Homework 4

萃英学院　2022 级　王一鑫

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Problem 1. (Exercise 2.38)

Find the shortest path from S to each other vertex in the weighted graph of Fig 1.

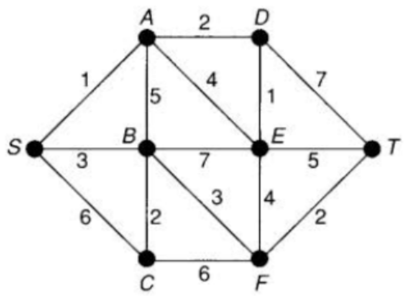


图 1: Weighted Graph

Solution.

Apply Dijkstra Algorithm to the weighted graph, we can label the graph as Fig 2.

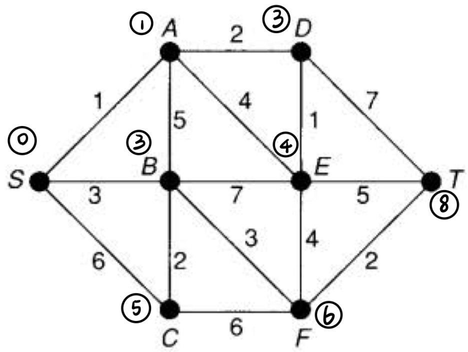


图 2: Labelled Graph

From the labelled graph we can see the shortest path from S to each other vertex.

(i) A. It's $S \to A$, which length is 1.

(ii) B. It's $S \to B$, which length is 3.

(iii) C. It's $S \to B \to C$, which length is 5.

(iv) D. It's $S \to A \to D$, which length is 3.

(v) E. It's $S \to A \to D \to E$, which length is 4.

(vi) F. It's $S \to B \to F$, which length is 6.

(vii) T. It's $S \to B \to F \to T$, which length is 8.

$\square$

PROBLEM 2. (Exercise 2.40) Find the longest path from A to G in the network of Fig 3.

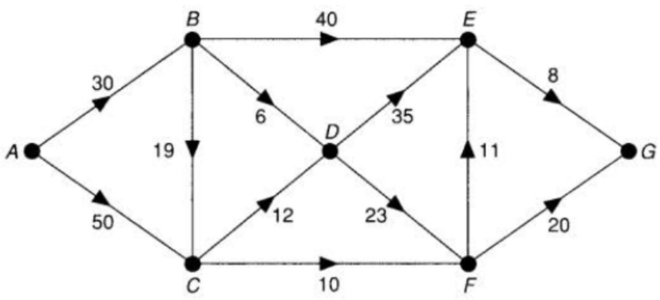


图 3: Network

Solution.

We assign as follows:

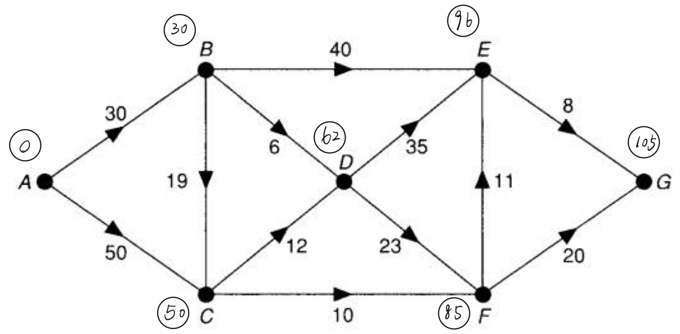


图 4: Labelled Graph

to vertex A, the number 0;

to vertex B, the number $l(A) + 30$, that is, 30;

to vertex C, the largest of the numbers $l(A) + 50$ and $l(B) + 19$, that is, 50;

to vertex D, the largest of the numbers $l(B) + 6$ and $l(C) + 12$, that is, 62;

to vertex F, the largest of the numbers $l(C) + 10$ and $l(D) + 23$, that is, 85;

to vertex E, the largest of the numbers $l(F)+11$, $l(B)+40$ and $l(D)+35$, that is, 96;

to vertex G, the largest of the numbers $l(E)+8$ and $l(F)+20$, that is, 105.

Then the longest path from A to G is $A \to C \to D \to F \to G$, and the length is 105. The whole process is shown as Fig 4.

□

PROBLEM 3. (Exercise 2.43)

Find the Hamiltonian cycle of greatest weight in the graph of Fig 5.

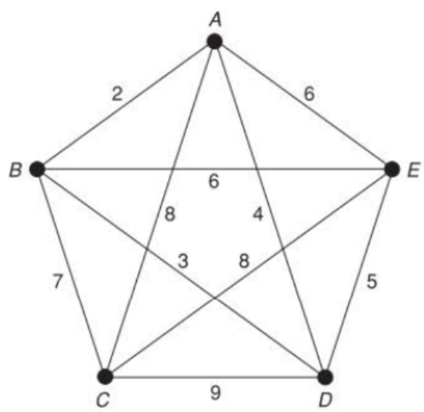


图 5: Graph for problem 3

SOLUTION.

There are $\frac{1}{2} \times 4! = 12$ possible cycles in the graph. After trying all the different Hamiltonian cycle, I find the cycle of greatest weight is

$$A \to C \to D \to B \to E \to A$$

and the weight is 32.

□

PROBLEM 4. (Exercise 2.45)

Let G be a connected graph with vertex-set $\{v_1, v_2, \ldots, v_n\}$, m edges and t triangles.

(i) If A is the adjacency matrix of G, prove that the number of walks of length 2 from v_i to v_j is the ijth entry of the matrix A^2. Deduce that $2m =$ the sum of the diagonal entries of A^2.

(ii) Obtain a corresponding result for the number of walks of length 3 from v_i to v_j, and deduce that $6t =$ the sum of the diagonal entries of A^3.

SOLUTION.

(i) Recall that if $A = (a_{ij})$ is the adjacency matrix of a graph G with n vertices $v_1, v_2, \ldots, v_n$, then A is an $n \times n$ symmetric matrix, and a_{ij} is the number of edges joining v_i, v_j (0 or 1 for a simple graph), i.e., the number of walks of length 1 from v_i to v_j. Since

$$(A^2)_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj}$$

We know that $(A^2)_{ij}$ is the number of walks of length 2 from v_i to v_j by definition.

$(A^2)_{ii}$ is the number of walks of length 2 starting at v_i and finishing at v_i. In a simple graph, we can only get back to the starting vertex in two steps by going to an adjacent vertex and back, and then we have the following equation.

$$\begin{aligned}(A^2)_{ii} &= a_{i1}a_{1i} + a_{i2}a_{2i} + \cdots + a_{in}a_{ni} \\ &= a_{i1}^2 + a_{i2}^2 + \cdots + a_{in}^2 \\ &= (0 \text{ or } 1) + (0 \text{ or } 1) + \cdots + (0 \text{ or } 1) \\ &= \text{the number of vertices } v_j \text{ adjacent to } v_i \\ &= \text{the degree of } v_i.\end{aligned}$$

Sum up on both sides we get

$$\operatorname{tr}(A^2) = \sum_{i=1}^{n} \deg(v_i) = 2m$$

(ii) Similarly we have

$$(A^3)_{ij} = \sum_{l=1}^{n}\sum_{k=1}^{n} a_{ik}a_{kl}a_{lj}$$

Thus the ijth entry of the matrix A^3 is the number of walks of length 3 from v_i to v_j.

$(A^3)_{ii}$ is the number of walks of length 3 starting at v_i and finishing at v_i. In a simple graph, we can only get back to the starting vertex in three steps by going to two adjacent vertices and back. Thus, $(A^3)_{ii}$ is the number of triangles which contains the vertex v_i without the order. For example, $v_iv_jv_k$ generates a triangle if $v_iv_jv_k$ are adjacent. When we count the number of triangles, since there are 6 permutations for $v_iv_jv_k$, every triangles will be counted 6 times. Therefore

$$\operatorname{tr}(A^3) = 6t$$

□

Problem 5. (Exercise 2.46)

(i) Prove that, if two distinct cycles of a graph G each contain an edge e, then G has a cycle that does not contain e.

(ii) Prove a similar result with 'cycles' replaced by 'cutsets'.

Solution.

(i) Given a graph $G = R \cup B$ composed of two simple cycles R and B, we can observe the behavior locally at each vertex. Each vertex will fall into one of the following five cases shown as Fig 6.

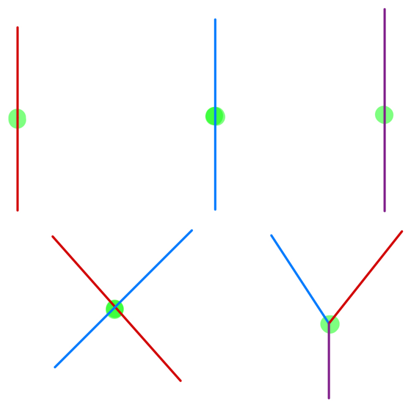


图 6: 5 Cases

where red edges belong to R and blue edges belong to B, the purple ones are shared.

We're given that a purple edge e exists and that $R \neq B$. Modify G by deleting all purple edges, which includes e. W're left with a non-empty graph since not all edges are shared. And in fact, by observing the effect on the above illustrated cases, we see that all vertices now have **even** degree. Hence, some connected component of our modified graph contains an Eulerian cycle. And furthermore, an Eulerian cycle trivially contains a simple cycle. Since we removed e in the first step, we've found a simple cycle in G not containing e.

(ii) Suppose that there are two cutsets (S_1, T_1) and (S_2, T_2) of G, then WLOG we assume vertices $u \in S_1 \cap S_2$ and $v \in T_1 \cap T_2$. Thus $e = uv$ is contained in both cutsets.

Now if $S_1 \cap T_2 \neq \emptyset$, then we claim that $((S_1 \cap T_2), G \setminus (S_1 \cap T_2))$ is a cutset. In fact, for any vertex $w \in S_1 \cap T_2$, w disconnects with any vertex in S_2 or T_1, which is also in $G \setminus (S_1 \cap T_2)$.

Furthermore, since both u and v are not in $S_1 \cap T_2$, $e \notin (S_1 \cap T_2)$. The statement holds true for $S_2 \cap T_1$, and $S_1 \cap T_2$ and $S_2 \cap T_1$ can not both be empty. Therefore, $((S_1 \cap T_2), G \setminus (S_1 \cap T_2))$ is a cutset that does not contain e.

□

PROBLEM 6. (Exercise 2.47)

(i) Prove that, if C is a cycle and C^* is a cutset of a connected graph G, then C and C^* have an even number of edges in common.

(ii) Prove that, if S is any set of edges of G with an even number of edges in common with each cutset of G, then S can be split into edge-disjoint cycles.

SOLUTION.

(i) Let U and W be the connected disjoint components of $G - C^*$. Then each of the vertices that C passes through is either in U or in W. We claim that an edge $\{v_i, v_{i+1}\}$ is common to C and C^* if and only if $v_i \in U$ and $v_{i+1} \in W$, or $v_i \in W$ and $v_{i+1} \in U$. In fact, if not, then after removing $\{v_i, v_{i+1}\}$ from C^* (i.e., 'putting back' this edge into G) the resulting graph is still disconnected, contradicting the minimality of C^*. Therefore the number of edges common to C and C^* is exactly equal to the number of times C traverses from U to W. Since C is a cycle, it returns to where it has started from, and this number therefore has to be even.

(ii) Let G' be the graph obtained from G by removing all edges of G that are not in S. Then S can be split into edge-disjoint cycles if and only if each of its components is Eulerian. This is equivalent to each vertex of G' having an even degree.

Assume S cannot be split into edge-disjoint cycles. By the above, some vertex v of G' has an odd degree. All vertices of G incident with v form a cutset, denoted as C^*, of G. The edges common to S and C^* are exactly the edges in S incident to v. Their number is equal to the degree of v in G', and therefore odd, a contradiction.

$\square$

Problem 7. (Exercise 2.52)

Prove that the Petersen graph is non-Hamiltonian.

Solution.

The Petersen graph has 10 vertices. If it has a Hamilton Cycle, it must go through each vertex. So then we can assume first that it is C_{10} which has

10 vertices and 10 edges. Now, since Petersen graph has 15 edges, we should add 5 more edges to this cycle.

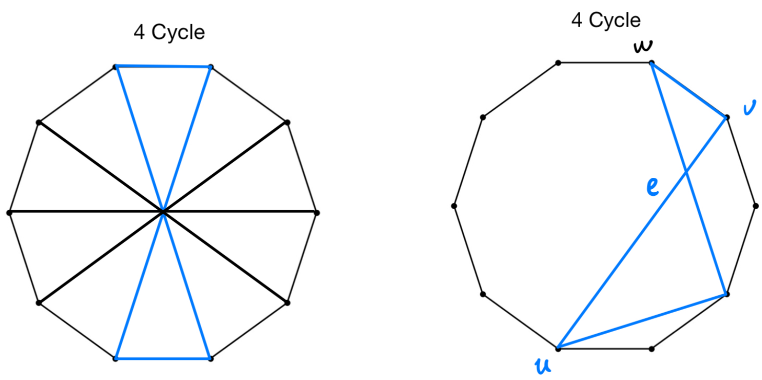
We discuss in different cases:

(i) **Case 1**: If each edge joins vertices opposite on C, then there is a 4 cycle.

(ii) **Case 2**: There exists an edge e joins vertices at distance 4 along C. Notice that no edge incident to a vertex opposite an endpoint of e on C can be added without creating a cycle with at most four vertices. Fig 7 shows an example.

(iii) **Case 3**: otherwise. There must exists a 3 cycle or 4 cycle.

Since Petersen graph has no 3 cycle or 4 cycle. Therefore, the Petersen graph is non-Hamiltonian.



Case 1 Case 2

图 7: Different Cases

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