Problem 1: #1

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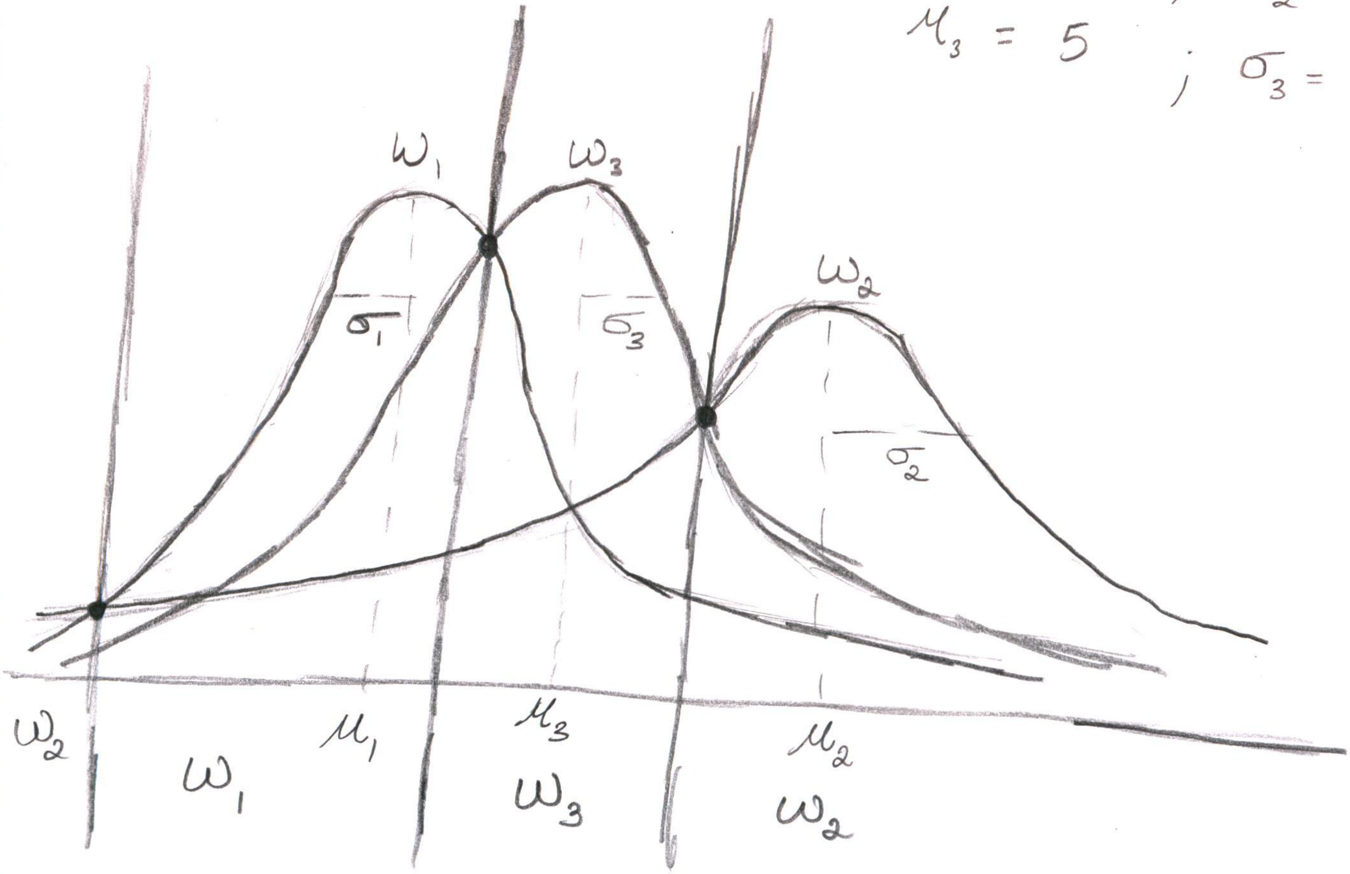
$\mu_1 = 4$; $\sigma_1 = 2$

$\mu_2 = 6$; $\sigma_2 = 3$

$\mu_3 = 5$; $\sigma_3 = 2$



Therefore, there should be 4 decision regions.

Problem 1: # 2a (logical)

Since the mean of ω_3 is 5 & the mean of ω_1 is 4 & they both have the same prior Probability and std. deviation then their intersection will be $(\mu_1 + \mu_3) \cdot \frac{1}{2} = 4.5$. Thus, since $x > 4.5$ then it will be in the region which belongs to ω_3. It cannot be ω_2 because the mean is too far from that point. Thus, by inspection, it is clear that the class to be chosen is ω_3.

Problem 1: # 2a (analytical)

- $p(4.7|\omega_1) = \frac{1}{\sqrt{2\pi} \cdot 2} \cdot e^{\left(-\frac{(4.7-4)^2}{2 \cdot 2^2}\right)} \approx 0.1876$

- $p(4.7|\omega_2) = \frac{1}{\sqrt{2\pi} \cdot 3} \cdot e^{\left(-\frac{(4.7-6)^2}{2 \cdot 3^2}\right)} \approx 0.12106$

- $p(4.7|\omega_3) = \frac{1}{\sqrt{2\pi} \cdot 2} \cdot e^{\left(-\frac{(4.7-5)^2}{2 \cdot 2^2}\right)} \approx 0.19724$

Since they have equal prior probabilities, then the Pdf determines the probabilities. Thus, $P(4.7/\omega_3)$ is the highest & therefore ω_3 is the chosen class.

Note: Prior not needed since they all have equal prior probabilities.

1 : #2b Let a be the left-most boundary, middle boundary, & c be the right-most boundary. As stated in 2a, $\boxed{b = 4.5}$

- To find a: $p(x|\omega_1) = p(x|\omega_2)$ and choose the smallest value of x.

$$\frac{1}{\sqrt{2\pi}\cdot 2}\cdot \exp\left(-\frac{(x-4)^2}{2\cdot 4}\right) = \frac{1}{\sqrt{2\pi}\cdot 3}\cdot \exp\left(-\frac{(x-6)^2}{2\cdot 9}\right)$$

$$\Rightarrow \frac{3}{2}\cdot \exp\left(-\frac{(x-4)^2}{8}\right) = \exp\left(-\frac{(x-6)^2}{18}\right) \Rightarrow \ln\left(\frac{3}{2}\right) - \frac{(x-4)^2}{8} = -\frac{(x-6)^2}{18}$$

$$\Rightarrow \ln\left(\frac{3}{2}\right) = -\frac{(x-6)^2}{18} + \frac{(x-4)^2}{8} \Rightarrow 2\ln\left(\frac{3}{2}\right) = \frac{-4(x-6)^2 + 9(x-4)^2}{36}$$

$$\Rightarrow 72\ln\left(\frac{3}{2}\right) = -4(x-6)^2 + 9(x-4)^2$$

$$\Rightarrow 72\ln\left(\frac{3}{2}\right) = -4(x^2 - 12x + 36) + 9(x^2 - 8x + 16)$$

$$\Rightarrow 72\ln\left(\frac{3}{2}\right) = -4x^2 + 48x \cancel{-144} + 9x^2 - 72x \cancel{+144}$$

$$\Rightarrow 72\ln\left(\frac{3}{2}\right) = 5x^2 - 24x \Rightarrow 5x^2 - 24x - 72\ln\left(\frac{3}{2}\right) = 0$$

$$\therefore x = \frac{24 \pm \sqrt{576 + 4\cdot 5\cdot 72\ln\left(\frac{3}{2}\right)}}{10} = \frac{24 \pm \sqrt{576 + 1440\ln\left(\frac{3}{2}\right)}}{10}$$

$$\therefore \boxed{a = \frac{24 - \sqrt{576 + 1440\ln\left(\frac{3}{2}\right)}}{10}}$$

Problem 1: #2 b continued

- To find c: $p(x|\omega_2) = p(x|\omega_3)$ and choose the highest value of x.

$$\cancel{\frac{1}{\sqrt{2\pi}}} \cdot \frac{1}{3} \cdot \exp\left(-\frac{(x-6)^2}{2\cdot 9}\right) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2} \cdot \exp\left(-\frac{(x-5)^2}{2\cdot 4}\right)$$

$$\Rightarrow \ln\left(\frac{2}{3}\right) = -\frac{(x-5)^2}{4\cdot 2} + \frac{(x-6)^2}{9\cdot 2}$$

$$\Rightarrow 2\ln\left(\frac{2}{3}\right) = \frac{-9(x-5)^2 + 4(x-6)^2}{36}$$

$$\Rightarrow 72\ln\left(\frac{2}{3}\right) = -9(x^2-10x+25) + 4(x^2-12x+36)$$

$$\Rightarrow 72\ln\left(\frac{2}{3}\right) = -9x^2 + 90x - 225 + 4x^2 - 48x + 144$$

$$\Rightarrow 72\ln\left(\frac{2}{3}\right) = -5x^2 + 42x - 81$$

$$\Rightarrow -5x^2 + 42x - \left(81 + 72\ln\left(\frac{2}{3}\right)\right) = 0$$

$$\therefore x = \frac{42 \mp \sqrt{1764 - 4(5)\left(81 + 72\ln\left(\frac{2}{3}\right)\right)}}{10}$$

$$\therefore \boxed{c = \frac{42 + \sqrt{1764 - 20\left(81 + 72\ln\left(\frac{2}{3}\right)\right)}}{10}}$$

Problem 1 : #2C | Assuming zero-one loss

$$P(\text{error}) = \frac{1}{3}\Big[\int_{-\infty}^{a} p(x|\omega_1)\,dx + \int_{-\infty}^{a} p(x|\omega_3)\,dx$$

$$+ \int_{a}^{b} p(x|\omega_2)\,dx + \int_{a}^{b} p(x|\omega_3)\,dx + \int_{b}^{c} p(x|\omega_1)\,dx + \int_{b}^{c} p(x|\omega_2)\,dx$$

$$+ \int_{c}^{+\infty} p(x|\omega_1)\,dx + \int_{c}^{+\infty} p(x|\omega_3)\,dx\Big]$$

$$= \frac{1}{3}\cdot\frac{1}{\sqrt{2\pi}}\Big[\int_{-\infty}^{a}\frac{1}{2}\cdot e^{\left(-\frac{(x-4)^2}{2\cdot 2^2}\right)}dx + \int_{-\infty}^{a}\frac{1}{2}\cdot e^{\left(-\frac{(x-5)^2}{2\cdot 2^2}\right)}dx$$

$$+ \int_{a}^{b}\frac{1}{3}\cdot e^{\left(-\frac{(x-6)^2}{2\cdot 3^2}\right)}dx + \int_{a}^{b}\frac{1}{2}\cdot e^{\left(-\frac{(x-5)^2}{2\cdot 2^2}\right)}dx + \int_{b}^{c}\frac{1}{2}\cdot e^{\left(-\frac{(x-4)^2}{2\cdot 2^2}\right)}dx$$

$$+ \int_{b}^{c}\frac{1}{3}\cdot e^{\left(-\frac{(x-6)^2}{2\cdot 3^2}\right)}dx + \int_{c}^{+\infty}\frac{1}{2}\cdot e^{\left(-\frac{(x-4)^2}{2\cdot 2^2}\right)}dx$$

$$+ \int_{c}^{+\infty}\frac{1}{2}\cdot e^{\left(-\frac{(x-5)^2}{2\cdot 2^2}\right)}dx\Big] \approx \boxed{0.812969}$$

Problem 1: 3b

$P(\omega_1) = 0.6$, $P(\omega_2) = 0.2$, $P(\omega_3) = 0.2$

$X = 4.7$

- $p(x|\omega_1) \cdot P(\omega_1) = \frac{1}{\sqrt{2\pi} \cdot 2} \cdot \exp\left(-\frac{(4.7-4)^2}{2 \cdot 2^2}\right) \cdot 0.6 \approx 0.11257$

- $p(x|\omega_2) \cdot P(\omega_2) = \frac{1}{\sqrt{2\pi} \cdot 3} \cdot \exp\left(-\frac{(4.7-6)^2}{2 \cdot 3^2}\right) \cdot 0.2 \approx 0.024213$

- $p(x|\omega_3) \cdot P(\omega_3) = \frac{1}{\sqrt{2\pi} \cdot 2} \cdot \exp\left(-\frac{(4.7-5)^2}{2 \cdot 2^2}\right) \cdot 0.2 \approx 0.03945$

Thus, with the new prior probabilities, ω_1 is the chosen class because it has the highest posterior probability.

Problem 1: 3c

The impact of prior probability made class 1 much more likely than the rest and thus enabled it to be selected much more often than assuming that it had equal prior probability with the other classes.

Problem 1:4 To get one decision region we would need $\sigma_1 \geq \sigma_2$ and $\sigma_3 < \sigma_1$ because $P(\omega_1)$ is high. Therefore $\sigma_1 = 3$, $\sigma_2 = 3$, $\sigma_3 = 2$ would work.

To get two decision regions $\sigma_1 = \sigma_2 = \sigma_3$.
Thus $\sigma_1 = \sigma_2 = \sigma_3 = 3$ would work.

Problem 1: #5 $p(x|\omega_1)\cdot P(\omega_1) = p(x|\omega_2)\cdot P(\omega_2)$

$$\frac{1}{2}\cdot \exp\left(-\frac{(x-4)^2}{2\cdot 4}\right)\cdot 0.6 = \frac{1}{3}\cdot \exp\left(-\frac{(x-6)^2}{2\cdot 9}\right)\cdot 0.2$$

$$\exp\left(-\frac{(x-4)^2}{8}\right) = \frac{2}{3}\cdot\frac{1}{3}\cdot\exp\left(-\frac{(x-6)^2}{18}\right)$$

$$-\frac{(x-4)^2}{8} = \ln\left(\frac{2}{9}\right) - \frac{(x-6)^2}{18}$$

$$\frac{-(x-4)^2}{8} + \frac{(x-6)^2}{18} = \ln\left(\frac{2}{9}\right)$$

$$-\frac{(x-4)^2}{4} + \frac{(x-6)^2}{9} = 2\ln\left(\frac{2}{9}\right)$$

$$-9(x-4)^2 + 4(x-6)^2 = 72\ln\left(\frac{2}{9}\right)$$

$$-9(x^2-8x+16) + 4(x^2-12x+36) = 72\ln\left(\frac{2}{9}\right)$$

$$-9x^2 + 72x \cancel{-144} + 4x^2 - 48x \cancel{+144} = 72\ln\left(\frac{2}{9}\right)$$

$$-5x^2 + 24x - 72\ln\left(\frac{2}{9}\right) = 0$$

$$x = \frac{+24 \pm \sqrt{576 + 4\cdot(+5)\left(-72\ln\left(\frac{2}{9}\right)\right)}}{+10}$$

$$x = \frac{24 \pm \sqrt{576 - 1440\ln\left(\frac{2}{9}\right)}}{10}$$

Problem 1: 5 continued

$$\therefore\ b = \frac{24 + \sqrt{576 - 1440 \ln\left(\frac{2}{9}\right)}}{10}$$

&

$$a = \frac{24 - \sqrt{576 - 1440 \ln\left(\frac{2}{9}\right)}}{10}$$

$$\boxed{\therefore\ p(\text{error} \mid \omega_1) = \frac{1}{5} \cdot \frac{1}{\sqrt{2\pi}} \left[\int_a^b \frac{1}{2} \cdot \exp\left(-\frac{(x-5)^2}{2 \cdot 4}\right) dx + \int_a^b \frac{1}{3} \cdot \exp\left(-\frac{(x-6)^2}{2 \cdot 9}\right) dx \right]}$$