

# Introduction to Computational Physics - Exercise 2

Simon Groß-Bölting, Lorenz Vogel, Sebastian Willenberg

8. April 2020

In the first exercise form the worksheet we worked on the following code:

```
1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 class Body:
5     def __init__(self, mass, position, velocity=np.array([0.,0.,0.])):
6         self.mass = mass          # mass of the body
7         self.position = position   # initial position vector
8         self.velocity = velocity  # initial velocity vector
9
10 def forward_euler(body1, body2, G, dt, N):
11     ''' This function computes the relative motion of two point-like bodies
12         under their mutual gravitational influence using a
13         forward Euler integration procedure
14         Input:  two bodies of the class "Body", gravitational constant G,
15                 time steps dt and number of time steps N
16         Output: location vector s and velocity vector w '''
17
18     # compute the total mass of the two bodies and set the
19     # characteristic length scale as the initial separation
20     M = body1.mass + body2.mass
21     R0 = np.linalg.norm(body1.position - body2.position)
22     h = dt/np.sqrt(R0**3/(G*M)) # compute the dimensionless step size
23
24     s = np.zeros((int(N), 3))    # create array for the location vectors
25     s[0] = (body1.position - body2.position)/R0
26
27     w = np.zeros((int(N), 3))    # create array for the velocity vectors
28     w[0] = (body1.velocity - body2.velocity)/np.sqrt(G*M/R0)
29
30     for i in range(1, int(N)):    # compute the relative motion
31         s[i] = s[i-1] + (w[i-1]*dt)
32         w[i] = w[i-1] - (s[i-1]/np.linalg.norm(s[i-1])**3)*dt
33     return (s, w)
34
35 def get_acceleration(body1, body2, G, position):
36     return -(body1.mass + body2.mass)*G*position/np.linalg.norm(position)**3
37
38 def leapfrog(body1, body2, G, dt, N):
39     ''' This function computes the relative motion of two point-like bodies
40         under their mutual gravitational influence using a Leapfrog scheme
41         Input:  two bodies of the class "Body", gravitational constant G,
42                 time steps dt and number of time steps N
43         Output: location vector s and velocity vector w '''
44
45     # compute the total mass of the two bodies and set the
46     # characteristic length scale as the initial separation
47     M = body1.mass + body2.mass
48     R0 = np.linalg.norm(body1.position - body2.position)
49     h = dt/np.sqrt(R0**3/(G*M)) # compute the dimensionless step size
```

```

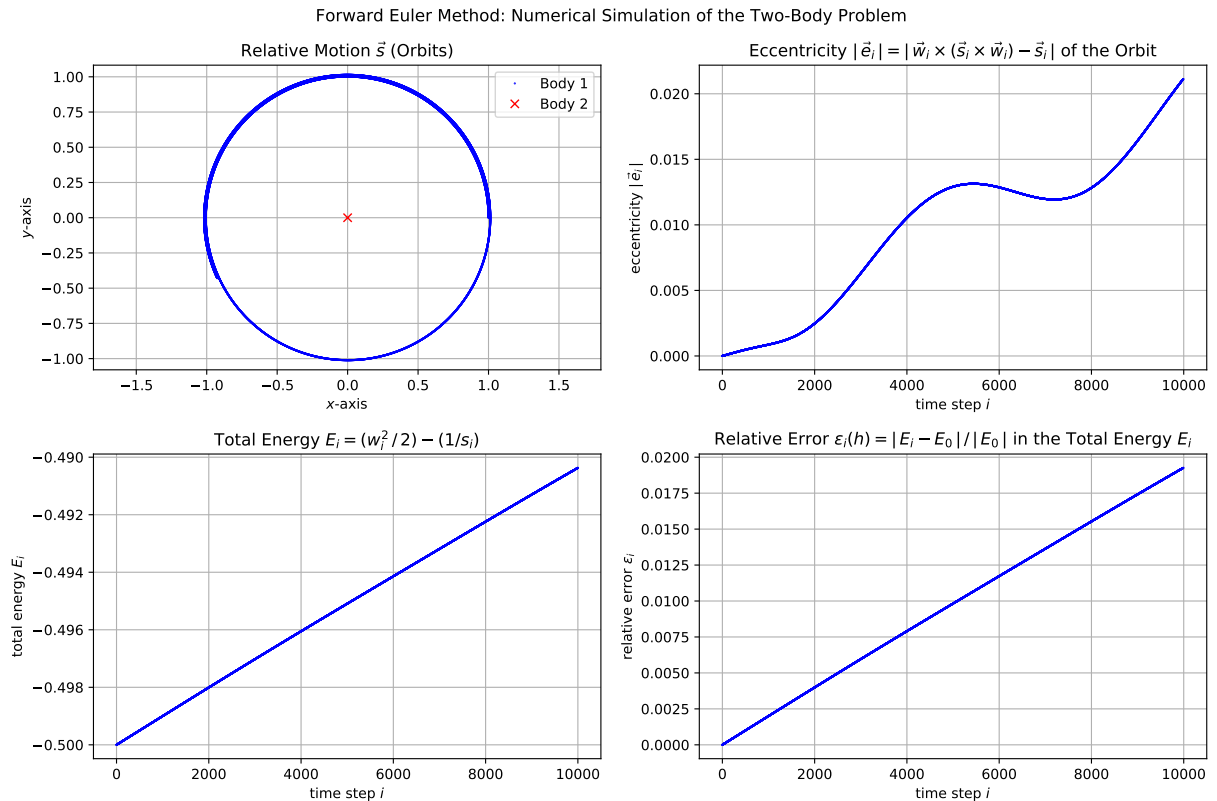
50
51 s = np.zeros((int(N),3))      # create array for the location vectors
52 s[0] = (body1.position-body2.position)/R0
53
54 w = np.zeros((int(N),3))      # create array for the velocity vectors
55 w[0] = (body1.velocity-body2.velocity)/np.sqrt(G*M/R0)
56
57 w[0] -= (s[0]/np.linalg.norm(s[0])**3)*h/2
58 for i in range(1,int(N)):
59     w[i] = w[i-1]+get_acceleration(body1, body2, G, s[i-1])*h
60     s[i] = s[i-1]+w[i]*h
61 return (s,w)
62
63
64 def total_energy(s,w):
65     ''' Function to compute the total energy of the system from the
66         dimensionless vectors s (location) and w (velocity) '''
67     energy = np.zeros(np.shape(s)[0])
68     for i in range(0,np.shape(s)[0]):
69         energy[i] = (0.5*np.linalg.norm(w[i])**2)-(1/np.linalg.norm(s[i]))
70     return energy
71
72 def relative_error(energy):
73     ''' Function to compute the relative error in the total energy at
74         step i compared to the initial value 0 '''
75     rel_error = np.zeros(np.shape(s)[0])
76     for i in range(0,np.shape(s)[0]):
77         rel_error[i] = abs(energy[i]-energy[0])/abs(energy[0])
78     return rel_error
79
80 def angular_momentum(s,w):
81     ''' Function to compute the angular momentum'''
82     angular_momentum = np.zeros((np.shape(s)[0],3))
83     for i in range(0,np.shape(s)[0]):
84         angular_momentum[i] = np.cross(s[i],w[i])
85     return angular_momentum
86
87 def eccentricity(s,w):
88     ''' Function to compute the eccentricity
89         from the Laplace-Runge-Lenz vector '''
90     LRL = np.zeros((np.shape(s)[0],3))
91     for i in range(0,np.shape(s)[0]):
92         LRL[i] = np.cross(w[i], np.cross(s[i],w[i]))-s[i]
93     return np.linalg.norm(LRL, axis=1)
94
95 # set the initial properties of the two-body system
96 G = 1. # set the gravitational constant
97 body1 = Body(1., np.array([1.,0.,0.]))
98 body2 = Body(1., np.array([0.,0.,0.]))
99 dt = 1e-3; N=1e4 # set step length and number of steps
100
101 # compute the initial velocity for a circular orbit
102 v0 = np.sqrt(G*(body1.mass+body2.mass)/np.linalg.norm(body1.position-body2.
103     position))
104 body1.velocity = np.array([0.,v0,0.])
105
106 ## Numerical Simulation of the Two-Body Problem using Forward Euler Method
107 fig, axs = plt.subplots(2,2,figsize=(12,8), constrained_layout=True)
108 fig.suptitle(r'Forward Euler Method: Numerical Simulation of the Two-Body
109     Problem')
110 s, w = forward_euler(body1, body2, G, dt, N)

```

```

111 energy = total_energy(s,w)                    # compute the total energy
112 rel_error = relative_error(energy)            # compute the relative error
113 angular_momentum = angular_momentum(s,w)     # compute the angular momentum
114 eccentricity = eccentricity(s,w)              # compute the eccentricity
115
116 axs[0,0].plot(s[:,0], s[:,1], 'b.', markersize=1, label='Body 1')
117 axs[0,0].plot([0], [0], 'rx', label='Body 2')
118 axs[0,0].set_title(r'Relative Motion $\vec{s}$ (Orbits)')
119 axs[0,0].set_xlabel(r'$x$-axis'); axs[0,0].set_ylabel(r'$y$-axis')
120 axs[0,0].axis('equal'); axs[0,0].legend(loc='best')
121
122 axs[0,1].plot(range(0,len(eccentricity)), eccentricity, 'b.', markersize=1)
123 axs[0,1].set_title(r'Eccentricity $\vert\vec{e}_i\vert$, $\vert\vec{w}_i\vert$ times')
124         +r'($\vec{s}_i\cdot\vec{w}_i)-\vec{s}_i\cdot\vec{w}_i$ of the Orbit'
125     )
126 axs[0,1].set_xlabel(r'time step $i$')
127 axs[0,1].set_ylabel(r'eccentricity $\vert\vec{e}_i\vert$, $\vert\vec{w}_i\vert$')
128
129 axs[1,0].plot(range(0,len(energy)), energy, 'b.', markersize=1)
130 axs[1,0].set_title(r'Total Energy $E_i=(w_i^2/2)-(1/s_i)$')
131 axs[1,0].set_xlabel(r'time step $i$'); axs[1,0].set_ylabel(r'total energy $E_i$')
132
133 axs[1,1].plot(range(0,len(rel_error)), rel_error, 'b.', markersize=1)
134 axs[1,1].set_title(r'Relative Error $\epsilon_i(h)=\vert E_i-E_0\vert/\vert E_0\vert$, $\epsilon_i$ in the Total Energy $E_i$')
135 axs[1,1].set_xlabel(r'time step $i$'); axs[1,1].set_ylabel(r'relative error $\epsilon_i$')
136
137 for ax in fig.get_axes():
138     ax.grid()
139 fig.savefig('figures/Two-Body-Problem-Euler-Method.pdf', format='pdf', dpi=100)

```



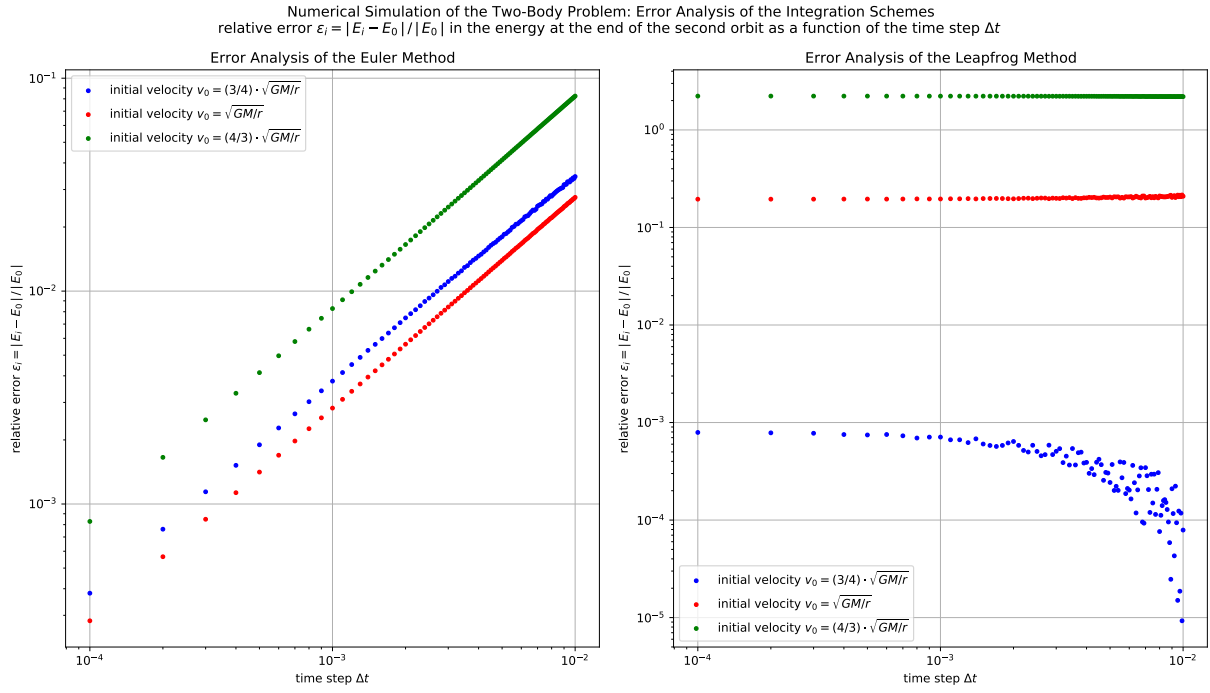
## 2 Error Analysis of Euler Scheme

- (a) After varying the initial velocity three times we can see that the double logarithmic plot of the function is always a straight line with the same slope but different starting points. Therefore all the lines are parallel. This differs quite a lot from what we would expect. The law of conservation of energy predicts, that the energy in the system should remain constant. But we actually see that the energy in the system increases.

```

1  ## Error Analysis of the Euler and the Leapfrog Scheme
2  v0_range = [(3/4)*v0, v0, (4/3)*v0]          # set the initial
    velocities
3  dt_range = np.linspace(1e-4, 1e-2, int(1e2))   # set different time steps
4
5  # compute the relative error in the energy for Euler and Leapfrog
6  rel_error_euler = np.zeros((len(v0_range),len(dt_range)))
7  rel_error_leapfrog = np.zeros((len(v0_range),len(dt_range)))
8
9  for i in range(0,len(v0_range)):
10     for j in range(0,len(dt_range)):
11         N = int(2*np.sqrt(1.**3/(1.*2.))/dt_range[j])
12         body1.velocity = np.array([0.,v0_range[i],0.])
13
14         s, w = forward_euler(body1, body2, G, dt_range[j], N)
15         rel_error_euler[i,j] = relative_error(total_energy(s,w))[N-1]
16
17         s, w = leapfrog(body1, body2, G, dt_range[j], N)
18         rel_error_leapfrog[i,j] = relative_error(total_energy(s,w))[N-1]
19
20 # plot the relative error in the energy for Euler and Leapfrog
21 fig, axs = plt.subplots(1,2,figsize=(14,8), constrained_layout=True)
22 fig.suptitle('Numerical Simulation of the Two-Body Problem: Error Analysis
    of the Integration Schemes\n'
    +r'relative error $\epsilon_i=\text{vert},E_i-E_0\text{,}\text{vert},/\text{,}\text{vert}
    \text{,}E_0\text{,}\text{vert}$ in the energy'
    +r' at the end of the second orbit as a function of the time
    step $\Delta t$')
23 label = [r'initial velocity $v_0=(3/4)\cdot\sqrt{GM/r}$',
24           r'initial velocity $v_0=\sqrt{GM/r}$',
25           r'initial velocity $v_0=(4/3)\cdot\sqrt{GM/r}$']
26 color = ['b.','r.','g.']
27 axs[0].set_title('Error Analysis of the Euler Method')
28 axs[1].set_title('Error Analysis of the Leapfrog Method')
29
30 for i in range(0,3):
31     axs[0].plot(dt_range, rel_error_euler[i,:], color[i], label=label[i])
32     axs[1].plot(dt_range, rel_error_leapfrog[i,:], color[i], label=label[i])
33
34 for ax in fig.get_axes():
35     ax.set_xlabel(r'time step $\Delta t$')
36     ax.set_ylabel(r'relative error $\epsilon_i=\text{vert},E_i-E_0\text{,}\text{vert},/\text{,}\text{vert}
37     \text{,}E_0\text{,}\text{vert}$')
38     ax.grid(); ax.legend(loc='best')
39     ax.set_xscale('log'); ax.set_yscale('log')
40 fig.savefig('figures/Two-Body-Problem_Error-Analysis.pdf', format='pdf',
41            dpi=100)
42 plt.show(); plt.clf(); plt.close()

```



- (b) After implementing the Leapfrog scheme we can see, that the energy remains constant for initial velocities that are smaller or equal to  $v_0 = 2\sqrt{\frac{GM}{R_0}}$ . This means that the conservation of energy applies here. If we plot the motion in this case we can see that the circular remains equidistant to the center. But if we look at initial velocities that are greater than  $v_0 = 2\sqrt{\frac{GM}{R_0}}$  we can see that the Leapfrog scheme also deviates from the constant energy. If we look at the plot we can see that it actually decreases.