Introduction to Computational Physics – Exercise 9

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The Lorenz Attractor

The Lorenz attractor problem is given by the following coupled set of differential equations:

$$\dot{x} = -\sigma(x - y) \tag{1}$$
$$\dot{y} = rx - y - xz \tag{2}$$
$$\dot{z} = xy - bz \tag{3}$$

As discussed in the lecture, the fixed points are $\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$ for all r, and (for $r > 1$) the points $C_\pm = \begin{pmatrix} \pm a_0 & \pm a_0 & r-1 \end{pmatrix}$ with $a_0 = \sqrt{b(r-1)}$. For the entire exercise, please use $\sigma = 10$ and $b = 8/3$. The value of r can be experimented with. When you create numerical solutions you can make plots in 2-D projection (e.g. in the (x, y)- or (x, z)-plane). You can also try a full 3-D plot.

Task: Solve numerically, using `rk4`, the above coupled set of equations for the values $r = 0.5$, 1.17, 1.3456, 25.0 and 29.0. Choose the initial conditions near one of the fixed points: $C_\pm$ for $r > 1$ and $\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$ for $r < 1$. Explain the behavior, as much as possible, with the stability properties of the fixed points.

Task: Determine the sequence z_k for $r = 26.5$, where z_k is a local maximum in z on the solution curve after k periods. Plot z_{k+1} as a function of z_k. When sufficient points are there, connect the points. The resulting function $z_{k+1} = f(z_k)$ has an intersection with the diagonal $z_{k+1} = z_k$. It is a fixed point of the function $f(z_k)$. Is the slope m of this function > 1, < -1 or between -1 and $+1$? Notice: The theory of discrete maps says that there is no periodic solution if $|m| > 1$. So, in such a case we can deduce that this solution of the Lorenz system is not periodic.