Introduction to Computational Physics SS2019

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Exercise 7 from June 3, 2020
Return before noon of June 12, 2019

1 Population dynamics (preparation)

The simple equation of growth of a population, as proposed by Malthus, has been improved by Verhulst including a growth limiting term, which represents the finite amount of resources available:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{K}\right) \tag{1.1}$$

Here the definitions of r, N, and K are as in the lecture (growth rate, population number, growth limiting number). By dimensional analysis we can renormalize the variables, such that the equation becomes dimensionless and does not depend on the parameters anymore ($\tau = rt$, $n = N/K$) and we get:

$$\frac{\mathrm{d}n}{\mathrm{d}\tau} = n(1 - n) \tag{1.2}$$

Solve this non-linear quadratic differential equation and discuss the solutions as function of the initial value $n_0 = n(\tau = 0)$. Try for fun also values $n_0 < 0$, $n_0 > 1$, even though they are not very realistic.
Can you find the analytic solution for $n(\tau)$?

2 Population dynamics (homework)

In this exercise we study the following equation for population dynamics:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN(1 - N/K) - \frac{BN^2}{A^2 + N^2} \tag{2.3}$$

where all parameters r, K, A and B are positive. It is a more complex example, in which the growth behaviour depends on whether N is smaller or larger than a critical populations size A.

1. (10 pt) Dimensional analysis: Determine the dimension of the parameters and rewrite the equation in dimensionless form. Note that there are different possibilities. Please formulate a dimensionless time τ that is *not* defined on the basis of r. Use $n = N/A$ as the dimensionless version of N.

2. (10 pt) Determine the stationary points n^* for $K/A = 7.3$. Note that for $n^* \neq 0$ these values are solutions of a cubic equation; it depends on n and the remaining free parameter. The cubic equation should be derived by yourself analytically; its zero points you can obtain numerically / graphically by using e.g. `Mathematica`. When do one or three real solutions exist as a function of the remaining free parameter? (Hint: we do not ask for some analytical formula here! It is enough to vary the free parameter and check using `Mathematica` which three solutions for the stationary points you get; as stationary points only real solutions are valid. Only one digit after the comma is enough, in other words you vary the free parameter by about 0.05.). Which of the stationary points is stable and unstable?