

Introduction to Computational Physics – Exercise 8

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Stability Analysis of Many Species Population Dynamics

In this exercise we study the evolution of 6 populations according to the following equations for population dynamics: 3 predator- (P_i) and 3 prey-species (N_i), all parameters positive, always $i, j = 1, \dots, 3$:

$$\frac{dN_i}{dt} = N_i \left(a_i - N_i - \sum_j b_{ij} P_j \right) \quad \text{and} \quad \frac{dP_i}{dt} = P_i \left(\sum_j c_{ij} N_j - d_i \right) \quad (1)$$

The parameters chosen are $a_1 = 56$, $a_2 = 12$, $a_3 = 35$, $d_1 = 85$, $d_2 = 9$ and $d_3 = 35$; the parameters b_{ij} and c_{ij} are given in matrix form here:

$$b_{ij} = \begin{pmatrix} 20 & 30 & 5 \\ 1 & 3 & 7 \\ 4 & 10 & 20 \end{pmatrix} \quad \text{and} \quad c_{ij} = \begin{pmatrix} 20 & 30 & 35 \\ 3 & 3 & 3 \\ 7 & 8 & 20 \end{pmatrix} \quad (2)$$

Notices: The unusual feature here in the equations is that the prey populations N_i have a Verhulst style growth limiting factor in their equations, which limits their growth even if there is no predator (model for limited resources even in absence of predators).

- Please do not try to make the equations dimensionless, just use the numbers given here.
- In this mathematical example the solutions can become negative in some cases, which is unphysical for population dynamics. Nevertheless let us please use this case to show the mathematical features.

Task: What are the fixed points (FP) for the system of equations given above? *Hint 1:* No complicated computations are necessary, the idea is that you should guess the fixed points very easily. Compare our previous examples. *Hint 2:* This time there are three fixed points! In addition to our “usual” ones, there is a third one related to the Verhulst growth limiting factor in the first three equations.

Task: What is the Jacobi matrix \mathbf{A} at the non-trivial fixed point? (non-trivial FP means here that *all* elements are unequal to zero. There is only one FP with this property)

Task: Determine the eigenvalues and eigenvectors λ_i and \mathbf{v}_i ($i = 1, \dots, 6$) of \mathbf{A} for this fixed point. Choose an initial state

$$\mathbf{n} = \sum_{i=1}^6 c_i \mathbf{v}_i \quad \text{with} \quad c_1 = c_2 = 3, \quad c_3 = c_4 = 1, \quad c_5 = -5, \quad c_6 = 0.1 \quad (3)$$

Plot and discuss the time dependent evolution of the six populations (given by this linear model); in particular what about oscillations, growth or decay?