# Homework: 10 Algorithms & Integrals

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# Homework 10: algorithms & integrals

To submit: on Tuesday, 19.12.2023, 7:30 a.m., online by the learning campus

**Exercise 1 (3 pts.)**
Calculate the square root of a positive real number $a$ by the Newton method, i.e. find a zero of

$$f(x) = 1 - \frac{a}{x^2}.$$

As initial value we consider $x_0 = \frac{1+a}{2}$

a) Derive the abstract steps of the Newton method for arbitrary $a$.

b) Solve two steps of the Newton method for $a = 2$.

c) Solve two steps of the Newton method for $a = 5$.

In b) and c) use a calculator or a mathematical software. Remark: In the case $x = 0$ to be excluded, we have directly $a = 0$ and no Newton method is required.

**Exercise 2 (6 bonus pts.)**
Go through the slides from the lecture on the golden-section search.

Write down an algorithm in pseudo code **or** implement the algorithm in a programming language of your choice.

(The solution sketches will be in MATLAB, the algorithm or the written code will be marked, not the running code.)

**Exercise 3 (9 pts.)**
Compute the following integrals using partial fraction expansion and equating coefficients:

a) [5 pts.]

$$\int_3^4 \frac{x+10}{x^2+5x-14}\,dx$$

b) [4 pts.]

$$\int_0^1 \frac{x-1}{(x-2)^3}\,dx$$

Exercise 1

$$f(x) = 1 - \frac{a}{x^2}$$

The general formula for Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{——— (i)}$$

for the given function $f(x) = 1 - \frac{a}{x^2}$, we need to find the first derivative $f'(x)$

$$f'(x) = \frac{2a}{x^3}$$

now, applying (i) and substituting the value of $f'(x)$ and $f(x)$

$$x_{n+1} = x_n - \frac{1 - a/x_n^2}{2a/x_n^3}$$

$$x_{n+1} = x_n - \frac{\dfrac{x_n^2 - a}{x_n^2}}{\dfrac{2a}{x_n^3}}$$

$$= x_n - \left\{ \frac{x_n^2 - a}{x_n^2} \cdot \frac{x_n^3}{2a} \right\}$$

$$= x_n - \left\{ \frac{(x_n^2 - a)\, x_n}{2a} \right\}$$

$$= x_n \left[ 1 - \frac{(x_n^2 - a)}{2a} \right]$$

$$= x_n \left[ \frac{2a - x_n^2 + a}{2a} \right]$$

$$x_{n+1} = x_n \left[ \frac{3a - x_n^2}{2a} \right]$$

(a) for an arbitrary 'a'

$$x_0 = \frac{a+1}{2}$$

$$x_1 = x_0 \left[ \frac{3a - x_0^2}{2a} \right]$$

$$= \left(\frac{a+1}{2}\right) \left[ \frac{3a - \left(\frac{a+1}{2}\right)^2}{2a} \right]$$

$$= \left(\frac{a+1}{2}\right) \left[ \frac{3a - \left(\frac{a^2+2a+1}{4}\right)}{2a} \right]$$

$$= \left(\frac{a+1}{2}\right) \left[ \frac{12a - a^2 - 2a - 1}{8a} \right]$$

$$= \frac{(a+1)(10a - a^2 - 1)}{16a}$$

(b) Netwon Method for $a = 2$

$$x_0 = \frac{1+2}{2} = 1,5$$

$$x_1 = x_0 \left[ \frac{3a - x_0^2}{2a} \right]$$

putting value of $x_0$ and $a$ in the above equation

$$= 1,5 \left( \frac{3(2) - (1,5)^2}{2(2)} \right)$$

$$= 1,5 \left( \frac{6 - 2,25}{4} \right)$$

$$= \frac{1,5 \times 3,75}{4}$$

$$x_1 = 1,40625$$

$$x_2 = x_1\left(\frac{3a - x_1^2}{2a}\right)$$

putting the value of $x_1$ & $a$ in the above equation

$$= (1{,}40625)\left(\frac{(3)(2) - (1{,}40625)^2}{2(2)}\right)$$

$$= 1{,}40625\left(\frac{6 - 1{,}97756}{4}\right)$$

$$= (1{,}40625)(1{,}00561)$$

$x_2 =$ 1,41421 (approximately)

(C) Newton Method for $a = 5$

$$x_0 = \frac{1+5}{2} = 3$$

$$x_1 = x_0\left(\frac{3a - x_0^2}{2a}\right)$$

now inserting the values of $x_0$ and $a$ in the above equation

$$x_1 = 3\left(\frac{(3)(5) - (3)^2}{2(5)}\right)$$

$$= 3\left(\frac{15 - 9}{10}\right)$$

$$= 3 \cdot \frac{6}{10}$$

$$x_1 = 1{,}8$$

$$x_2 = x_1\left(\frac{3a - x_1^2}{2a}\right)$$

putting value of $x_1$ & $a$

$$x_2 = (1{,}8)\left(\frac{(3)(5) - (1{,}8)^2}{(2)(5)}\right)$$

$$= 1{,}8\left(\frac{15 - 3{,}24}{10}\right)$$

$$= (1{,}8)\left(\frac{11{,}76}{10}\right)$$

$$= 1{,}8 \cdot 1{,}176$$

$x_2 = 2{,}1168$ approximately

# Exercise 2

pseudo- Code

```
Algorithm Golden-Section Search(f, a, b, tolerance)
  Input: f (function to minimize), a (lower interval bound), b (upper interval bound), tolerance (convergence criterion)
  Output: x_min (approximation of the point where f attains its minimum in the interval [a, b])

  1. Define τ = (sqrt(5) - 1) / 2
  2. Initialize x1 = b - τ * (b - a)
  3. Initialize x2 = a + τ * (b - a)
  4. Compute f1 = f(x1)
  5. Compute f2 = f(x2)

  6. While (b - a) > tolerance do:
       a. If f1 > f2 then:
            i. Set a = x1
           ii. Set x1 = x2
          iii. Set x2 = b - τ * (b - a)
           iv. Compute f1 = f2
            v. Compute f2 = f(x2)
       b. Else:
            i. Set b = x2
           ii. Set x2 = x1
          iii. Set x1 = a + τ * (b - a)
           iv. Compute f2 = f1
            v. Compute f1 = f(x1)

  7. Set x_min = (a + b) / 2 (or choose the point x1 or x2 with the lower function value)

  8. Return x_min

End Algorithm
```

# Java code

```java
public class GoldenSectionSearch {
    // Function to minimize (replace this with the actual function)
    private static double f(double x) {
        // Placeholder function definition
        return (x * x) - 4 * x + 4; // Example: f(x) = x^2 - 4x + 4
    }

    public static double goldenSectionSearch(double a, double b, double tolerance) {
        double tau = (Math.sqrt(5) - 1) / 2;
        double x1 = b - tau * (b - a);
        double x2 = a + tau * (b - a);
        double f1 = f(x1);
        double f2 = f(x2);

        while ((b - a) > tolerance) {
            if (f1 > f2) {
                a = x1;
                x1 = x2;
                x2 = b - (x1 - a);
                f1 = f2;
                f2 = f(x2);
            } else {
                b = x2;
                x2 = x1;
                x1 = a + (b - x2);
                f2 = f1;
                f1 = f(x1);
            }
        }

        // The minimum lies within the interval [a, b]
        // We return the midpoint as the best estimate of the minimum location
        return (a + b) / 2;
    }

    public static void main(String[] args) {
        double a = 0; // lower bound of the interval
        double b = 2; // upper bound of the interval
        double tolerance = 1e-5; // tolerance

        double minimum = goldenSectionSearch(a, b, tolerance);
        System.out.println("The minimum is at: " + minimum);
    }
}
```

Golden ratio constant ($\tau$) : This constant, approximately 0,618 comes from the golden ratio. The golden section search uses this value to divide the interval.

Inital point calculation ($x_1$ and $x_2$): The interval $[a,b]$ is divided into two parts, $x_1$ and $x_2$, which are not in the middle but at position according to the golden ratio. $x_1$ is closer to a and $x_2$ is closer to b.

Please look at the additional files that we have uploaded which contains a Text file Golden_Ratio_Pseudo_Code and a Java Code file GoldenSectionSearch.

Exercise 3

(a) $$\int_3^4 \frac{x+10}{x^2+5x-14} \cdot dx$$

we can re-write $x^2+5x-14$ as $x^2+7x-2x-14$

$x(x+7)-2(x+7)$

$(x-2)(x+7)$

$$\frac{x+10}{(x+7)(x-2)} = \frac{A}{(x+7)} + \frac{B}{(x-2)}$$

$$\frac{x+10}{(x+7)(x-2)} = \frac{A(x-2) + B(x+7)}{(x+7)(x-2)}$$

$$x+10 = A(x-2)+B(x+7)$$

$$x+10 = Ax-2A+Bx+7B$$

$$(x+10) = (A+B)x + (-2A+7B)$$

Now equating the coffecient of the like terms

$$x = (A+B)x$$

$$1 = A+B \quad \Rightarrow A = 1-B \quad \text{---(i)}$$

$$10 = -2A + 7B$$

putting the value in of (i)

$$10 = -2(1-B) + 7B$$

$$10 = -2 + 2B + 7B$$

$$12 = 9B$$

$$\boxed{B = 4/3}$$

now putting value of B in (i)

$$A = 1 - 4/3$$

$$\boxed{A = -1/3}$$

now replacing the value of A & B

$$\frac{x+10}{(x+7)(x-2)} = -\frac{1}{3(x+7)} + \frac{4}{3(x-2)}$$

$$\int_3^4 \frac{x+10}{(x+7)(x-2)} \cdot dx = \int_3^4 \left( -\frac{1}{3(x+7)} + \frac{4}{3(x-2)} \right) \cdot dx$$

$$\int \frac{1}{1+x} = \ln(1+x)$$ where $x$ is a constant

$$-\frac{1}{3}\Big[\ln|x+7|\Big]_3^4 + \frac{4}{3}\Big[\ln|x-2|\Big]_3^4$$

$$-\frac{1}{3}\Big[\ln|4+7| - \ln|3+7|\Big] + \frac{4}{3}\Big[\ln|4-2| - \ln|3-2|\Big]$$

$$-\frac{1}{3}\Big[\ln 11 - \ln 10\Big] + \frac{4}{3}\Big[\ln 2 - \ln 1\Big]$$

$\therefore \ln 1 = 0$

$$-\frac{1}{3}\ln 11 + \frac{1}{3}\ln 10 + \frac{4}{3}\ln 2$$

(b) $$\int_0^1 \frac{x-1}{(x-2)^3} \cdot dx$$

$$\frac{x-1}{(x-2)^3} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}$$

$$\frac{x-1}{(x-2)^3} = \frac{A(x-2)^2 + B(x-2) + C}{(x-2)^3}$$

$$\frac{x-1}{(x-2)^3} = \frac{A(x^2+4-4x) + Bx - 2B + C}{(x-2)^3}$$

$$\frac{x-1}{(x-2)^3} = \frac{Ax^2 + 4A - 4Ax + Bx - 2B + C}{(x-2)^3}$$

$$\frac{x-1}{(x-2)^2} = \frac{Ax^2 + (B-4A)x + 4A - 2B + C}{(x-2)^3}$$

Comparing the cofficients.

$x^2 = 0$ hence $A=0$

$x = 1$ hence $1 = -4A + B$

$x^0 = -1$ hence $-1 = 4A - 2B + C$
Constant

$A=0$, $B=1$, $C=1$

now putting the values of A, B and C

$$\frac{x-1}{(x-2)^3} = \cancel{\frac{0}{(x-2)}}^{\,0} + \frac{1}{(x-2)^2} + \frac{1}{(x-2)^3}$$

$$\int_0^1 \frac{(x-1)}{(x-2)^3} \cdot dx = \int_0^1 \left(\frac{1}{(x-2)^2} + \frac{1}{(x-2)^3}\right) \cdot dx$$

$$\int \frac{1}{(x-c)^2} = \frac{-1}{x-c}$$ where c is constant

$$\int \frac{1}{(x-c)^3} = \frac{-1}{2(x+c)^2}$$

$$\int_0^1 \frac{1}{(x-2)^2} \cdot dx + \int_0^1 \frac{1}{(x-2)^3} \cdot dx$$

$$\left[\frac{-1}{(x-2)}\right]_0^1 + \left[\frac{-1}{2(x-2)^2}\right]_0^1$$

$$\left[\frac{-1}{(1-2)} - \frac{-1}{(0-2)}\right] + \left[\frac{-1}{2(1-2)^2} - \frac{-1}{2(0-2)^2}\right]$$

$$\left[1 - \frac{1}{2}\right] + \left[-\frac{1}{2} + \frac{1}{8}\right]$$

$$\frac{1}{2} - \frac{3}{8} = \frac{8-6}{16} = \frac{2}{16} = \frac{1}{8}$$

Dankeschön für Ihre Zeit ☺