WA 1

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1 Recurrence

1.1 $T(n) = 16T(\frac{n}{4}) + 32n\log^{128} n$

Define $R(n) = \frac{T(n)}{n^2}$:

$$\begin{aligned} R(n) &= \frac{T(\frac{n}{4})}{(\frac{n}{4})^2} + \frac{32\log^{128} n}{n} \\ &= R(\frac{n}{4}) + \frac{32\log^{128} n}{n} \end{aligned}$$

Define $S(m) = R(4^n)$:

$$\begin{aligned} S(m) &= S(m-1) + \frac{32 \cdot 4^m (\log(4^m))^{128}}{4^{m\cdot\log_4(16)}} \\ &= S(m-1) + \frac{32 \cdot 4^m m^{128}}{16^m} \\ &= S(m-1) + \frac{32(m)^{128}}{4^m} \\ S(m) &\in \Theta(S(0) + 32\sum_{i=1}^{m} \frac{i^{128}}{4^i}) \end{aligned}$$

It's easy to show that $f(n) = \sum_{i=0}^{m} \frac{i^{128}}{4^i} \in O(1)$ as

$$\lim_{n\to\infty} \frac{n^{128}}{4^n} = \lim_{n\to\infty} \frac{128n^{127}}{\ln(4)4^n} = \cdots = \lim_{n\to\infty} \frac{c_1}{c_2 4^n} = 0 \quad \text{for some constant } c_1, c_2$$

$\lim_{n\to\infty} \frac{\frac{n^{128}}{4^n}}{1} = 0 \implies f(n) = \frac{n^{128}}{4^n} \in o(1) \implies f(n) \in O(1)$

$$\begin{aligned} S(m) &\in O(1) \\ R(n) &\in O(1) \\ T(n) &\in O(n^2) \end{aligned}$$

1.2 $T(n) = 16T(\frac{n}{4}) + 64n^2\log^8 n + 32n\log^{128} n$

Define $R(n) = \frac{T(n)}{n^2}$:

$$\begin{aligned} R(n) &= \frac{16 \cdot T(n)}{(\frac{n^2}{4})} + \frac{64n^2(\log(n))^8}{n^2} + \frac{32n\log^{128} n}{n^2} \\ R(n) &= \frac{16 \cdot T(n)}{(\frac{n^2}{4})} + 64(\log(n))^8 + \frac{32\log^{128} n}{n} \end{aligned}$$

Define $m = 4^n$, $n = \log(m)$, $S(m) = R(4^n)$

$$S(m) = S(m-1) + \frac{}{4}$$

1.3 $T(n) = T(\sqrt{n}) + \log(n)$

Define $n = 2^m$, $m = \log(n)$

$$\begin{aligned} T(n) &= T(n^{\frac{1}{2}}) + \log(n) \\ T(2^m) &= T(2^{\frac{1}{2}m}) + \log(2^m) \\ &= T(2^{\frac{1}{2}m}) + m \\ R(m) &= R(\frac{m}{2}) + m \end{aligned}$$

Define $m = 2^k$, $k = \log(m)$

$$\begin{aligned} R(2^k) &= R(\frac{2^k}{2}) + 2^k \\ S(k) &= S(k-1) + 2^k \\ S(k) &\in O(S(0) + \sum_{i=1}^{k} 2^i) = O(1 + 2^{k+1} + 2) = O(2^k) \end{aligned}$$

Map back to $R(m)$ and $T(n)$:

$$\begin{aligned} & S(k) \in O(2^k) \\ \implies & R(m) \in O(2^{\log(m)}) = O(m) \\ \implies & T(n) \in O(\log(n)) \end{aligned}$$

1.4 $T(n) = T(\sqrt{n}) + \log\log(n)$

Define $n = 2^m$, $m = \log(n)$

$$\begin{aligned} T(n) &= T(n^{\frac{1}{2}}) + \log\log(n) \\ T(2^m) &= T(2^{\frac{1}{2}m}) + \log\log(2^m) \\ &= T(2^{\frac{1}{2}m}) + \log(m) \\ R(m) &= R(\frac{m}{2}) + \log(m) \end{aligned}$$

Define $m = 2^k$, $k = \log(m)$

$$\begin{aligned} R(2^k) &= R(\frac{2^k}{2}) + \log(2^k) \\ S(k) &= S(k-1) + k \\ S(k) &\in O(S(0) + \sum_{i=1}^{k} i) = O(\frac{k(k+1)}{2}) = O(k^2) \end{aligned}$$

Map back to $R(m)$ and $T(n)$:

$$\begin{aligned} & S(k) \in O(k^2) \\ \implies & R(m) \in O(\log^2(m)) \\ \implies & T(n) \in O((\log\log(n))^2) \end{aligned}$$