\overline{u}_\alpha + \Phi + \overline{I}\right) + \overline{u}_\mu h^{\mu\nu}\overline{p}, \tag{50}$$

$$\mathcal{C} = \mathcal{C}(q, \theta), \tag{51}$$

$$\overline{\mathcal{C}} = \mathcal{C}(\overline{q}, \overline{\theta}). \tag{52}$$

The function $\mathcal{C}$ is called a Casimir invariant; it depends on q and θ only (Shepherd, 1990) and is defined such that the pseudo-energy $\mathcal{A}^\nu$ has no first-order dependence on departure terms. This requirement leads to an equation for the Casimir invariant:

$$\begin{aligned}\overline{\rho}\,\overline{u}^\nu&\left[\frac{\partial\mathcal{C}}{\partial q}(\overline{q},\overline{\theta})q' + h^{\mu\alpha}\overline{u}_\mu u'_\alpha + \left(\frac{\overline{c_\mathrm{p}}\overline{T}}{\overline{\theta}} + \frac{\partial\mathcal{C}}{\partial\theta}(\overline{q},\overline{\theta})\right)\theta'\right.\\ &\left.+\frac{\overline{p}}{\overline{\rho}^2}\rho'\right] + h^{\mu\nu}\left(\overline{u}_\mu p' + u'_\mu\overline{p}\right)\\ &+\left(\overline{\rho}u'^\nu + \rho'\overline{u}^\nu\right)\left[\frac{1}{2}h^{\mu\alpha}\overline{u}_\mu\overline{u}_\alpha + \Phi + \overline{I} + \overline{\mathcal{C}}\right] = 0.\end{aligned} \tag{53}$$

To find an explicit expression for the pseudo-energy, the preceding equation must be solved for $\mathcal{C}$.

6. First application: Rotating spherical coordinates

Spherical coordinates are useful in meteorology and geophysical fluid dynamics since the Earth is approximately a sphere. The means by which to obtain the dynamical equations in this coordinate system from the mass–momentum tensor is exemplified below. The origin of the coordinate system is chosen to coincide with the centre of the body producing the gravitational field. An inertial, Cartesian coordinate system with axes x^μ is related to a spherical coordinate system with axes $\tilde{x}^\mu$, which rotates around the axis x^3, following

$$x^0 = \tilde{x}^0, \tag{54}$$

$$x^1 = \tilde{x}^3\cos\tilde{x}^2\cos(\tilde{x}^1 + \Omega\tilde{x}^0), \tag{55}$$

$$x^2 = \tilde{x}^3\cos\tilde{x}^2\sin(\tilde{x}^1 + \Omega\tilde{x}^0), \tag{56}$$

$$x^3 = \tilde{x}^3\sin\tilde{x}^2, \tag{57}$$

where $\tilde{x}^1 = \lambda$ is longitude, $\tilde{x}^2 = \phi$ is latitude, $\tilde{x}^3 = r$ is the distance from the origin and Ω is the coordinate system's angular velocity divided by $\tilde{u}^0 = 1$. The metric tensor $\tilde{g}_{\mu\nu}$ can be calculated from the invariance of $\mathrm{d}s^2$. Its components are written

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} 1 + (\Omega r\cos\phi)^2 & \Omega(r\cos\phi)^2 & 0 & 0\\ \Omega(r\cos\phi)^2 & (r\cos\phi)^2 & 0 & 0\\ 0 & 0 & r^2 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \tag{58}$$

with $\sqrt{\tilde{g}} = r^2\cos\phi$. Its inverse $\tilde{g}^{\mu\nu}$ is

$$\tilde{g}^{\mu\nu} = \begin{pmatrix} 1 & -\Omega & 0 & 0\\ -\Omega & \Omega^2 + (r\cos\phi)^{-2} & 0 & 0\\ 0 & 0 & r^{-2} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{59}$$

The tensor $\tilde{h}^{\mu\nu}$ is

$$\tilde{h}^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & (r\cos\phi)^{-2} & 0 & 0\\ 0 & 0 & r^{-2} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{60}$$

The Christoffel symbols can be calculated from either (16) or (17). They are written $\tilde{\Gamma}^0_{\mu\nu} = 0$,

$$\tilde{\Gamma}^1_{\mu\nu} = \begin{pmatrix} 0 & 0 & -\Omega\tan\phi & \Omega r^{-1}\\ 0 & 0 & -\tan\phi & r^{-1}\\ -\Omega\tan\phi & -\tan\phi & 0 & 0\\ \Omega r^{-1} & r^{-1} & 0 & 0\end{pmatrix}, \tag{61}$$

$$\tilde{\Gamma}^2_{\mu\nu} = \begin{pmatrix} \Omega^2\cos\phi\sin\phi & \Omega\cos\phi\sin\phi & 0 & 0\\ \Omega\cos\phi\sin\phi & \cos\phi\sin\phi & 0 & 0\\ 0 & 0 & 0 & r^{-1}\\ 0 & 0 & r^{-1} & 0\end{pmatrix}, \tag{62}$$

$$\tilde{\Gamma}^3_{\mu\nu} = \begin{pmatrix} -r(\Omega\cos\phi)^2 & -r\Omega\cos^2\phi & 0 & 0\\ -r\Omega\cos^2\phi & -r\cos^2\phi & 0 & 0\\ 0 & 0 & -r & 0\\ 0 & 0 & 0 & 0\end{pmatrix}. \tag{63}$$

Equation (38) then gives the continuity and momentum equations referred to a rotating spherical coordinate system (Pedlosky, 1987). The Coriolis terms and centrifugal acceleration appear through various Christoffel symbols. It is customary in meteorology to write wind fields as (u, v, w) to represent physical distances per unit of time. This can be achieved by multiplying the square root of $\tilde{g}_{ii}$ with $\tilde{u}^i$ (no summation implied):

$$u \equiv \tilde{u}^1 r\cos\phi, \tag{64}$$

$$v \equiv \tilde{u}^2 r, \tag{65}$$

$$w \equiv \tilde{u}^3. \tag{66}$$

The components (u, v, w) are not tensors. Appendix A treats more general coordinate systems, including the similar oblate and confocal oblate spheroidal coordinates.

7. Second application: A generalized vertical coordinate

A particular class of coordinate transformations especially useful in meteorology concerns a general transformation of the vertical coordinate. The method to obtain the explicit form of the equations from (38) with an arbitrary vertical coordinate is presented below. It is done in spherical geometry using results from the previous section.

A general single coordinate transformation is written

$$\tilde{x}^0 = x^0, \tag{67}$$
$$\tilde{x}^1 = x^1, \tag{68}$$
$$\tilde{x}^2 = x^2, \tag{69}$$
$$\tilde{x}^3 = \eta(x^0, x^1, x^2, x^3). \tag{70}$$

The coordinates $\tilde{x}^\mu$ of the previous section become x^μ in this section: $x^0 = u^0 t$, $x^1 = \lambda$, $x^2 = \phi$, $x^3 = r$. The inverse transformation is written

$$x^0 = \tilde{x}^0, \tag{71}$$
$$x^1 = \tilde{x}^1, \tag{72}$$
$$x^2 = \tilde{x}^2, \tag{73}$$
$$x^3 = r(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3). \tag{74}$$

The tensor $\tilde{g}^{\mu\nu}$ in the new coordinate system is calculated from the $g^{\mu\nu}$ of the spherical coordinate system (59) using (7). After using the identity $\eta_{,\mu} = -(\partial r/\partial\eta)^{-1}\partial r/\partial\tilde{x}^\mu$ (for $\mu = 0, 1, 2$) from the chain rule for derivatives, the non-zero components are

$$\tilde{g}^{00} = 1, \tag{75}$$
$$\tilde{g}^{01} = \tilde{g}^{10} = -\Omega, \tag{76}$$
$$\tilde{g}^{03} = \tilde{g}^{30} = -\left(\frac{\partial r}{\partial\eta}\right)^{-1}\left(\frac{\partial r}{\partial\tilde{x}^0} - \Omega\frac{\partial r}{\partial\tilde{x}^1}\right), \tag{77}$$
$$\tilde{g}^{11} = \Omega^2 + (r\cos\phi)^{-2}, \tag{78}$$
$$\tilde{g}^{13} = \tilde{g}^{31} = \Omega\left(\frac{\partial r}{\partial\eta}\right)^{-1}\left(\frac{\partial r}{\partial\tilde{x}^0} - \Omega\frac{\partial r}{\partial\tilde{x}^1}\right) - (r\cos\phi)^{-2}\left(\frac{\partial r}{\partial\eta}\right)^{-1}\frac{\partial r}{\partial\tilde{x}^1}, \tag{79}$$
$$\tilde{g}^{22} = r^{-2}, \tag{80}$$
$$\tilde{g}^{23} = \tilde{g}^{32} = -r^{-2}\left(\frac{\partial r}{\partial\eta}\right)^{-1}\frac{\partial r}{\partial\tilde{x}^2}, \tag{81}$$
$$\tilde{g}^{33} = \left(\frac{\partial r}{\partial\eta}\right)^{-2}\left[1 + \left(\frac{\partial r}{\partial\tilde{x}^0} - \Omega\frac{\partial r}{\partial\tilde{x}^1}\right)^2 + (r\cos\phi)^{-2}\left(\frac{\partial r}{\partial\tilde{x}^1}\right)^2 + r^{-2}\left(\frac{\partial r}{\partial\tilde{x}^2}\right)^2\right], \tag{82}$$

with $\sqrt{\tilde{g}} = (\partial r/\partial\eta)r^2\cos\phi$. The non-zero components of the tensor $\tilde{h}^{\mu\nu}$ are

$$\tilde{h}^{11} = (r\cos\phi)^{-2}, \tag{83}$$
$$\tilde{h}^{13} = \tilde{h}^{31} = -(r\cos\phi)^{-2}\left(\frac{\partial r}{\partial\eta}\right)^{-1}\frac{\partial r}{\partial\tilde{x}^1}, \tag{84}$$
$$\tilde{h}^{22} = r^{-2}, \tag{85}$$
$$\tilde{h}^{23} = \tilde{h}^{32} = -r^{-2}\left(\frac{\partial r}{\partial\eta}\right)^{-1}\frac{\partial r}{\partial\tilde{x}^2}, \tag{86}$$
$$\tilde{h}^{33} = \left(\frac{\partial r}{\partial\eta}\right)^{-2}\left[1 + (r\cos\phi)^{-2}\left(\frac{\partial r}{\partial\tilde{x}^1}\right)^2 + r^{-2}\left(\frac{\partial r}{\partial\tilde{x}^2}\right)^2\right]. \tag{87}$$

The metric $\tilde{g}_{\mu\nu}$ is obtained by inverting $\tilde{g}^{\mu\nu}$ or more directly from using the invariance property of ds^2:

$$\tilde{g}_{00} = 1 + (\Omega r\cos\phi)^2 + \left(\frac{\partial r}{\partial\tilde{x}^0}\right)^2, \tag{88}$$
$$\tilde{g}_{01} = \tilde{g}_{10} = \Omega(r\cos\phi)^2 + \frac{\partial r}{\partial\tilde{x}^0}\frac{\partial r}{\partial\tilde{x}^1}, \tag{89}$$
$$\tilde{g}_{02} = \tilde{g}_{20} = \frac{\partial r}{\partial\tilde{x}^0}\frac{\partial r}{\partial\tilde{x}^2}, \tag{90}$$
$$\tilde{g}_{03} = \tilde{g}_{30} = \frac{\partial r}{\partial\tilde{x}^0}\frac{\partial r}{\partial\eta}, \tag{91}$$
$$\tilde{g}_{11} = (r\cos\phi)^2 + \left(\frac{\partial r}{\partial\tilde{x}^1}\right)^2, \tag{92}$$
$$\tilde{g}_{12} = \tilde{g}_{21} = \frac{\partial r}{\partial\tilde{x}^1}\frac{\partial r}{\partial\tilde{x}^2}, \tag{93}$$
$$\tilde{g}_{13} = \tilde{g}_{31} = \frac{\partial r}{\partial\tilde{x}^1}\frac{\partial r}{\partial\eta}, \tag{94}$$
$$\tilde{g}_{22} = r^2 + \left(\frac{\partial r}{\partial\tilde{x}^2}\right)^2, \tag{95}$$
$$\tilde{g}_{23} = \tilde{g}_{32} = \frac{\partial r}{\partial\tilde{x}^2}\frac{\partial r}{\partial\eta}, \tag{96}$$
$$\tilde{g}_{33} = \left(\frac{\partial r}{\partial\eta}\right)^2. \tag{97}$$

The Christoffel symbols can be obtained from (17) and the non-zero components are

$$\tilde{\Gamma}^1_{00} = \frac{2\Omega}{r}\frac{\partial r}{\partial\tilde{x}^0}, \tag{98}$$
$$\tilde{\Gamma}^1_{01} = \tilde{\Gamma}^1_{10} = \frac{1}{r}\frac{\partial r}{\partial\tilde{x}^0} + \frac{\Omega}{r}\frac{\partial r}{\partial\tilde{x}^1}, \tag{99}$$
$$\tilde{\Gamma}^1_{02} = \tilde{\Gamma}^1_{20} = \frac{\Omega}{r}\frac{\partial r}{\partial\tilde{x}^2} - \Omega\tan\phi, \tag{100}$$
$$\tilde{\Gamma}^1_{03} = \tilde{\Gamma}^1_{30} = \frac{\Omega}{r}\frac{\partial r}{\partial\eta}, \tag{101}$$
$$\tilde{\Gamma}^1_{11} = \frac{2}{r}\frac{\partial r}{\partial\tilde{x}^1}, \tag{102}$$
$$\tilde{\Gamma}^1_{12} = \tilde{\Gamma}^1_{21} = \frac{1}{r}\frac{\partial r}{\partial\tilde{x}^2} - \tan\phi, \tag{103}$$
$$\tilde{\Gamma}^1_{13} = \tilde{\Gamma}^1_{31} = \frac{1}{r}\frac{\partial r}{\partial\eta}, \tag{104}$$
$$\tilde{\Gamma}^2_{00} = \Omega^2\cos\phi\sin\phi, \tag{105}$$
$$\tilde{\Gamma}^2_{01} = \tilde{\Gamma}^2_{10} = \Omega\cos\phi\sin\phi, \tag{106}$$
$$\tilde{\Gamma}^2_{02} = \tilde{\Gamma}^2_{20} = \frac{1}{r}\frac{\partial r}{\partial\tilde{x}^0}, \tag{107}$$
$$\tilde{\Gamma}^2_{11} = \cos\phi\sin\phi, \tag{108}$$
$$\tilde{\Gamma}^2_{12} = \tilde{\Gamma}^2_{21} = \frac{1}{r}\frac{\partial r}{\partial\tilde{x}^1}, \tag{109}$$
$$\tilde{\Gamma}^2_{22} = \frac{2}{r}\frac{\partial r}{\partial\tilde{x}^2}, \tag{110}$$
$$\tilde{\Gamma}^2_{23} = \tilde{\Gamma}^2_{32} = \frac{1}{r}\frac{\partial r}{\partial\eta}, \tag{111}$$
$$\tilde{\Gamma}^3_{\mu\nu} = \left(\frac{\partial r}{\partial\eta}\right)^{-1}\left[\frac{\partial^2 r}{\partial\tilde{x}^\mu\partial\tilde{x}^\nu} - \frac{\partial r}{\partial\tilde{x}^1}\tilde{\Gamma}^1_{\mu\nu} - \frac{\partial r}{\partial\tilde{x}^2}\tilde{\Gamma}^2_{\mu\nu} + \Gamma^3_{\mu\nu}\right], \tag{112}$$

where, in the last equation, the non-zero $\tilde{\Gamma}^1_{\mu\nu}$ are from (98)–(104), $\tilde{\Gamma}^2_{\mu\nu}$ from (105)–(111) and $\Gamma^3_{\mu\nu}$ from (63). One can then write the full expression for the equations of motion using these $\tilde{\Gamma}$ and $\tilde{h}$ values in (36) and (38).

It is worth noting that the third component of the momentum equation is fairly complicated in this coordinate system. To solve the third component of momentum numerically, in practice one can choose to simplify its treatment by keeping $u^3 = \mathrm{d}r/\mathrm{d}t$ as a third component of velocity and work with the coordinate system $\tilde{S}$. The zeroth, first and second components of velocity are identical in both coordinate systems, while the third component is related to the 4-velocities in $\tilde{S}$ by

$$u^3 = \frac{\partial r}{\partial \tilde{x}^\mu}\tilde{u}^\mu. \tag{113}$$

In the coordinate system S, the third component of the inviscid momentum equation is written

$$\begin{aligned}\rho u^\nu u^3{}_{,\nu} + \rho\Gamma^3_{\mu\nu}u^\mu u^\nu &= -h^{3\nu}\left(p_{,\nu} + \rho\Phi_{,\nu}\right)\\ &= -\left(\frac{\partial p}{\partial r} + \rho\frac{\partial \Phi}{\partial r}\right)\end{aligned} \tag{114}$$

after using the continuity equation $(\rho u^\nu)_{;\nu} = 0$. The first term can be rewritten $\rho\tilde{u}^\nu(\partial u^3/\partial\tilde{x}^\nu)$, the second term can become $\rho\Gamma^3_{\mu\nu}\tilde{u}^\mu\tilde{u}^\nu$, given that $\Gamma^3_{3\nu} = \Gamma^3_{\nu 3} = 0$, and that $u^0 = \tilde{u}^0$, $u^1 = \tilde{u}^1$ and $u^2 = \tilde{u}^2$. The right-hand side can also be rewritten with derivatives calculated in $\tilde{S}$. An equation is then obtained for u^3 in $\tilde{S}$:

$$\rho\tilde{u}^\nu\frac{\partial u^3}{\partial\tilde{x}^\nu} + \rho\Gamma^3_{\mu\nu}\tilde{u}^\mu\tilde{u}^\nu = -\left(\frac{\partial r}{\partial\eta}\right)^{-1}\left(\frac{\partial p}{\partial\eta} + \rho\frac{\partial\Phi}{\partial\eta}\right). \tag{115}$$

Due to the presence of two radial velocities (u^3 and $\tilde{u}^3$), (113) then becomes an essential part of the governing equations.

8. Geometric approximations

It is sometimes desirable to simplify the complete dynamical equations and deal with approximate versions of them. When these approximations are of geometric nature, the metric tensor $g_{\mu\nu}$ and consequently $h^{\mu\nu}$ as well as $\Gamma^\alpha_{\mu\nu}$ are modified. One such example commonly used in meteorology is the 'thin-shell' approximation, which consists of treating the atmosphere as a shallow fluid compared with the mean Earth's radius a. Two approaches leading to different sets of equations under the thin-shell approximation are briefly discussed below. They are not necessarily new, but they help illustrate the clarity and straightforwardness that the tensor formalism brings to the analysis of geometric approximations. Müller (1989) presents a similar discussion.

The dynamical equations can be simplified by modifyng the metrics (58) and (59), replacing the variable r with the constant a within the metrics. An approach ensuring that conservation laws automatically preserve their divergent-free form under the thin-shell approximation consists of using (16) to obtain the Christoffel symbols consistently with the approximate metrics, keeping in mind that $g_{\mu\nu}$ now has no explicit dependence on $x^3 = r$. For spherical geometry, one obtains

$$\Gamma^0_{\mu\nu} = 0 = \Gamma^3_{\mu\nu}, \tag{116}$$

$$\Gamma^1_{\mu\nu} = -\tan\phi\begin{pmatrix} 0 & 0 & \Omega & 0\\ 0 & 0 & 1 & 0\\ \Omega & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \tag{117}$$

$$\Gamma^2_{\mu\nu} = \cos\phi\sin\phi\begin{pmatrix} \Omega^2 & \Omega & 0 & 0\\ \Omega & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}. \tag{118}$$

If one uses in (38) this approximate metric ($r = a$ in (58)) and the above Christoffel symbols, one indeed obtains the thin-shell dynamical equations commonly used in meteorology. For a historical perspective on the thin-shell approximation, the reader is referred to Phillips (1966), Veronis (1968) and Phillips (1968).

In general, when geometric approximations are made, tensor equations describing fundamental conservation laws are automatically preserved in divergent-free form when following the approach of first defining the approximate metric $g_{\mu\nu}$ and then obtaining $g^{\mu\nu}$ and the Christoffel symbols from (16) consistently with the approximate metric. This property follows naturally from writing the conservation laws in tensor form. An additional assumption is needed when using this approach, however: one must assume that (38) remains valid for the approximate metric. Recall that the validity of (38) has been demonstrated in Euclidean space only, whereas the approximate metric can in general become non-Euclidean. White *et al.* (2005), White and Wood (2012) and Thuburn and White (2013) provide related discussions of geometric approximations in a non-tensorial framework.

Alternatively, if the metric tensors and Christoffel symbols are approximated indpendently, it can be shown that conservation laws in divergent-free form are not preserved. Due to this drawback, this method is not much used nowadays. The approach presented by Veronis (1968), which does not preserve conservation laws in divergent-free form, is equivalent to approximating the metric tensors and the Christoffel symbols independently.

Note that (115) is further simplified under the thin-shell approximation, since $\Gamma^3_{\mu\nu} = 0$. In the fourth generation of Environment Canada's Global Environmental Multiscale (GEM) model for weather forecasting, the approach of solving (115) under the thin-shell approximation has been followed.

Approximations that are of dynamical nature (for example the hydrostatic approximation) are not directly linked to the metric and are obtained from dynamical considerations beyond the scope of this work.

9. Discussion

Using the formalism presented in this article, the equations for a classical fluid in an external gravitational field were written in a concise tensor form, allowing for unambiguous transformations to arbitrary time-varying coordinates. The applicability of this formalism was illustrated with the derivation of the equations of motion in coordinates relevant to meteorology: rotating spherical coordinates with a generalized vertical coordinate, as well as a class of 'spheroidal-like' coordinates (see the Appendix) that includes the similar and confocal limits. The tensor formalism also provides a systematic approach to geometric approximations that preserves the divergent-free form of the conservation laws.

In practice, four-dimensional tensor equations have so far not been employed in meteorological research and therefore the benefits from the tensor formalism, such as its systematic nature, generality, brevity and even elegance, are missed. Vinokur (1974) and Avis (1976) do propose a four-dimensional tensor formalism similar to the one presented here, but their technical notes seem to be widely ignored, since they are rarely cited in meteorology literature. Other authors have previously proposed tensor equations for classical fluids with time-dependent spatial coordinate transformations (Carlson *et al.*, 1995; Luo and Bewley, 2004; Dutton, 2002). They start from an existing set of equations in a fixed (i.e. not varying in time) three-dimensional curvilinear coordinate system and propose an extension to time-varying coordinates by adding *ad hoc* vectors and additional terms to the equations. A drawback in this approach is that these *ad hoc* quantities are tensors only in fixed spatial coordinates and do not behave as tensors under four-dimensional synchronous transformations. The equations presented in Luo and Bewley (2004) are valid but, because they are non-tensorial equations in space–time, their physical interpretation is obscured. Meanwhile, the formalism and equations proposed here are truly tensorial in space–time, hence no *ad hoc* terms are needed, transformation rules are well-established and the physical interpretation of all terms is clear.

To make the above argument clearer, consider the limit in which the transformations (25)–(27) have no time dependence and its impact on the Christoffel symbols. In this case, it would be sufficient to work with a purely spatial metric tensor that coincides with the previously defined h_{ij}. Therefore, Christoffel symbols would be written as $G^i_{jk} = \frac{1}{2}h^{ia}(h_{aj,k} + h_{ak,j} - h_{jk,a})$. One can demonstrate that $G^i_{jk} = \Gamma^i_{jk}$, where Γ^i_{jk} are the spatial components of the four-dimensional Christoffel symbols defined for synchronous coordinates in section 3. If one now wishes to generalize the formalism to time-varying coordinates, one needs to prescribe how to introduce the $\Gamma^i_{\mu 0}$ terms. The tensor formalism proposed in this article is the only non *ad hoc* way of achieving this.

The reader familiar with Landau and Lifshitz (1975) may have also noted that their three-dimensional metric $\gamma_{ij} = -g_{ij} + g_{0i}g_{0j}/g_{00}$ (chapter X, section 84) is reminiscent of the tensor $h^{\mu\nu}$ from (32). Recall that γ_{ij} is the appropriate metric used to define spatial distances between simultaneous events in general relativity. Meanwhile, the non-relativistic quantity $h^{\mu\nu}$ was shown to be a four-dimensional contravariant tensor, the properties of which follow from the invariance of ds^2 and $(dx^0)^2$ only, so any resemblance to γ_{ij} is inconsequential.

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Appendix A: Beyond spherical coordinates

This Appendix presents the metric tensors and associated Christoffel symbols of a coordinate system allowing a closer representation of the oblate shape of the Earth. The coordinate transformations treated here are of the form

$$x^0 = \tilde{x}^0, \tag{A1}$$

$$x^1 = \tilde{x}^3 \cos\tilde{x}^2 \cos(\tilde{x}^1 + \Omega\tilde{x}^0), \tag{A2}$$

$$x^2 = \tilde{x}^3 \cos\tilde{x}^2 \sin(\tilde{x}^1 + \Omega\tilde{x}^0), \tag{A3}$$

$$x^3 = \tilde{x}^3 F(\tilde{x}^3) \sin\tilde{x}^2, \tag{A4}$$

where the x^μ represent inertial, Cartesian coordinates and the $\tilde{x}^\mu$ are the coordinates in $\tilde{S}$. The function F can be any differentiable function of $\tilde{x}^3$. As before, one writes $\tilde{x}^0 = \tilde{u}^0 t$, $\tilde{x}^1 = \lambda$, $\tilde{x}^2 = \phi$ and $\tilde{x}^3 = r$. However, the variable r is not in general the distance from the centre of the coordinate system. The metric tensor in $\tilde{S}$ is

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix}, \tag{A5}$$

where

$$\mathbf{A} = \begin{pmatrix} 1 + (\Omega r\cos\phi)^2 & \Omega(r\cos\phi)^2 \\ \Omega(r\cos\phi)^2 & (r\cos\phi)^2 \end{pmatrix}, \tag{A6}$$

$$\mathbf{B} = \begin{pmatrix} r^2(\sin^2\phi + F^2\cos^2\phi) & -r(1-FG)\sin\phi\cos\phi \\ -r(1-FG)\sin\phi\cos\phi & G^2\sin^2\phi + \cos^2\phi \end{pmatrix}, \tag{A7}$$

$$\mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{A8}$$

with $G(r) \equiv d(rF(r))/dr$, $\sqrt{\tilde{g}} = r^2 J\cos\phi$ and $J \equiv F\cos^2\phi + G\sin^2\phi$.

An orthogonal coordinate system (a diagonal metric tensor when $\Omega = 0$) from the transformations (A1)–(A4) is obtained by solving $FG = 1$. The only possible choice for an orthogonal coordinate system is then

$$F(r) = \sqrt{1 - \frac{R^2}{r^2}}, \tag{A9}$$

where R is a non-negative constant and the coordinate r is defined in the range $r \geqslant R$. Note that the special case $R = 0$ reduces to spherical coordinates. For $R > 0$, one obtains the confocal oblate spheroidal coordinate system (Gates, 2004), where $2R$ is the distance between the two foci. Any F other than (A9) will produce a non-orthogonal coordinate system. In particular, the (non-orthogonal) similar oblate spheroidal coordinate system is obtained by choosing $F = e$, where e is a constant chosen in the range $]0, 1[$. Surfaces of the similar oblate spheroidal coordinates resemble the Earth's effective geopotential surfaces more than do surfaces of the confocal oblate spheroidal coordinates. White *et al.* (2008) propose other definitions of latitude (transformations different from (A1)–(A4)) that can provide orthogonal coordinates with the similar oblate spheroidal system. However, they do not provide analytical expressions for latitude.

The inverse of the covariant metric tensor can be written

$$\tilde{g}^{\mu\nu} = \begin{pmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^{-1} \end{pmatrix}, \tag{A10}$$

where

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & -\Omega \\ -\Omega & \Omega^2 + (r\cos\phi)^{-2} \end{pmatrix}, \tag{A11}$$

$$\mathbf{B}^{-1} = \frac{1}{J^2}\begin{pmatrix} r^{-2}(\cos^2\phi + G^2\sin^2\phi) & r^{-1}(1-FG)\sin\phi\cos\phi \\ r^{-1}(1-FG)\sin\phi\cos\phi & F^2\cos^2\phi + \sin^2\phi \end{pmatrix}. \tag{A12}$$

The Christoffel symbols are written

$$\tilde{\Gamma}^0_{\mu\nu} = 0, \tag{A13}$$

$$\tilde{\Gamma}^1_{\mu\nu} = \begin{pmatrix} \mathbf{0} & \mathbf{C}_1 \\ \mathbf{C}_1^{\mathrm{T}} & \mathbf{0} \end{pmatrix}, \tag{A14}$$

where

$$\mathbf{C}_1 = \begin{pmatrix} -\Omega\tan\phi & \Omega r^{-1} \\ -\tan\phi & r^{-1} \end{pmatrix}, \tag{A15}$$

$$\tilde{\Gamma}^2_{\mu\nu} = \begin{pmatrix} \mathbf{C}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_3 \end{pmatrix}, \tag{A16}$$

$$\mathbf{C}_2 = \frac{G}{J}\sin\phi\cos\phi \begin{pmatrix} \Omega^2 & \Omega \\ \Omega & 1 \end{pmatrix}, \tag{A17}$$

$$\mathbf{C}_3 = \frac{1}{rJ}\begin{pmatrix} r(G-F)\sin\phi\cos\phi & G \\ G & G'\sin\phi\cos\phi \end{pmatrix}, \tag{A18}$$

$$\tilde{\Gamma}^3_{\mu\nu} = \begin{pmatrix} \mathbf{C}_4 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_5 \end{pmatrix}, \tag{A19}$$

and where

$$\mathbf{C}_4 = \frac{-rF}{J}\cos^2\phi \begin{pmatrix} \Omega^2 & \Omega \\ \Omega & 1 \end{pmatrix}, \tag{A20}$$

$$\mathbf{C}_5 = \frac{1}{J}\begin{pmatrix} -rF & (G-F)\sin\phi\cos\phi \\ (G-F)\sin\phi\cos\phi & G'\sin^2\phi \end{pmatrix}. \tag{A21}$$

The term G' is dG/dr. For the similar oblate spheroidal coordinate system, $F = G = J = e$ and $G' = 0$.

Coordinate transformations (68)–(70) and associated tensor and Christoffel symbol transformations can then provide a generalized vertical coordinate for this non-spherical coordinate system.

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