WA 1

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1 Asymptotic Analysis

1.1 $f(n) = 3n + 1,\ g(n) = 4n$

$f(n) \in \Theta(g(n))$

Proof. $\Theta(g) : \{f : \exists c_1, c_2 > 0 \wedge n_0 > 0, \forall n \geq n_0, c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)\}$

Let $c_1 = \frac{3}{4}, c_2 = 1, n_0 = 1$, trivial $\square$

1.2 $f(n) = 3^n,\ g(n) = 2^n \cdot n^{10000}$

$f(n) \in \omega(g(n))$

Proof. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{3^n}{2^n \cdot n^{10000}} = \lim_{n\to\infty} \frac{1.5^n}{n^{10000}} = \lim_{n\to\infty} \frac{\ln(1.5)1.5^n}{10000n^{9999}}$
$= ... = \lim_{n\to\infty} \frac{(\ln 1.5)^{10000} 1.5^n}{10000!} = \infty \implies f(n) \in \omega(g(n))$ $\square$

1.3 $f(n) = 2^n,\ g(n) = 2^{n+1}$

$f(n) \in \Theta(g(n))$

Proof. $g(n) = 2 \cdot 2^n$, let $c_1 = \frac{1}{2}, c_2 = 1, n_0 = 2$,
$\forall n \geq n_0, 1 \cdot 2^n \leq 2^n \leq 2 \cdot 2^n \implies f(n) \in \Theta(g(n))$ $\square$

1.4 $f(n) = 2^{(2^n)},\ g(n) = 2^{(2^{n+1})}$

$f(n) \in o(g(n))$

Proof. Let $u = 2^n$, $f(n) = 2^u, g(n) = 2^{2\cdot u}, g(n) = (f(n))^2. \forall n \in \mathbb{Z}^+, f(n) > 1$,
$f(n) < (f(n))^2, 0 < f(n) < g(n) \implies f(n) \in o(g(n))$ $\square$

1.5 $f(n) = \sum_{i=1}^{n} \frac{1}{2^i},\ g(n) = \sum_{i=1}^{n} \frac{i^{10000}}{2^i}$

$f(n) \in \Theta(g(n))$

Proof. $f(n) = \frac{\frac{1}{2}(\frac{1}{2^n} - 1)}{\frac{1}{2} - 1} = 1 - \frac{1}{2^n}$, trivial to show this is tightly upper bounded by constant. $f(n) \in \Theta(1)$
Next we will show that the summation $g(n)$ converges.

$$\lim_{n\to\infty} \frac{\frac{n^{10000}}{2^n}}{\frac{(n-1)^{10000}}{2^{n-1}}} = \lim_{n\to\infty} \frac{n^{10000}}{2(n-1)^{10000}} = \lim_{n\to\infty} \frac{10000n^{9999}}{20000(n-1)^{9999}} = \dots$$

$$= \lim_{n\to\infty} \frac{10000!}{2(10000!)} = \frac{1}{2} < 1 \implies \sum_{i=1}^{n} \frac{i^{10000}}{2^i} \quad \text{converges}$$

$g(n) \in \Theta(1) \implies h(n) \in \Theta(g(n))$ by symmetry $\implies f(n) \in \Theta(g(n))$ by transitivity. $\square$

1.6 $f(n) = \log\log(n^n),\ g(n) = \log\log(2^{\sqrt{n}})$

$f(n) \in \Theta(g(n))$

Proof. $f(n) = \log(n\log(n)) = \log(n) + \log(\log(n))$
$g(n) = \log(\sqrt{n}\log 2) = \log(\sqrt{n}) + \log(\log 2) = \frac{1}{2}\log(n) + \log\log(2)$
$L = \frac{f(n)}{g(n)} = \frac{\log n + \log(\log n)}{0.5\log n + \log(\log 2)} = \frac{1+\frac{\log\log(n)}{\log(n)}}{\frac{1}{2}\frac{\log\log(2)}{\log(n)}}$
$\lim_{n\to\infty} L = 2$
$0 < \lim_{n\to\infty}\frac{f(n)}{g(n)} < \infty \implies f(n) \in \Theta(g(n))$ $\square$

1.7 $f(n) = 2^{\sqrt{\log n}},\ g(n) = \sqrt{n}$

$f(n) \in o(g(n))$

Proof. $L = \frac{f(n)}{g(n)}, \log(L) = \log(f(n)) - \log(g(n)) = \sqrt{\log n}\log(2) - \frac{1}{2}\log(n)$
$= -\frac{1}{2}\log n(1 - \frac{\log 2\sqrt{\log n}}{\log n}) = -\frac{1}{2}\log n(1 - \frac{\log 2}{\sqrt{\log n}})$
$\lim_{n\to\infty}\log(L) = -\infty \implies \lim_{n\to\infty} L = e^{-\infty} = 0^+ \implies f(n) \in o(g(n))$ $\square$

1.8 $f(n) = 2^{\sqrt{\log n}},\ g(n) = 2^{2^{\sqrt{\log\log n}}}$

$f(n) \in \omega(g(n))$

Proof. Let $u = \log\log n,\ f(u) = 2^{\sqrt{2^u}}, g(u) = 2^{\sqrt{u}}$
$L = \frac{f(u)}{g(u)} = \frac{2^{\sqrt{2^u}}}{2^{\sqrt{u}}}, \log(L) = \log(2)(\sqrt{2^u} - \sqrt{u}), \lim_{u\to\infty}\log(L) = \infty \implies \lim_{u\to\infty} L = \infty$
$u = \log\log n$ is monotonous and unbounded, $\lim\frac{f(u)}{g(u)} = \infty \implies \lim\frac{f(n)}{g(n)} = \infty \implies f(n) \in \omega(g(n))$ $\square$

1.9 $f(n) = n^{\frac{1}{\log n}},\ g(n) = n^{\frac{1}{(\log n)^2}}$

$f(n) \in \Theta(g(n))$

Proof. Let $L = \frac{f(n)}{g(n)} = n^{\frac{1}{\log n} - \frac{1}{(\log n)^2}} = n^{\frac{\log n - 1}{(\log n)^2}}$
$\log(L) = \log n(\frac{\log n - 1}{(\log n)^2}) = \frac{\log n - 1}{\log n} = 1 - \frac{1}{\log n}$,
$\lim_{n\to\infty}\log(L) = 1 \implies \lim_{n\to\infty} L = 2 \implies f(n) \in \Theta(g(n))$ $\square$

1.10 $f(n) = n^{\frac{1}{\log n}},\ g(n) = n^{\frac{1}{\sqrt{\log n}}}$

$f(n) \in o(g(n))$

Proof. Let $L = \frac{f(n)}{g(n)} = n^{\frac{1}{\log n} - \frac{1}{(\log n)^{\frac{1}{2}}}} = n^{\frac{1-(\log n)^{\frac{1}{2}}}{\log n}}$
$\log(L) = \log n(\frac{1-\sqrt{\log n}}{\log n}) = 1 - \sqrt{\log n}$
$\lim_{n\to\infty}\log(L) = -\infty \implies \lim_{n\to\infty} L = 0^+ \implies f(n) \in o(g(n))$ $\square$

2 Recurrence

2.1 $T(n) = 16T(\frac{n}{4}) + 32n\log^{128} n$

Define $R(n) = \frac{T(n)}{n^2}$:

$$\begin{aligned} R(n) &= \frac{T(\frac{n}{4})}{(\frac{n}{4})^2} + \frac{32\log^{128} n}{n} \\ &= R(\frac{n}{4}) + \frac{32\log^{128} n}{n} \end{aligned}$$

Define $S(m) = R(4^m)$:

$$\begin{aligned} S(m) &= S(m-1) + \frac{32 \cdot 4^m (\log(4^m))^{128}}{4^{m\cdot\log_4(16)}} \\ &= S(m-1) + \frac{32 \cdot 4^m m^{128}}{16^m} \\ &= S(m-1) + \frac{32(m)^{128}}{4^m} \\ S(m) &\in \Theta(S(0) + 32\sum_{i=1}^{m} \frac{i^{128}}{4^i}) \end{aligned}$$

It's easy to show that $f(n) = \sum_{i=0}^{m} \frac{i^{128}}{4^i} \in O(1)$ as

$$\lim_{n\to\infty} \frac{n^{128}}{4^n} = \lim_{n\to\infty} \frac{128n^{127}}{\ln(4)4^n} = \cdots = \lim_{n\to\infty} \frac{c_1}{c_2 4^n} = 0 \quad \text{for some constant } c_1, c_2$$

$\lim_{n\to\infty} \frac{\frac{n^{128}}{4^n}}{1} = 0 \implies f(n) = \frac{n^{128}}{4^n} \in o(1) \implies f(n) \in O(1)$

$$\begin{aligned} S(m) &\in O(1) \\ R(n) &\in O(1) \\ T(n) &\in O(n^2) \end{aligned}$$

2.2 $T(n) = 16T(\frac{n}{4}) + 64n^2\log^8 n + 32n\log^{128} n$

Define $R(n) = \frac{T(n)}{n^2}$:

$$\begin{aligned} R(n) &= \frac{16 \cdot T(\frac{n}{4})}{(\frac{n^2}{4})} + \frac{64n^2(\log(n))^8}{n^2} + \frac{32n\log^{128} n}{n^2} \\ R(n) &= \frac{16 \cdot T(\frac{n}{4})}{(\frac{n^2}{4})} + 64(\log(n))^8 + \frac{32\log^{128} n}{n} \end{aligned}$$

Define $m = 4^n$, $n = \log(m)$, $S(m) = R(4^m)$

$$\begin{aligned} S(m) &= S(m-1) + \frac{64(4^m)^2(\log(4^m))^8}{4^{m\cdot\log_4 16}} + \frac{32(4^m)(\log(4^m))^{128}}{4^{m\cdot\log_4 16}} \\ &= S(m-1) + 64(m\log 4)^8 + \frac{32(m(\log 4))^{128}}{4^m} \\ S(m) &\in O\left(S(0) + 64 \cdot (\log 4)^8 \sum_{i=1}^{m} i^8 + 32 \cdot (\log 4)^{128} \sum_{i=1}^{m} \frac{i^{128}}{4^i}\right) \end{aligned}$$

We have shown in 1.1 that $g(m) = \sum_{i=1}^{m} \frac{i^{128}}{4^i} \in O(1)$.

$$\sum_{i=1}^{n} i^p = \frac{1}{p+1} \sum_{j=0}^{p} (-1)^j \binom{p+1}{j} B_j \, n^{p+1-j}, \quad p \in \mathbb{N}$$

where B_j are the Bernoulli numbers. (https://en.wikipedia.org/wiki/Faulhaber)

$$h(k) = \sum_{i=1}^{k} i^8 \in O(k^9)$$

$$\begin{aligned} S(m) &\in O(S(0) + m^9) \\ &\in O(m^8) \\ R(n) &\in O((\log_4 n)^9) \\ T(n) &\in O(n^2 \log^9 n) \end{aligned}$$

2.3 $T(n) = T(\sqrt{n}) + \log(n)$

Define $n = 2^m$, $m = \log(n)$

$$\begin{aligned} T(n) &= T(n^{\frac{1}{2}}) + \log(n) \\ T(2^m) &= T(2^{\frac{1}{2}m}) + \log(2^m) \\ &= T(2^{\frac{1}{2}m}) + m \\ R(m) &= R(\frac{m}{2}) + m \end{aligned}$$

Define $m = 2^k$, $k = \log(m)$

$$\begin{aligned} R(2^k) &= R(\frac{2^k}{2}) + 2^k \\ S(k) &= S(k-1) + 2^k \\ S(k) &\in O(S(0) + \sum_{i=1}^{k} 2^i) = O(1 + 2^{k+1} + 2) = O(2^k) \end{aligned}$$

Map back to $R(m)$ and $T(n)$:

$$\begin{aligned} &S(k) \in O(2^k) \\ \implies &R(m) \in O(2^{\log(m)}) = O(m) \\ \implies &T(n) \in O(\log(n)) \end{aligned}$$

2.4 $T(n) = T(\sqrt{n}) + \log\log(n)$

Define $n = 2^m$, $m = \log(n)$

$$\begin{aligned} T(n) &= T(n^{\frac{1}{2}}) + \log\log(n) \\ T(2^m) &= T(2^{\frac{1}{2}m}) + \log\log(2^m) \\ &= T(2^{\frac{1}{2}m}) + \log(m) \\ R(m) &= R(\frac{m}{2}) + \log(m) \end{aligned}$$

Define $m = 2^k$, $k = \log(m)$

$$\begin{aligned} R(2^k) &= R(\frac{2^k}{2}) + \log(2^k) \\ S(k) &= S(k-1) + k \\ S(k) &\in O(S(0) + \sum_{i=1}^{k} i) = O(\frac{k(k+1)}{2}) = O(k^2) \end{aligned}$$

Map back to $R(m)$ and $T(n)$:

$$\begin{aligned} &S(k) \in O(k^2) \\ \implies &R(m) \in O(\log^2(m)) \\ \implies &T(n) \in O((\log\log(n))^2) \end{aligned}$$