Introduction to Computational Physics – Exercise 7

Simon Groß-Bölting, Lorenz Vogel, Sebastian Willenberg

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Population dynamics

The simple equation of growth of a population, as proposed by Malthus (1798), has been improved by Verhulst (1836) including a growth limiting term, which represents the finite amount of resources available. In this exercise we study a modified form of Verhulst's equation for population dynamics:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN(1 - N/K) - \frac{BN^2}{A^2 + N^2} \tag{1}$$

where all parameters r, K, A and B are positive. It is a more complex example, in which the growth behaviour depends on whether N is smaller or larger than a critical populations size A.

Dimensional analysis

Task: Determine the dimension of the parameters and rewrite the equation in dimensionless form. Note that there are different possibilities. Please formulate a dimensionless time τ that is not defined on the basis of r. Use $n = N/A$ as the dimensionless version of N.

Table 1: Dimension of the parameters

description	parameter	dimension
time	t	$[t] = \mathrm{s}$
population number	N	$[N] = 1$
growth rate	r	$[r] = \mathrm{s}^{-1}$
growth limiting number	K	$[K] = 1$
critical population size	A	$[A] = 1$
—	B	$[B] = \mathrm{s}^{-1}$

By dimensional analysis we can renormalize the variables, such that the above equation for population dynamics becomes dimensionless: The steps required for this are well described in the Wikipedia article "Nondimensionalization". First of all we have to identify all the independent and dependent variables in the given equation:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{K}\right) - \frac{BN^2}{A^2 + N^2} = rN - \frac{rN^2}{K} - \frac{BN^2}{A^2 + N^2} \tag{2}$$

In this equation the independent variable is t (time) and the dependent variable is $N = N(t)$ (population number). Now we replace each of the two variables with the product of a dimensionless variable (τ and n) and a characteristic unit (N_c and t_c):

$$N := N_\mathrm{c} n \quad \Leftrightarrow \quad n := N/N_\mathrm{c} \qquad \text{and} \qquad t := t_\mathrm{c}\tau \quad \Leftrightarrow \quad \tau := t/t_\mathrm{c} \tag{3}$$

The subscripted "c" added to a variable-name is used to denote the characteristic unit used to scale that variable. As required in the task, we set $N_\mathrm{c} := A$, so that we use $n := N/A$ as the dimensionless version of the population number N. Then we obtain the following equation:

$$\frac{A}{t_\mathrm{c}}\frac{\mathrm{d}n}{\mathrm{d}\tau} = rAn - \frac{rA^2}{K}n^2 - \frac{BA^2n^2}{A^2 + A^2n^2} = rAn - \frac{rA^2}{K}n^2 - B\frac{n^2}{1 + n^2} \tag{4}$$

Dividing the entire equation by the coefficient A/t_c in front of the first derivative term and using the relation

$$\frac{n^2}{1+n^2} = \left[\frac{1+n^2}{n^2}\right]^{-1} = \left[1+\frac{1}{n^2}\right]^{-1} = \left[1+n^{-2}\right]^{-1} = \frac{1}{1+n^{-2}} \tag{5}$$

gives us:

$$\frac{\mathrm{d}n}{\mathrm{d}\tau} = t_c r n - \frac{t_c r A}{K} n^2 - \frac{t_c B}{A} \frac{1}{1+n^{-2}} \tag{6}$$

Now the characteristic unit for each variable can be defined, such that the coefficients of as many terms as possible become 1. Since we have already determined the dimensionless version of N, we only have to define the characteristic unit t_c of the time t. Because we are supposed to formulate a dimensionless time $\tau = t/t_c$ that is not defined on the basis of the growth rate r, we have to use the coefficient in front of the last term:

$$\frac{t_c B}{A} \stackrel{!}{=} 1 \qquad \Longrightarrow \qquad t_c := \frac{A}{B} \tag{7}$$

Now we can rewrite the given equation for population dynamics in its dimensionless form:

$$\frac{\mathrm{d}n}{\mathrm{d}\tau} = \frac{rA}{B} n - \frac{rA^2}{BK} n^2 - \frac{1}{1+n^{-2}} \tag{8}$$

By defining the dimensionless parameter $D := rA/B$, we can further simplify the dimensionless equation:

$$\frac{\mathrm{d}n}{\mathrm{d}\tau} = Dn - \frac{DA}{K} n^2 - \frac{1}{1+n^{-2}} \qquad \text{with} \qquad D := \frac{rA}{B} \tag{9}$$

Stationary points

Task: Determine the stationary points n^* for $K/A = 7.3$. Note that for $n^* \neq 0$ these values are solutions of a cubic equation; it depends on n and the remaining free parameter. The cubic equation should be derived by yourself analytically; its zero points you can obtain numerically / graphically by using e.g. Mathematica. When do one or three real solutions exist as a function of the remaining free parameter? (Hint: we do not ask for some analytical formula here! It is enough to vary the free parameter and check using Mathematica which three solutions for the stationary points you get; as stationary points only real solutions are valid. Only one digit after the comma is enough, in other words you vary the free parameter by about 0.05.).
Which of the stationary points is stable and unstable?

Definition: A fixed point (FP) or stationary point of a differentiable function of one variable is a point on the graph of the function where the function's derivative is zero.

$$n = n^* \text{ stationary point of } n = n(\tau) \qquad \Longleftrightarrow \qquad \frac{\mathrm{d}n}{\mathrm{d}\tau} = 0 \tag{10}$$

$$\begin{aligned}
0 &\stackrel{!}{=} Dn - \frac{DA}{K} n^2 - \frac{1}{1+n^{-2}} && \Big| \cdot (1+n^{-2}) && (11)\\
\Leftrightarrow \quad &= D(1+n^{-2})n - \frac{DA}{K}(1+n^{-2})n^2 - 1 && && (12)\\
\Leftrightarrow \quad &= D(n+n^{-1}) - \frac{DA}{K}(n^2+1) - 1 && && (13)\\
\Leftrightarrow \quad &= D\left(\frac{n^2+1}{n}\right) - \frac{DA}{K}(n^2+1) - 1 && \Big| \cdot n && (14)\\
\Leftrightarrow \quad &= D(n^2+1) - \frac{DA}{K}(n^3+n) - n && && (15)\\
\Leftrightarrow \quad &= -\frac{DA}{K} n^3 + Dn^2 - \frac{DA}{K} n - n + D && && (16)\\
\Leftrightarrow \quad &= -\frac{DA}{K} n^3 + Dn^2 - \left(\frac{DA}{K} + 1\right) n + D && && (17)
\end{aligned}$$

Finally, we multiply the entire equation by $-K/(DA)$. So we get that for $n \neq 0$ the stationary points n^* are solutions of a cubic equation:

$$0 \overset{!}{=} n^3 - \frac{K}{A} n^2 + \left(1 + \frac{1}{D}\frac{K}{A}\right) n - \frac{K}{A} \tag{18}$$

If we set $K/A = 7.3$ as required in the task, the cubic equation above only depends on n and the remaining free parameter $D = rA/B$.