

Lecture 1: Asymptotic Analysis

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1 Definition

$$\begin{aligned} O(g) &: \{f : \exists c > 0 \wedge n_0 > 0, \forall n \geq n_0, 0 \leq f(n) \leq c \cdot g(n)\} \\ \Omega(g) &: \{f : \exists c > 0 \wedge n_0 > 0, \forall n \geq n_0, 0 \leq c \cdot g(n) \leq f(n)\} \\ \Theta(g) &: \{f : \exists c_1, c_2 > 0 \wedge n_0 > 0, \forall n \geq n_0, c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)\} \\ o(g) &: \{f : \forall c > 0, \exists n_0 > 0, \forall n \geq n_0, 0 \leq f(n) < c \cdot g(n)\} \\ \omega(g) &: \{f : \forall c > 0, \forall n \leq n_0, 0 \leq c \cdot g(n) < f(n)\} \end{aligned}$$

2 Proof with limit

3 Secondary conclusions

$$\Theta(f) = O(f) \cap \Omega(f)$$

Proof. 1. prove: $O(f) \cap \Omega(f) \subseteq \Theta(f)$

From definition:

$$\begin{aligned} \Theta(f(n)) &: \{g : \exists c_1, c_2 > 0, \exists n_0 \geq 0, \forall n \geq n_0, c_1 \cdot f(n) \leq g(n) \leq c_2 \cdot f(n)\} \\ O(f(n)) \cap \Omega(f(n)) &: \{g : \exists c_1 > 0, \exists n_1 \geq 0, \forall n \geq n_1, 0 \leq g(n) \leq c_1 \cdot f(n) \wedge \exists c_2 > 0, \exists n_2 \geq 0, \forall n \geq n_2, 0 \leq c_2 \cdot f(n) \leq g(n)\} \end{aligned}$$

$$\begin{aligned} \text{When } 0 \leq c_1 \leq c_2, O(f(n)) \cap \Omega(f(n)) &: \{g : \exists n_1, n_2 \geq 0, \forall n \geq \{\max(n_1, n_2)\}, 0 \leq c_1 f(n) \leq g(n) \leq c_2 f(n)\} \\ \implies O(f) \cap \Omega(f) &\subseteq \Theta(f) \end{aligned}$$

2. prove: $\Theta(f) \subseteq O(f) \cap \Omega(f)$

$$(f(n) > 0 \wedge, c_1 \leq, c_2) \implies (0 \leq c_1 f(n) \leq g(n) \wedge 0 \leq g(n) \leq c_2 f(n))$$

$$\implies \Theta(f) \subseteq O(f) \cap \Omega(f)$$

$$\implies \Theta(f) = O(f) \cap \Omega(f)$$

□

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0 \implies f(n) \in o(g(n))$$

Proof. From limit definition, $\forall \epsilon > 0, \exists n_0 \geq 0, \forall n \geq n_0, \frac{f(n)}{g(n)} < \epsilon$. Let $c = \epsilon, \frac{f(n)}{g(n)} < \epsilon, f(n) < c \cdot g(n)$.
 $f(n) \geq 0, f(n) \in o(g(n))$ □

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty \implies f(n) \in \omega(g(n))$$

Proof. From $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty \iff \lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = 0$, as

$$\lim_{x \rightarrow \infty} \frac{1}{f(x)} = \frac{1}{\lim_{x \rightarrow \infty} f(x)}, \lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = \frac{1}{\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)}} = \frac{1}{\infty} = 0.$$

Similarly, $\forall \epsilon > 0, \exists n_0 \geq 0, \forall n \geq n_0, \frac{g(n)}{f(n)} < \epsilon$. Let $c = \frac{1}{\epsilon}$, $\frac{g(n)}{f(n)} < \epsilon, c \cdot g(n) < f(n)$. □

$$0 < \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} < \infty \implies f(n) \in \Theta(g(n))$$

Proof. Definitions:

$$\Theta(g) : \{f : \exists c_1, c_2 > 0 \wedge n_0 > 0, \forall n \geq n_0, c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)\}$$

$$O(g) : \{f : \exists c > 0 \wedge n_0 > 0, \forall n \geq n_0, 0 \leq f(n) \leq c \cdot g(n)\}$$

$$o(g) : \{f : \forall c > 0, \exists n_0 > 0, \forall n \geq n_0, 0 \leq f(n) < c \cdot g(n)\}$$

Prove: $\Theta(f) = O(f) \setminus o(f)$

Proof. 1. $\Theta(f) \subseteq O(f) \setminus o(f)$

Suppose $g(n) \in \Theta(f), \Theta(f) = O(f) \cap \Omega(f) \implies g(n) \in O(f)$.

Suppose $g(n) \in o(f), (\exists c_1, c_2 > 0, \exists n_1 > 0, \forall n \geq n_1, 0 \leq c_1 \cdot f(n) \leq g(n) \leq c_2 \cdot f(n)) \wedge (\forall c_3 > 0, \exists n_2 > 0, \forall n \geq n_2, 0 \leq g(n) < c_3 \cdot f(n))$

Let $c_3 < c_1, c_1 \cdot f(n) \leq g(n) < c_3 \cdot f(n)$, contradiction,

$\implies g(n) \in \Theta(f) \rightarrow g(n) \notin o(f)$

$\implies \Theta(f) \subseteq O(f) \setminus o(f)$

2.

$$O(f) \setminus o(f) \subseteq \Theta(f) =$$

$$\begin{aligned} & \{g : (\exists c_1 > 0, \exists n_1 > 0, \forall n \geq n_1, 0 \leq g(n) \leq c_1 \cdot f(n)) \\ & \quad \wedge \neg(\forall c_2 > 0, \exists n_2 > 0, \forall n \geq n_2, 0 \leq g(n) < c_2 \cdot f(n))\} \\ & \{g : (\exists c_1 > 0, \exists n_1 > 0, \forall n \geq n_1, 0 \leq g(n) \leq c_1 \cdot f(n)) \\ & \quad \wedge (\exists c_2 > 0, \forall n_2 > 0, \exists n \geq n_2, (g(n) < 0 \vee g(n) \geq c_2 \cdot f(n)))\} \\ & \{g : \exists c_1 > c_2 > 0, \forall n_2 > 0, \exists n_1 \geq n_2, \forall n \geq n_1, 0 \leq c_2 \cdot f(n) \leq g(n) \leq c_1 \cdot f(n)\} \\ & \{g : \exists c_1 > c_2 > 0, \exists n_0 > 0, \forall n \geq n_0, 0 \leq c_2 \cdot f(n) \leq g(n) \leq c_1 \cdot f(n)\} \end{aligned}$$

□

□

4 Reflexivity

4.1 $f(n) \in O(f(n))$

Proof. Suppose $f(n) > 0$

Let $c = 1, n_0 = 1, \forall n \geq n_0, 0 \leq f(n) \leq f(n), \implies f(n) \in O(f(n))$ □

4.2 $f(n) \in \Omega(f(n)), f(n) \in \Theta(f(n))$

Similarly, by picking the right c and n_0 it's easy to prove existential statements.

5 Transitivity

$$5.1 \quad f(n) \in O(g(n)) \wedge g(n) \in O(h(n)) \implies f(n) \in O(h(n))$$

Proof.

$$\exists c_1 > 0, n_1 > 0, \forall n \geq n_1, 0 \leq f(n) \leq c_1 \cdot g(n)$$

$$\exists c_2 > 0, n_2 > 0, \forall n \geq n_2, 0 \leq g(n) \leq c_2 \cdot h(n)$$

$$\text{Let } c_1 = 1, n_0 = \max(n_1, n_2), 0 \leq f(n) \leq g(n) \leq c_2 \cdot h(n) \implies \exists c_2 > 0, \forall n \geq n_0, 0 \leq f(n) \leq c_2 \cdot h(n) \implies f(n) \in O(h(n)) \quad \square$$

$$5.2 \quad f(n) \in \Omega(g(n)) \wedge g(n) \in \Omega(h(n)) \implies f(n) \in \Omega(h(n))$$

Similar to $O(f(n))$ proof

$$5.3 \quad f(n) \in \Theta(g(n)) \wedge g(n) \in \Theta(h(n)) \implies f(n) \in \Theta(h(n))$$

$$\exists c_1^{[1]}, c_2^{[1]} > 0, n_1 > 0, \forall n \geq n_1, 0 \leq c_1^{[1]} \cdot g(n) \leq f(n) \leq c_2^{[1]} \cdot g(n)$$

$$\exists c_1^{[2]}, c_2^{[2]} > 0, n_2 > 0, \forall n \geq n_2, 0 \leq c_1^{[2]} \cdot h(n) \leq g(n) \leq c_2^{[2]} \cdot h(n)$$

$$\text{Let } c_1^{[1]} = c_2^{[1]} = 1, f(n) = g(n),$$

$$0 \leq c_1^{[2]} \cdot h(n) \leq f(n) \leq c_2^{[2]} \cdot h(n) \implies f(n) \in \Theta(h(n))$$

$$5.4 \quad f(n) \in o(g(n)) \wedge g(n) \in o(h(n)) \implies f(n) \in o(h(n))$$

Proof.

$$\forall c_1 > 0, \exists n_1 > 0, \forall n \geq n_1, 0 \leq f(n) < c_1 \cdot g(n) \quad (1)$$

$$\forall c_2 > 0, \exists n_2 > 0, \forall n \geq n_2, 0 \leq g(n) < c_2 \cdot h(n) \quad (2)$$

$$\text{From (2), } \forall c_2 > 0, \exists n_2 > 0, \forall n \geq n_2, 0 \leq c_1 \cdot g(n) < c_1 \cdot c_2 \cdot h(n)$$

$$\forall c_1, c_2 > 0, \exists n_1, n_2 > 0, \forall n \geq \max(n_1, n_2), 0 \leq f(n) < c_1 \cdot g(n) < c_1 \cdot c_2 \cdot h(n)$$

$$\forall c_1, c_2 > 0, \exists n_1, n_2 > 0, \forall n \geq \max(n_1, n_2), 0 \leq f(n) < c_1 \cdot c_2 \cdot h(n) \quad \square$$

$$5.5 \quad f(n) \in \omega(g(n)) \wedge g(n) \in \omega(h(n)) \implies f(n) \in \omega(h(n))$$

$$\forall c_1 > 0, \exists n_1 > 0, \forall n \geq n_1, 0 \leq c_1 \cdot g(n) < f(n)$$

$$\forall c_2 > 0, \exists n_2 > 0, \forall n \geq n_2, 0 \leq c_2 \cdot h(n) < g(n)$$

$$\forall c_1, c_2 > 0, \exists n_1, n_2 > 0, \forall n \geq \max(n_1, n_2), 0 \leq c_1 \cdot c_2 \cdot h(n) < c_1 \cdot g(n) < f(n)$$

$$6 \quad \text{Symmetry: } f(n) \in \Theta(g(n)) \iff g(n) \in \Theta(f(n))$$

Proof.

$$f(n) \in \Theta(g(n)) \implies$$

$$\exists c_1, c_2 > 0, n_1 > 0, \forall n \geq n_1, 0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n) \implies$$

$$\left(0 \leq g(n) \leq \frac{1}{c_1} \cdot f(n) \leq \frac{c_2}{c_1} \cdot g(n)\right) \wedge \left(0 \leq \frac{c_1}{c_2} \cdot g(n) \leq \frac{1}{c_2} \cdot f(n) \leq g(n)\right) \implies$$

$$0 \leq \frac{1}{c_2} \cdot f(n) \leq g(n) \leq \frac{1}{c_1} \cdot f(n)$$

Identical proof for the other direction □

7 Complementary

$$7.1 \quad f(n) \in O(g(n)) \iff g(n) \in \Omega(f(n))$$

Proof.

$$\begin{aligned} f(n) \in O(g(n)) &\implies \\ \exists c > 0 \wedge n_0 > 0, \forall n \geq n_0, 0 \leq f(n) \leq c \cdot g(n) &\implies \\ 0 \leq \frac{1}{c} \cdot f(n) \leq g(n) \end{aligned}$$

Proof for the other direction and the next statement are identical. □

$$7.2 \quad f(n) \in o(g(n)) \iff g(n) \in \omega(f(n))$$

8 Problems

$$3^{n+1} \in O(3^n)$$

Proof.

$$\begin{aligned} 3^{n+1} &= 3 \cdot 3^n \\ \text{let } c &= 4, n_0 = 1, \forall n \geq n_0, \\ 3^{n+1} &< c \cdot 3^n \end{aligned}$$

□

$$4^n \notin O(2^n)$$

Proof. Suppose $f(n) = 4^n \in O(2^n)$,

$$\begin{aligned} \exists c > 0, n_0 > 0, \forall n \geq n_0, 0 \leq 2^{2 \times n} &\leq c \cdot 2^n \implies \\ 0 \leq 2^n &\leq c \\ 0 \leq n &\leq \log_2 c(\text{const}) \\ n &\text{ is upper bounded by a constant, a contradiction} \end{aligned}$$

□

$$2^{\lfloor \log_2 n \rfloor} \in \Theta(n)$$

Proof.

$$\begin{aligned} \text{Let } k &= \lfloor \log_2 n \rfloor, 2^k \leq n < 2^{k+1} = 2 \cdot 2^k \\ \text{Let } c_1 &= \frac{1}{2}, c_2 = 1, n_0 = 2, \\ \frac{n}{2} &< 2^k \leq n \end{aligned}$$

□

For constant i , $a > 0$, $(n + a)^i \in O(n^i)$

Proof.

$$\begin{aligned} \forall a > 0, \exists n_0 > a, \forall n \geq n_0, f(n) = (n + a)^i &\leq (2n)^i = 2^i \cdot n^i, \\ \text{Let } c = 2^i, n_0 = a + 1, \forall n \geq n_0, 0 \leq f(n) &\leq c \cdot n^i \end{aligned}$$

□

Prove: $2^{\log_2 n} \in O(n)$

Proof. Let $c = 2, n_0 = 2, \forall n \geq n_0, 2^{\log_2 n} = n \implies 0 \leq n \leq 2n$

□

Prove: $2^{\log_2 n} \in \Omega(n)$

Proof. Let $c = 0.5, n_0 = 2, \forall n \geq n_0, 2^{\log_2 n} = n \implies 0 \leq 0.5n \leq n$

□

Prove: $2^{\log_2 n} \notin \Theta(\sqrt{n})$

Proof. Suppose otherwise, $\exists c_1, c_2 > 0, n_0 > 0, \forall n \geq n_0, 0 \leq c_1 \cdot \sqrt{n} \leq n \leq c_2 \sqrt{n}$,

$$c_1 \leq \sqrt{n} \leq c_2$$

However, $\lim_{n \rightarrow \infty} \sqrt{n} = \infty$, thus cannot be bound by constant c_2 , a contradiction.

□

Prove: $2^{\log_2 n} \notin \omega(n)$

Proof. Suppose otherwise, $\forall c > 0, \exists n_0 > 0, \forall n \geq n_0, 0 \leq c \cdot n < n$. When $c = 2, \forall n_0 > 0, \forall n \geq n_0, 2n \geq n$, a contradiction. □

Rank order of growth

$$f_1(n) = \log n$$

By reflexivity,

$$f_1(n) \in \Theta(\log n)$$

$$f_2(n) = n!$$

By reflexivity,

$$f_2(n) \in \Theta(n!)$$

$$f_3(n) = 2^n + n$$

$$f_3(n) \in \Theta(2^n)$$

Proof.

□

$$f_4(n) = n^{2.3} + 16n$$

$$f_4(n) \in \Theta(n^{2.3})$$

Proof.

□

$$f_5(n) = \log(n^2)$$

$$f_5(n) \in \Theta(\log(n))$$

Proof. Assume base 2, $f_5(n) = \log(n^2) = 2 \log(n)$, let $c_1 = 1, c_2 = 3, n_0 = 2$,
 $\forall n \geq n_0, 0 \leq \log(n) \leq \log(n) \leq 3 \log(n)$ □

$$\Theta(\log(n)) \subseteq o(n^{2.3}) \subseteq o(2^n) \subseteq o(n!)$$

Proof.

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\log(n)}{n^{2.3}} &= \lim_{n \rightarrow \infty} \frac{(\log(n))'}{(n^{2.3})'} = \lim_{n \rightarrow \infty} \frac{1}{\ln(2)n^{1.3}} = 0, \\ \implies \forall f(n) \in \Theta(\log(n)), f(n) &\in o(n) \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{n^{2.3}}{2^n} &= \lim_{n \rightarrow \infty} \frac{2.3 \ln(n)}{n \cdot \ln(2)} = \lim_{n \rightarrow \infty} \frac{2.3}{n \cdot \ln(2)} = 0, \\ \implies \forall f(n) \in \Theta(n^{2.3}), f(n) &\in o(2^n) \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{2^n}{n!} &= \lim_{n \rightarrow \infty} \frac{\prod_{r=1}^n 2}{\prod_{r=1}^n r} = 0, \forall n > 2 \\ \implies \forall f(n) \in \Theta(2^n), f(n) &\in o(n!) \end{aligned}$$

□

9 Recurrence

master theorem

Theorem 9.1 (Master Method). *Consider the recurrence*

$$T(n) = aT\left(\frac{n}{b}\right) + f(n),$$

where a, b are constants. Then:

(A) If $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then

$$T(n) = O(n^{\log_b a}).$$

(B) If $f(n) = \Theta(n^{\log_b a})$, then

$$T(n) = \Theta(n^{\log_b a} \log n).$$

(C) If $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if f satisfies the smoothness condition

$$af(n/b) \leq cf(n) \quad \text{for some constant } c < 1,$$

then

$$T(n) = \Theta(f(n)).$$

Telescoping

For recurrence relation

$$T : \mathbb{N} \rightarrow \mathbb{R}, T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

1. Define operator $N_k : (N_k T)(n) = \frac{T(n)}{n^k}$, where $k = \log_b a$
2. Transform $T(n) \rightarrow R(n) := \frac{T(n)}{n^k}$
3. Obtain the recurrence relation of $R(n) = R\left(\frac{n}{b}\right) + \frac{f(n)}{n^k}$
4. Define map $E_b : \mathbb{Z} \rightarrow \mathbb{N}, E_b(m) = b^m$
5. Define operator $P_b : (P_b H)(m) = H(E_b(m)) = H(b^m)$
6. Transform $R(n) \rightarrow S(m) := P_b(R) = R \circ E_b, S(m) = R(b^m)$

$$\begin{array}{ccccc}
 T(n) & \xrightarrow{N_k} & R(n) & \xrightarrow{P_b} & S(m) \\
 \downarrow n \mapsto n/b & & \downarrow n \mapsto n/b & & \downarrow m \mapsto m-1 \\
 a \cdot T\left(\frac{n}{b}\right) + f(n) & \longleftarrow & R\left(\frac{n}{b}\right) + \frac{f(n)}{n^k} & \longleftarrow & S(m-1) + \frac{f(b^m)}{b^{m \cdot k}} \\
 & & & & \vdots \\
 \boxed{T(n) = n^k R(n) \in \Theta(n^k H(n))} & \longleftarrow & R(n) \in \Theta(G(\log_b n)) = H(n) & \longleftarrow & S(m) \in \Theta(S(0) + \sum_{i=1}^m \frac{f(b^i)}{b^{i \cdot k}}) = G(m)
 \end{array}$$

$$T(n) \sim n^{\log_b a} \sum_{i=1}^{\lfloor \log_b n \rfloor} \frac{f(b^i)}{b^{i \log_b a}}$$

Problems

$$T(n) = 4T\left(\frac{n}{4}\right) + \frac{n}{\log(n)}$$

$$\text{Define } R(n) = \frac{T(n)}{n},$$

$$\begin{aligned}
 R(n) &= \frac{4 \cdot T\left(\frac{n}{4}\right)}{n} + \frac{1}{\log(n)} \\
 &= R\left(\frac{n}{4}\right) + \frac{1}{\log(n)}
 \end{aligned}$$

$$\text{Define } S(m) = R(4^m),$$

$$\begin{aligned}
 S(m) &= S(m-1) + \frac{\frac{4^m}{\log(4^m)}}{4^m \cdot \log_4 4} \\
 &= S(m-1) + \frac{1}{m \log(4)}
 \end{aligned}$$

$$S(m) \sim \frac{1}{\log(4)} \sum_{i=1}^m \frac{1}{i}$$

$$S(m) \in \Theta(\log m)$$

Map back to $R(n)$ and $T(n)$,

$$\begin{aligned}
 S(m) \in \Theta(\log m) &\implies R(n) \in \Theta(\log(\log_4 n)) \\
 &\implies T(n) \in n \log(\log(n))
 \end{aligned}$$

$$T(n) = 4T\left(\frac{n}{2}\right) + \sqrt{n}$$

Define $R(n) = \frac{T(n)}{n^2}$,

$$\begin{aligned} R(n) &= \frac{4 \cdot T\left(\frac{n}{2}\right)}{n^2} + \frac{\sqrt{n}}{n^2} \\ &= R\left(\frac{n}{2}\right) + n^{-\frac{3}{2}} \end{aligned}$$

Define $S(m) = R(2^n)$,

$$\begin{aligned} S(m) &= S(m-1) + \frac{(2^m)^{-\frac{1}{2}}}{2^{m \cdot \log_2 4}} \\ &= S(m-1) + \frac{2^{-\frac{m}{2}}}{2^{2m}} \\ &= S(m-1) + \frac{1}{2^{\frac{3m}{2}}} \\ S(m) &\sim \sum_{i=1}^m \frac{1}{2^{\frac{3i}{2}}}, \text{ clearly converges} \\ S(m) &\in \Theta(1) \end{aligned}$$

2 Maps back,

$$\begin{aligned} S(m) \in \Theta(1) &\implies R(n) \in \Theta(1) \\ &\implies T(n) \in \Theta(n^2) \end{aligned}$$

$$T(n) = T\left(\frac{n}{2}\right) + 2^{\sqrt{\log n}}$$

1. Using telescoping

$a = 0 \rightarrow \log_b a = 0$, there is no need for $T(n) \mapsto R(n)$.

Define $S(m) = T(2^n)$, we have:

$$\begin{aligned} S(m) &= S(m-1) + 2^{\sqrt{\log 2^m}} \\ &= S(m-1) + 2^{\sqrt{m}} \\ S(m) &\sim \sum_{i=1}^m 2^{\sqrt{i}} \approx \int_1^m 2^{\sqrt{x}} dx \end{aligned}$$

Solve for $\int_1^m 2^{\sqrt{x}} dx$ by substitution: let $u = x^{\frac{1}{2}}$

$$\begin{aligned} \frac{du}{dx} &= \frac{1}{2} x^{-\frac{1}{2}} \\ dx &= \frac{1}{2u} \\ \int 2^{\sqrt{x}} dx &= \int 2u \cdot 2^u du = 2 \int u 2^u du \\ &= \frac{2^u (2u \ln(2) - 1)}{(\ln(2))^2} + C \\ &= \frac{2^{\sqrt{x}} (2\sqrt{x} \ln(2) - 1)}{(\ln(2))^2} + C \\ \int_1^m 2^{\sqrt{x}} dx &= \frac{1}{(\ln(2))^2} \left(2^{\sqrt{m}} (2\sqrt{m} \ln(2) - 1) \right) \end{aligned}$$

Ignoring the constants, we have

$$S(m) \sim \sum_{i=1}^m 2^{\sqrt{i}} \approx c \cdot 2^{\sqrt{m}} \sqrt{m}$$

$$T(n) \in \Theta(\sqrt{\log n} \cdot 2^{\sqrt{\log n}})$$

Recurrence relation of merging k arrays of size n

$$T(k, n) = 2T\left(\frac{k}{2}, n\right) + \frac{nk}{2}$$

$$T\left(\frac{k}{2}, n\right) = 2T\left(\frac{k}{4}, n\right) + \frac{nk}{4}$$

Observed that the non-recursive function can be tightly bounded by nk ,

$$T(k, n) = 2T\left(\frac{k}{2}, n\right) + cnk$$

With respect to k , Define $R(k, n) = \frac{T(k, n)}{k}$

$$R(k, n) = \frac{2T\left(\frac{k}{2}, n\right)}{k} + cn$$

$$= R\left(\frac{k}{2}, n\right) + cn$$

Define $S(m, n) = R(2^m, n)$,
 $f(k, n) = cnk \rightarrow f(2^m, n) = cn2^m$,

$$S(m, n) = S(m-1, n) + \frac{cn2^m}{2^m}$$

$$\sim cn \sum_{i=1}^m 1$$

$$S(m, n) \in \Theta(nm)$$

Maps back to $R(n)$,

$$S(m, n) \in \Theta(n) \implies R(k, n) \in \Theta(n \log(k))$$

$$\implies T(k, n) \in \Theta(nk \log(k))$$

10 Proof of Correctness

Iterative algorithms

Steps

1. Loop invariant
2. Initialization
3. Maintenance
4. Termination

```
Dijkstra( $G(V, E), s \in V$ ):  
   $d(s) = 0$   
   $R = \{s\}$   
  while( $R \neq V$ ):  
     $v \in V \setminus R, v = \min_{u \in R} (d(u) + w(u, v))$   
     $d(v) = \min_{u \in R} (d(u) + w(u, v))$   
     $R \leftarrow v$ 
```

Loop invariant:

$$\forall u \in R, d(u) = \text{dist}(s, u)$$

Initialization: