Introduction to Computational Physics SS2020

Lecturers: Rainer Spurzem & Ralf Klessen
Tutors: Mischa Breuhaus, Li-Hsin Chen, Ismael Pessa Gutierrez, Da Eun Kang, Giancarlo Mattia, Toni Peter

Exercise 9 from June 17, 2020
Return before noon of June 26, 2020

1 Fixed Points of the Lorenz dynamical system

The Lorenz dynamical equations (given below), as discussed in the lecture, have one trivial fixed point $\lambda_1 = (x, y, z) = (0, 0, 0)$, which is stable for $0 \leq r < 1$, and gets unstable for $r > 1$. And there are two more non-trivial fixed points $\lambda_{2,3}$ given below in the homework part; for some range of $r > 1$ they are all real. The stability of $\lambda_{2,3}$ should be examined by the Jacobian taken at the fixed points, and then looking for its eigenvalues by means of finding the zero points of the following characteristic polynomial (same for both fixed points):

$$P(\lambda) = \lambda^3 + (1 + b + \sigma)\lambda^2 + b(\sigma + r)\lambda + 2\sigma b(r - 1) \tag{1.1}$$

1. Plot $P(\lambda)$ as a function of λ (negative and positive real values, $\sigma = 10$, $b = 8/3$) for $0 \leq r \leq 1.8$ (choose 3-4 discrete values of r). Recapitulate from the zero points which you see in the plots, what can you say about the stability of the fixed points.

2. Find all zero-points (including complex ones) of the characteristic polynomial for $r > r_1 \approx 1.34561$. Plot the zero points in the complex plane for some discrete values of $r_1 < r < 28$ and connect them by lines. Once again recap from the lecture, what does the occurrence of complex conjugate imaginary parts mean for the solution near the fixed point? What the positive or negative real part?

2 The Lorenz attractor (homework)

The Lorenz attractor problem is given by the following coupled set of differential equations:

$$\begin{aligned} \dot{x} &= -\sigma(x - y) && (2.2) \\ \dot{y} &= rx - y - xz && (2.3) \\ \dot{z} &= xy - bz && (2.4) \end{aligned}$$

As discussed in the lecture, the fixed points are $(0, 0, 0)$ for all r, and (for $r > 1$) the points $C_\pm = (\pm a_0, \pm a_0, r - 1)$ with $a_0 = \sqrt{b(r-1)}$. For the entire exercise, please use $\sigma = 10$ and $b = 8/3$. The value of r can be experimented with. When you create numerical solutions you can make plots in 2-D projection (e.g. in the $x-y$ or $x-z$ plane). You can also try a full 3-D plot.

1. (13 pt) Solve numerically, using `rk4`, the above coupled set of equations for the values $r = 0.5$, 1.17, 1.3456, 25.0 and 29.0 Choose the initial conditions near one of the fixed points: $C_{\pm}$ for $r > 1$ and $(0, 0, 0)$ for $r < 1$. Explain the behavior, as much as possible, with the stability properties of the fixed points.

2. (7 pt) Determine the sequence z_k for $r = 26.5$, where z_k is a local maximum in z on the solution curve after k periods. Plot z_{k+1} as a function of z_k. When sufficient points are there, connect the points. The resulting function $z_{k+1} = f(z_k)$ has an intersection with the diagonal $z_{k+1} = z_k$. It is a fixed point of the function $f(z_k)$. Is the slope m of this function > 1, < -1 or between -1 and $+1$? Notice: the theory of discrete maps says that there is **NO** periodic solution if $|m| > 1$. So, in such a case we can deduce that this solution of the Lorenz system is not periodic.