Document Title

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Contents

1

Show by induction that

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution:

To prove this by induction,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Step 1: Base Case

For $n = 1$:

$$\sum_{i=1}^{1} i^2 = 1^2 = 1$$

$$\frac{1(1+1)(2(1)+1)}{6} = \frac{1 \times 2 \times 3}{6} = 1$$

The base case holds.

Step 2: Inductive Hypothesis

Suppose that it holds for some $n = k$:

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Step 3: Inductive Step

We need to show the formula holds for $n = k+1$:

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$$

Using the inductive hypothesis:

$$\sum_{i=1}^{k+1} i^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

Factor $k+1$ from both terms:

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)}{6}\left[k(2k+1) + 6(k+1)\right]$$

Simplify the expression inside the brackets:

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

Step 4: Conclusion

Since the formula holds for $n = 1$ and assuming it holds for $n = k$ implies it holds for $n = k+1$, by the principle of mathematical induction, the formula

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

is true for all positive integers n.

2

For a particular alphabet set Σ, how many strings of length n are there in Σ^*? How many strings in Σ^* have length $\leq$ n?

2.1: $|s| = n$

It is obvious that

$$\forall n \in \mathbb{N}, \forall s \in \Sigma^*, (|s| = n) \implies \forall i \in \{1, 2, \ldots, n\}, \exists \sigma \in \Sigma \text{ such that } s_i = \sigma$$

The number of choices per postion is

$$|\Sigma|$$

The number of strings of length n is given by:

$$\text{Number of strings of length } k = |\Sigma| \times |\Sigma| \times \cdots \times |\Sigma| = |\Sigma|^n$$

2.2: $|s| \leq n$

From (2.1),

$$\text{Number of strings of length } k = |\Sigma| \times |\Sigma| \times \cdots \times |\Sigma| = |\Sigma|^n$$

$$\text{Number of strings of length} \leq k = |\Sigma|^0 \times |\Sigma|^1 \times \cdots \times |\Sigma|^n = \sum_{i=0}^{n} |\Sigma|^i$$

Simplifying this geometric series gives:

$$\sum_{i=0}^{n} |\Sigma|^i = \frac{|\Sigma|^{n+1} - 1}{|\Sigma| - 1}$$

3

Prove:

$$A \cdot \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A \cdot B_i$$

Solution:

Supose

$$s \in A \cdot \bigcup_{i=1}^{\infty} B_i$$

By the definition of concatenation,

$$s = a \cdot b \text{ such that } \exists a, b(a \in A, b \in \bigcup_{i=1}^{\infty} B_i)$$

$$b \in \bigcup_{i=1}^{\infty} B_i \implies \exists j \in \mathbb{N}(b \in B_j)$$

$$s = a \cdot b \implies s \in \bigcup_{j=1}^{\infty} A \cdot B_j$$

Thus,

$$A \cdot \bigcup_{i=1}^{\infty} B_i \subseteq \bigcup_{i=1}^{\infty} A \cdot B_i$$

Prove:

$$(A^*)^+ = (A^+)^*$$

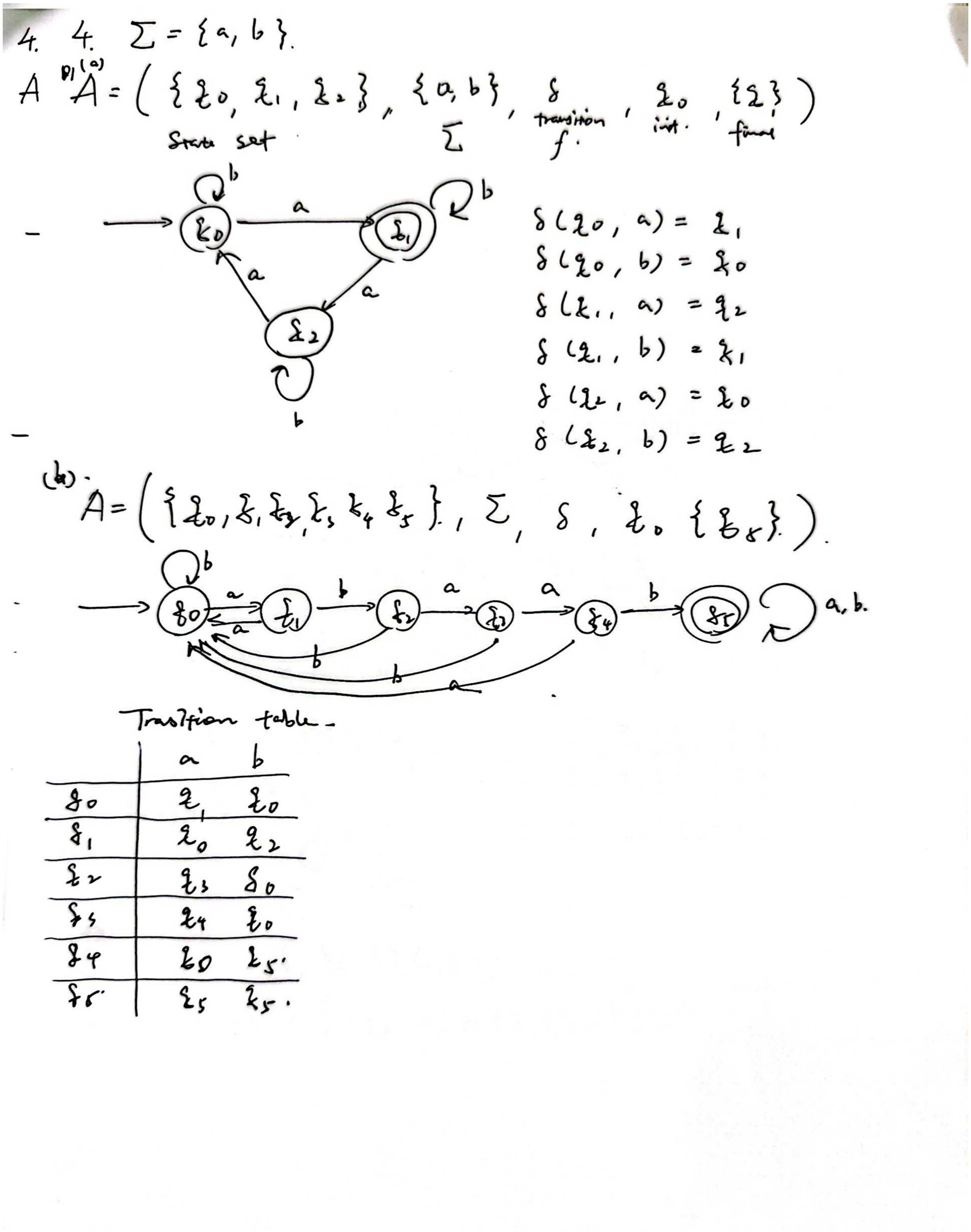
Solution:

$$(A^*)^+ = A^* \cup (A^*)^2 \cup (A^*)^3 \cup ...$$
$$= (\{\epsilon\} \cup A \cup A^2 \cup A^3 \cup ...) \cup (\{\epsilon\} \cup A \cup A^2 \cup A^3 \cup ...) \cup ...$$
$$= \{\epsilon\} \cup A \cup A^2 \cup A^3 \cup ... \text{ (LHS)}$$

$$(A^+)^* = \{\epsilon\} \cup A^+ \cup (A^+)^2 \cup ...$$
$$= \{\epsilon\} \cup (A \cup A^2 \cup A^3 \cup ...) \cup (A \cup A^2 \cup A^3 \cup ...) \cup ...$$
$$= \{\epsilon\} \cup A \cup A^2 \cup A^3 \cup ... \text{ (RHS)}$$

4

I'm too lazy to do this drawing here please refer to the photo attached below.



4. 4. $\Sigma = \{a, b\}$.

(a) $A = (\{q_0, q_1, q_2\}, \{a, b\}, \delta, q_0, \{q_1\})$

State set, Σ, transition f., init., final

$\delta(q_0, a) = q_1$

$\delta(q_0, b) = q_0$

$\delta(q_1, a) = q_2$

$\delta(q_1, b) = q_1$

$\delta(q_2, a) = q_0$

$\delta(q_2, b) = q_2$

(b). $A = (\{q_0, q_1, q_2, q_3, q_4, q_5\}, \Sigma, \delta, q_0, \{q_5\})$

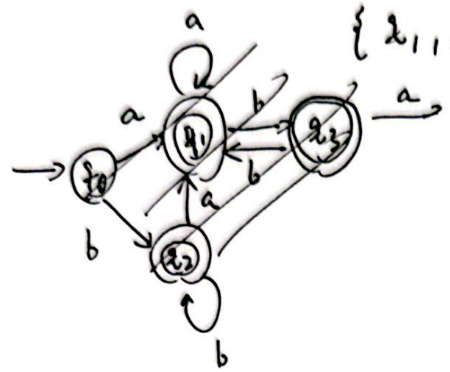
Transition table.

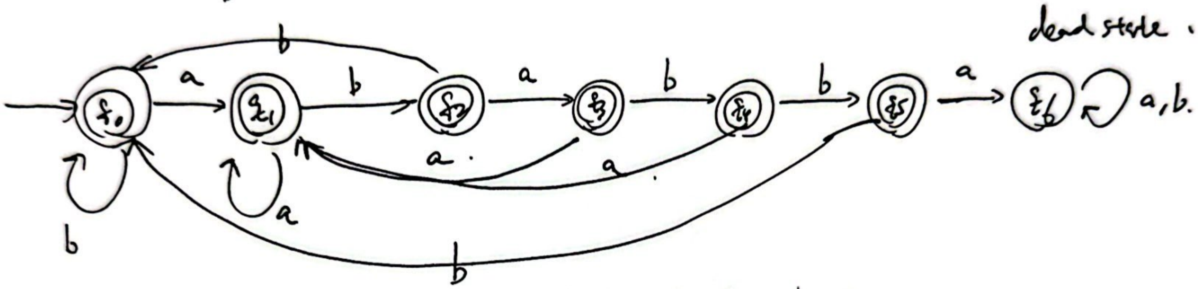
	a	b
q_0	q_1	q_0
q_1	q_0	q_2
q_2	q_3	q_0
q_3	q_4	q_0
q_4	q_0	q_5
q_5	q_5	q_5

Figure 1: Full-size image example

4. c). $\Sigma = \{a, b\}$.

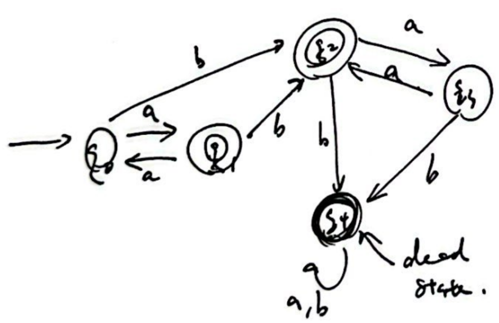
$A = (\{q_0, q_1, q_2, q_3, q_4, q_5, q_6, \cancel{q_7}\}, \Sigma, \delta, q_0,$
$\{q_1, q_2, q_3, q_4, q_5, \cancel{q_6}\}.)$.





	q_0	q_1	q_2	q_3	q_4	q_5	q_6
a	q_1	q_1	q_3	q_1	q_1	q_6	q_6
b	q_0	q_2	q_0	q_4	q_5	q_0	q_6

d.



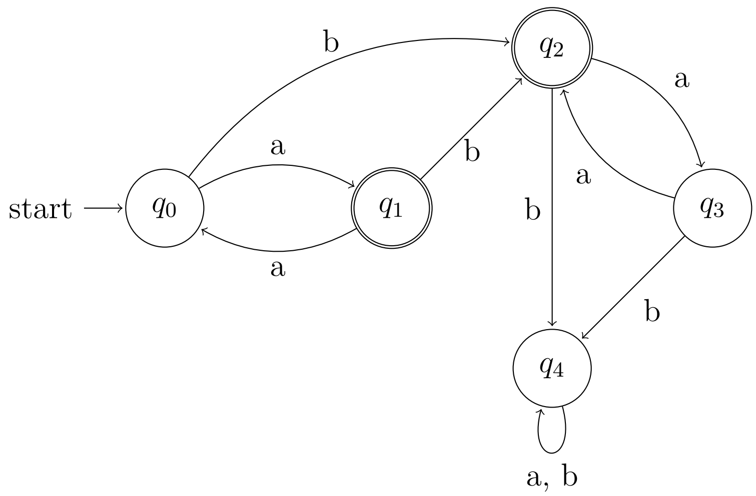
δ:

	q_0	q_1	q_2	q_3	q_4
a	q_1	q_0	q_3	q_2	q_4
b	q_2	q_2	q_4	q_4	q_4

$A = (\{q_0 \cdots q_4\}, \Sigma, \delta, q_0, \{q_1, q_2, q_3\})$

$L(A) = \{ w \in \{a,b\}^* : \hat{\delta}(q_0, w) \in \{q_1, q_2, q_3\} \}$.

5



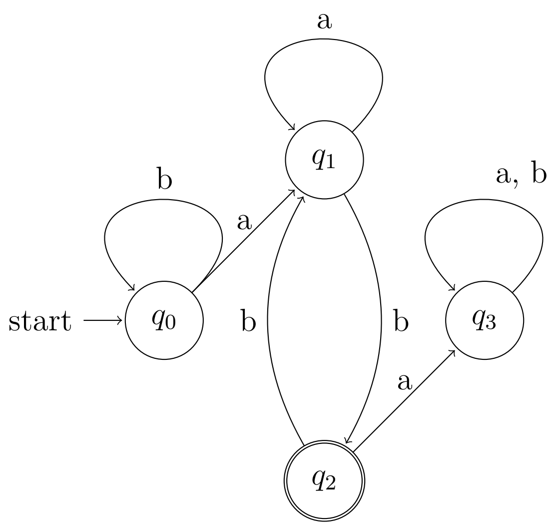
This DFA contains 2 accepting states, suggesting the union of the language accepted at state 1 and state 2.

- The language accepted at state 1 is
  - Case 1-1: $a(aa)^*$.
- The language accepted at state 2 can be
  - Case 2-1: $(aa)^*b$: ((0 - 1 - 0)* - 2) $(Case\ 2-1 \subseteq Case\ 2-3)$
  - Case 2-2: $(aa)^*b(aa)^*$: ((0 - 1 - 0)* - (2 - 3 - 2)*) $(Case\ 2-2 \subseteq Case\ 2-4)$
  - Case 2-3: $a^*(aa)^*b$: (0 - (1 - 0 - 1)* - 2) $(Case\ 2-3 \subseteq Case\ 2-2)$
  - Case 2-4: $a^*(aa)^*b(aa)^*$: (0 - (1 - 0 - 1)* - (2 - 3 - 2)*)
  - Case 2-5: $b(aa)^*$: (0 - (2 - 3 - 2)*) $(Case\ 2-5 \subseteq Case\ 2-2)$

All unioned together, language accepted by this DFA is

$$L = a(aa)^* \cup a^*(aa)^*b(aa)^*$$

6



- The language accepted at state 2 is
  - Case 1: Looping on q_0, looping on q_1, enter q_2:

$$L_1 = b^*(a)(a)^*b$$

  - Case 2: From q_2, looping on q_1 and eventually go back to q_2:

$$L_2 = b^*(a)(a)^*b(b(a)^*b)^*$$

Unioned together,

$$L = L_1 \cup L_2 = b^*aa^*b(ba^*b)^*$$