Homework 11

萃英学院　　2022 级　　王一鑫

2025 年 6 月 6 日

PROBLEM 1. (Exercise 5.41) Let G be a simple graph with an odd number of vertices. Prove that if G is regular of degree Δ, then

$$\chi'(G) = \Delta + 1.$$

SOLUTION. Since G is regular of degree Δ, the number of edges in G is

$$|E| = \frac{\Delta |V|}{2}.$$

Let $|V| = 2m + 1$ be odd, where $m \in \mathbb{N}$. Then

$$|E| = \frac{\Delta(2m+1)}{2}.$$

We consider an edge coloring $c : E \to \{1, \dots, k\}$ using k colors. Assume for contradiction that $\chi'(G) \leq \Delta$, i.e., that G is properly edge-colorable with at most Δ colors.

Then the total number of edges is partitioned among $k \leq \Delta$ color classes. By the pigeonhole principle, there exists at least one color used on at least

$$\left\lceil \frac{|E|}{k} \right\rceil \geq \left\lceil \frac{\Delta(2m+1)/2}{\Delta} \right\rceil = \left\lceil \frac{2m+1}{2} \right\rceil = m + 1$$

edges.

Therefore, at least one color class contains $m+1$ edges. But G has only $2m+1$ vertices, and each color class must consist of pairwise non-adjacent edges (i.e., a matching). The maximum size of a matching in a graph with $2m+1$ vertices is at most m, since each edge uses two distinct vertices. Hence, a matching with $m+1$ edges must involve at least $2(m+1) = 2m+2 > 2m+1$ vertices, which is impossible.

This contradiction implies that G cannot be properly edge-colored using only Δ colors. Thus, we must have $\chi'(G) \geq \Delta + 1$.

On the other hand, Vizing's Theorem tells us that for any simple graph,

$$\chi'(G) \leq \Delta + 1.$$

Therefore, we conclude that $\chi'(G) = \Delta + 1$. □

> Problem 2. (Exercise 6.5)
>
> Prove that, if $G = G(V_1, V_2)$ is a bipartite graph in which the degree of each vertex in V_1 is not less than the degree of each vertex in V_2, then G has a complete matching.

Solution. Let $\varphi(A) \subseteq V_2$ denote the set of neighbors of a subset $A \subseteq V_1$, i.e., those vertices in V_2 adjacent to at least one vertex in A. According to Corollary 6.2, a complete matching from V_1 to V_2 exists if and only if

$$|A| \leq |\varphi(A)| \quad \text{for all subsets } A \subseteq V_1.$$

We will prove this condition holds under the hypothesis that all vertices in V_1 have degree at least as large as those in V_2.

Suppose, for the sake of contradiction, that there exists a subset $A \subseteq V_1$ such that $|A| > |\varphi(A)|$. Let $B = \varphi(A) \subseteq V_2$. Consider the number of edges between A and B. We count these edges in two ways:

On one hand, since each vertex in A has degree at least d_1, and all its neighbors are in B, the total number of edges from A to B is at least

$$e(A, B) \geq |A| \cdot d_1.$$

On the other hand, since the degree of every vertex in V_2 is at most $d_2 \leq d_1$, the total number of edges from B to A is at most

$$e(A, B) \leq |B| \cdot d_2.$$

Combining these, we obtain:

$$|A| \cdot d_1 \leq |B| \cdot d_2.$$

Since $d_1 \geq d_2$, this implies $|A| \leq |B| = |\varphi(A)|$, contradicting our assumption that $|A| > |\varphi(A)|$.

Therefore, for all $A \subseteq V_1$, it must hold that $|A| \leq |\varphi(A)|$. By Corollary 6.2, this implies that a complete matching from V_1 to V_2 exists. □

Problem 3. (Exercise 6.7)

Decide which of the following families of subsets of $E = \{G, R, A, P, H, S\}$ have transversals, find a transversal for those that have them, and list all the partial transversals of those that have no transversal:

(i) $(\{R\}, \{R, G\}, \{A, G\}, \{A, R\})$

(ii) $(\{G, R\}, \{R, P, H\}, \{G, S\}, \{R, H\})$

Solution.

(i) No trasversal.

We now list all possible partial transversals: $\emptyset$, $\{R\}$, $\{G\}$, $\{A\}$, $\{R, G\}$, $\{R, A\}$, $\{G, R\}$, $\{G, R, A\}$.

(ii) Transversal: $\{G, P, S, H\}$.

$\square$

PROBLEM 4. (Exercise 6.9)

Let E be the set $\{1, 2, \ldots, 50\}$. How many different transversals has the family

$$(\{1, 2\}, \{2, 3\}, \{3, 4\}, \ldots, \{50, 1\})?$$

SOLUTION.

There is only one transversal - namely, $\{1, 2, \cdots, 50\}$.

$\square$

PROBLEM 5. (Exercise 6.14)

Verify Theorems 6.5 and 6.6 for the Petersen graph in the two cases:

1. when v and w are adjacent vertices;
2. when v and w are not adjacent.

SOLUTION.

(i) For this case, we verify Theorem 6.5 by counting the maximum number of edge-disjoint paths connecting two distinct adjacent vertices v and w of a connected graph and the minimum number of edges in a vw-disconnecting set. See Figure (1).

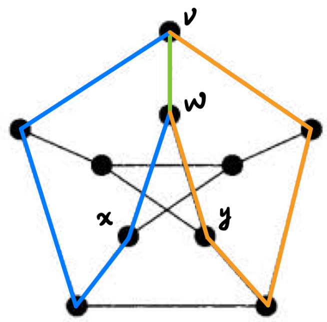

图 1: Edge-disjoint paths of the Petersen Graph

We know from the figure that there are three edge-disjoint paths which are coloured green, blue and orange respectively, and $E = \{vw, xw, yw\}$ is a vw-disconnecting set. Thus Theorem 6.5 holds.

(ii) See Figure (2).

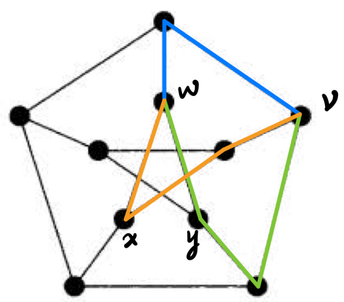

图 2: Edge-disjoint paths of the Petersen Graph

We know from the figure that there are three vertex-disjoint paths which are coloured green, blue and orange respectively, and $V = \{w, x, y\}$ is a vw-separating set. Thus Theorem 6.6 holds.

$\square$

PROBLEM 6.

Compute the number of different perfect matchings of complete bipartite graph $K_{n,n}$ and complete graph K_{2n}.

SOLUTION.

(i) To count the number of perfect matchings in $K_{n,n}$, we proceed as follows. Each perfect matching corresponds to a bijection $f : X \to Y$, where each vertex $x_i \in X$ is matched with a distinct vertex $y_j \in Y$. Since there are $n!$ such bijections from an n-element set to another n-element set, the number of perfect matchings in $K_{n,n}$ is exactly

$$M_{n,n} = n!$$

(ii) Let K_{2n} denote the complete graph on $2n$ vertices. We aim to compute the number of perfect matchings in K_{2n}, denoted by M_{2n}.

To form a perfect matching, we begin by choosing a pair of vertices from the $2n$ available. The number of ways to choose the first vertex is $2n$, and the number of ways to choose its partner is $2n-1$. After forming this pair, we have $2n-2$ vertices remaining. We then select a vertex from these and pair it with one of the remaining $2n-3$ vertices, and so on. Proceeding in this way, we construct n pairs, and the number of such ordered selections is:

$$(2n-1)(2n-3)(2n-5)\cdots 1$$

This is $(2n-1)!!$.

□