MATH 2393 Calculus III

Summer 2022

List of Projects

Project 1. Derivatives of Vector Functions (Section 13.2)

(a) Show that if $\mathbf{r}$ is a vector function such that $\mathbf{r}''$ exists, then

$$\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t)$$

(b) If $\mathbf{r}(t) \neq 0$, show that

$$\frac{d}{dt}|\mathbf{r}(t)| = \frac{1}{|\mathbf{r}(t)|}\mathbf{r}(t) \cdot \mathbf{r}'(t)$$

(c) If $\mathbf{u}(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}''(t)]$, show that

$$\mathbf{u}'(t) = \mathbf{r}(t) \cdot [\mathbf{r}'(t) \times \mathbf{r}''(t)]$$

Project 2. Unit Normal and Binormal vectors (Section 13.3)

Show that at every point on the curve

$$\mathbf{r}(t) = \left\langle e^t \cos t, e^t \sin t, e^t \right\rangle$$

the angle between the unit tangent vector and the z-axis is the same. Then show that the same result holds true for the unit normal and binormal vectors.

Project 3. Application of Clairaut's Theorem (Section 14.3)

$$f(x, y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(i) Use a computer program to graph f.

(ii) Find $f_x(x, y)$ and $f_y(x, y)$ when $(x, y) \neq (0, 0)$.

(iii) Find $f_x(0, 0)$ and $f_y(0, 0)$ using the definition of partials.

(iv) Show that $f_{xy}(0, 0) = -1$ and $f_{yx}(0, 0) = 1$.

(v) Does the result of (d) contradict Clairaut's Theorem? Use graphs of f_{xy} and f_{yx} to illustrate your answer.

Project 4. Designing A Dumpster (Section 14.7)

For this project we locate a rectangular trash Dumpster in order to study its shape and construction. We then attempt to determine the dimensions of a container of similar design that minimize the construction cost.

1. First locate a trash Dumpster in your area. Carefully study and describe all details of its construction, and determine its volume. Include the sketch of the container.

2. While maintaining the general shape and method of construction, determine the dimension such a container of the same volume should have in order to minimize the cost of construction. Use the following assumptions in your analysis:

   - The sides, back, and front are to be made from 12-gauge (0.1046 inch thick) steel sheets, which cost \$0.70 per square foot (including any required cuts or bends).
   - The base is to be made from a 10-gauge (0.1345 inch thick) steel sheet, which costs \$0.90 per square foot.
   - Lids cost approximately \$50.00 each, regardless of dimensions.
   - Welding costs approximately \$0.18 per foot for material and labor combined.

   Give justification of any further assumptions or simplifications may affect the final result.

3. Describe how any of your assumptions or simplifications may affect the final result.

4. If you were hired as a consultant on this investigation, what would your conclusions be? Would you recommend altering the design of the Dumpster? If so, describe the savings that would result.

Project 5. Improper Integral (Section 15.3)

(a) We define the improper integral (over the entire plane $\mathbb{R}^2$)

$$\begin{aligned} I &= \iint\limits_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dydx \\ &= \lim_{a\to\infty}\iint_{D_a} e^{-(x^2+y^2)}\, dA \end{aligned}$$

where D_a is the disk with radius a and center the origin. Show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dA = \pi$$

(b) An equivalent definition of the improper integral in part (a) is

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \lim_{a\to\infty} \iint_{S_a} e^{-(x^2+y^2)}\, dA$$

where S_a is the square with vertices $\{\pm a, \pm a\}$. Use this to show that

$$\int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \pi$$

(c) Deduce that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

(d) By making the change of variable $t = \sqrt{2}x$, show that

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}$$

Explain the significance of this result in Statistics.

Project 6. Centroid/ Triple Integral (Section 15.7)

(a) Find the volume of the region E that lies between the paraboloid $z = 24 - x^2 - y^2$ and the cone $z = 2\sqrt{x^2 + y^2}$.

(b) Find the centroid of E (the center of mass in the case where the density is constant).

Project 7. Cylindrical and Spherical Coordinates (Section 15.7, 15.8)

(a) Evaluate $\iiint_E (x^2 + y^2)\, dV$, where E lies between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$.

(b) Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane, and below the cone $z = \sqrt{x^2 + y^2}$.

Project 8. Line Integrals(Section 16.2)

(a) Find the work done by the force field $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$ on a particle that moves once around the circle $x^2 + y^2 = 4$ oriented in the counter-clockwise direction.

(b) Use a computer program to graph the force field and circle on the same screen. Use the graph to explain your answer to part (a).

Project 9. Area enclosed by Epicycloid (Section 16.4)

If a circle C with radius 1 rolls along the outside of the circle $x^2 + y^2 = 16$, a fixed point P on C traces out a curve called an *epicycloid*, with parametric equations

$$x = 5\cos t - \cos 5t, \quad y = 5\sin t - \sin 5t$$

(i) Sketch the graph of epicycloid.

(ii) Use Green's Theorem to find the area enclosed by the curve.

Project 10. Stoke's Theorem (Section 16.6)

Evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = \left\langle x^2 y, \frac{1}{3}x^3, xy \right\rangle$$

and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counter-clockwise as viewed from above.

(i) Sketch the graph of both the hyperbolic paraboloid and cylinder with domains chosen.

(ii) Find parametric equations for C and use them to sketch the graph of C .