Lecture 1: Asymptotic Analysis

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1 Definition

$$
\begin{aligned}
&O(g) : \{f : \exists c > 0 \wedge n_0 > 0, \forall n \geq n_0, 0 \leq f(n) \leq c \cdot g(n)\} \\
&\Omega(g) : \{f : \exists c > 0 \wedge n_0 > 0, \forall n \geq n_0, 0 \leq c \cdot g(n) \leq f(n)\} \\
&\Theta(g) : \{f : \exists c_1, c_2 > 0 \wedge n_0 > 0, \forall n \geq n_0, c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)\} \\
&o(g) : \{f : \forall c > 0, \exists n_0 > 0, \forall n \geq n_0, 0 \leq f(n) < c \cdot g(n)\} \\
&\omega(g) : \{f : \forall c > 0, \forall n \leq n_0, 0 \leq c \cdot g(n) < f(n)\}
\end{aligned}
$$

2 Proof with limit

3 Secondary conclusions

$$\Theta(f) = O(f) \cap \Omega(f)$$

Proof. 1. prove: $O(f) \cap \Omega(f) \subseteq \Theta(f)$
From definition:

$$
\begin{aligned}
&\Theta(f(n)) : \{g : \exists c_1, c_2 > 0, \exists n_0 \geq 0, \forall n \geq n_0, c_1 \cdot f(n) \leq g(n) \leq c_2 \cdot f(n)\} \\
&O(f(n)) \cap \Omega(f(n)) : \{g : \exists c_1 > 0, \exists n_1 \geq 0, \forall n \geq n_1, 0 \leq g(n) \leq c_1 \cdot f(n) \wedge \exists c_2 > 0, \exists n_2 \geq 0, \forall n \geq n_2, 0 \leq c_2 \cdot f(n) \leq g(n)\}
\end{aligned}
$$

When $0 \leq c_1 \leq c_2, O(f(n)) \cap \Omega(f(n)) : \{g : \exists n_1, n_2 \geq 0, \forall n \geq \{\max(n_1, n_2)\}, 0 \leq c_1 f(n) \leq g(n) \leq c_2 f(n)\}$
$\implies O(f) \cap \Omega(f) \subseteq \Theta(f)$
2. prove: $\Theta(f) \subseteq O(f) \cap \Omega(f)$
$(f(n) > 0 \wedge, c_1 \leq, c_2) \implies (0 \leq c_1 f(n) \leq g(n) \wedge 0 \leq g(n) \leq c_2 f(n))$
$\implies \Theta(f) \subseteq O(f) \cap \Omega(f)$
$\implies \Theta(f) = O(f) \cap \Omega(f)$

□

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \implies f(n) \in o(g(n))$$

Proof. From lmit definition, $\forall \epsilon > 0, \exists n_0 \geq 0, \forall n \geq n_0, \frac{f(n)}{g(n)} < \epsilon$. Let $c = \epsilon, \frac{f(n)}{g(n)} < \epsilon, f(n) < c \cdot g(n)$. $f(n) \geq 0, f(n) \in o(g(n))$ □

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty \implies f(n) \in \omega(g(n))$$

Proof. From $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \iff \lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$, as
$\lim_{x\to\infty} \frac{1}{f(x)} = \frac{1}{\lim_{x\to\infty} f(x)}, \lim_{n\to\infty} \frac{g(n)}{f(n)} = \frac{1}{\lim_{n\to\infty} \frac{f(n)}{g(n)}} = \frac{1}{\infty} = 0$.
Similarly, $\forall \epsilon > 0, \exists n_0 \geq 0, \forall n \geq n_0, \frac{g(n)}{f(n)} < \epsilon$. Let $c = \frac{1}{\epsilon}, \frac{g(n)}{f(n)} < \epsilon, c \cdot g(n) < f(n)$. □

$$0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty \implies f(n) \in \Theta(g(n))$$

Proof. Definitions:
$\Theta(g) : \{f : \exists c_1, c_2 > 0 \wedge n_0 > 0, \forall n \geq n_0, c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)\}$
$O(g) : \{f : \exists c > 0 \wedge n_0 > 0, \forall n \geq n_0, 0 \leq f(n) \leq c \cdot g(n)\}$
$o(g) : \{f : \forall c > 0, \exists n_0 > 0, \forall n \geq n_0, 0 \leq f(n) < c \cdot g(n)\}$
Prove: $\Theta(f) = O(f) \setminus o(f)$

Proof. 1. $\Theta(f) \subseteq O(f) \setminus o(f)$
Suppose $g(n) \in \Theta(f), \Theta(f) = O(f) \cap \Omega(f) \implies g(n) \in O(f)$.
Suppose $g(n) \in o(f), (\exists c_1, c_2 > 0, \exists n_1 > 0, \forall n \geq n_1, 0 \leq c_1 \cdot f(n) \leq g(n) \leq c_2 \cdot f(n)) \wedge (\forall c_3 > 0, \exists n_2 > 0, \forall n \geq n_2, 0 \leq g(n) < c_3 \cdot f(n))$
Let $c_3 < c_1, c_1 \cdot f(n) \leq g(n) < c_3 \cdot f(n)$, contradiction,
$\implies g(n) \in \Theta(f) \to g(n) \notin o(f)$
$\implies \Theta(f) \subseteq O(f) \setminus o(f)$
2.

$$O(f) \setminus o(f) \subseteq \Theta(f) =$$

$$\begin{aligned}
&\{g : (\exists c_1 > 0, \exists n_1 > 0, \forall n \geq n_1,\ 0 \leq g(n) \leq c_1 \cdot f(n)) \\
&\qquad \wedge\ \neg(\forall c_2 > 0, \exists n_2 > 0, \forall n \geq n_2,\ 0 \leq g(n) < c_2 \cdot f(n))\} \\
&\{g : (\exists c_1 > 0, \exists n_1 > 0, \forall n \geq n_1,\ 0 \leq g(n) \leq c_1 \cdot f(n)) \\
&\qquad \wedge\ (\exists c_2 > 0, \forall n_2 > 0, \exists n \geq n_2,\ (g(n) < 0 \vee g(n) \geq c_2 \cdot f(n)))\} \\
&\{g : \exists c_1 > c_2 > 0, \forall n_2 > 0, \exists n_1 \geq n_2, \forall n \geq n_1,\ 0 \leq c_2 \cdot f(n) \leq g(n) \leq c_1 \cdot f(n)\} \\
&\{g : \exists c_1 > c_2 > 0, \exists n_0 > 0, \forall n \geq n_0,\ 0 \leq c_2 \cdot f(n) \leq g(n) \leq c_1 \cdot f(n)\}
\end{aligned}$$

□

□

4 Reflexivity

4.1 $f(n) \in O(f(n))$

Proof. Suppose $f(n) > 0$
Let $c = 1, n_0 = 1, \forall n \geq n_0, 0 \leq f(n) \leq f(n), \implies f(n) \in O(f(n))$ □

4.2 $f(n) \in \Omega(f(n))$, $f(n) \in \Theta(f(n))$

Similarly, by picking the right c and n_0 it's easy to prove existential statements.

5 Transitivity

5.1 $f(n) \in O(g(n)) \land g(n) \in O(h(n)) \implies f(n) \in O(h(n))$

Proof.

$$\exists c_1 > 0, n_1 > 0, \forall n \geq n_1, 0 \leq f(n) \leq c_1 \cdot g(n)$$
$$\exists c_2 > 0, n_2 > 0, \forall n \geq n_2, 0 \leq g(n) \leq c_2 \cdot h(n)$$

Let $c_1 = 1, n_0 = \max(n_1, n_2)$, $0 \leq f(n) \leq g(n) \leq c_2 \cdot h(n) \implies \exists c_2 > 0, \forall n \geq n_0, 0 \leq f(n) \leq c_2 \cdot h(n)$
$\implies f(n) \in O(h(n))$ □

5.2 $f(n) \in \Omega(g(n)) \land g(n) \in \Omega(h(n)) \implies f(n) \in \Omega(h(n))$

Similar to $O(f(n))$ proof

5.3 $f(n) \in \Theta(g(n)) \land g(n) \in \Theta(h(n)) \implies f(n) \in \Theta(h(n))$

$$\exists c_1^{[1]}, c_2^{[1]} > 0, n_1 > 0, \forall n \geq n_1, 0 \leq c_1^{[1]} \cdot g(n) \leq f(n) \leq c_2^{[1]} \cdot g(n)$$
$$\exists c_1^{[2]}, c_2^{[2]} > 0, n_2 > 0, \forall n \geq n_2, 0 \leq c_1^{[2]} \cdot h(n) \leq g(n) \leq c_2^{[2]} \cdot h(n)$$

Let $c_1^{[1]} = c_2^{[1]} = 1, f(n) = g(n)$,
$0 \leq c_1^{[2]} \cdot h(n) \leq f(n) \leq c_2^{[2]} \cdot h(n) \implies f(n) \in \Theta(h(n))$

5.4 $f(n) \in o(g(n)) \land g(n) \in o(h(n)) \implies f(n) \in o(h(n))$

Proof.

$$\forall c_1 > 0, \exists n_1 > 0, \forall n \geq n_1, 0 \leq f(n) < c_1 \cdot g(n) \tag{1}$$
$$\forall c_2 > 0, \exists n_2 > 0, \forall n \geq n_2, 0 \leq g(n) < c_2 \cdot h(n) \tag{2}$$

From (2), $\forall c_2 > 0, \exists n_2 > 0, \forall n \geq n_2, 0 \leq c_1 \cdot g(n) < c_1 \cdot c_2 \cdot h(n)$

$$\forall c_1, c_2 > 0, \exists n_1, n_2 > 0, \forall n \geq \max(n_1, n_2), 0 \leq f(n) < c_1 \cdot g(n) < c_1 \cdot c_2 \cdot h(n)$$
$$\forall c_1, c_2 > 0, \exists n_1, n_2 > 0, \forall n \geq \max(n_1, n_2), 0 \leq f(n) < c_1 \cdot c_2 \cdot h(n)$$

□

5.5 $f(n) \in \omega(g(n) \land g(n) \in \omega(h(n))) \implies f(n) \in \omega(h(n))$

$$\forall c_1 > 0, \exists n_1 > 0, \forall n \geq n_1, 0 \leq c_1 \cdot g(n) < f(n)$$
$$\forall c_2 > 0, \exists n_2 > 0, \forall n \geq n_2, 0 \leq c_2 \cdot h(n) < g(n)$$
$$\forall c_1, c_2 > 0, \exists n_1, n_2 > 0, \geq \max(n_1, n_2), 0 \leq c_1 \cdot c_2 \cdot h(n) < c_1 \cdot g(n) < f(n)$$

6 Symmetry: $f(n) \in \Theta(g(n)) \iff g(n) \in \Theta(f(n))$

Proof.

$$\begin{aligned}
&f(n) \in \Theta(g(n)) \implies \\
&\exists c_1, c_2 > 0, n_1 > 0, \forall n \geq n_1, 0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n) \implies \\
&\left(0 \leq g(n) \leq \frac{1}{c_1} \cdot f(n) \leq \frac{c_2}{c_1} \cdot g(n)\right) \land \left(0 \leq \frac{c_1}{c_2} \cdot g(n) \leq \frac{1}{c_2} \cdot f(n) \leq g(n)\right) \implies \\
&0 \leq \frac{1}{c_2} \cdot f(n) \leq g(n) \leq \frac{1}{c_1} \cdot f(n)
\end{aligned}$$

Identical proof for the other direction ☐

7 Complementary

7.1 $f(n) \in O(g(n) \iff g(n) \in \Omega(f(n)))$

Proof.

$$
\begin{aligned}
&f(n) \in O(g(n)) \implies \\
&\exists c > 0 \wedge n_0 > 0, \forall n \geq n_0, 0 \leq f(n) \leq c \cdot g(n) \implies \\
&0 \leq \frac{1}{c} \cdot f(n) \leq g(n)
\end{aligned}
$$

Proof for the other direction and the next statement are identical. ☐

7.2 $f(n) \in o(g(n) \iff g(n) \in \omega(f(n)))$

8 Problems

$3^{n+1} \in O(3^n)$

Proof.

$$
\begin{aligned}
&3^{n+1} = 3 \cdot 3^n \\
&letc = 4, n_0 = 1, \forall n \geq n_0, \\
&3^{n+1} < c \cdot 3^n
\end{aligned}
$$

☐

$4^n \notin O(2^n)$

Proof. Suppose $f(n) = 4^n \in O(2^n)$,

$$
\begin{aligned}
&\exists c > 0, n_0 > 0, \forall n \geq n_0, 0 \leq 2^{2 \times n} \leq c \cdot 2^n \implies \\
&0 \leq 2^n \leq c \\
&0 \leq n \leq \log_2 c (const) \\
&\text{n is upper bounded by a constant, a contradiction}
\end{aligned}
$$

☐

$2^{\lfloor \log_2 n \rfloor} \in \Theta(n)$

Proof.

$$
\begin{aligned}
&\text{Let } k = \lfloor \log_2 n \rfloor, 2^k \leq n < 2^{k+1} = 2 \cdot 2^k \\
&\text{Let } c_1 = \frac{1}{2}, c_2 = 1, n_0 = 2, \\
&\frac{n}{2} < 2^k \leq n
\end{aligned}
$$

☐

For constant i, $a > 0$, $(n+a)^i \in O(n^i)$

Proof.

$$\forall a > 0, \exists n_0 > a, \forall n \geq n_0, f(n) = (n+a)^i \leq (2n)^i = 2^i \cdot n^i,$$
$$\text{Let } c = 2^i, n_0 = a+1, \forall n \geq n_0, 0 \leq f(n) \leq c \cdot n^i$$

□

Prove: $2^{\log_2 n} \in O(n)$

Proof. Let $c = 2, n_0 = 2, \forall n \geq n_0, 2^{\log_2 n} = n \implies 0 \leq n \leq 2n$ □

Prove: $2^{\log_2 n} \in \Omega(n)$

Proof. Let $c = 0.5, n_0 = 2, \forall n \geq n_0, 2^{\log_2 n} = n \implies 0 \leq 0.5n \leq n$ □

Prove: $2^{\log_2 n} \notin \Theta(\sqrt{n})$

Proof. Suppose otherwise, $\exists c_1, c_2 > 0, n_0 > 0, \forall n \geq n_0, 0 \leq c_1 \cdot \sqrt{n} \leq n \leq c_2\sqrt{n}$,

$$c_1 \leq \sqrt{n} \leq c_2$$
$$\text{However, } \lim_{n\to\infty} \sqrt{n} = \infty, \text{ thus cannot be bound by constant } c_2, \text{a contradiction.}$$

□

Prove: $2^{\log_2 n} \notin \omega(n)$

Proof. Suppose otherwise, $\forall c > 0, \exists n_0 > 0, \forall n \geq n_0, 0 \leq c \cdot n < n$. When $c = 2, \forall n_0 > 0, \forall n \geq n_0, 2n \geq n$, a contradiction. □

Rank order of growth

$f_1(n) = \log n$

By reflexivity,

$$f_1(n) \in \Theta(\log n)$$

$f_2(n) = n!$

By reflexivity,

$$f_2(n) \in \Theta(n!)$$

$f_3(n) = 2^n + n$

$$f_3(n) \in \Theta(2^n)$$

Proof. □

$f_4(n) = n^{2.3} + 16n$

$$f_4(n) \in \Theta(n^{2.3})$$

Proof. □

$f_5(n) = \log(n^2)$

$$f_5(n) \in \Theta(\log(n))$$

Proof. Assume base 2, $f_5(n) = \log(n^2) = 2\log(n)$, let $c_1 = 1, c_2 = 3, n_0 = 2$,
$\forall n \geq n_0, 0 \leq \log(n) \leq \log(n) \leq 3\log(n)$ □

$$\Theta(\log(n)) \subseteq o(n^{2.3}) \subseteq o(2^n) \subseteq o(n!)$$

Proof.

$$\lim_{n\to\infty} \frac{\log(n)}{n^{2.3}} = \lim_{n\to\infty} \frac{(\log(n))'}{(n^{2.3})'} = \lim_{n\to\infty} \frac{1}{\ln(2)n^{1.3}} = 0,$$
$$\implies \forall f(n) \in \Theta(\log(n)), f(n) \in o(n)$$

$$\lim_{n\to\infty} \frac{n^{2.3}}{2^n} = \lim_{n\to\infty} \frac{2.3\ln(n)}{n \cdot \ln(2)} = \lim_{n\to\infty} \frac{2.3}{n \cdot \ln(2)} = 0,$$
$$\implies \forall f(n) \in \Theta(n^{2.3}), f(n) \in o(2^n)$$

$$\lim_{n\to\infty} \frac{2^n}{n!} = \lim_{n\to\infty} \frac{\prod_{r=1}^{n} 2}{\prod_{r=1}^{n} r} = 0, \forall n > 2$$
$$\implies \forall f(n) \in \Theta(2^n), f(n) \in o(n!)$$

□

9 Recurrence

master theorem

Theorem 9.1 (Master Method)**.** *Consider the recurrence*

$$T(n) = aT\left(\tfrac{n}{b}\right) + f(n),$$

where a, b are constants. Then:

(A) If $f(n) = O(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$, then

$$T(n) = O(n^{\log_b a}).$$

(B) If $f(n) = \Theta(n^{\log_b a})$, then

$$T(n) = \Theta(n^{\log_b a} \log n).$$

(C) If $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$, and if f satisfies the smoothness condition

$$af(n/b) \leq cf(n) \quad \textit{for some constant } c < 1,$$

then

$$T(n) = \Theta(f(n)).$$

Telescoping

For recurrence relation

$$T : \mathbb{N} \to \mathbb{R}, T(n) = a \cdot T(\frac{n}{b}) + f(n)$$

1. Define operator $N_k : (N_k T)(n) = \frac{T(n)}{n^k}$, where $k = \log_b a$
2. Transform $T(n) \to R(n) := \frac{T(n)}{n^k}$
3. Obtain the recurrence relation of $R(n) = R(\frac{n}{b}) + \frac{f(n)}{n^k}$
4. Define map $E_b : \mathbb{Z} \to \mathbb{N}, E_b(m) = b^m$
5. Define operator $P_b : (P_b H)(m) = H(E_b(m)) = H(b^m)$
6. Transform $R(n) \to S(m) := P_b(R) = R \circ E_b, S(m) = R(b^m)$

$$\begin{tikzcd}
T(n) \arrow[r, "N_k"] \arrow[d, "n \mapsto n/b"] & R(n) \arrow[r, "P_b"] \arrow[d, "n \mapsto n/b"] & S(m) \arrow[d, "m \mapsto m-1"] \\
a \cdot T(\frac{n}{b}) + f(n) \arrow[r, leftrightarrow] & R(\frac{n}{b}) + \frac{f(n)}{n^k} \arrow[r, leftrightarrow] & S(m-1) + \frac{f(b^m)}{b^{m \cdot k}} \arrow[d, dashed] \\
\boxed{T(n) = n^k R(n) \in \Theta(n^k H(n))} & R(n) \in \Theta(G(\log_b n)) = H(n) \arrow[l] & S(m) \in \Theta(S(0) + \sum_{i=1}^{m} \frac{f(b^i)}{b^{i \cdot k}}) = G(m) \arrow[l]
\end{tikzcd}$$

$$T(n) \sim n^{\log_b a} \sum_{i=1}^{\lfloor \log_b n \rfloor} \frac{f(b^i)}{b^{i \log_b a}}$$

Problems

$T(n) = 4T(\frac{n}{4}) + \frac{n}{\log(n)}$

Define $R(n) = \frac{T(n)}{n}$,

$$\begin{aligned}
R(n) &= \frac{4 \cdot T(\frac{n}{4})}{n} + \frac{1}{\log(n)} \\
&= R(\frac{n}{4}) + \frac{1}{\log(n)}
\end{aligned}$$

Define $S(m) = R(4^n)$,

$$\begin{aligned}
S(m) &= S(m-1) + \frac{\frac{4^m}{\log(4^m)}}{4^{m \cdot \log_4 4}} \\
&= S(m-1) + \frac{1}{m \log(4)} \\
S(m) &\sim \frac{1}{\log(4)} \sum_{i=1}^{m} \frac{1}{i} \\
S(m) &\in \Theta(\log m)
\end{aligned}$$

Map back to $R(n)$ and $T(n)$,

$$\begin{aligned}
S(m) \in \Theta(\log m) &\implies R(n) \in \Theta(\log(\log_4 n)) \\
&\implies T(n) \in n \log(\log(n))
\end{aligned}$$

$T(n) = 4T(\frac{n}{2}) + \sqrt{n}$

Define $R(n) = \frac{T(n)}{n^2}$,

$$\begin{aligned} R(n) &= \frac{4 \cdot T(\frac{n}{2})}{n^2} + \frac{\sqrt{n}}{n^2} \\ &= R(\frac{n}{2}) + n^{-\frac{3}{2}} \end{aligned}$$

Define $S(m) = R(2^n)$,

$$\begin{aligned} S(m) &= S(m-1) + \frac{(2^m)^{-\frac{1}{2}}}{2^{m \cdot \log_2 4}} \\ &= S(m-1) + \frac{2^{-\frac{m}{2}}}{2^{2m}} \\ &= S(m-1) + \frac{1}{2^{\frac{3m}{2}}} \\ S(m) &\sim \sum_{i=1}^{m} \frac{1}{2^{\frac{3i}{2}}} \text{ , clearly converges} \\ S(m) &\in \Theta(1) \end{aligned}$$

2 Maps back,

$$\begin{aligned} S(m) \in \Theta(1) &\implies R(n) \in \Theta(1) \\ &\implies T(n) \in \Theta(n^2) \end{aligned}$$

$T(n) = T(\frac{n}{2}) + 2^{\sqrt{\log n}}$

1. Using telescoping
$a = 0 \to \log_b a = 0,$ there is no need for $T(n) \mapsto R(n)$.
Define $S(m) = T(2^n)$, we have:

$$\begin{aligned} S(m) &= S(m-1) + 2^{\sqrt{\log 2^m}} \\ &= S(m-1) + 2^{\sqrt{m}} \\ S(m) &\sim \sum_{i=1}^{m} 2^{\sqrt{i}} \approx \int_1^m 2^{\sqrt{x}} dx \end{aligned}$$

Solve for $\int_1^m 2^{\sqrt{x}} dx$ by substitution: let $u = x^{\frac{1}{2}}$

$$\begin{aligned} \frac{du}{dx} &= \frac{1}{2} x^{-\frac{1}{2}} \\ dx &= \frac{1}{2u} \\ \int 2^{\sqrt{x}} dx &= \int 2u \cdot 2^u du = 2 \int u 2^u du \\ &= \frac{2^u (2u \ln(2) - 1)}{(\ln(2))^2} + C \\ &= \frac{2^{\sqrt{x}} (2\sqrt{x} \ln(2) - 1)}{(\ln(2))^2} + C \\ \int_1^m 2^{\sqrt{x}} dx &= \frac{1}{(\ln(2))^2} \left(2^{\sqrt{m}} (2\sqrt{m} \ln(2) - 1) \right) \end{aligned}$$

Ignoring the constants, we have

$$S(m) \sim \sum_{i=1}^{m} 2^{\sqrt{i}} \approx c \cdot 2^{\sqrt{m}} \sqrt{m}$$
$$T(n) \in \Theta(\sqrt{\log n} \cdot 2^{\sqrt{\log n}})$$

Recurrence relation of merging k arrays of size n

$$T(k,n) = 2T(\frac{k}{2}, n) + \frac{nk}{2}$$
$$T(\frac{k}{2}, n) = 2T(\frac{k}{4}, n) + \frac{nk}{4}$$

Observed that the non-recursive funcion can be tightly bounded by nk,

$$T(k,n) = 2T(\frac{k}{2}, n) + cnk$$

With respect to k, Define $R(k,n) = \frac{T(k,n)}{k}$

$$\begin{aligned} R(k,n) &= \frac{2T(\frac{k}{2}, n)}{k} + cn \\ &= R(\frac{k}{2}, n) + cn \end{aligned}$$

Define $S(m,n) = R(2^m, n)$,
$f(k,n) = cnk \to f(2^m, n) = cn2^m$,

$$\begin{aligned} S(m,n) &= S(m-1, n) + \frac{cn2^m}{2^m} \\ &\sim cn \sum_{i=1}^{m} 1 \\ S(m,n) &\in \Theta(nm) \end{aligned}$$

Maps back to $R(n)$,

$$\begin{aligned} S(m,n) \in \Theta(n) &\implies R(k,n) \in \Theta(n \log(k)) \\ &\implies T(k,n) \in \Theta(nk \log(k)) \end{aligned}$$

10 Proof of Correctness

Iterative algorithms

Steps

1. Loop invariant: desireable conditions that holds at the start of every iteration
2. Initialization: the loop invariant is true at the start of the first iteration
3. Maintenance: If the loop invariant is true at the start of the current iteration, it is true at the start of the next iteration
4. Termination: The loop invariant is true at the end of the last iteration $\to$ the algorithm outputs a correct answer

Example: Dijkstra

Setting: Let $G = (V, E)$ be a directed graph with strictly positive edge weights $w : E \mapsto \mathbb{R}^+$. Fix source vertex $s \in V$, $\forall v \in V$,let $dist(s, v)$ be the length of a shortest path from s to v. If v is unreachable from s, $dist(s, v) = \infty$.
Notation: $dist(s, v)$: the actual shortest distance from s to v; $d(v)$: the algorithm variable of current shortest possible distance from s to v. $d(v) \geq dist(s, v)$

```
Dijkstra(G(V,E), s ∈ V):
    d(s) = 0
    R = {s}
    while(R ≠ V):
        v ∈ V \ R, v = min_{u∈R}(d(u) + w(u,v))
        d(v) = min_{u∈R}(d(u) + w(u,v))
        R ← v
```

Loop invariant:

$$\forall v \in R, d(v) = dist(s, v)$$
$$R \subseteq V \text{ is the set of settled vertices}$$

Initialization:

$$d(s) = dist(s, s) = 0$$

Maintenance:

Suppose $\forall u \in R, d(u) = dist(s, u)$, select $v \in V \setminus R$ such that,

$$v = \min_{u \in R}(d(u) + w(u, v))$$
$$d(v) = \min_{u \in R}(d(u) + w(u, v))$$

Prove: $d(v) = dist(s, v)$
1. $d(v) \geq dist(s, v)$

$$dist(s, v) = min_{u \in V}(d(u) + w(u, v)) \leq min_{u \in R}(d(u) + w(u, v)) = d(v)$$

2. $d(v) \leq dist(s, v)$
Suppose otherwise, for the shortest path $P = s \vdash^* v$, $\exists x \in V, x \notin R$,