The Numerov algorithm is an efficient approach to numerically solve the time-independet Schrödinger equation. We want to use the Numerov algorithm to calculate the stationary states $\Psi(z)$ of neutrons in the gravitational field of the Earth. For small changes in the vertical amplitude z the potential can be expressed as $V(z) = mgz$ for $z \geq 0$. By placing a perfectly reflecting horizontal mirror at $z = 0$ the potential becomes $V(z) = \infty$ for $z < 0$. Neutrons that fall onto the mirror are reflected upwards, and so we only seek solutions for $z \geq 0$. The stationary Schrödinger equation for this problem is given by

$$\left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + mgz\right\}\Psi(z) = E\Psi(z) \quad , \tag{0.1}$$

with E being the energy eigenvalue of the Hamilton operator. We can rewrite this equation to:

$$\frac{\partial^2}{\partial z^2}\Psi(z) - \frac{2m}{\hbar^2}\left(E - mgz\right)\Psi(z) = 0 \qquad \Longleftrightarrow \qquad \frac{\partial^2}{\partial z^2}\Psi(z) - \frac{2m^2 g}{\hbar^2}\left(\frac{E}{mg} - z\right)\Psi(z) = 0 \tag{0.2}$$

By looking at the units of the factor in front of the bracket

$$\left[\frac{2m^2 g}{\hbar^2}\right] = \frac{\mathrm{kg}^2\,\mathrm{m}\,\mathrm{s}^{-2}}{(\mathrm{J\,s})^2} = \frac{\mathrm{kg}^2\,\mathrm{m}}{\mathrm{s}^4}\left(\frac{\mathrm{kg}\,\mathrm{m}^2}{\mathrm{s}^2}\right)^{-2} = \frac{\mathrm{kg}^2\,\mathrm{m}}{\mathrm{s}^4}\,\frac{\mathrm{s}^4}{\mathrm{kg}^2\,\mathrm{m}^4} = \mathrm{m}^{-3} \quad , \tag{0.3}$$

we find a suitable definition for the characteristic length scale z_0:

$$\frac{1}{z_0} := \left(\frac{2m^2 g}{\hbar^2}\right)^{1/3} \tag{0.4}$$

If we use the notation $x := z/z_0$ and $\Psi(z) := \psi(x)/\sqrt{z_0}$, the second order partial derivative becomes:

$$\frac{\partial^2}{\partial z^2}\Psi(z) = \frac{1}{\sqrt{z_0}}\,\frac{1}{z_0^2}\,\frac{\partial^2}{\partial x^2}\psi(x) = \frac{1}{z_0^{5/2}}\,\frac{\partial^2}{\partial x^2}\psi(x) \tag{0.5}$$

The differential equation then becomes:

$$\frac{1}{z_0^{5/2}}\,\frac{\partial^2}{\partial x^2}\psi(x) - \frac{1}{z_0^3}\,\frac{1}{\sqrt{z_0}}\left(\frac{E}{mg} - z\right)\psi(x) = 0 \qquad \Longleftrightarrow \qquad \frac{\partial^2}{\partial x^2}\psi(x) - \frac{1}{z_0}\left(\frac{E}{mg} - z\right)\psi(x) = 0 \tag{0.6}$$

We now define the characteristic energy scale as $E_0 := mgz_0$. The unit of E_0 is then given by:

$$[E_0] = [mgz_0] = \mathrm{kg\,m\,s^{-2}\,m} = \mathrm{kg\,m^2\,s^{-2}} = \mathrm{J} \tag{0.7}$$

After this proper choice of length and energy units

$$x := \frac{z}{z_0} = z\left(\frac{2m^2 g}{\hbar^2}\right)^{1/3} \qquad \text{and} \qquad \epsilon := \frac{E}{E_0} = \frac{E}{mgz_0} = \frac{E}{mg}\left(\frac{2m^2 g}{\hbar^2}\right)^{1/3} \tag{0.8}$$

the Schrödinger equation for neutrons in the gravitational field of the Earth can be rewritten as:

$$\frac{\partial^2}{\partial x^2}\psi(x) + (\epsilon - x)\psi(x) \qquad \Longleftrightarrow \qquad \psi''(x) + (\epsilon - x)\psi(x) = 0 \tag{0.9}$$