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CS-556 HW3

1)

a) $\mathbb{P}(\text{exactly 3 aces}) = \frac{A \times B}{C}$

A: # of ways to draw exactly 3 aces out of the 4 in the deck
B: # of ways to draw 4 non-aces out of the 48 in the deck
C: # of ways to draw 7 cards from a deck of 52

$A = {}_4C_3 = \frac{4!}{3!(4-3)!} = 4$

$B = {}_{48}C_4 = \frac{48!}{4!(48-4)!} = 194580$

$C = {}_{52}C_7 = \frac{52!}{7!(52-7)!} = 133784560$

$\Rightarrow \mathbb{P}(\text{exactly 3 aces}) = \frac{(4)(194580)}{(133784560)} \approx 5.81\%$

b) $\mathbb{P}(\text{exactly 2 kings}) = \frac{AB}{C}$

A: # of ways to draw 2 kings out of the 4 in the deck
B: # of ways to draw 5 non-king cards out of the 48 in the deck
C: # of ways to draw 7 cards out of the 52 in the deck

$A = {}_4C_2 = \frac{4!}{2!(4-2)!} = 6$

$B = {}_{48}C_5 = \frac{48!}{5!(48-5)!} = 1712304$

$C = {}_{52}C_7 = \frac{52!}{7!(52-7)!} = 133784560$

$\Rightarrow \mathbb{P} = 7.689\%$

1 (cont.)

c)

$$\mathbb{P}(\text{3 aces OR 2 kings OR both}) = \frac{A_1B_1 + A_2B_2 + A_1A_2B_3}{C}$$

A_1: # of ways to choose 3 aces from the 4 in the deck

B_1: # of ways to choose 4 non-aces from the 48 in the deck

A_2: # of ways to choose 2 kings from the 4 in the deck

B_2: # of ways to choose 5 non-kings from the 48 in the deck

B_3: # of ways to choose 2 non-aces/non-kings from the 44 in the deck

C: # of ways to choose 7 cards from the 52 in the deck

$A_1 = {}_4C_3 = 4$

$B_1 = {}_{48}C_4 = 194580$

$A_2 = {}_4C_2 = 6$

$B_2 = {}_{48}C_5 = 1712304$

$B_3 = {}_{44}C_2 = 946$

$C = {}_{52}C_7 = 133784560$

$$\Rightarrow \mathbb{P} = \mathbb{P}(\text{exactly 3 aces}) + \mathbb{P}(\text{exactly 2 kings}) + \frac{(4)(6)(946)}{133784560}$$

$$\Rightarrow \mathbb{P} \approx 5.81\% + 7.689\% + 0.017\%$$

$$= 13.52\%$$

numb... should be ...

2)

A: # of heads Alice gets, B: # of heads Bob gets

$A \sim \text{bin}(n, 1/2)$, $B \sim \text{bin}(n+1, 1/2)$

$C = B - A$ ← this is a random variable w/ expectation & variance:

$$\mathbb{E}(C) = \mathbb{E}(B) - \mathbb{E}(A) = \frac{(n+1)}{2} - \frac{n}{2} = \frac{1}{2}$$

$$\text{Var}(C) = \text{Var}(B) + \text{Var}(A) = \frac{(n+1)}{4} + \frac{n}{4} = \frac{3n+1}{4}$$

now $\mathbb{P}(B > Y) = \mathbb{P}(C > 0)$ → use central limit theorem to approximate C as a normal distribution when n is large

$C \sim \text{norm}\left(\frac{1}{2}, \frac{3n+1}{4}\right)$

⇓ standardize

$Z = \frac{C - (1/2)}{\sqrt{\frac{3n+1}{4}}} \sim \text{norm}(0,1)$ → since the normal distribution is symmetric around 0, $\mathbb{P}(Z > 0) = 1/2$

$\mathbb{P}(Z > 0) = \mathbb{P}(C > 0) = 1/2$ for large n

3)

$\mathbb{P}(\overset{A}{\text{holding heads/tails coin}} \mid \overset{B}{\text{heads}}) = \mathbb{P}(A|B) = \dfrac{\mathbb{P}(B|A)\,\mathbb{P}(A)}{\mathbb{P}(B)}$

Ω:

	heads coin	tails coin	heads/tail coin
opt. 1	heads	tails	~~x~~ heads
opt. 2	heads	tails	tails

$\mathbb{P}(A) = 1/3$

$\mathbb{P}(B) = (1)(1/3) + (0)(1/3) + (1/2)(1/3) = 1/2$

$\mathbb{P}(B|A) = 1/2$

$\Rightarrow \mathbb{P}(A|B) = \dfrac{(1/2)(1/3)}{(1/2)} = 1/3$

4)

$$\Omega_{ij} = \begin{cases} (m+1) \text{ white balls in jar } i \text{ if } j=1 \\ (m) \text{ white balls in jar } i \text{ if } j=2 \\ m \text{ white} + n \text{ black balls in jar } i \text{ if } j=0 \end{cases}$$

$$\begin{tikzcd}
 & \Omega_{21} \arrow[r] \arrow[dr] & \Omega_{31} \arrow[r, dotted] \arrow[dr, dotted] & \Omega_{k1} \\
\Omega_{10} \arrow[ur] \arrow[r] & \Omega_{22} \arrow[r] \arrow[ur] & \Omega_{32} \arrow[r, dotted] \arrow[ur, dotted] & \Omega_{k2}
\end{tikzcd}$$

W_i : event where white ball is drawn from jar i

B_i : event where black ball is drawn from jar i

$$\mathbb{P}(W_1) = \frac{m}{m+n}$$

$$\mathbb{P}(W_i) = \mathbb{P}(W_{i-1})\,\mathbb{P}(W_i \mid W_{i-1}) + \mathbb{P}(B_{i-1})\,\mathbb{P}(W_i \mid B_{i-1})$$

(i+1) step:

$$\Rightarrow \mathbb{P}(W_2) = \left(\frac{m}{m+n}\right)\left(\frac{m+1}{m+n+1}\right) + \left(\frac{n}{m+n}\right)\left(\frac{m}{m+n+1}\right) = \left(\frac{m}{m+n}\right)\left(\frac{m+1}{m+n+1}\right) + \frac{n}{m+n+1}\left(\frac{m}{n+m}\right)$$

$$= \left(\frac{m}{m+n}\right)\left(\frac{m+1}{m+n+1} + \frac{n}{m+n+1}\right) = \frac{m}{m+n}(1) = \mathbb{P}(W_1)$$

$$\Rightarrow \mathbb{P}(W_{i+1}) = \mathbb{P}(W_i)$$

$$\Rightarrow \mathbb{P}(W_k) = \mathbb{P}(W_{k-1}) = \cdots = \mathbb{P}(W_2) = \mathbb{P}(W_1) = \frac{m}{m+n}$$

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5) n power plants, ith powerplant fails w/ probability p_i

a) $\mathbb{P}(\text{all plants working}) = \prod_i^n (1-p_i)$

$\Rightarrow \mathbb{P}(\text{blackout}) = 1 - \mathbb{P}(\text{all plants working}) = 1 - \prod_i^n (1-p_i)$

b) $\mathbb{P}(\text{no blackout}) = \prod_{(i,j)}^n (p_i p_j) + \sum_{(i,j)}^n \left(p_i p_j \prod_{(k,l) \neq (i,j)}^n (1 - p_k p_l) \right)$

$\Rightarrow \mathbb{P}(\text{blackout}) = 1 - \mathbb{P}(\neg \text{blackout})$
$= 1 - \prod_{(i,j)}^n (p_i p_j) + \sum_{(i,j)} \left(p_i p_j \prod_{(k,l) \neq (i,j)}^n (1 - p_k p_l) \right)$

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