LEARNING GUIDE

Week No.: __6-7__

TOPIC/S

DERIVATION OF TRIGONOMETRIC IDENTITIES

- Reciprocal Identities
- Pythagorean Identities
- Quotient Identities
- Negative-Angle Identities
- Cofunction Identities
- Sum and Difference Identities for sine, cosine and tangent
- Double-Angle Identities
- Product-to-Sum and Sum-to-Product Identities
- Half-Angle Identities

EXPECTED COMPETENCIES

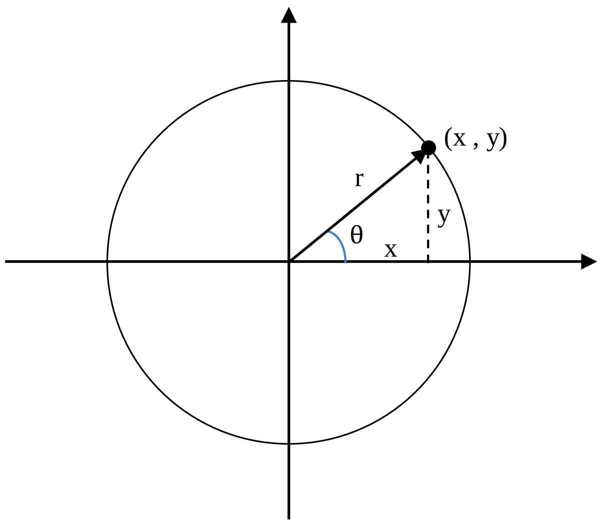
Upon completing this Learning Module, you will be able to:

1. Derive the different trigonometric identities.
2. Verify trigonometric identities
3. Solve functions involving trigonometric identities.

CONTENT/TECHNICAL INFORMATION

- **DERIVATION OF TRIGONOMETRIC IDENTITIES**

Recall the six trigonometric functions of angle θ are **sine, cosine, tangent, cotangent, secant** and **cosecant**. In the following definitions, we use the customary abbreviations for the names of these functions: **sin, cos, tan, cot, sec** and **csc.**



Let (x , y) be a point other than the origin on the terminal side of an angle θ in standard position. The distance from the point to the origin is $r = \sqrt{x^2 + y^2}$. The six trigonometric functions of θ are defined as follows:

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}\ (x \neq 0)$$

$$\csc\theta = \frac{r}{y}\ (y \neq 0) \qquad \sec\theta = \frac{r}{x}\ (x \neq 0) \qquad \cot\theta = \frac{x}{y}\ (y \neq 0)$$

EXAMPLE 1 Finding Function Values of an Angle

The terminal side of an angle θ in standard position passes through the point $(8, 15)$. Find the values of the six trigonometric functions of angle θ.

SOLUTION **Figure 25** shows angle θ and the triangle formed by dropping a perpendicular from the point $(8, 15)$ to the x-axis. The point $(8, 15)$ is 8 units to the right of the y-axis and 15 units above the x-axis, so $x = 8$ and $y = 15$. Now use $r = \sqrt{x^2 + y^2}$.

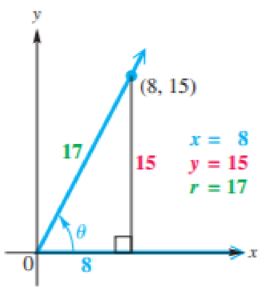


Figure 25

$$r = \sqrt{8^2 + 15^2} = \sqrt{64 + 225} = \sqrt{289} = 17$$

We can now use these values for x, y, and r to find the values of the six trigonometric functions of angle θ.

$$\sin\theta = \frac{y}{r} = \frac{15}{17} \qquad \cos\theta = \frac{x}{r} = \frac{8}{17} \qquad \tan\theta = \frac{y}{x} = \frac{15}{8}$$

$$\csc\theta = \frac{r}{y} = \frac{17}{15} \qquad \sec\theta = \frac{r}{x} = \frac{17}{8} \qquad \cot\theta = \frac{x}{y} = \frac{8}{15}$$

EXAMPLE 2 Finding Function Values of an Angle

The terminal side of an angle θ in standard position passes through the point $(-3, -4)$. Find the values of the six trigonometric functions of angle θ.

SOLUTION As shown in **Figure 26,** $x = -3$ and $y = -4$.

$r = \sqrt{(-3)^2 + (-4)^2}$ $r = \sqrt{x^2 + y^2}$

$r = \sqrt{25}$ Simplify the radicand.

$r = 5$ $r > 0$

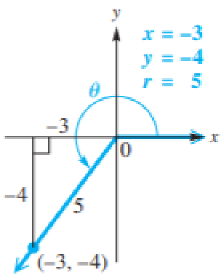


Figure 26

Now we use the definitions of the trigonometric functions.

$$\sin\theta = \frac{-4}{5} = -\frac{4}{5} \qquad \cos\theta = \frac{-3}{5} = -\frac{3}{5} \qquad \tan\theta = \frac{-4}{-3} = \frac{4}{3}$$

$$\csc\theta = \frac{5}{-4} = -\frac{5}{4} \qquad \sec\theta = \frac{5}{-3} = -\frac{5}{3} \qquad \cot\theta = \frac{-3}{-4} = \frac{3}{4}$$

▪ RECIPROCAL IDENTITIES

Recall the definition of a reciprocal: the reciprocal of the nonzero number x is $\frac{1}{x}$. Notice that the equations above; $\sin\boldsymbol{\theta}$ & $\csc\boldsymbol{\theta}$, $\cos\boldsymbol{\theta}$ & $\sec\boldsymbol{\theta}$ and $\tan\boldsymbol{\theta}$ & $\cot\boldsymbol{\theta}$ are reciprocals of each other. The reciprocal identities hold for any angle $\boldsymbol{\theta}$ that does not lead to a zero denominator. Thus for all angles $\boldsymbol{\theta}$ for which both functions are defined,

$$\sin\theta = \frac{1}{\csc\theta} \qquad \cos\theta = \frac{1}{\sec\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

EXAMPLE 1 Using the Reciprocal Identities

Find each function value.

(a) $\cos\theta$, given that $\sec\theta = \frac{5}{3}$ **(b)** $\sin\theta$, given that $\csc\theta = -\frac{\sqrt{12}}{2}$

SOLUTION

(a) We use the fact that $\cos\theta$ is the reciprocal of $\sec\theta$.

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\frac{5}{3}} = 1 \div \frac{5}{3} = 1 \cdot \frac{3}{5} = \frac{3}{5} \quad \text{Simplify the complex fraction.}$$

(b) $\sin\theta = \frac{1}{\csc\theta}$ — $\sin\theta$ is the reciprocal of $\csc\theta$.

$= \frac{1}{-\frac{\sqrt{12}}{2}}$ — Substitute $\csc\theta = -\frac{\sqrt{12}}{2}$.

$= -\frac{2}{\sqrt{12}}$ — Simplify the complex fraction as in part (a).

$= -\frac{2}{2\sqrt{3}}$ — $\sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3}$

$= -\frac{1}{\sqrt{3}}$ — Divide out the common factor 2.

$= -\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$ — Rationalize the denominator.

$= -\frac{\sqrt{3}}{3}$ — Multiply.

EXAMPLE 2 Finding All Function Values Given One Value and the Quadrant

Suppose that angle θ is in quadrant II and $\sin\theta = \frac{2}{3}$. Find the values of the five remaining trigonometric functions.

SOLUTION Choose any point on the terminal side of angle θ. For simplicity, since $\sin\theta = \frac{y}{r}$, choose the point with $r = 3$.

$$\sin\theta = \frac{2}{3} \quad \text{Given value}$$

$$\frac{y}{r} = \frac{2}{3} \quad \text{Substitute } \tfrac{y}{r} \text{ for } \sin\theta.$$

Because $\frac{y}{r} = \frac{2}{3}$ and $r = 3$, it follows that $y = 2$. We must find the value of x.

$$x^2 + y^2 = r^2 \quad \text{Pythagorean theorem}$$

$$x^2 + 2^2 = 3^2 \quad \text{Substitute.}$$

$$x^2 + 4 = 9 \quad \text{Apply exponents.}$$

$$x^2 = 5 \quad \text{Subtract 4.}$$

Remember *both* roots.

$$x = \sqrt{5} \quad \text{or} \quad x = -\sqrt{5} \quad \text{Square root property: If } x^2 = k, \text{ then } x = \sqrt{k} \text{ or } x = -\sqrt{k}.$$

Because θ is in quadrant II, x must be negative. Choose $x = -\sqrt{5}$ so that the point $\left(-\sqrt{5}, 2\right)$ is on the terminal side of θ. See **Figure 36.**

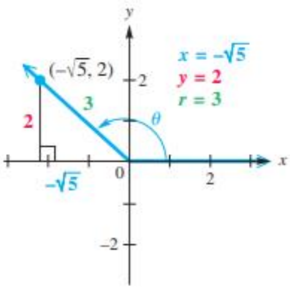


Figure 36

$$\cos\theta = \frac{x}{r} = \frac{-\sqrt{5}}{3} = -\frac{\sqrt{5}}{3}$$

$$\sec\theta = \frac{r}{x} = \frac{3}{-\sqrt{5}} = -\frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{3\sqrt{5}}{5}$$

$$\tan\theta = \frac{y}{x} = \frac{2}{-\sqrt{5}} = -\frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$$

These have rationalized denominators.

$$\cot\theta = \frac{x}{y} = \frac{-\sqrt{5}}{2} = -\frac{\sqrt{5}}{2}$$

$$\csc\theta = \frac{r}{y} = \frac{3}{2}$$

- **PYTHAGOREAN IDENTITIES**

We derive three new identities from the relationship $x^2 + y^2 = r^2$.

$x^2 + y^2 = r^2$; divide by r^2

$\frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2}$; simplify using power rule for exponents

$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$; recall $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$

$(\cos\theta)^2 + (\sin\theta)^2 = 1$

or $\boldsymbol{sin^2\,\theta + cos^2\,\theta = 1}$

Starting again with $x^2 + y^2 = r^2$.

$x^2 + y^2 = r^2$; divide by x^2

$\frac{x^2}{x^2} + \frac{y^2}{x^2} = \frac{r^2}{x^2}$; simplify using power rule for exponents

$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2$; recall $\tan\theta = \frac{y}{x}$, $\sec\,\theta = \frac{r}{x}$

$1 + (\tan\theta)^2 = (\sec\theta)^2$

or $\boldsymbol{tan^2\,\theta + 1 = sec^2\,\theta}$

Similarly, dividing $x^2 + y^2 = r^2$ by y^2 leads to:

$\frac{x^2}{y^2} + \frac{y^2}{y^2} = \frac{r^2}{y^2}$; simplify using power rule for exponents

$\left(\frac{x}{y}\right)^2 + 1 = \left(\frac{r}{y}\right)^2$; recall $\cot\theta = \frac{x}{y}$, $\csc\,\theta = \frac{r}{y}$

$(\cot\theta)^2 + 1 = (\csc\theta)^2$

or $\boldsymbol{1 + cot^2\,\theta = csc^2\,\theta}$

These three identities are called the Pythagorean identities since the original equation that led to them, $x^2 + y^2 = r^2$, comes from the Pythagorean theorem.

For all angles θ for which the function values are defined,

$$\boldsymbol{sin^2\,\theta + cos^2\,\theta = 1} \qquad \boldsymbol{tan^2\,\theta + 1 = sec^2\,\theta} \qquad \boldsymbol{1 + cot^2\,\theta = csc^2\,\theta}$$

- **QUOTIENT IDENTITIES**

Consider the quotient of $\sin\theta$ and $\cos\theta$

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{r} * \frac{r}{x} = \frac{y}{x} = \tan\theta \qquad ,\text{for } \sin\theta \neq 0$$

$$\frac{\cos\theta}{\sin\theta} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{x}{r} * \frac{r}{y} = \frac{x}{y} = \cot\theta \qquad ,\text{for } \sin\theta \neq 0$$

For all angles θ for which the denominators are not zero,

$$\boldsymbol{\frac{\sin\theta}{\cos\theta} = \tan\theta} \qquad \boldsymbol{\frac{\cos\theta}{\sin\theta} = \cot\theta}$$

Looking ahead to calculus:

The reciprocal, Pythagorean and quotient identities are used in calculus to find derivatives and integrals of trigonometric functions. A standard technique of integration called ***trigonometric substitution*** relies on the Pythagorean identities.

EXAMPLE 1 **Using Identities to Find Function Values**

Find $\sin\theta$ and $\tan\theta$, given that $\cos\theta = -\frac{\sqrt{3}}{4}$ and $\sin\theta > 0$.

SOLUTION Start with the Pythagorean identity that includes $\cos\theta$.

$$\sin^2\theta + \cos^2\theta = 1 \qquad \text{Pythagorean identity}$$

$$\sin^2\theta + \left(-\frac{\sqrt{3}}{4}\right)^2 = 1 \qquad \text{Replace } \cos\theta \text{ with } -\tfrac{\sqrt{3}}{4}.$$

$$\sin^2\theta + \frac{3}{16} = 1 \qquad \text{Square } -\tfrac{\sqrt{3}}{4}.$$

$$\sin^2\theta = \frac{13}{16} \qquad \text{Subtract } \tfrac{3}{16}.$$

$$\sin\theta = \pm\frac{\sqrt{13}}{4} \qquad \text{Take square roots.}$$

Choose the correct sign here.

$$\sin\theta = \frac{\sqrt{13}}{4} \qquad \text{Choose the positive square root because } \sin\theta \text{ is positive.}$$

To find $\tan\theta$, use the values of $\cos\theta$ and $\sin\theta$ and the quotient identity for $\tan\theta$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\sqrt{13}}{4}}{-\frac{\sqrt{3}}{4}} = \frac{\sqrt{13}}{4}\left(-\frac{4}{\sqrt{3}}\right) = -\frac{\sqrt{13}}{\sqrt{3}}$$

$$= -\frac{\sqrt{13}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{39}}{3} \qquad \text{Rationalize the denominator.}$$

EXAMPLE 2 Using Identities to Find Function Values

Find $\sin\theta$ and $\cos\theta$, given that $\tan\theta = \frac{4}{3}$ and θ is in quadrant III.

SOLUTION Because θ is in quadrant III, $\sin\theta$ and $\cos\theta$ will both be negative. It is tempting to say that since $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and $\tan\theta = \frac{4}{3}$, then $\sin\theta = -4$ and $\cos\theta = -3$. This is *incorrect,* however—both $\sin\theta$ and $\cos\theta$ must be in the interval $[-1, 1]$.

We use the Pythagorean identity $\tan^2\theta + 1 = \sec^2\theta$ to find $\sec\theta$, and then the reciprocal identity $\cos\theta = \frac{1}{\sec\theta}$ to find $\cos\theta$.

$$\tan^2\theta + 1 = \sec^2\theta \qquad \text{Pythagorean identity}$$

$$\left(\frac{4}{3}\right)^2 + 1 = \sec^2\theta \qquad \tan\theta = \tfrac{4}{3}$$

$$\frac{16}{9} + 1 = \sec^2\theta \qquad \text{Square } \tfrac{4}{3}.$$

$$\frac{25}{9} = \sec^2\theta \qquad \text{Add.}$$

Be careful to choose the correct sign here.

$$-\frac{5}{3} = \sec\theta \qquad \text{Choose the negative square root because } \sec\theta \text{ is negative when } \theta \text{ is in quadrant III.}$$

$$-\frac{3}{5} = \cos\theta \qquad \text{Secant and cosine are reciprocals.}$$

Now we use this value of $\cos\theta$ to find $\sin\theta$.

$$\sin^2\theta = 1 - \cos^2\theta \qquad \text{Pythagorean identity (alternative form)}$$

$$\sin^2\theta = 1 - \left(-\frac{3}{5}\right)^2 \qquad \cos\theta = -\tfrac{3}{5}$$

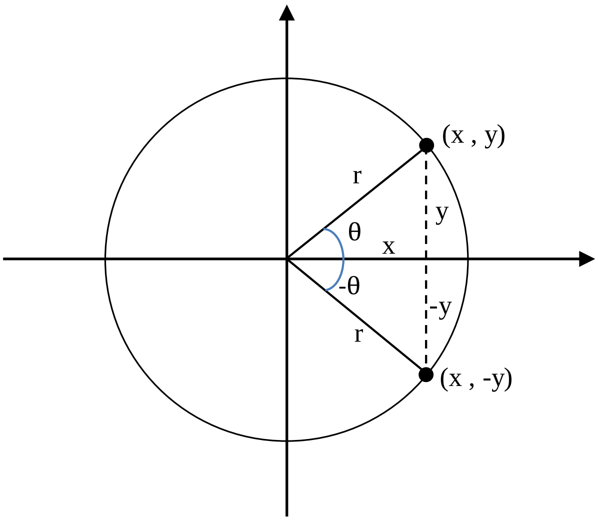
$$\sin^2\theta = 1 - \frac{9}{25} \qquad \text{Square } -\tfrac{3}{5}.$$

$$\sin^2\theta = \frac{16}{25} \qquad \text{Subtract.}$$

Again, be careful.

$$\sin\theta = -\frac{4}{5} \qquad \text{Choose the negative square root.}$$

- **NEGATIVE-ANGLE IDENTITIES**



As suggested by the circle shown above, an angle θ having the point (x,y) on its terminal side has a corresponding angle $-\theta$ with the point (x,-y) on its terminal side.

From the definition of sine,

$$\sin(-\theta) = \frac{-y}{r} \qquad \text{and} \qquad \sin\theta = \frac{y}{r}$$

So $\sin(-\theta)$ and $\sin\theta$ are negatives of each other, or

$$\boldsymbol{\sin(-\theta) = -\sin\theta}$$

From the definition of cosine,

$$\cos(-\theta) = \frac{x}{r} \qquad \text{and} \qquad \cos\theta = \frac{x}{r}$$

$$\boldsymbol{\cos(-\theta) = \cos\theta}$$

From the definition of tangent,

$$\tan(-\theta) = \frac{-y}{x} \qquad \text{and} \qquad \tan\theta = \frac{y}{x}$$

$$\boldsymbol{\tan(-\theta) = -\tan\theta}$$

From the definition of cosecant,

$$\csc(-\theta) = \frac{r}{-y} \qquad \text{and} \qquad \csc\theta = \frac{r}{y}$$

$$\boldsymbol{\csc(-\theta) = -\csc\theta}$$

From the definition of secant,

$$\sec(-\theta) = \frac{r}{x} \qquad \text{and} \qquad \sec\theta = \frac{r}{x}$$

$$\boldsymbol{\sec(-\theta) = \sec\ \theta}$$

From the definition of cotangent,

$$\cot(-\theta) = \frac{x}{-y} \qquad \text{and} \qquad \cot\theta = \frac{x}{y}$$

$$\boldsymbol{\cot(-\theta) = -\cot\,\theta}$$

Here's the summary of negative-angle identities:

$$\boldsymbol{\sin(-\theta) = -\sin\theta} \qquad \boldsymbol{\cos(-\theta) = \cos\theta} \qquad \boldsymbol{\tan(-\theta) = -\tan\theta}$$

$$\boldsymbol{\csc(-\theta) = -\csc\theta} \qquad \boldsymbol{\sec(-\theta) = \sec\,\theta} \qquad \boldsymbol{\cot(-\theta) = -\cot\,\theta}$$

EXAMPLE 1 Finding Trigonometric Function Values Given One Value and the Quadrant

If $\tan\theta = -\frac{5}{3}$ and θ is in quadrant II, find each function value.

(a) $\sec\theta$ **(b)** $\sin\theta$ **(c)** $\cot(-\theta)$

SOLUTION

(a) We use an identity that relates the tangent and secant functions.

$$\tan^2\theta + 1 = \sec^2\theta \qquad \text{Pythagorean identity}$$

$$\left(-\frac{5}{3}\right)^2 + 1 = \sec^2\theta \qquad \tan\theta = -\tfrac{5}{3}$$

$$\frac{25}{9} + 1 = \sec^2\theta \qquad \text{Square } -\tfrac{5}{3}.$$

$$\frac{34}{9} = \sec^2\theta \qquad \text{Add; } 1 = \tfrac{9}{9}$$

$$-\sqrt{\frac{34}{9}} = \sec\theta \qquad \text{Take the negative square root because } \theta \text{ is in quadrant II.}$$

Choose the correct sign.

$$\sec\theta = -\frac{\sqrt{34}}{3} \qquad \text{Simplify the radical: } -\sqrt{\tfrac{34}{9}} = -\tfrac{\sqrt{34}}{\sqrt{9}} = -\tfrac{\sqrt{34}}{3}, \text{ and rewrite.}$$

(b)

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{Quotient identity}$$

$$\cos\theta\tan\theta = \sin\theta \quad \text{Multiply each side by } \cos\theta.$$

$$\left(\frac{1}{\sec\theta}\right)\tan\theta = \sin\theta \quad \text{Reciprocal identity}$$

$$\left(-\frac{3\sqrt{34}}{34}\right)\left(-\frac{5}{3}\right) = \sin\theta$$

$\tan\theta = -\frac{5}{3}$, and from part (a), $\frac{1}{\sec\theta} = \frac{1}{-\frac{\sqrt{34}}{3}} = -\frac{3}{\sqrt{34}} = -\frac{3}{\sqrt{34}} \cdot \frac{\sqrt{34}}{\sqrt{34}} = -\frac{3\sqrt{34}}{34}$.

$$\sin\theta = \frac{5\sqrt{34}}{34} \quad \text{Multiply and rewrite.}$$

(c)

$$\cot(-\theta) = \frac{1}{\tan(-\theta)} \quad \text{Reciprocal identity}$$

$$\cot(-\theta) = \frac{1}{-\tan\theta} \quad \text{Even-odd identity}$$

$$\cot(-\theta) = \frac{1}{-\left(-\frac{5}{3}\right)} \quad \tan\theta = -\frac{5}{3}$$

$$\cot(-\theta) = \frac{3}{5} \quad \frac{1}{-\left(-\frac{5}{3}\right)} = 1 \div \frac{5}{3} = 1 \cdot \frac{3}{5} = \frac{3}{5}$$

CAUTION ***When taking the square root, be sure to choose the sign based on the quadrant of θ and the function being evaluated.***

EXAMPLE 2 Writing One Trignometric Function in Terms of Another

Write $\cos x$ in terms of $\tan x$.

SOLUTION By identities, $\sec x$ is related to both $\cos x$ and $\tan x$.

$$1 + \tan^2 x = \sec^2 x \quad \text{Pythagorean identity}$$

$$\frac{1}{1+\tan^2 x} = \frac{1}{\sec^2 x} \quad \text{Take reciprocals.}$$

$$\frac{1}{1+\tan^2 x} = \cos^2 x \quad \text{The reciprocal of } \sec^2 x \text{ is } \cos^2 x.$$

Remember both the positive and negative roots.

$$\pm\sqrt{\frac{1}{1+\tan^2 x}} = \cos x \quad \text{Take the square root of each side.}$$

$$\cos x = \frac{\pm 1}{\sqrt{1+\tan^2 x}} \quad \text{Quotient rule for radicals: } \sqrt[n]{\tfrac{a}{b}} = \tfrac{\sqrt[n]{a}}{\sqrt[n]{b}}; \text{ rewrite.}$$

$$\cos x = \frac{\pm\sqrt{1+\tan^2 x}}{1+\tan^2 x} \quad \text{Rationalize the denominator.}$$

The choice of the + sign or the − sign is made depending on the quadrant of x.

The functions $\tan\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$ can easily be expressed in terms of $\sin\theta$, $\cos\theta$, or both. We make such substitutions in an expression to simplify it.

EXAMPLE 3 Rewriting an Expression in Terms of Sine and Cosine

Write $\frac{1+\cot^2\theta}{1-\csc^2\theta}$ in terms of $\sin\theta$ and $\cos\theta$, and then simplify the expression so that no quotients appear.

SOLUTION

$$\frac{1+\cot^2\theta}{1-\csc^2\theta} \qquad \text{Given expression}$$

$$=\frac{1+\dfrac{\cos^2\theta}{\sin^2\theta}}{1-\dfrac{1}{\sin^2\theta}} \qquad \text{Quotient identities}$$

$$=\frac{\left(1+\dfrac{\cos^2\theta}{\sin^2\theta}\right)\sin^2\theta}{\left(1-\dfrac{1}{\sin^2\theta}\right)\sin^2\theta} \qquad \text{Simplify the complex fraction by multiplying both numerator and denominator by the LCD.}$$

$$=\frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta-1} \qquad \text{Distributive property: } (b+c)a = ba+ca$$

$$=\frac{1}{-\cos^2\theta} \qquad \text{Pythagorean identities}$$

$$=-\sec^2\theta \qquad \text{Reciprocal identity}$$

- **SUM AND DIFFERENCE IDENTITIES FOR COSINE**

Difference Identity for Cosine

"$\cos(A-B)$ *does not equal* $\cos A - \cos B$ "

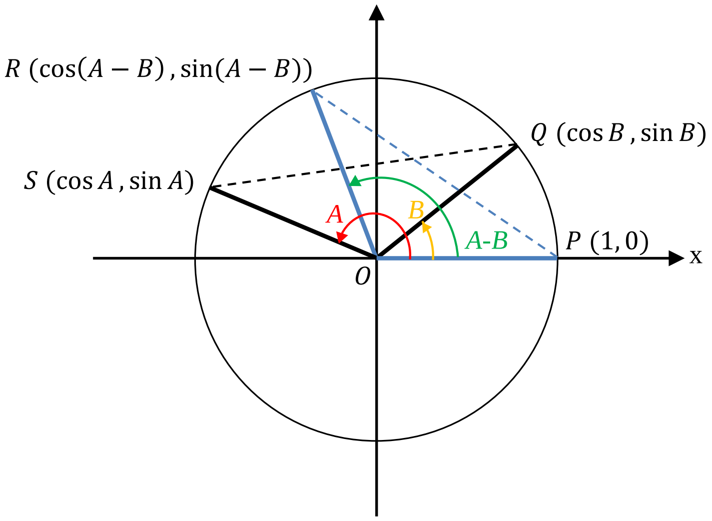
For example, if $A=\frac{\pi}{2}$ and $B=0$, then

$$\cos(A-B) \quad = \cos\left(\frac{\pi}{2}-0\right) = \cos\frac{\pi}{2} = \mathbf{0}$$

while $\quad \cos A - \cos B = \cos\frac{\pi}{2} - \cos 0 = 0 - 1 = \mathbf{-1}$

We can now derive a formula for $\cos(A-B)$. We start by locating angles A and B in standard position on a unit circle, with $B < A$. Let S and Q be the points where the terminal sides of angles A and B, respectively, intersect the circle. Let P be the point (1,0), and locate point R on the unit circle so that angle POR equals the difference $A-B$. See the figure below.

y



Point Q is on the unit circle, thus the x-coordinate of Q is the cosine of angle B, while the y-coordinate of Q is the sine of angle B.

$$Q \text{ has coordinates } (\cos B, \sin B).$$

In the same way,

$$S \text{ has coordinates } (\cos A, \sin A),$$

and $$R \text{ has coordinates } (cos(A-B), sin(A-B)).$$

Angle SOQ also equals $A - B$. Since the central angles SOQ and POR are equal, chords PR and SQ are equal. By the distance formula, since $PR = SQ$,

$$\sqrt{[\cos(A-B)-1]^2 + [\sin(A-B)-0]^2} = \sqrt{(\cos A - \cos B)^2 + (\sin A - \sin B)^2}$$

Squaring both sides and clearing parentheses gives

$$\cos^2(A-B) - 2\cos(A-B) + 1 + \sin^2(A-B)$$
$$= \cos^2 A - 2\cos A \cos B + \cos^2 B + \sin^2 A - 2\sin A \sin B + \sin^2 B$$

Since $\sin^2\theta + \cos^2\theta = 1$ for any value of θ, we can rewrite the equation as

$$2 - 2\cos(A-B) = 2 - 2\cos A \cos B - 2\sin A \sin B \quad ;\text{subtract 2 both sides}$$
$$-2\cos(A-B) = -2\cos A \cos B - 2\sin A \sin B \quad ;\text{divide by} -2 \text{ both sides}$$
$$\mathbf{\cos(A-B) = \cos A \cos B + \sin A \sin B}$$

This is the identity for $\cos(A-B)$. Although the figure above shows angles A and B in the second and first quadrants, respectively, this result is the same for any values of these angles.

Sum Identity for Cosine To find a similar expression for $\cos(A+B)$, rewrite $A+B$ as $A-(-B)$ and use the identity for $\cos(A-B)$.

$$\cos(A+B) = \cos[A-(-B)]$$
$$\cos(A+B) = \cos A\cos(-B) + \sin A\sin(-B) \quad \text{; Cosine difference identity}$$
$$\cos(A+B) = \cos A\cos B + \sin A\,(-\sin B) \quad \text{; Negative angle identities}$$
$$\boldsymbol{\cos(A+B) = \cos A\cos B - \sin A\sin B}$$

SUM AND DIFFERENCE IDENTITIES FOR COSINE:

$$\boldsymbol{\cos(A-B) = \cos A\cos B + \sin A\sin B}$$
$$\boldsymbol{\cos(A+B) = \cos A\cos B - \sin A\sin B}$$

EXAMPLE 1 **Finding Exact Cosine Function Values**

Find the *exact* value of each expression.

(a) $\cos 15^\circ$ **(b)** $\cos \frac{5\pi}{12}$ **(c)** $\cos 87^\circ \cos 93^\circ - \sin 87^\circ \sin 93^\circ$

SOLUTION

(a) To find $\cos 15^\circ$, we write 15° as the sum or difference of two angles with known function values, such as 45° and 30°, because

$$15^\circ = 45^\circ - 30^\circ. \quad \text{(We could also use } 60^\circ - 45^\circ\text{.)}$$

Then we use the cosine difference identity.

$\cos 15^\circ$

$= \cos(45^\circ - 30^\circ)$ $\quad 15^\circ = 45^\circ - 30^\circ$

$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$ $\quad$ Cosine difference identity

$= \dfrac{\sqrt{2}}{2}\cdot\dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{2}}{2}\cdot\dfrac{1}{2}$ $\quad$ Substitute known values.

$= \dfrac{\sqrt{6}+\sqrt{2}}{4}$ $\quad$ Multiply, and then add fractions.

(b) $\cos \frac{5\pi}{12}$

$$= \cos\left(\frac{\pi}{6} + \frac{\pi}{4}\right) \qquad \frac{\pi}{6} = \frac{2\pi}{12} \text{ and } \frac{\pi}{4} = \frac{3\pi}{12}$$

$$= \cos\frac{\pi}{6}\cos\frac{\pi}{4} - \sin\frac{\pi}{6}\sin\frac{\pi}{4} \qquad \text{Cosine sum identity}$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{1}{2}\cdot\frac{\sqrt{2}}{2} \qquad \text{Substitute known values.}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4} \qquad \text{Multiply, and then subtract fractions.}$$

(c) $\cos 87^\circ \cos 93^\circ - \sin 87^\circ \sin 93^\circ$

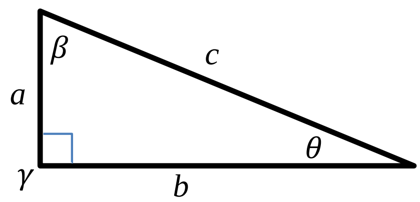
$$= \cos(87^\circ + 93^\circ) \qquad \text{Cosine sum identity}$$

$$= \cos 180^\circ \qquad \text{Add.}$$

$$= -1 \qquad \cos 180^\circ = -1$$

- **COFUNCTION IDENTITIES**

Figure below shows a right triangle with acute angles θ and β and a right angle at γ. The length of the opposite side of θ is a, and the length of the side opposite angle B is b. The length of the hypotenuse is c.



Recall: **Right-Triangle-Based Definitions of Trigonometric Functions**
For any acute angle θ in standard position

$$\sin\theta = \frac{a}{c} \qquad \cos\theta = \frac{b}{c} \qquad \tan\theta = \frac{a}{b}$$

$$\csc\theta = \frac{c}{a} \qquad \sec\theta = \frac{c}{b} \qquad \cot\theta = \frac{b}{a}$$

By the preceding definitions, $\sin\theta = \frac{a}{c}$. Since $\cos\beta$ is also equal to $\frac{a}{c}$,

$$\sin\theta = \frac{a}{c} = \cos\beta$$

Similarly, $\quad \boldsymbol{\tan\theta = \frac{a}{b} = \cot\beta} \quad$ and $\quad \boldsymbol{\sec\theta = \frac{c}{b} = \csc\beta}$

Since the sum of the three angles in any triangle is 180^{O} and angle γ equals 90^{O}, angles θ and β must have a sum of $180^{\text{O}} - 90^{\text{O}} = 90^{\text{O}}$. As mentioned in our previous learning modules, angles with a sum of 90^{O} are complementary angles. Since angles θ and β are complementary and $\sin\theta = \cos\beta$, the functions sine and cosine are called **cofunctions**. Tangent and cotangent are also functions, as are secant and cosecant. And since the angles θ and β are complementary, $\theta + \beta = 90^{\text{O}}$, or $\beta = 90^{\text{O}} - \theta$, giving

$$\sin\theta = \cos\beta$$
$$\boldsymbol{\sin\theta = \cos(90^{\text{O}} - \theta)}$$

Similar results, called the **cofunctions identities,** are true for the other trigonometric functions.

$$\boldsymbol{\sin\theta = \cos(90^{\text{O}} - \theta)} \qquad \boldsymbol{\sec\theta = \csc(90^{\text{O}} - \theta)} \qquad \boldsymbol{\tan\theta = \cot(90^{\text{O}} - \theta)}$$

$$\boldsymbol{\cos\theta = \sin(90^{\text{O}} - \theta)} \qquad \boldsymbol{\csc\theta = \sec(90^{\text{O}} - \theta)} \qquad \boldsymbol{\cot\theta = \tan(90^{\text{O}} - \theta)}$$

EXAMPLE 1 **Writing Functions in Terms of Cofunctions**

Write each function in terms of its cofunction.

(a) $\cos 52°$ **(b)** $\tan 71°$ **(c)** $\sec 24°$

SOLUTION

(a)

Cofunctions

$$\cos 52° = \sin(90° - 52°) = \sin 38° \qquad \cos A = \sin(90° - A)$$

Complementary angles

(b) $\tan 71° = \cot(90° - 71°) = \cot 19°$ **(c)** $\sec 24° = \csc 66°$

EXAMPLE 2 **Solving Equations Using Cofunction Identities**

Find one solution for each equation. Assume all angles involved are acute angles.

(a) $\cos(\theta + 4^\circ) = \sin(3\theta + 2^\circ)$ **(b)** $\tan(2\theta - 18^\circ) = \cot(\theta + 18^\circ)$

SOLUTION

(a) Sine and cosine are cofunctions, so $\cos(\theta + 4^\circ) = \sin(3\theta + 2^\circ)$ is true if the sum of the angles is 90°.

$$(\theta + 4^\circ) + (3\theta + 2^\circ) = 90^\circ \quad \text{Complementary angles}$$
$$4\theta + 6^\circ = 90^\circ \quad \text{Combine like terms.}$$
$$4\theta = 84^\circ \quad \text{Subtract } 6^\circ \text{ from each side.}$$
$$\theta = 21^\circ \quad \text{Divide by 4.}$$

(b) Tangent and cotangent are cofunctions.

$$(2\theta - 18^\circ) + (\theta + 18^\circ) = 90^\circ \quad \text{Complementary angles}$$
$$3\theta = 90^\circ \quad \text{Combine like terms.}$$
$$\theta = 30^\circ \quad \text{Divide by 3.}$$

EXAMPLE 3 **Using Cofunction Identities to Find θ**

Find one value of θ or x that satisfies each of the following.

(a) $\cot\theta = \tan 25^\circ$ **(b)** $\sin\theta = \cos(-30^\circ)$ **(c)** $\csc\frac{3\pi}{4} = \sec x$

SOLUTION

(a) Because tangent and cotangent are cofunctions, $\tan(90^\circ - \theta) = \cot\theta$.

$$\cot\theta = \tan 25^\circ$$
$$\tan(90^\circ - \theta) = \tan 25^\circ \quad \text{Cofunction identity}$$
$$90^\circ - \theta = 25^\circ \quad \text{Set angle measures equal.}$$
$$\theta = 65^\circ \quad \text{Solve for } \theta.$$

(b)

$$\sin\theta = \cos(-30^\circ)$$
$$\cos(90^\circ - \theta) = \cos(-30^\circ) \quad \text{Cofunction identity}$$
$$90^\circ - \theta = -30^\circ \quad \text{Set angle measures equal.}$$
$$\theta = 120^\circ \quad \text{Solve for } \theta.$$

(c)

$$\csc\frac{3\pi}{4} = \sec x$$
$$\csc\frac{3\pi}{4} = \csc\left(\frac{\pi}{2} - x\right) \quad \text{Cofunction identity}$$
$$\frac{3\pi}{4} = \frac{\pi}{2} - x \quad \text{Set angle measures equal.}$$
$$x = -\frac{\pi}{4} \quad \text{Solve for } x\text{: } \tfrac{\pi}{2} - \tfrac{3\pi}{4} = \tfrac{2\pi}{4} - \tfrac{3\pi}{4} = -\tfrac{\pi}{4}.$$

EXAMPLE 4 Finding cos(*s* + *t*) Given Information about *s* and *t*

Suppose that $\sin s = \frac{3}{5}$, $\cos t = -\frac{12}{13}$, and both s and t are in quadrant II. Find $\cos(s+t)$.

SOLUTION By the cosine sum identity,

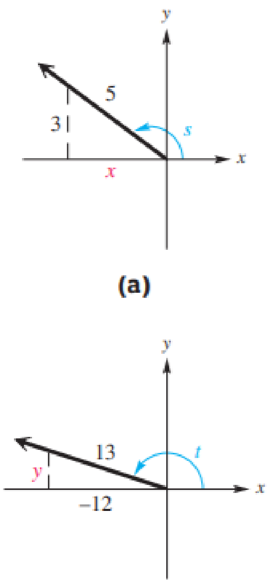
$$\cos(s+t) = \cos s \cos t - \sin s \sin t.$$

The values of $\sin s$ and $\cos t$ are given, so we can find $\cos(s+t)$ if we know the values of $\cos s$ and $\sin t$. There are two ways to do this.

Method 1 We use angles in standard position. To find $\cos s$ and $\sin t$, we sketch two reference triangles in the second quadrant, one with $\sin s = \frac{3}{5}$ and the other with $\cos t = -\frac{12}{13}$. Notice that for angle t, we use -12 to denote the length of the side that lies along the x-axis. See **Figure 5.**

In **Figure 5(a),** $y = 3$ and $r = 5$. We must find x.

$$x^2 + y^2 = r^2 \quad \text{Pythagorean theorem}$$
$$x^2 + 3^2 = 5^2 \quad \text{Substitute.}$$
$$x^2 = 16 \quad \text{Isolate } x^2.$$
$$x = -4 \quad \text{Choose the negative square root here.}$$



(a)

(b)

Figure 5

Thus, $\cos s = \frac{x}{r} = -\frac{4}{5}$.

In **Figure 5(b),** $x = -12$ and $r = 13$. We must find y.

$$x^2 + y^2 = r^2 \quad \text{Pythagorean theorem}$$
$$(-12)^2 + y^2 = 13^2 \quad \text{Substitute.}$$
$$y^2 = 25 \quad \text{Isolate } y^2.$$
$$y = 5 \quad \text{Choose the positive square root here.}$$

Thus, $\sin t = \frac{y}{r} = \frac{5}{13}$.

Now we can find $\cos(s+t)$.

$$\cos(s+t) = \cos s \cos t - \sin s \sin t \quad \text{Cosine sum identity (1)}$$
$$= -\frac{4}{5}\left(-\frac{12}{13}\right) - \frac{3}{5}\cdot\frac{5}{13} \quad \text{Substitute.}$$
$$= \frac{48}{65} - \frac{15}{65} \quad \text{Multiply.}$$
$$\cos(s+t) = \frac{33}{65} \quad \text{Subtract.}$$

Method 2 We use Pythagorean identities here. To find $\cos s$, recall that $\sin^2 s + \cos^2 s = 1$, where s is in quadrant II.

$$\left(\frac{3}{5}\right)^2 + \cos^2 s = 1 \qquad \sin s = \tfrac{3}{5}$$

$$\frac{9}{25} + \cos^2 s = 1 \qquad \text{Square } \tfrac{3}{5}.$$

$$\cos^2 s = \frac{16}{25} \qquad \text{Subtract } \tfrac{9}{25}.$$

$$\cos s = -\frac{4}{5}$$

$\cos s < 0$ because s is in quadrant II.

To find $\sin t$, we use $\sin^2 t + \cos^2 t = 1$, where t is in quadrant II.

$$\sin^2 t + \left(-\frac{12}{13}\right)^2 = 1 \qquad \cos t = -\tfrac{12}{13}$$

$$\sin^2 t + \frac{144}{169} = 1 \qquad \text{Square } -\tfrac{12}{13}.$$

$$\sin^2 t = \frac{25}{169} \qquad \text{Subtract } \tfrac{144}{169}.$$

$$\sin t = \frac{5}{13}$$

$\sin t > 0$ because t is in quadrant II.

From this point, the problem is solved using the same steps beginning with the equation marked (1) in Method 1 on the previous page. The result is

$$\cos(s + t) = \frac{33}{65}. \qquad \text{Same result as in Method 1}$$

- **SUM AND DIFFERENCE IDENTITIES FOR SINE**

We can use the cosine sum and difference identities to derive similar identities for sine. Since $\sin\theta = \cos(90^{o} - \theta)$, we replace θ with $A + B$ to get

$$\sin\theta = \cos(90^{o} - \theta)$$
$$\sin(A + B) = \cos[90^{o} - (A + B)]$$
$$= \cos[90^{o} - A - B]$$
$$= \cos[(90^{o} - A) - B] \qquad \text{; use cosine difference identity}$$
$$\sin(A + B) = \cos(90^{o} - A)\cos B + \sin(90^{o} - A)\sin B$$

Then use cofunctions identities for sine and cosine. Thus,

$$\mathbf{\sin(A + B) = \sin A \cos B + \cos A \sin B}$$

Now we write $\sin(A-B)$ as $\sin[A+(-B)]$ and use the identity for $\sin(A+B)$.

$$\begin{aligned}\sin(A-B) &= \sin[A+(-B)] && \text{; use sine sum identity}\\ &= \sin A\cos(-B)+\cos A\sin(-B) && \text{; use negative angle indentities}\\ &= \sin A\cos B+\cos A\,(-\sin B) && \text{; simplify}\end{aligned}$$

$$\boldsymbol{\sin(A-B) = \sin A\cos B - \cos A\sin B}$$

SUM AND DIFFERENCE IDENTITIES FOR SINE:

$$\boldsymbol{\sin(A+B) = \sin A\cos B + \cos A\sin B}$$
$$\boldsymbol{\sin(A-B) = \sin A\cos B - \cos A\sin B}$$

- **SUM AND DIFFERENCE IDENTITIES FOR TANGENT**

We can use the sine and cosine sum and difference identities to derive similar identities for tangent. Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, we replace θ with $A+B$ to get

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} \qquad \text{; use sine and cosine sum identities}$$

$$\tan(A+B) = \frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

We express this result in terms of the tangent function by multiplying both numerator and denominator by $\frac{1}{\cos A\cos B}$.

$$\tan(A+B) = \frac{\frac{\sin A\cos B+\cos A\sin B}{1}}{\frac{\cos A\cos B-\sin A\sin B}{1}} * \frac{\frac{1}{\cos A\cos B}}{\frac{1}{\cos A\cos B}} \qquad \text{; Simplify}$$

$$\tan(A+B) = \frac{\frac{\sin A\cos B}{\cos A\cos B}+\frac{\cos A\sin B}{\cos A\cos B}}{\frac{\cos A\cos B}{\cos A\cos B}-\frac{\sin A\sin B}{\cos A\cos B}} \qquad \text{; Simplify}$$

$$\tan(A+B) = \frac{\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}}{1-\frac{\sin A}{\cos A}*\frac{\sin B}{\cos B}} \qquad \text{; Simplify}$$

$$\boldsymbol{\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}}$$

Replacing B with $-B$ and using the fact that $\tan(-B) = -\tan B$ gives the identity for the tangent of the difference of two angles.

$$\tan(A-B)=\frac{\tan A+\tan(-B)}{1-\tan A\tan(-B)}$$

$$\boldsymbol{\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}}$$

SUM AND DIFFERENCE IDENTITIES FOR TANGENT:

$$\boldsymbol{\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}}$$

$$\boldsymbol{\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}}$$

EXAMPLE 1 **Finding Exact Sine and Tangent Function Values**

Find the *exact* value of each expression.

(a) $\sin 75^\circ$ **(b)** $\tan \frac{7\pi}{12}$ **(c)** $\sin 40^\circ \cos 160^\circ - \cos 40^\circ \sin 160^\circ$

SOLUTION

(a) $\sin 75^\circ$

$= \sin(45^\circ + 30^\circ)$ — $75^\circ = 45^\circ + 30^\circ$

$= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$ — Sine sum identity

$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}$ — Substitute known values.

$= \frac{\sqrt{6}+\sqrt{2}}{4}$ — Multiply, and then add fractions.

(b) $\tan \frac{7\pi}{12}$

$$= \tan\left(\frac{\pi}{3} + \frac{\pi}{4}\right) \qquad \frac{\pi}{3} = \frac{4\pi}{12} \text{ and } \frac{\pi}{4} = \frac{3\pi}{12}$$

$$= \frac{\tan \frac{\pi}{3} + \tan \frac{\pi}{4}}{1 - \tan \frac{\pi}{3} \tan \frac{\pi}{4}} \qquad \text{Tangent sum identity}$$

$$= \frac{\sqrt{3} + 1}{1 - \sqrt{3} \cdot 1} \qquad \text{Substitute known values.}$$

$$= \frac{\sqrt{3} + 1}{1 - \sqrt{3}} \cdot \frac{1 + \sqrt{3}}{1 + \sqrt{3}} \qquad \text{Rationalize the denominator.}$$

$$= \frac{\sqrt{3} + 3 + 1 + \sqrt{3}}{1 - 3} \qquad (a + b)(c + d) = ac + ad + bc + bd;\ (x - y)(x + y) = x^2 - y^2$$

$$= \frac{4 + 2\sqrt{3}}{-2} \qquad \text{Combine like terms.}$$

Factor first. Then divide out the common factor.

$$= \frac{2(2 + \sqrt{3})}{2(-1)} \qquad \text{Factor out 2.}$$

$$= -2 - \sqrt{3} \qquad \text{Write in lowest terms.}$$

(c) $\sin 40° \cos 160° - \cos 40° \sin 160°$

$$= \sin(40° - 160°) \qquad \text{Sine difference identity}$$

$$= \sin(-120°) \qquad \text{Subtract.}$$

$$= -\sin 120° \qquad \text{Even-odd identity}$$

$$= -\frac{\sqrt{3}}{2} \qquad \text{Substitute the known value.}$$

EXAMPLE 2 Writing Functions as Expressions Involving Functions of θ

Write each function as an expression involving functions of θ alone.

(a) $\sin(30° + \theta)$ **(b)** $\tan(45° - \theta)$ **(c)** $\sin(180° - \theta)$

SOLUTION

(a) $\sin(30° + \theta)$

$$= \sin 30° \cos\theta + \cos 30° \sin\theta \qquad \text{Sine sum identity}$$

$$= \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta \qquad \sin 30° = \frac{1}{2} \text{ and } \cos 30° = \frac{\sqrt{3}}{2}$$

$$= \frac{\cos\theta + \sqrt{3}\sin\theta}{2} \qquad \frac{a}{b} \cdot c = \frac{ac}{b}\text{; Add fractions.}$$

(b) $\tan(45° - \theta)$

$$= \frac{\tan 45° - \tan\theta}{1 + \tan 45° \tan\theta} \quad \text{Tangent difference identity}$$

$$= \frac{1 - \tan\theta}{1 + 1 \cdot \tan\theta} \quad \tan 45° = 1$$

$$= \frac{1 - \tan\theta}{1 + \tan\theta} \quad \text{Multiply.}$$

(c) $\sin(180° - \theta)$

$$= \sin 180° \cos\theta - \cos 180° \sin\theta \quad \text{Sine difference identity}$$

$$= 0 \cdot \cos\theta - (-1)\sin\theta \quad \sin 180° = 0 \text{ and } \cos 180° = -1$$

$$= \sin\theta \quad \text{Simplify.}$$

EXAMPLE 3 **Finding Function Values and the Quadrant of $A + B$**

Suppose that A and B are angles in standard position such that $\sin A = \frac{4}{5}$, $\frac{\pi}{2} < A < \pi$, and $\cos B = -\frac{5}{13}$, $\pi < B < \frac{3\pi}{2}$. Find each of the following.

(a) $\sin(A + B)$ **(b)** $\tan(A + B)$ **(c)** the quadrant of $A + B$

SOLUTION

(a) The identity for $\sin(A + B)$ involves $\sin A$, $\cos A$, $\sin B$, and $\cos B$. We are given values of $\sin A$ and $\cos B$. We must find values of $\cos A$ and $\sin B$.

$$\sin^2 A + \cos^2 A = 1 \quad \text{Fundamental identity}$$

$$\left(\frac{4}{5}\right)^2 + \cos^2 A = 1 \quad \sin A = \frac{4}{5}$$

$$\frac{16}{25} + \cos^2 A = 1 \quad \text{Square } \frac{4}{5}.$$

$$\cos^2 A = \frac{9}{25} \quad \text{Subtract } \frac{16}{25}.$$

Pay attention to signs.

$$\cos A = -\frac{3}{5} \quad \text{Take square roots. Because } A \text{ is in quadrant II, } \cos A < 0.$$

In the same way, $\sin B = -\frac{12}{13}$. Now find $\sin(A + B)$.

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \quad \text{Sine sum identity}$$

$$= \frac{4}{5}\left(-\frac{5}{13}\right) + \left(-\frac{3}{5}\right)\left(-\frac{12}{13}\right) \quad \text{Substitute the given values for } \sin A \text{ and } \cos B \text{ and the values found for } \cos A \text{ and } \sin B.$$

$$= -\frac{20}{65} + \frac{36}{65} \quad \text{Multiply.}$$

$$\sin(A + B) = \frac{16}{65} \quad \text{Add.}$$

(b) To find $\tan(A+B)$, use the values of sine and cosine from part (a), $\sin A = \frac{4}{5}$, $\cos A = -\frac{3}{5}$, $\sin B = -\frac{12}{13}$, and $\cos B = -\frac{5}{13}$, to obtain $\tan A$ and $\tan B$.

$$\tan A = \frac{\sin A}{\cos A} = \frac{\frac{4}{5}}{-\frac{3}{5}} = \frac{4}{5} \div \left(-\frac{3}{5}\right) = \frac{4}{5} \cdot \left(-\frac{5}{3}\right)$$

$$\tan A = -\frac{4}{3}$$

$$\tan B = \frac{\sin B}{\cos B} = \frac{-\frac{12}{13}}{-\frac{5}{13}} = -\frac{12}{13} \div \left(-\frac{5}{13}\right) = -\frac{12}{13} \cdot \left(-\frac{13}{5}\right)$$

$$\tan B = \frac{12}{5}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$ Tangent sum identity

$$= \frac{\left(-\frac{4}{3}\right) + \frac{12}{5}}{1 - \left(-\frac{4}{3}\right)\left(\frac{12}{5}\right)}$$ Substitute.

$$= \frac{\frac{16}{15}}{1 + \frac{48}{15}}$$ Perform the indicated operations.

$$= \frac{\frac{16}{15}}{\frac{63}{15}}$$ Add terms in the denominator.

$$= \frac{16}{15} \div \frac{63}{15}$$ Simplify the complex fraction.

$$= \frac{16}{15} \cdot \frac{15}{63}$$ Definition of division

$$\tan(A+B) = \frac{16}{63}$$ Multiply.

(c) $\sin(A+B) = \frac{16}{65}$ and $\tan(A+B) = \frac{16}{63}$ See parts (a) and (b).

Both are positive. Therefore, $A + B$ must be in quadrant I, because it is the only quadrant in which both sine and tangent are positive.

EXAMPLE 4 **Verifying an Identity**

Verify that the equation is an identity.

$$\sin\left(\frac{\pi}{6}+\theta\right)+\cos\left(\frac{\pi}{3}+\theta\right)=\cos\theta$$

SOLUTION Work on the left side, using the sine and cosine sum identities.

$$\sin\left(\frac{\pi}{6}+\theta\right)+\cos\left(\frac{\pi}{3}+\theta\right)$$

$$=\left(\sin\frac{\pi}{6}\cos\theta+\cos\frac{\pi}{6}\sin\theta\right)+\left(\cos\frac{\pi}{3}\cos\theta-\sin\frac{\pi}{3}\sin\theta\right)$$

Sine sum identity; cosine sum identity

$$=\left(\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta\right)+\left(\frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta\right)$$

$\sin\frac{\pi}{6}=\frac{1}{2}$; $\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$; $\cos\frac{\pi}{3}=\frac{1}{2}$; $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$

$$=\frac{1}{2}\cos\theta+\frac{1}{2}\cos\theta \quad \text{Simplify.}$$

$$=\cos\theta \quad \text{Add.}$$

- **DOUBLE-ANGLE IDENTITIES**

When $A = B$ in the identities for the sum of two angles, these identities are called **double-angle identities.** For example, to derive an expression for $\cos 2A$, we let $B = A$ in the identity $\cos(A + B) = \cos A \cos B - \sin A \sin B$.

$\cos 2A = \cos(A + A)$; use cosine sum identity
$\cos 2A = \cos A \cos A - \sin A \sin A$
$\boldsymbol{\cos 2A = \cos^2 A - \sin^2 A}$

Two other useful forms of this identity can be obtained by substituting either $\cos^2 A = 1 - \sin^2 A$ or $\sin^2 A = 1 - \cos^2 A$. Replace $\cos^2 A$ with the expression $1 - \sin^2 A$ to get

$\cos 2A = \cos^2 A - \sin^2 A$
$\cos 2A = (1 - \sin^2 A) - \sin^2 A$
$\boldsymbol{\cos 2A = 1 - 2\sin^2 A}$,

or replace $\sin^2 A$ with $1 - \cos^2 A$ to get

$\cos 2A = \cos^2 A - \sin^2 A$
$\cos 2A = \cos^2 A - (1 - \cos^2 A)$
$\cos 2A = \cos^2 A - 1 + \cos^2 A$
$\boldsymbol{\cos 2A = 2\cos^2 A - 1}$.

We find $\sin 2A$ with the identity $\sin(A+B) = \sin A \cos B + \cos A \sin B$, letting $B = A$.

$\sin 2A = \sin(A + A)$; use sine sum identity
$\sin 2A = \sin A \cos A + \cos A \sin A$
$\mathbf{\sin 2A = 2 \sin A \cos A}$

Using the identity for $\tan(A+B)$, we find $\tan 2A$.

$\tan 2A = \tan(A + A)$; use tangent sum identity

$$\tan 2A = \frac{\tan A + \tan A}{1 - \tan A \tan A}$$

$$\mathbf{\tan 2A = \frac{2\tan A}{1 - \tan^2 A}}$$

DOUBLE-ANGLE IDENTITIES:

$$\mathbf{\cos 2A = \cos^2 A - \sin^2 A} \qquad \mathbf{\cos 2A = 1 - 2\sin^2 A}$$

$$\mathbf{\cos 2A = 2\cos^2 A - 1} \qquad \mathbf{\sin 2A = 2 \sin A \cos A}$$

$$\mathbf{\tan 2A = \frac{2\tan A}{1 - \tan^2 A}}$$

EXAMPLE 1 Finding Function Values of 2θ Given Information about θ

Given $\cos\theta = \frac{3}{5}$ and $\sin\theta < 0$, find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$.

SOLUTION To find $\sin 2\theta$, we must first find the value of $\sin\theta$.

$\sin^2\theta + \cos^2\theta = 1$ Pythagorean identity

$\sin^2\theta + \left(\frac{3}{5}\right)^2 = 1$ $\cos\theta = \frac{3}{5}$

$\sin^2\theta = \frac{16}{25}$ $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$; Subtract $\frac{9}{25}$.

Pay attention to signs here. $\sin\theta = -\frac{4}{5}$ Take square roots. Choose the negative square root because $\sin\theta < 0$.

Now use the double-angle identity for sine.

$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(-\frac{4}{5}\right)\left(\frac{3}{5}\right) = -\frac{24}{25}$ $\sin\theta = -\frac{4}{5}$ and $\cos\theta = \frac{3}{5}$

Now we find $\cos 2\theta$, using the first of the double-angle identities for cosine.

Any of the three forms may be used. $\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{9}{25} - \frac{16}{25} = -\frac{7}{25}$ $\cos\theta = \frac{3}{5}$ and $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$; $\sin\theta = -\frac{4}{5}$ and $\left(-\frac{4}{5}\right)^2 = \frac{16}{25}$

The value of tan 2θ can be found in either of two ways. We can use the double-angle identity and the fact that $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{4}{5}}{\frac{3}{5}} = -\frac{4}{5} \div \frac{3}{5} = -\frac{4}{5} \cdot \frac{5}{3} = -\frac{4}{3}$.

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(-\frac{4}{3}\right)}{1-\left(-\frac{4}{3}\right)^2} = \frac{-\frac{8}{3}}{-\frac{7}{9}} = \frac{24}{7}$$

Alternatively, we can find tan 2θ by finding the quotient of sin 2θ and cos 2θ.

$$\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{-\frac{24}{25}}{-\frac{7}{25}} = \frac{24}{7} \qquad \text{Same result as above}$$

EXAMPLE 2 **Finding Function Values of θ Given Information about 2θ**

Find the values of the six trigonometric functions of θ given $\cos 2\theta = \frac{4}{5}$ and $90° < \theta < 180°$.

SOLUTION We must obtain a trigonometric function value of θ alone.

$$\cos 2\theta = 1 - 2\sin^2\theta \qquad \text{Double-angle identity}$$

$$\frac{4}{5} = 1 - 2\sin^2\theta \qquad \cos 2\theta = \tfrac{4}{5}$$

$$-\frac{1}{5} = -2\sin^2\theta \qquad \text{Subtract 1 from each side.}$$

$$\frac{1}{10} = \sin^2\theta \qquad \text{Multiply by } -\tfrac{1}{2}.$$

$$\sin\theta = \sqrt{\frac{1}{10}} \qquad \text{Take square roots. Choose the positive square root because } \theta \text{ terminates in quadrant II.}$$

$$\sin\theta = \frac{1}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} \qquad \text{Quotient rule for radicals; rationalize the denominator.}$$

$$\sin\theta = \frac{\sqrt{10}}{10} \qquad \sqrt{a}\cdot\sqrt{a} = a$$

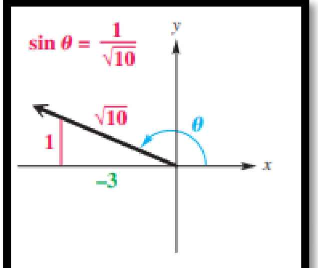


Figure 7

Now find values of cos θ and tan θ by sketching and labeling a right triangle in quadrant II. Because $\sin\theta = \frac{1}{\sqrt{10}}$, the triangle in **Figure 7** is labeled accordingly. The Pythagorean theorem is used to find the remaining leg.

$$\cos\theta = \frac{-3}{\sqrt{10}} = -\frac{3\sqrt{10}}{10} \quad \text{and} \quad \tan\theta = \frac{1}{-3} = -\frac{1}{3} \qquad \cos\theta = \tfrac{x}{r} \text{ and } \tan\theta = \tfrac{y}{x}$$

We find the other three functions using reciprocals.

$$\csc\theta = \frac{1}{\sin\theta} = \sqrt{10}, \quad \sec\theta = \frac{1}{\cos\theta} = -\frac{\sqrt{10}}{3}, \quad \cot\theta = \frac{1}{\tan\theta} = -3$$

EXAMPLE 3 Verifying an Identity

Verify that the following equation is an identity.

$$\cot x \sin 2x = 1 + \cos 2x$$

SOLUTION We start by working on the left side, writing all functions in terms of sine and cosine and then simplifying the result.

$$\begin{aligned}
\cot x \sin 2x &= \frac{\cos x}{\sin x} \cdot \sin 2x && \text{Quotient identity} \\
&= \frac{\cos x}{\sin x}(2 \sin x \cos x) && \text{Double-angle identity} \\
&= 2\cos^2 x && \text{Multiply.} \\
&= 1 + \cos 2x && \cos 2x = 2\cos^2 x - 1\text{, so } 2\cos^2 x = 1 + \cos 2x.
\end{aligned}$$

Be able to recognize alternative forms of identities.

EXAMPLE 4 Simplifying Expressions Using Double-Angle Identities

Simplify each expression.

(a) $\cos^2 7x - \sin^2 7x$

(b) $\sin 15° \cos 15°$

SOLUTION

(a) This expression suggests one of the double-angle identities for cosine: $\cos 2A = \cos^2 A - \sin^2 A$. Substitute $7x$ for A.

$$\cos^2 7x - \sin^2 7x = \cos 2(7x) = \cos 14x$$

(b) If the expression $\sin 15° \cos 15°$ were

$$2 \sin 15° \cos 15°,$$

we could apply the identity for $\sin 2A$ directly because $\sin 2A = 2 \sin A \cos A$.

$$\begin{aligned}
&\sin 15° \cos 15° \\
&= \frac{1}{2}(2) \sin 15° \cos 15° && \text{Multiply by 1 in the form } \tfrac{1}{2}(2). \\
&= \frac{1}{2}(2 \sin 15° \cos 15°) && \text{Associative property} \\
&= \frac{1}{2}\sin(2 \cdot 15°) && 2 \sin A \cos A = \sin 2A\text{, with } A = 15° \\
&= \frac{1}{2}\sin 30° && \text{Multiply.} \\
&= \frac{1}{2} \cdot \frac{1}{2} && \sin 30° = \tfrac{1}{2} \\
&= \frac{1}{4} && \text{Multiply.}
\end{aligned}$$

This is not an obvious way to begin, but it is indeed valid.

EXAMPLE 5 **Deriving a Multiple-Angle Identity**

Write $\sin 3x$ in terms of $\sin x$.

SOLUTION

$\sin 3x$

$= \sin(2x + x)$ Use the simple fact that 3 = 2 + 1 here.	$3x = 2x + x$
$= \sin 2x \cos x + \cos 2x \sin x$	Sine sum identity
$= (2 \sin x \cos x)\cos x + (\cos^2 x - \sin^2 x)\sin x$	Double-angle identities
$= 2 \sin x \cos^2 x + \cos^2 x \sin x - \sin^3 x$	Multiply.
$= 2 \sin x(1 - \sin^2 x) + (1 - \sin^2 x)\sin x - \sin^3 x$	$\cos^2 x = 1 - \sin^2 x$
$= 2 \sin x - 2 \sin^3 x + \sin x - \sin^3 x - \sin^3 x$	Distributive property
$= 3 \sin x - 4 \sin^3 x$	Combine like terms.

- **PRODUCT-TO-SUM IDENTITIES**

The identities for $\cos(A+B)$ and $\cos(A-B)$ can be added to derive an identity useful in calculus.

$$\begin{array}{rl} & \cos(A+B) = \cos A\cos B - \sin A\sin B \\ + & \cos(A-B) = \cos A\cos B + \sin A\sin B \\ \hline \end{array}$$

$$\cos(A+B) + \cos(A-B) = 2\cos A\cos B$$

Or $\mathbf{\cos A\cos B} = \frac{1}{2}[\mathbf{\cos(A+B) + \cos(A-B)}]$

Similarly, subtracting $\cos(A+B)$ from $\cos(A-B)$ gives

$$\begin{array}{rl} & \cos(A-B) = \cos A\cos B + \sin A\sin B \\ - & \cos(A+B) = \cos A\cos B - \sin A\sin B \\ \hline \end{array}$$

$$\cos(A-B) - \cos(A+B) = 2\sin A\sin B$$

Or $\mathbf{\sin A\sin B} = \frac{1}{2}[\mathbf{\cos(A-B) - \cos(A+B)}]$

Using the identities for $\sin(A+B)$ and $\sin(A-B)$ in the same way, we get two more identities.

$$\begin{array}{rl} & \sin(A+B) = \sin A\cos B + \cos A\sin B \\ + & \sin(A-B) = \sin A\cos B - \cos A\sin B \\ \hline \end{array}$$

$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B$$

Or $\mathbf{\sin A \cos B} = \frac{1}{2}[\mathbf{\sin(A+B) + \sin(A-B)}]$

Similarly, subtracting $\sin(A+B)$ from $\sin(A-B)$ gives

$$\begin{array}{rl} & \sin(A+B) = \sin A \cos B + \cos A \sin B \\ - & \sin(A-B) = \sin A \cos B - \cos A \sin B \\ \hline \end{array}$$

$$\sin(A+B) - \sin(A-B) = 2\cos A \sin B$$

Or $\mathbf{\cos A \sin B} = \frac{1}{2}[\mathbf{\sin(A+B) - \sin(A-B)}]$

PRODUCT-TO-SUM IDENTITIES:

$$\mathbf{\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]}$$

$$\mathbf{\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]}$$

$$\mathbf{\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]}$$

$$\mathbf{\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]}$$

EXAMPLE 1 **Using a Product-to-Sum Identity**

Write 4 cos 75° sin 25° as the sum or difference of two functions.

SOLUTION

$4\cos 75^\circ \sin 25^\circ$

$= 4\left[\frac{1}{2}(\sin(75^\circ + 25^\circ) - \sin(75^\circ - 25^\circ))\right]$ Use the identity for $\cos A \sin B$, with $A = 75^\circ$ and $B = 25^\circ$.

$= 2\sin 100^\circ - 2\sin 50^\circ$ Simplify.

▪ SUM-TO-PRODUCT IDENTITIES

Another group of identities allows us to write sums as products. We can transform the product-to-sum identities into equivalent useful forms – the sum-to-product identities- using substitution.

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

EXAMPLE 2 Using a Sum-to-Product Identity

Write $\sin 2\theta - \sin 4\theta$ as a product of two functions.

SOLUTION $\sin 2\theta - \sin 4\theta$

$= 2\cos\left(\frac{2\theta + 4\theta}{2}\right)\sin\left(\frac{2\theta - 4\theta}{2}\right)$ Use the identity for $\sin A - \sin B$, with $A = 2\theta$ and $B = 4\theta$.

$= 2\cos\frac{6\theta}{2}\sin\left(\frac{-2\theta}{2}\right)$ Simplify the numerators.

$= 2\cos 3\theta \sin(-\theta)$ Divide.

$= -2\cos 3\theta \sin\theta$ $\sin(-\theta) = -\sin\theta$

▪ HALF-ANGLE IDENTITIES

From the alternative forms of the identity for $\cos 2A$, we derive three additional identities for $\sin\frac{A}{2}$, $\cos\frac{A}{2}$ and $\tan\frac{A}{2}$. These are known as **half-angle identities.**

To derive the identity for $\sin\frac{A}{2}$, start with the following double-angle identity for cosine and solve for $\sin x$.

$\cos 2x = 1 - 2\sin^2 x$	; add $2\sin^2 x$ both sides
$\cos 2x + 2\sin^2 x = 1$	; subtract $\cos 2x$ both sides
$2\sin^2 x = 1 - \cos 2x$	; divide by 2 both sides
$\sin^2 x = \dfrac{1-\cos 2x}{2}$	; take square roots

$$\sin x = \pm\sqrt{\frac{1-\cos 2x}{2}} \qquad \text{; let } 2\text{x} = \text{A, so x} = \frac{A}{2} \text{ then substitute}$$

$$\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos 2\left(\frac{A}{2}\right)}{2}} \qquad \text{; simplify}$$

$$\mathbf{\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}}$$

The $\pm$ sign in this identity indicates that the appropriate sign is chosen depending on the quadrant of $\frac{A}{2}$. For example, if $\frac{A}{2}$ is a quadrant III angle, we choose negative sign since function is negative in quadrant III.

We derive the identity for $\cos\frac{A}{2}$ using the double-angle identity $\cos 2x = 2\cos^2 x - 1$.

$$\cos 2x = 2\cos^2 x - 1 \qquad \text{; add 1 both sides}$$
$$1 + \cos 2x = 2\cos^2 x \qquad \text{; rewrite then divide by 2}$$
$$\cos^2 x = \frac{1+\cos 2x}{2} \qquad \text{; take square roots}$$
$$\cos x = \pm\sqrt{\frac{1+\cos 2x}{2}} \qquad \text{; replace x with } \frac{A}{2} \text{ and simplify}$$

$$\mathbf{\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}}$$

An identity for $\tan\frac{A}{2}$ comes from the identities for $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$.

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{\pm\sqrt{\frac{\mathbf{1-\cos A}}{\mathbf{2}}}}{\pm\sqrt{\frac{\mathbf{1+\cos A}}{\mathbf{2}}}}$$

$$\mathbf{\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}}$$

We derive an alternative identity for $\tan\frac{A}{2}$ using the double-angle identities.

$$\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} \qquad \text{; multiply by } 2\cos\frac{A}{2} \text{ in numerator and denominator}$$

$$\tan\frac{A}{2} = \frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{2\cos^2\frac{A}{2}} \qquad ;\text{simplify and apply double} - \text{angle identities}$$

$$\tan\frac{A}{2} = \frac{\sin 2\left(\frac{A}{2}\right)}{1+\cos 2\left(\frac{A}{2}\right)} \qquad ;\text{simplify}$$

$$\boldsymbol{\tan\frac{A}{2} = \frac{\sin A}{1+\cos A}}$$

From the identity $\tan\frac{A}{2} = \frac{\sin A}{1+\cos A}$, we can also derive

$$\boldsymbol{\tan\frac{A}{2} = \frac{1-\cos A}{\sin A}}$$

HALF-ANGLE IDENTITIES:

$$\boldsymbol{\sin\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}}} \qquad \boldsymbol{\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}}$$

$$\boldsymbol{\tan\frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}} \qquad \boldsymbol{\tan\frac{A}{2} = \frac{\sin A}{1+\cos A}} \qquad \boldsymbol{\tan\frac{A}{2} = \frac{1-\cos A}{\sin A}}$$

EXAMPLE 1 **Using a Half-Angle Identity to Find an Exact Value**

Find the exact value of cos 15° using the half-angle identity for cosine.

SOLUTION

$$\cos 15^\circ = \cos\frac{30^\circ}{2} = \sqrt{\frac{1+\cos 30^\circ}{2}}$$

Choose the positive square root.

$$= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\left(1+\frac{\sqrt{3}}{2}\right)\cdot 2}{2\cdot 2}} = \frac{\sqrt{2+\sqrt{3}}}{2}$$

Simplify the radicals.

EXAMPLE 2 **Using a Half-Angle Identity to Find an Exact Value**

Find the exact value of tan 22.5° using the identity $\tan \frac{A}{2} = \frac{\sin A}{1 + \cos A}$.

SOLUTION Because $22.5° = \frac{45°}{2}$, replace A with 45°.

$$\tan 22.5° = \tan \frac{45°}{2} = \frac{\sin 45°}{1 + \cos 45°} = \frac{\frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}} = \frac{\frac{\sqrt{2}}{2}}{1 + \frac{\sqrt{2}}{2}} \cdot \frac{2}{2}$$

$$= \frac{\sqrt{2}}{2 + \sqrt{2}} = \frac{\sqrt{2}}{2 + \sqrt{2}} \cdot \frac{2 - \sqrt{2}}{2 - \sqrt{2}} = \frac{2\sqrt{2} - 2}{2}$$ Rationalize the denominator.

Factor first, and then divide out the common factor.

$$= \frac{2(\sqrt{2} - 1)}{2} = \sqrt{2} - 1$$

✔ **Now Try Exercise 13.**

EXAMPLE 3 **Finding Function Values of $\frac{s}{2}$ Given Information about s**

Given $\cos s = \frac{2}{3}$, with $\frac{3\pi}{2} < s < 2\pi$, find $\sin \frac{s}{2}$, $\cos \frac{s}{2}$, and $\tan \frac{s}{2}$.

SOLUTION The angle associated with $\frac{s}{2}$ terminates in quadrant II because

$$\frac{3\pi}{2} < s < 2\pi \quad \text{and} \quad \frac{3\pi}{4} < \frac{s}{2} < \pi. \quad \text{Divide by 2.}$$

See **Figure 9.** In quadrant II, the values of $\cos \frac{s}{2}$ and $\tan \frac{s}{2}$ are negative and the value of $\sin \frac{s}{2}$ is positive. Use the appropriate half-angle identities and simplify.

$$\sin \frac{s}{2} = \sqrt{\frac{1 - \frac{2}{3}}{2}} = \sqrt{\frac{1}{6}} = \frac{\sqrt{1}}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

Rationalize all denominators.

$$\cos \frac{s}{2} = -\sqrt{\frac{1 + \frac{2}{3}}{2}} = -\sqrt{\frac{5}{6}} = -\frac{\sqrt{5}}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = -\frac{\sqrt{30}}{6}$$

$$\tan \frac{s}{2} = \frac{\sin \frac{s}{2}}{\cos \frac{s}{2}} = \frac{\frac{\sqrt{6}}{6}}{-\frac{\sqrt{30}}{6}} = \frac{\sqrt{6}}{-\sqrt{30}} = -\frac{\sqrt{6}}{\sqrt{30}} \cdot \frac{\sqrt{30}}{\sqrt{30}} = -\frac{\sqrt{180}}{30} = -\frac{6\sqrt{5}}{6 \cdot 5} = -\frac{\sqrt{5}}{5}$$

Notice that it is not necessary to use a half-angle identity for $\tan \frac{s}{2}$ once we find $\sin \frac{s}{2}$ and $\cos \frac{s}{2}$. However, using this identity provides an excellent check.

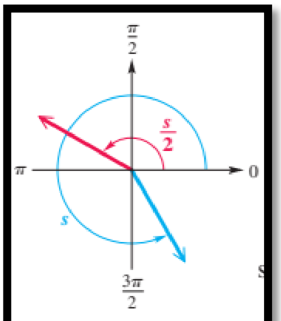


Figure 9

EXAMPLE 4 **Simplifying Expressions Using Half-Angle Identities**

Simplify each expression.

(a) $\pm\sqrt{\dfrac{1+\cos 12x}{2}}$ **(b)** $\dfrac{1-\cos 5\alpha}{\sin 5\alpha}$

SOLUTION

(a) This matches part of the identity for $\cos\frac{A}{2}$. Replace A with $12x$.

$$\cos\frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}} = \pm\sqrt{\frac{1+\cos 12x}{2}} = \cos\frac{12x}{2} = \cos 6x$$

(b) Use the identity $\tan\frac{A}{2} = \frac{1-\cos A}{\sin A}$ with $A = 5\alpha$.

$$\frac{1-\cos 5\alpha}{\sin 5\alpha} = \tan\frac{5\alpha}{2}$$

EXAMPLE 5 **Verifying an Identity**

Verify that the following equation is an identity.

$$\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2 = 1 + \sin x$$

SOLUTION We work on the more complicated left side.

Remember the middle term when squaring a binomial.

$$\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2$$

$$= \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2} + \cos^2\frac{x}{2} \qquad (x+y)^2 = x^2 + 2xy + y^2$$

$$= 1 + 2\sin\frac{x}{2}\cos\frac{x}{2} \qquad \sin^2\tfrac{x}{2} + \cos^2\tfrac{x}{2} = 1$$

$$= 1 + \sin 2\left(\frac{x}{2}\right) \qquad 2\sin\tfrac{x}{2}\cos\tfrac{x}{2} = \sin 2\left(\tfrac{x}{2}\right)$$

$$= 1 + \sin x \qquad \text{Multiply.}$$

Recall:

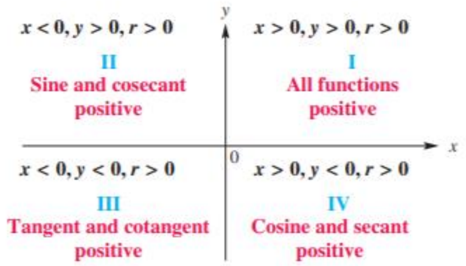
Signs and Ranges of Function Values In the definitions of the trigonometric functions, r is the distance from the origin to the point (x, y). This distance is undirected, so $r > 0$. If we choose a point (x, y) in quadrant I, then both x and y will be positive, and the values of all six functions will be positive.

A point (x, y) in quadrant II satisfies $x < 0$ and $y > 0$. This makes the values of sine and cosecant positive for quadrant II angles, while the other four functions take on negative values. Similar results can be obtained for the other quadrants.

This important information is summarized here.

Signs of Trigonometric Function Values

θ in Quadrant	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
I	+	+	+	+	+	+
II	+	−	−	−	−	+
III	−	−	+	+	−	−
IV	−	+	−	−	+	−



PROGRESS CHECK

SET 1 Use the appropriate reciprocal identity to find each function value. Rationalize denominators when applicable.

1 $\csc\theta$, given that $\sin\theta = -\frac{3}{7}$

2 $\tan\theta$, given that $\cot\theta = -2.5$

SET 2 Use identities to solve each of the following. Rationalize denominators when applicable.

1 Find $\csc\theta$, given that $\cot\theta = -\frac{1}{2}$ and θ is in quadrant IV.

2 Find $\sec\theta$, given that $\tan\theta = \frac{\sqrt{7}}{3}$ and θ is in quadrant III.

SET 3 Give all six trigonometric function values for each angle θ. Rationalize denominators when applicable.

1 $\tan\theta = -\frac{15}{8}$, and θ is in quadrant II.

2 $\cot\theta = \frac{\sqrt{3}}{8}$, and θ is in quadrant I.

SET 4 Use identities to correctly complete each sentence.

1 If $\cos\theta = 0.8$, and $\sin\theta = 0.6$, then $\tan(-\theta) =$ ____________.

2 If $\sin\theta = \frac{2}{3}$, then $-\sin(-\theta) =$ ____________.

SET 5 For each expression in Column I, choose the expression from Column II that completes an identity. One or both expressions may need to be rewritten.

I	II
1 $-\tan x\cos x =$ _____	$A.\ \frac{\sin^2 x}{\cos^2 x}$
2 $\sec^2 x - 1 =$ _____	$B.\ \frac{\sin^2 x}{\cos^2 x}$
3 $\frac{\sec x}{\csc x} =$ _____	$C.\ \sin(-x)$
4 $1 + \sin^2 x =$ _____	$D.\ \csc^2 x - \cot^2 x + \sin^2 x$
5 $\cos^2 x =$ _____	$E.\ \tan x$

SET 6 Write each expression in terms of sine and cosine, and then simplify the expression so that no quotients appear and all functions are of θ only.

1 $(1 - \cos\theta)(1 + \sec\theta)$

2 $\frac{\sec^2\theta - 1}{\csc^2\theta - 1}$

3 $-\sec^2(-\theta) + \sin^2(-\theta) + \cos^2(-\theta)$

SET 7 Verify that each equation is an identity.

1 $1 + \cos 2x - \cos^2 x = \cos^2 x$

2 $\sec 2x = \frac{\cot^2 x + 1}{\cot^2 x - 1}$

3 $\sin(x + y) + \sin(x - y) = 2\sin x\cos y$

4 $\frac{\sin(x-y)}{\sin(x+y)} = \frac{\tan x - \tan y}{\tan x + \tan y}$

5 $\sin 4x = 4\sin x\cos x\cos 2x$

6 $1 + \tan x\tan 2x = \sec 2x$

7 $\tan\frac{\theta}{2} = \csc\theta - \cot\theta$

8 $\sec^2\frac{x}{2} = \frac{2}{1+\cos x}$

SET 8 Find $\cos(s+t)$ and $\cos(s-t)$.

1 $\sin s = \frac{3}{5}$ and $\sin t = -\frac{12}{13}$, s in quadrant I and t in quadrant III

2 $\cos s = -\frac{1}{5}$ and $\sin t = \frac{3}{5}$, s and t in quadrant II

SET 9 Use the given information to find a) $\sin(s+t)$, b) $\tan(s+t)$ and c) the quadrant of $s+t$.

1 $\cos s = -\frac{8}{17}$ and cosn $t = -\frac{3}{5}$, s and t in quadrant III

2 $\sin s = \frac{2}{3}$ and $\sin t = -\frac{1}{3}$, s in quadrant II and t in quadrant IV

REFERENCES

Lial, M. L., Hornsby, J., & Schneider, D. I. (2009). *Trigonometry Ninth Edition.* Quezon City, Philippines: C&E Publishing, Inc.

LEARNING GUIDE

Week No.: 8

TOPIC/S

INVERSE CIRCULAR FUNCTIONS

EXPECTED COMPETENCIES

Upon completing this Learning Module, you will be able to:

1. Analyze the graphs of inverse circular functions.
2. Solve equations/problems involving inverse circular functions.

CONTENT/TECHNICAL INFORMATION

- **INVERSE CIRCULAR FUNCTIONS**

Inverse Functions We now review some basic concepts from algebra. ***For a function f, every element x in the domain corresponds to one and only one element y, or $f(x)$, in the range.*** This means the following:

1. If point (a, b) lies on the graph of f, then there is no other point on the graph that has a as first coordinate.
2. Other points may have b as second coordinate, however, because the definition of function allows range elements to be used more than once.

If a function is defined so that ***each range element is used only once,*** then it is a **one-to-one function.** For example, the function

$$f(x) = x^3 \text{ is a one-to-one function}$$

because every real number has exactly one real cube root. However,

$$g(x) = x^2 \text{ is not a one-to-one function}$$

because $g(2) = 4$ and $g(-2) = 4$. There are two domain elements, 2 and -2, that correspond to the range element 4.

The **horizontal line test** helps determine graphically whether a function is one-to-one.

Horizontal Line Test

A function is one-to-one if every horizontal line intersects the graph of the function at most once.

This test is applied to the graphs of $f(x) = x^3$ and $g(x) = x^2$ in **Figure 1.**

By interchanging the components of the ordered pairs of a one-to-one function f, we obtain a new set of ordered pairs that satisfies the definition of a function. This new function is the *inverse function,* or *inverse,* of f.

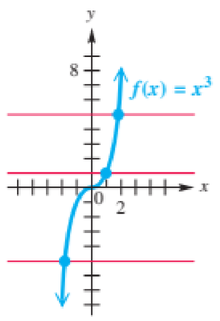
Inverse Function

The **inverse function** of a one-to-one function f is defined as follows.

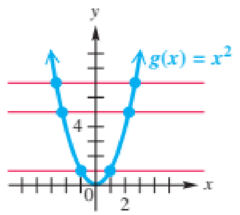
$$f^{-1} = \{(y, x) \mid (x, y) \text{ belongs to } f\}$$

The special notation used for inverse functions is f^{-1} (read **"f-inverse"**). It represents the function created by interchanging the input (domain) and the output (range) of a one-to-one function.

CAUTION ***Do not confuse the -1 in f^{-1} with a negative exponent.*** The symbol $f^{-1}(x)$ represents the inverse function of f, *not* $\frac{1}{f(x)}$.



$f(x) = x^3$ is a one-to-one function. It satisfies the conditions of the horizontal line test.



$g(x) = x^2$ is not one-to-one. It does not satisfy the conditions of the horizontal line test.

Figure 1

The following statements summarize the concepts of inverse functions.

Summary of Inverse Functions

1. In a one-to-one function, each x-value corresponds to only one y-value and each y-value corresponds to only one x-value.
2. If a function f is one-to-one, then f has an inverse function f^{-1}.
3. The domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1}. That is, if the point (a, b) lies on the graph of f, then the point (b, a) lies on the graph of f^{-1}.
4. The graphs of f and f^{-1} are reflections of each other across the line $y = x$.
5. To find $f^{-1}(x)$ for $f(x)$, follow these steps.

   Step 1 Replace $f(x)$ with y and interchange x and y.

   Step 2 Solve for y.

   Step 3 Replace y with $f^{-1}(x)$.

Inverse Sine Function Refer to the graph of the sine function in **Figure 4** on the next page. Applying the horizontal line test, we see that $y = \sin x$ does not define a one-to-one function. If we restrict the domain to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is the part of the graph in **Figure 4** shown in color, this restricted function is one-to-one and has an inverse function. The range of $y = \sin x$ is $[-1, 1]$, so the domain of the inverse function will be $[-1, 1]$, and its range will be $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

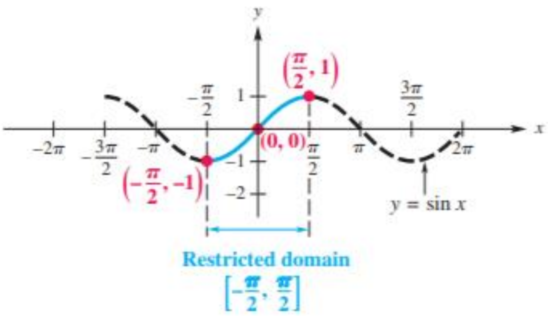


Figure 4

Reflecting the graph of $y = \sin x$ on the restricted domain, shown in **Figure 5(a),** across the line $y = x$ gives the graph of the inverse function, shown in **Figure 5(b).** Some key points are labeled on the graph. The equation of the inverse of $y = \sin x$ is found by interchanging x and y to obtain

$$x = \sin y.$$

This equation is solved for y by writing

$$\mathbf{y = \sin^{-1} x} \quad \text{(read \textbf{"inverse sine of } \boldsymbol{x}\textbf{"}).}$$

As **Figure 5(b)** shows, the domain of $y = \sin^{-1} x$ is $[-1, 1]$, while the restricted domain of $y = \sin x$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, is the range of $y = \sin^{-1} x$. ***An alternative notation for*** $\sin^{-1} x$ ***is*** $\arcsin x$.

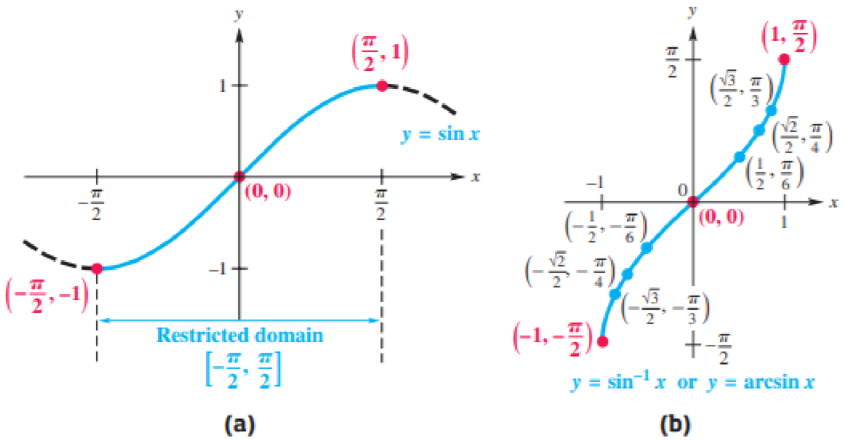


Figure 5

Inverse Sine Function

$y = \sin^{-1} x$ or $y = \arcsin x$ means that $x = \sin y$, for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

We can think of $y = \sin^{-1} x$ ***or*** $y = \arcsin x$ ***as***
"y is the number (angle) in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ ***whose sine is x."***

Thus, we can write $y = \sin^{-1} x$ as $\sin y = x$ to evaluate it. We must pay close attention to the domain and range intervals.

EXAMPLE 1 Finding Inverse Sine Values

Find the value of each real number y if it exists.

(a) $y = \arcsin \frac{1}{2}$ **(b)** $y = \sin^{-1}(-1)$ **(c)** $y = \sin^{-1}(-2)$

ALGEBRAIC SOLUTION

(a) The graph of the function defined by $y = \arcsin x$ (**Figure 5(b)**) includes the point $\left(\frac{1}{2}, \frac{\pi}{6}\right)$. Therefore, $\arcsin \frac{1}{2} = \frac{\pi}{6}$.

Alternatively, we can think of $y = \arcsin \frac{1}{2}$ as "y is the number in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $\frac{1}{2}$." Then we can write the given equation as $\sin y = \frac{1}{2}$. Because $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\frac{\pi}{6}$ is in the range of the arcsine function, $y = \frac{\pi}{6}$.

(b) Writing the equation $y = \sin^{-1}(-1)$ in the form $\sin y = -1$ shows that $y = -\frac{\pi}{2}$. Notice that the point $\left(-1, -\frac{\pi}{2}\right)$ is on the graph of $y = \sin^{-1} x$.

(c) Because -2 is not in the domain of the inverse sine function, $\sin^{-1}(-2)$ does not exist.

CAUTION In **Example 1(b),** it is tempting to give the value of $\sin^{-1}(-1)$ as $\frac{3\pi}{2}$ because $\sin \frac{3\pi}{2} = -1$. However, $\frac{3\pi}{2}$ is not in the range of the inverse sine function. ***Be certain that the number given for an inverse function value is in the range of the particular inverse function being considered.***

Inverse Sine Function $y = \sin^{-1} x$ or $y = \arcsin x$

Domain: $[-1, 1]$ Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

x	y
-1	$-\frac{\pi}{2}$
$-\frac{\sqrt{2}}{2}$	$-\frac{\pi}{4}$
0	0
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$
1	$\frac{\pi}{2}$

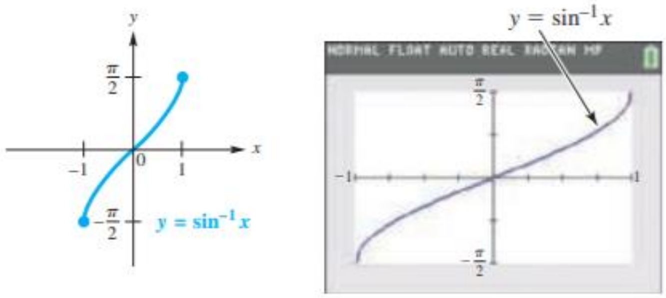


Figure 7

- The inverse sine function is increasing on the open interval $(-1, 1)$ and continuous on its domain $[-1, 1]$.
- Its x- and y-intercepts are both $(0, 0)$.
- Its graph is symmetric with respect to the origin, so the function is an odd function. For all x in the domain, $\sin^{-1}(-x) = -\sin^{-1} x$.

Inverse Cosine Function The function

$$y = \cos^{-1} x \quad \text{or} \quad y = \arccos x$$

is defined by restricting the domain of the function $y = \cos x$ to the interval $[0, \pi]$ as in **Figure 8.** This restricted function, which is the part of the graph in **Figure 8** shown in color, is one-to-one and has an inverse function. The inverse function, $y = \cos^{-1} x$, is found by interchanging the roles of x and y. Reflecting the graph of $y = \cos x$ across the line $y = x$ gives the graph of the inverse function shown in **Figure 9.** Some key points are shown on the graph.

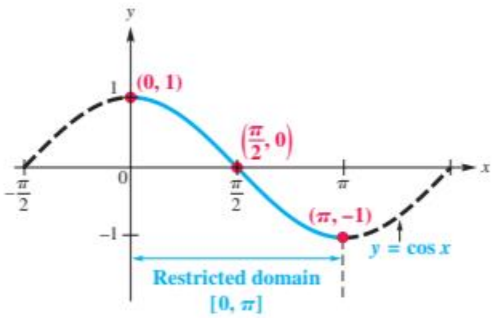


Figure 8

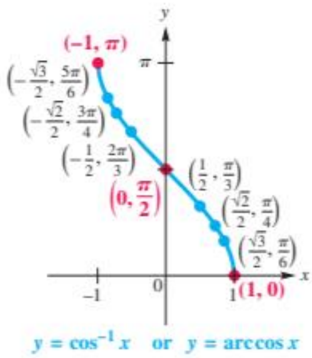


Figure 9

Inverse Cosine Function

$y = \cos^{-1} x$ or $y = \arccos x$ means that $x = \cos y$, for $0 \le y \le \pi$.

We can think of $y = \cos^{-1} x$ or $y = \arccos x$ as

"y is the number (angle) in the interval $[0, \pi]$ whose cosine is x."

EXAMPLE 2 Finding Inverse Cosine Values

Find the value of each real number y if it exists.

(a) $y = \arccos 1$ **(b)** $y = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

SOLUTION

(a) Because the point $(1, 0)$ lies on the graph of $y = \arccos x$ in **Figure 9,** the value of y, or arccos 1, is 0. Alternatively, we can think of $y = \arccos 1$ as

"y is the number in $[0, \pi]$ whose cosine is 1," or $\cos y = 1$.

Thus, $y = 0$, since $\cos 0 = 1$ and 0 is in the range of the arccosine function.

(b) We must find the value of y that satisfies

$$\cos y = -\frac{\sqrt{2}}{2}, \quad \text{where } y \text{ is in the interval } [0, \pi],$$

which is the range of the function $y = \cos^{-1} x$. The only value for y that satisfies these conditions is $\frac{3\pi}{4}$. Again, this can be verified from the graph in **Figure 9.**

Inverse Cosine Function $y = \cos^{-1} x$ or $y = \arccos x$

Domain: $[-1, 1]$ Range: $[0, \pi]$

x	y
-1	π
$-\frac{\sqrt{2}}{2}$	$\frac{3\pi}{4}$
0	$\frac{\pi}{2}$
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$
1	0

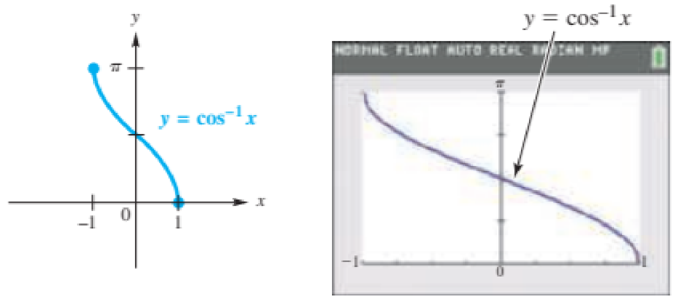


Figure 10

- The inverse cosine function is decreasing on the open interval $(-1, 1)$ and continuous on its domain $[-1, 1]$.
- Its x-intercept is $(1, 0)$ and its y-intercept is $\left(0, \frac{\pi}{2}\right)$.
- Its graph is not symmetric with respect to either the y-axis or the origin.

Inverse Tangent Function Restricting the domain of the function $y = \tan x$ to the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ yields a one-to-one function. By interchanging the roles of x and y, we obtain the inverse tangent function given by

$$y = \tan^{-1} x \quad \text{or} \quad y = \arctan x.$$

Figure 11 shows the graph of the restricted tangent function. **Figure 12** gives the graph of $y = \tan^{-1} x$.

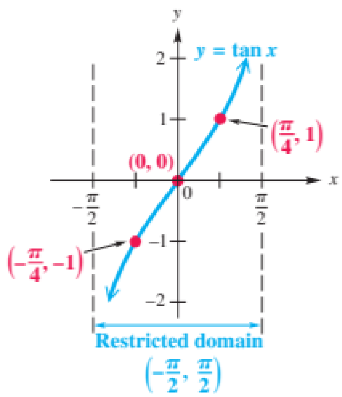


Figure 11

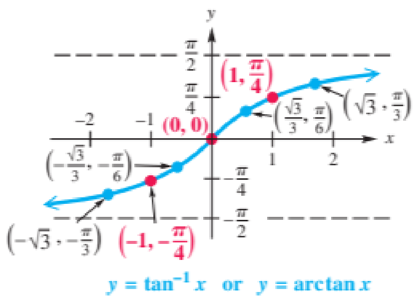


Figure 12

Inverse Tangent Function

$y = \tan^{-1} x$ or $y = \arctan x$ means that $x = \tan y$, for $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

We can think of $y = \tan^{-1} x$ or $y = \arctan x$ as

"y is the number (angle) in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is x."

Inverse Tangent Function $y = \tan^{-1} x$ or $y = \arctan x$

Domain: $(-\infty, \infty)$ Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

x	y
-1	$-\frac{\pi}{4}$
$-\frac{\sqrt{3}}{3}$	$-\frac{\pi}{6}$
0	0
$\frac{\sqrt{3}}{3}$	$\frac{\pi}{6}$
1	$\frac{\pi}{4}$

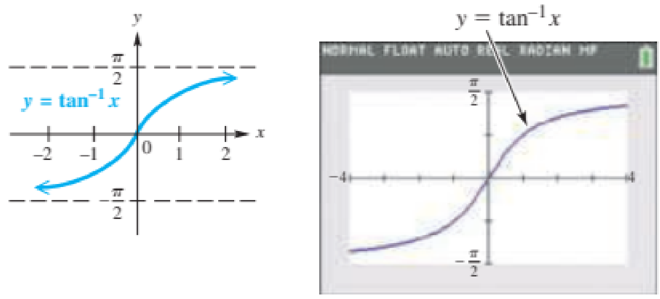


Figure 13

- The inverse tangent function is increasing on $(-\infty, \infty)$ and continuous on its domain $(-\infty, \infty)$.
- Its x- and y-intercepts are both $(0, 0)$.
- Its graph is symmetric with respect to the origin, so the function is an odd function. For all x in the domain, $\tan^{-1}(-x) = -\tan^{-1} x$.
- The lines $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are horizontal asymptotes.

Other Inverse Circular Functions The other three inverse trigonometric functions are defined similarly. Their graphs are shown in **Figure 14.**

Inverse Cotangent, Secant, and Cosecant Functions*

$y = \cot^{-1} x$ or $y = \operatorname{arccot} x$ means that $x = \cot y$, for $0 < y < \pi$.

$y = \sec^{-1} x$ or $y = \operatorname{arcsec} x$ means that $x = \sec y$, for $0 \le y \le \pi$, $y \ne \frac{\pi}{2}$.

$y = \csc^{-1} x$ or $y = \operatorname{arccsc} x$ means that $x = \csc y$, for $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$.

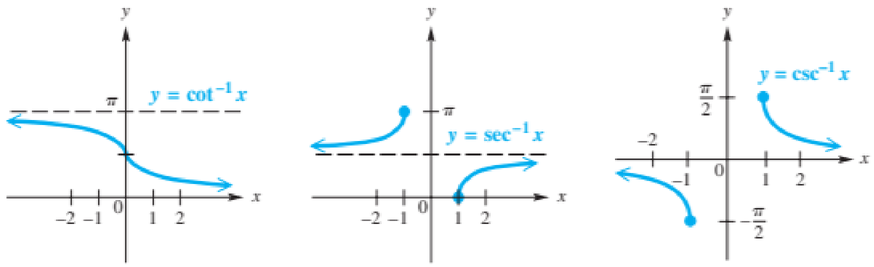


Figure 14

The table gives all six inverse circular functions with their domains and ranges.

Summary of Inverse Circular Functions

Inverse Function	Domain	Range: Interval	Range: Quadrants of the Unit Circle
$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	I and IV
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	I and II
$y = \tan^{-1} x$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	I and IV
$y = \cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$	I and II
$y = \sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$	I and II
$y = \csc^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$	I and IV

Inverse Function Values The inverse circular functions are formally defined with real number ranges. However, there are times when it may be convenient to find degree-measured angles equivalent to these real number values. It is also often convenient to think in terms of the unit circle and choose the inverse function values on the basis of the quadrants given in the preceding table.

EXAMPLE 3 Finding Inverse Function Values (Degree-Measured Angles)

Find the *degree measure* of θ if it exists.

(a) $\theta = \arctan 1$ **(b)** $\theta = \sec^{-1} 2$

SOLUTION

(a) Here θ must be in $(-90°, 90°)$, but because 1 is positive, θ must be in quadrant I. The alternative statement, $\tan \theta = 1$, leads to $\theta = 45°$.

(b) Write the equation as $\sec \theta = 2$. For $\sec^{-1} x$, θ is in quadrant I or II. Because 2 is positive, θ is in quadrant I and $\theta = 60°$, since $\sec 60° = 2$. Note that $60°$ $\left(\text{the degree equivalent of } \frac{\pi}{3}\right)$ is in the range of the inverse secant function.

✔ **Now Try Exercises 37 and 45.**

The inverse trigonometric function keys on a calculator give correct results for the inverse sine, inverse cosine, and inverse tangent functions.

$$\sin^{-1} 0.5 = 30°, \qquad \sin^{-1}(-0.5) = -30°,$$
$$\tan^{-1}(-1) = -45°, \quad \text{and} \quad \cos^{-1}(-0.5) = 120° \qquad \text{Degree mode}$$

However, finding $\cot^{-1} x$, $\sec^{-1} x$, and $\csc^{-1} x$ with a calculator is not as straightforward because these functions must first be expressed in terms of $\tan^{-1} x$, $\cos^{-1} x$, and $\sin^{-1} x$, respectively. If $y = \sec^{-1} x$, for example, then we have $\sec y = x$, which must be written in terms of cosine as follows.

$$\text{If} \quad \sec y = x, \quad \text{then} \quad \frac{1}{\cos y} = x, \quad \text{or} \quad \cos y = \frac{1}{x}, \quad \text{and} \quad y = \cos^{-1} \frac{1}{x}.$$

Use the following to evaluate these inverse functions on a calculator.

$$\sec^{-1} x \textit{ is evaluated as } \cos^{-1}\frac{1}{x}; \quad \csc^{-1} x \textit{ is evaluated as } \sin^{-1}\frac{1}{x};$$

$$\cot^{-1} x \textit{ is evaluated as } \begin{cases} \tan^{-1}\frac{1}{x} & \text{if } x > 0 \\ 180^\circ + \tan^{-1}\frac{1}{x} & \text{if } x < 0. \end{cases} \quad \text{Degree mode}$$

EXAMPLE 4 Finding Inverse Function Values with a Calculator

Use a calculator to approximate each value.

(a) Find y in radians if $y = \csc^{-1}(-3)$.

(b) Find θ in degrees if $\theta = \text{arccot}(-0.3541)$.

SOLUTION

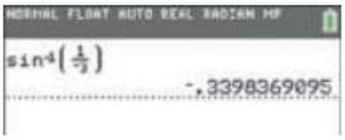
(a) With the calculator in radian mode, enter $\csc^{-1}(-3)$ as $\sin^{-1}\left(\frac{1}{-3}\right)$ to obtain $y \approx -0.3398369095$. See **Figure 15(a).**

(b) A calculator in degree mode gives the inverse tangent value of a negative number as a quadrant IV angle. The restriction on the range of arccotangent implies that θ must be in quadrant II.

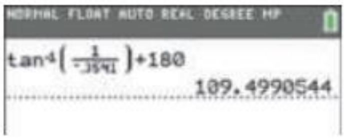
$$\text{arccot}(-0.3541) \quad \text{is entered as} \quad \tan^{-1}\left(\frac{1}{-0.3541}\right) + 180^\circ.$$

As shown in **Figure 15(b),**

$$\theta \approx 109.4990544^\circ.$$



(a)



(b)

Figure 15

CAUTION ***Be careful when using a calculator to evaluate the inverse cotangent of a negative quantity.*** Enter the inverse tangent of the *reciprocal* of the negative quantity, which returns an angle in quadrant IV. Because inverse cotangent is negative in quadrant II, adjust the calculator result by adding π or 180° accordingly. (Note that $\cot^{-1} 0 = \frac{\pi}{2}$ or 90°.)

EXAMPLE 5 Finding Function Values Using Definitions of the Trigonometric Functions

Evaluate each expression without using a calculator.

(a) $\sin\left(\tan^{-1}\frac{3}{2}\right)$ **(b)** $\tan\left(\cos^{-1}\left(-\frac{5}{13}\right)\right)$

SOLUTION

(a) Let $\theta = \tan^{-1}\frac{3}{2}$, and thus $\tan\theta = \frac{3}{2}$. The inverse tangent function yields values only in quadrants I and IV, and because $\frac{3}{2}$ is positive, θ is in quadrant I. Sketch θ in quadrant I, and label a triangle, as shown in **Figure 16** on the next page. By the Pythagorean theorem, the hypotenuse is $\sqrt{13}$. The value of sine is the quotient of the side opposite and the hypotenuse.

$$\sin\left(\tan^{-1}\frac{3}{2}\right) = \sin\theta = \frac{3}{\sqrt{13}} = \frac{3}{\sqrt{13}}\cdot\frac{\sqrt{13}}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$$

Rationalize the denominator.

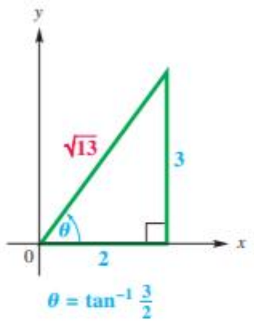


Figure 16

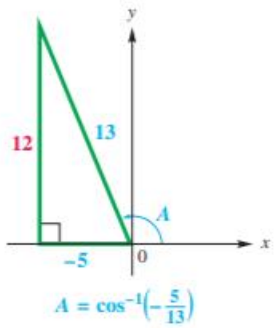


Figure 17

(b) Let $A = \cos^{-1}\left(-\frac{5}{13}\right)$. Then, $\cos A = -\frac{5}{13}$. Because $\cos^{-1}x$ for a negative value of x is in quadrant II, sketch A in quadrant II. See **Figure 17.**

$$\tan\left(\cos^{-1}\left(-\frac{5}{13}\right)\right) = \tan A = -\frac{12}{5}$$

EXAMPLE 6 Finding Function Values Using Identities

Evaluate each expression without using a calculator.

(a) $\cos\left(\arctan\sqrt{3} + \arcsin\frac{1}{3}\right)$ **(b)** $\tan\left(2\arcsin\frac{2}{5}\right)$

SOLUTION

(a) Let $A = \arctan\sqrt{3}$ and $B = \arcsin\frac{1}{3}$. Therefore, $\tan A = \sqrt{3}$ and $\sin B = \frac{1}{3}$. Sketch both A and B in quadrant I, as shown in **Figure 18,** and use the Pythagorean theorem to find the unknown side in each triangle.

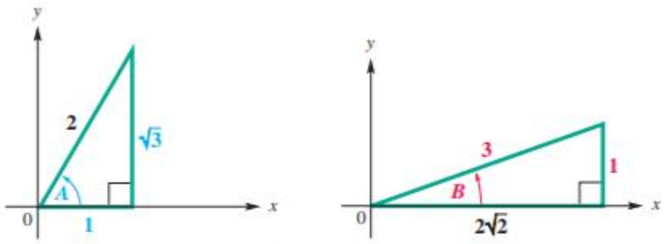


Figure 18

$$\cos\left(\arctan\sqrt{3} + \arcsin\frac{1}{3}\right) \qquad \text{Given expression}$$

$$= \cos(A + B) \qquad \text{Let } A = \arctan\sqrt{3} \text{ and } B = \arcsin\tfrac{1}{3}.$$

$$= \cos A \cos B - \sin A \sin B \qquad \text{Cosine sum identity}$$

$$= \frac{1}{2} \cdot \frac{2\sqrt{2}}{3} - \frac{\sqrt{3}}{2} \cdot \frac{1}{3} \qquad \text{Substitute values using Figure 18.}$$

$$= \frac{2\sqrt{2} - \sqrt{3}}{6} \qquad \text{Multiply and write as a single fraction.}$$

(b) Let $B = \arcsin\frac{2}{5}$, so that $\sin B = \frac{2}{5}$. Sketch angle B in quadrant I, and use the Pythagorean theorem to find the length of the third side of the triangle. See **Figure 19.**

$$\tan\left(2\arcsin\frac{2}{5}\right) \qquad \text{Given expression}$$

$$= \frac{2\left(\frac{2}{\sqrt{21}}\right)}{1 - \left(\frac{2}{\sqrt{21}}\right)^2} \qquad \text{Use } \tan 2B = \tfrac{2\tan B}{1-\tan^2 B} \text{ with } \tan B = \tfrac{2}{\sqrt{21}} \text{ from Figure 19.}$$

$$= \frac{\frac{4}{\sqrt{21}}}{1 - \frac{4}{21}} \qquad \text{Multiply and apply the exponent.}$$

$$= \frac{\frac{4}{\sqrt{21}} \cdot \frac{\sqrt{21}}{\sqrt{21}}}{\frac{17}{21}} \qquad \text{Rationalize in the numerator. Subtract in the denominator.}$$

$$= \frac{\frac{4\sqrt{21}}{21}}{\frac{17}{21}} \qquad \text{Multiply in the numerator.}$$

$$= \frac{4\sqrt{21}}{17} \qquad \text{Divide; } \tfrac{\frac{a}{b}}{\frac{c}{d}} = \tfrac{a}{b} \div \tfrac{c}{d} = \tfrac{a}{b} \cdot \tfrac{d}{c}.$$

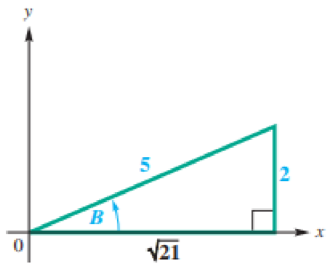


Figure 19

While the work shown in **Examples 5 and 6** does not rely on a calculator, we can use one to support our algebraic work. By entering $\cos\left(\arctan\sqrt{3}+\arcsin\frac{1}{3}\right)$ from **Example 6(a)** into a calculator, we find the approximation 0.1827293862, the same approximation as when we enter $\frac{2\sqrt{2}-\sqrt{3}}{6}$ (the exact value we obtained algebraically). Similarly, we obtain the same approximation when we evaluate $\tan\left(2\arcsin\frac{2}{5}\right)$ and $\frac{4\sqrt{21}}{17}$, supporting our answer in **Example 6(b).**

Write each trigonometric expression as an algebraic expression in u.

(a) $\sin(\tan^{-1} u)$ **(b)** $\cos(2\sin^{-1} u)$

SOLUTION

(a) Let $\theta = \tan^{-1} u$, so $\tan\theta = u$. Because

$$-\frac{\pi}{2} < \tan^{-1} u < \frac{\pi}{2},$$

sketch θ in quadrants I and IV and label two triangles, as shown in **Figure 20.** Sine is given by the quotient of the side opposite and the hypotenuse, so we have the following.

$$\sin(\tan^{-1} u) = \sin\theta = \frac{u}{\sqrt{u^2+1}} = \frac{u}{\sqrt{u^2+1}} \cdot \frac{\sqrt{u^2+1}}{\sqrt{u^2+1}} = \frac{u\sqrt{u^2+1}}{u^2+1}$$

Rationalize the denominator.

The result is positive when u is positive and negative when u is negative.

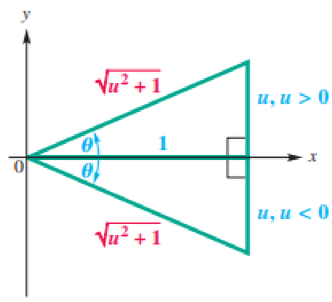


Figure 20

(b) Let $\theta = \sin^{-1} u$, so that $\sin\theta = u$. To find $\cos 2\theta$, use the double-angle identity $\cos 2\theta = 1 - 2\sin^2\theta$.

$$\cos(2\sin^{-1} u) = \cos 2\theta = 1 - 2\sin^2\theta = 1 - 2u^2$$

EXAMPLE 8 Finding Optimal Angle of Elevation of a Shot Put

The optimal angle of elevation θ for a shot-putter to achieve the greatest distance depends on the velocity v of the throw and the initial height h of the shot. See **Figure 21.** One model for θ that attains this greatest distance is

$$\theta = \arcsin\left(\sqrt{\frac{v^2}{2v^2 + 64h}}\right).$$

(*Source*: Townend, M. S., *Mathematics in Sport*, Chichester, Ellis Horwood Ltd.)

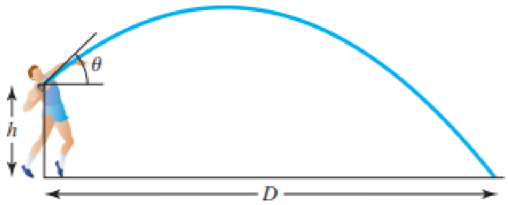


Figure 21

An athlete can consistently put the shot with $h = 6.6$ ft and $v = 42$ ft per sec. At what angle should he release the ball to maximize distance?

SOLUTION To find this angle, substitute and use a calculator in degree mode.

$$\theta = \arcsin\left(\sqrt{\frac{42^2}{2(42^2) + 64(6.6)}}\right) \approx 42^\circ \quad \text{Use } h = 6.6,\ v = 42, \text{ and a calculator.}$$

PROGRESS CHECK

CONCEPT PREVIEW *Fill in the blank(s) to correctly complete each sentence.*

1. For a function to have an inverse, it must be ________-to-________.
2. The domain of $y = \arcsin x$ equals the ________ of $y = \sin x$.
3. $y = \cos^{-1} x$ means that $x =$ ________ for $0 \le y \le \pi$.
4. The point $\left(\frac{\pi}{4}, 1\right)$ lies on the graph of $y = \tan x$. Therefore, the point ________ lies on the graph of $y = \tan^{-1} x$.
5. If a function f has an inverse and $f(\pi) = -1$, then $f^{-1}(-1) =$ ________.
6. To evaluate $\sec^{-1} x$, use the value of $\cos^{-1} \dfrac{1}{\_\_\_\_}$.

CONCEPT PREVIEW *Write a short answer for each of the following.*

7. Consider the inverse sine function $y = \sin^{-1} x$, or $y = \arcsin x$.
   (a) What is its domain? **(b)** What is its range?
   (c) Is this function increasing or decreasing?
   (d) Why is $\arcsin(-2)$ not defined?

8. Consider the inverse cosine function $y = \cos^{-1} x$, or $y = \arccos x$.

(a) What is its domain?

(b) What is its range?

(c) Is this function increasing or decreasing?

(d) $\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$. Why is $\arccos\left(-\frac{1}{2}\right)$ not equal to $-\frac{4\pi}{3}$?

9. Consider the inverse tangent function $y = \tan^{-1} x$, or $y = \arctan x$.

(a) What is its domain?

(b) What is its range?

(c) Is this function increasing or decreasing?

(d) Is there any real number x for which $\arctan x$ is not defined? If so, what is it (or what are they)?

10. Give the domain and range of each inverse trigonometric function, as defined in this section.

(a) inverse cosecant function

(b) inverse secant function

(c) inverse cotangent function

Find the exact value of each real number y if it exists. Do NOT use a calculator.

1. $y = \arccos(-\frac{\sqrt{3}}{2})$
2. $y = \arcsin(-\sqrt{2})$
3. $y = \mathrm{arcsec}(\frac{2\sqrt{3}}{3})$

Give the degree measure of θ if it exits. Do NOT use a calculator.

1. $\theta = \arctan(\sqrt{3})$
2. $\theta = csc^{-1}(\frac{\sqrt{3}}{3})$
3. $\theta = \mathrm{arccot}(\frac{\sqrt{3}}{3})$

Use a calculator to approximate each value in decimal degrees.

1. $\theta = \arcsin(-0.13349122)$
2. $\theta = csc^{-1}(1.9422833)$
3. $\theta = \mathrm{arcsec}(-5.1180378)$

Use a calculator to approximate each real number value. (Be sure the calculator is in radian mode.)

1. $y = \arcsin(0.92837781)$
2. $y = cot^{-1}(-0.92170128)$
3. $y = \mathrm{arcsec}(-4.7963825)$

Graph each inverse circular function by hand.

1. $y = \text{arccsc}(2x)$
2. $y = \text{arcsec}(0.5x)$

Evaluate each expression without using a calculator.

1. $\tan(\arccos \frac{3}{4})$
2. $\cos(2\arctan(-2))$
3. $\sin(\arcsin\left(\frac{1}{2}\right) + \arctan(-3))$

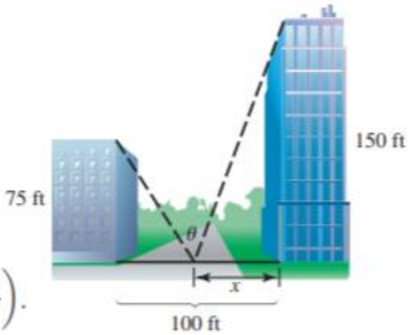
Write each trigonometric expression as an algebraic expression in u, for u > 0.

1. $\sin(\arccos u)$
2. $\cot(2\arcsin u)$
3. $\sec(\text{arcscpt}\left(\frac{\sqrt{4-u^2}}{u}\right))$

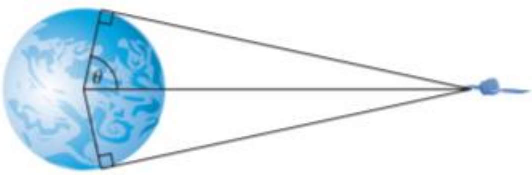
Applications:

Landscaping Formula A shrub is planted in a 100-ft-wide space between buildings measuring 75 ft and 150 ft tall. The location of the shrub determines how much sun it receives each day. Show that if θ is the angle in the figure and x is the distance of the shrub from the taller building, then the value of θ (in radians) is given by

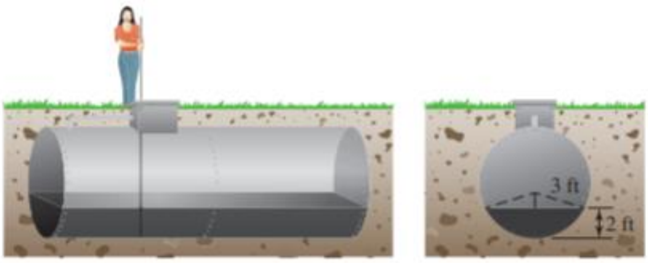
$$\theta = \pi - \arctan\left(\frac{75}{100 - x}\right) - \arctan\left(\frac{150}{x}\right).$$



Communications Satellite Coverage The figure shows a stationary communications satellite positioned 20,000 mi above the equator. What percent, to the nearest tenth, of the equator can be seen from the satellite? The diameter of Earth is 7927 mi at the equator.



Oil in a Storage Tank The level of oil in a storage tank buried in the ground can be found in much the same way as a dipstick is used to determine the oil level in an automobile crankcase. Suppose the ends of the cylindrical storage tank in the figure are circles of radius 3 ft and the cylinder is 20 ft long. Determine the volume of oil in the tank to the nearest cubic foot if the rod shows a depth of 2 ft. (*Hint*: The volume will be 20 times the area of the shaded segment of the circle shown in the figure on the right.)



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Lial, M. L., Hornsby, J., & Schneider, D. I. (2009). *Trigonometry Ninth Edition.* Quezon City, Philippines: C&E Publishing, Inc.