Homework 1

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PROBLEM 1. Let $\mathbb{X} \coloneqq C[a,b]$, where $-\infty < a < b < \infty$ and $\|u\| \coloneqq \underset{a\le x\le b}{max} |u(x)|$. Show that $\{u \in \mathbb{X} \colon \int_a^b u(x)\mathrm{d}x = 0\}$ is a closed linear subspace of $\mathbb{X}$. This set is not dense in $\mathbb{X}$.

SOLUTION. (Subspace) First we show that $\mathbb{A} \coloneqq \{u \in \mathbb{X} \colon \int_a^b u(x)\mathrm{d}x = 0\}$ is a linear subspace of $\mathbb{X}$. For $\forall u, v \in \mathbb{A}$, we have

$$\int_a^b u(x) + v(x)\mathrm{d}x = \int_a^b u(x)\mathrm{d}x + \int_a^b v(x)\mathrm{d}x = 0 + 0 = 0$$

thus $u + v \in \mathbb{A}$. For $\forall c \in \mathbb{R}$, $\forall u \in \mathbb{A}$, we have

$$\int_a^b cu(x)\mathrm{d}x = c\int_a^b u(x)\mathrm{d}x = 0$$

then $cu \in \mathbb{A}$.

(Closed) Let $u_n \to u$ as $n \to \infty$, where $u_n \in \mathbb{A}$ for all n. Since $u_n \in \mathbb{A}$ is bounded, by Lebesgue's Dominated Convergence Theorem, we have

$$\int_a^b u(x)\mathrm{d}x = \int_a^b \lim_{n\to\infty} u_n(x)\mathrm{d}x = \lim_{n\to\infty}\int_a^b u_n(x)\mathrm{d}x = 0$$

Then $u \in \mathbb{A}.\mathbb{A}$ is closed.

(Not Dense) Let $v(x) \equiv 1 \in \mathbb{X}$ in $[a, b]$. Choose $\varepsilon_0 = \frac{1}{2}$. Then for $\forall u \in \mathbb{A}$, we have

$$|\int_a^b (u(x) - v(x))\mathrm{d}x| = b - a \leq (b-a)\|u - v\|$$

So

$$\|u - v\| \geq 1 > \varepsilon_0$$

Thus $\mathbb{A}$ is not dense in $\mathbb{X}$.

PROBLEM 2. Show that $\{u \in \mathbb{X}\colon u(a)^2 = u(b)\}$ is a closed subset of $\mathbb{X}$, but not a linear subspace of $\mathbb{X}$.

SOLUTION. (Closed) Let $u_n \to u$ as $n \to \infty$, where $u_n \in \mathbb{A}$ for all n. Since $u_n(a)^2 = u_n(b)$, take $n \to \infty$ on both sides, we have $u(a)^2 = u(b)$.Then $u \in \mathbb{A}$, so $\mathbb{A}$ is closed.

(Not linear) For $\forall u, v \in \mathbb{A}$, we have

$$[(u+v)(a)]^2 = [u(a) + v(a)]^2 = u(a)^2 + v(a)^2 + 2u(a)v(a) \neq u(b) + v(b)$$

Thus $u + v \notin \mathbb{A}$. So $\mathbb{A}$ is not linear.

PROBLEM 3. Show that $\{u \in \mathbb{X}\colon u(a) > 0\}$ is an open, convex, not dense subset of $\mathbb{X}$.

SOLUTION. (Open) $\forall u \in \mathbb{A}$, choose $\varepsilon = \frac{u}{2}$, then $(u - \varepsilon)(a) = \frac{u}{2}(a) > 0$, so $B(u, \varepsilon) \subset \mathbb{A}$, then $\mathbb{A}$ is open.

(Convex) For $\forall u, v \in \mathbb{A}$, $0 \leq \alpha \leq 1$, we have

$$[\alpha u + (1-\alpha)v](a) = \alpha u(a) + (1-\alpha)v(a) > 0$$

Thus $\alpha u + (1-\alpha)v \in \mathbb{A}$. So $\mathbb{A}$ is convex.

(Not Dense) Let $v(x) \equiv -1 \in \mathbb{X}$ in $[a, b]$. Choose $\varepsilon_0 = 1$. Then for $\forall u \in \mathbb{A}$, we have

$$|(u(x) - v(x))(a)| = u(a) + 1 \leq \|u - v\|$$

So

$$\|u - v\| \geq 1 + u(a) > \varepsilon_0$$

Thus $\mathbb{A}$ is not dense in $\mathbb{X}$.

PROBLEM 4. Show that $\{u \in \mathbb{X}\colon u(a) = 1\}$ is a closed, convex, not dense subset of $\mathbb{X}$.

SOLUTION. (Closed) Let $u_n \to u$ as $n \to \infty$, where $u_n \in \mathbb{A}$ for all n. Since $u_n(a)^2 = 1$, take $n \to \infty$ on both sides, we have $u^2(a) = 1$.Then $u \in \mathbb{A}$, so $\mathbb{A}$ is closed.

(Convex) For $\forall u, v \in \mathbb{A}$, $0 \leq \alpha \leq 1$, we have

$$[\alpha u + (1 - \alpha)v](a) = \alpha u(a) + (1 - \alpha)v(a) = \alpha + 1 - \alpha = 1$$

Thus $\alpha u + (1 - \alpha)v \in \mathbb{A}$. So $\mathbb{A}$ is convex.

(Not Dense) Let $v(x) \equiv -1 \in \mathbb{X}$ in $[a, b]$. Choose $\varepsilon_0 = 1$. Then for $\forall u \in \mathbb{A}$, we have

$$|(u(x) - v(x))(a)| = u(a) + 1 \leq \|u - v\|$$

So

$$\|u - v\| \geq 1 + u(a) > \varepsilon_0$$

Thus $\mathbb{A}$ is not dense in $\mathbb{X}$.

Problem 5. Show that $\{u \in \mathbb{X}\colon \|u\| \leq 1\}$ is not a compact subset of $\mathbb{X}$.

Solution. To show that the set

$$\mathbb{A} = \{u \in \mathbb{X} : \|u\| \leq 1\}$$

is not a compact subset of the Banach space $\mathbb{X} = C[a, b]$, we will construct a sequence in $\mathbb{A}$ that does not contain a convergent subsequence.

WLOG, consider the sequence of functions $f_n = x^n \in \mathbb{A}$ defined on $[0, 1]$. Observe that f_n converges pointwise to f.

$$f(x) = \begin{cases} 1, & \text{if } 0 \leq x < 1 \\ 0, & \text{if } x = 1. \end{cases}$$

Since f is not continuous, no subsequence of f_n converges. Then $\mathbb{A}$ is not a compact subset of $\mathbb{X}$.

Problem 6. Show that $\{u \in \mathbb{X}\colon u(x) = 0 \text{ on } [c, d]\}$ is not dense in $\mathbb{X}$ provided $a \leq c \leq d \leq b$.

Solution. Let $v(x) \in \mathbb{X}$ in $[a, b]$ and $v(x) = 1$ in $[c, d]$. Choose $\varepsilon_0 = 1$. Then for $\forall u \in \mathbb{A}$, we have

$$\|u - v\| \geq 1 \geq \varepsilon$$

Thus $\mathbb{A}$ is not dense in $\mathbb{X}$.

PROBLEM 7. Show that $\mathbb{A} = \{u \in \mathbb{X} \colon u(a) \geq 0\}$ is the closure of the set $\mathbb{B} = \{u \in \mathbb{X} \colon u(a) > 0\}$.

SOLUTION. Recall that the closure of a set includes all the points of the set as well as all its limit points. Therefore, we need to show that for any $u \in \mathbb{A}$, there exists a sequence $\{u_n\} \subseteq \mathbb{B}$ such that $u_n \to u$ with the norm $\|\cdot\|$.

Consider $u \in \mathbb{A}$, then $u(a) \geq 0$. If $u(a) > 0$, then $u \in \mathbb{B}$. If $u(a) = 0$, we can construct a sequence $\{u_n\}$, define

$$u_n(x) = u(x) + \frac{1}{n}$$

for all $x \in [a, b]$. Clearly, $u_n(a) = u(a) + \frac{1}{n} > 0$, so $u_n \in \mathbb{B}$. As $n \to \infty$, we have $u_n \to u$ because

$$\|u_n - u\| = \max_{a \leq x \leq b} \left| u(x) + \frac{1}{n} - u(x) \right| = \frac{1}{n} \to 0.$$

This shows that u is a limit point of $\mathbb{B}$.

Therefore, $\mathbb{A}$ is the closure of $\mathbb{B}$.

PROBLEM 8. If we set $\phi(u) \coloneqq |u(a)|$, then ϕ is not a norm on $\mathbb{X}$.

SOLUTION. We will check the properties of the norm. Let $\phi(u) = |u(a)| = 0$, which implies $u(a) \equiv 0$, but we can choose u such that $u(x) \neq 0$ in $(a, b]$. Thus $\phi(u)$ is not a norm on $\mathbb{X}$.

PROBLEM 9. If we set

$$\|u\|_1 \coloneqq \int_a^b |u(x)| \mathrm{d}x$$

then $\|\cdot\|_1$ is a norm on $\mathbb{X}$, but $\mathbb{X}$ is not a Banach space with respect to $\|\cdot\|_1$.

SOLUTION. We will check the properties of the norm.

- $\|u\|_1 = \int_a^b |u(x)|\mathrm{d}x \geq 0$ and $\|u\|_1 = 0$ iff $u = 0$.
- $\forall \alpha \in \mathbb{R}$, we have $\|\alpha u\|_1 = \int_a^b |\alpha u(x)|\mathrm{d}x = |\alpha| \int_a^b |u(x)|\mathrm{d}x = |\alpha|\|u\|_1$.
- $\forall u, v \in \mathbb{X}$, we have

  $$\|u+v\|_1 = \int_a^b |u(x)+v(x)|\mathrm{d}x \leq \int_a^b |u(x)|\mathrm{d}x + \int_a^b |v(x)|\mathrm{d}x = \|u\|_1 + \|v\|_1$$

  then $\|\cdot\|_1$ is a norm on $\mathbb{X}$. Define a discontinuous function $w : [a,b] \to \mathbb{R}$, say,

  $$w(x) := \begin{cases} 1, & \text{if } a \leq x \leq c < b \\ 0, & \text{if } c < x \leq b \end{cases}$$

  Construct a sequence $\{u_n\}$ in $\mathbb{X}$

  $$u_n(x) := \begin{cases} 1, & \text{if } a \leq x \leq c - \frac{1}{n}, \\ n(c-x), & \text{if } c - \frac{1}{n} < x < c, \\ 0, & \text{if } c \leq x \leq b. \end{cases}$$

  such that

  $$\|u_n - w\|_1 = \int_{c-\frac{1}{n}}^c |n(c-x) - 1|\mathrm{d}x \leq \frac{1}{n} \to 0 \quad \text{as } n \to \infty$$

  $\forall \varepsilon > 0$, choose $n_\varepsilon > \varepsilon$, then for $\forall n > m \geq n_\varepsilon$, we have

  $$\|u_n - u_m\|_1 \leq \int_{c-\frac{1}{m}}^c 1\mathrm{d}x \leq \frac{1}{m} < \varepsilon$$

  Then $\{u_n\}$ is Cauchy with respect to $\|\cdot\|_1$. Suppose that $\|u_n - u\|_1 \to 0$ as $n \to \infty$, where $u \in \mathbb{X}$. Then,

  $$\|u - w\|_1 \leq \|u - u_n\|_1 + \|u_n - w\|_1 \to 0$$

  Hece $u(x) = w(x)$ on $[a,b]$, contradicting the continuity of the function u.

Problem 10. The operators $A : \mathbb{X} \to \mathbb{X}$ and $B : \mathbb{X} \to \mathbb{X}$ defined through

$$(Au)(x) \coloneqq u(a) \qquad \text{and} \qquad (Bu)(x) \coloneqq \int_a^x u(y)\mathrm{d}y$$

are linear and continuous with $\|A\| = 1$ and $\|B\| = b - a$.

Solution. For all $u, v \in \mathbb{X}$ and $\alpha, \beta \in \mathbb{R}$, we have

$$(A(\alpha u + \beta v))(x) = \alpha u(a) + \beta v(a) = \alpha(Au)(x) + \beta(Av)(x)$$

and

$$(B(\alpha u + \beta v))(x) = \alpha \int_a^x u(y)\mathrm{d}y + \beta \int_a^x v(y)\mathrm{d}y = \alpha(Bu)(x) + \beta(Bv)(x)$$

imply that the operators are linear. Notice that

$$\|Au\| = \underset{a \le x \le b}{max} |(Au)(x)| = u(a) \le \underset{a \le x \le b}{max} |u(x)| = \|u\|$$

and

$$\|Bu\| = \underset{a \le x \le b}{max} |(Bu)(x)| \le (b - a) \underset{a \le x \le b}{max} |u(x)| = (b - a)\|u\|$$

Thus the operators are continuous. Further, $\|A\| = \underset{\|u\|=1}{sup} \|Au\| = 1$ and $\|B\| = \underset{\|u\|=1}{sup} \|Bu\| = b - a$.

Problem 11. If we set

$$f(u) \coloneqq \int_a^b yu(y)\mathrm{d}y \quad \text{for all } u \in \mathbb{X}$$

then $f \in \mathbb{X}^*$ with $\|f\| = \frac{(b-a)^2}{2}$.

Solution. For all $u \in \mathbb{X}$,

$$|f(u)| \le \underset{a \le x \le b}{max} |u(x)| \int_a^b y\mathrm{d}y = \frac{(b-a)^2}{2}\|u\|$$

Thus $\|f\| = \underset{\|u\|\leq 1}{sup}\ |f(u)| = \frac{(b-a)^2}{2}$

PROBLEM 12. Let $\alpha \in \mathbb{R}$ with $|\alpha|(b-a) < 1$. For each given $u_0 \in \mathbb{X}$, the iteration method

$$u_{n+1}(x) = \alpha \int_a^b \sin u_n(x)\mathrm{d}x + 1 \quad n = 0, 1, \cdots \ x \in [a, b]$$

converges uniformly on $[a, b]$ to the unique solution $u \in \mathbb{X}$ of the integral equation

$$u(x) = \alpha \int_a^b \sin u(x)\mathrm{d}x + 1 \quad x \in [a, b]$$

SOLUTION. Define the operator

$$(Au)(x) := \alpha \int_a^b \sin u(x)\mathrm{d}x + 1 \quad x \in [a, b]$$

Then, the original equation corresponds to the fixed-point problem

$$u = Au$$

If $u \in \mathbb{X}$, then so is the function $Au : [a, b] \to R$. This way we get the operator

$$A : \mathbb{X} \to \mathbb{X}$$

For $\forall u, v \in \mathbb{X}$,we have

$$\begin{aligned}
\|Au - Av\| &= \underset{a\leq x\leq b}{max} |(Au)(x) - (Av)(x)| \\
&\leq |\alpha| \int_a^b \underset{a\leq x\leq b}{max} |\sin u(x) - \sin v(x)|\mathrm{d}x \\
&\leq |\alpha| \int_a^b \underset{a\leq x\leq b}{max} |cos w(x)||u(x) - v(x)|\mathrm{d}x \\
&\leq |\alpha|(b-a)\|u - v\|
\end{aligned}$$

Since $|\alpha|(b-a) < 1$, by Banach Fixed Point Theorem we show the conclusion.

Problem 13. Let $\alpha \in \mathbb{R}$ with $|\alpha| < 1$. For each given $u_0 \in \mathbb{R}$, the iteration method

$$u_{n+1} = \alpha \sin u_n + 1 \quad n = 0, 1, \cdots$$

converges to the unique solution $u \in \mathbb{R}$ of the equation $u \in \mathbb{R}$ of the equation $u = \alpha \sin u + 1$.

Solution. Define the operator

$$f(u) \coloneqq \alpha \sin u + 1$$

Then, the original equation corresponds to the fixed-point problem

$$u = f(u)$$

If $u_0 \in \mathbb{R}$, then except for u_0, $u_n \in \mathbb{I} = [0, 2]$ for $n \geq 1$, so is the function $f(u)$. This way we get the operator

$$f : \mathbb{I} \to \mathbb{I}$$

Where $\mathbb{I}$ is a closed nonempty set in the Banach space $\mathbb{R}$. For $\forall u, v \in \mathbb{I}$,we have

$$\begin{aligned}
\|Au - Av\| &= \underset{0 \leq x \leq 2}{max} |(Au)(x) - (Av)(x)| \\
&\leq |\alpha| \underset{0 \leq x \leq 2}{max} |\sin u(x) - \sin v(x)| \\
&\leq |\alpha| \underset{0 \leq x \leq 2}{max} |cosw(x)||u(x) - v(x)| \\
&\leq |\alpha| \|u - v\|
\end{aligned}$$

Since $|\alpha| < 1$, by Banach Fixed Point Theorem we show the conclusion.

Problem 14. Let $K(x, y) : [a, b] \times [a, b] \to \mathbb{R}$ be continuous with $0 \leq K(x, y) \leq d$ for all $x, y \in [a, b]$. Let $2(b-a)d \leq 1$ along with $u_0(x) \equiv 0$ and $v_0(x) \equiv 2$. Then, the two iteration methods

$$u_{n+1}(x) = \int_a^b K(x, y)u_n(y)\mathrm{d}y + 1 \quad n = 0, 1, \cdots \ x \in [a, b]$$
$$v_{n+1}(x) = \int_a^b K(x, y)v_n(y)\mathrm{d}y + 1$$

converge uniformly on $[a, b]$ to the unique solution $u \in \mathbb{X}$ of the integral equation

$$u(x) = \int_a^b K(x, y)u(y)\mathrm{d}y + 1 \quad x \in [a, b]$$

where $u_0(x) \leq u_1(x) \leq \cdots \leq v_1(x) \leq v_0(x)$ for all $x \in [a, b]$

Solution. Define the operator

$$(Au)(x) \coloneqq \int_a^b K(x, y)u(y)\mathrm{d}y + 1$$

If $u \in \mathbb{X}$, then so is the function $Au : [a, b] \to R$. This way we get the operator

$$A : \mathbb{X} \to \mathbb{X}$$

A is continuous, $\mathbb{X}$ is bounded, it suffices to show

- A($\mathbb{X}$) is bounded.
- A($\mathbb{X}$) is equicontinuous.

For all $u \in \mathbb{X}$,

$$\|Au\| = \max_{a \leq x \leq b} |\int_a^b K(x, y)u(y)\mathrm{d}y + 1| \leq (b-a)d\|u\| + 1$$

Since $K(x, y)$ is uniformly continuous, then for $\forall \varepsilon > 0$, $\exists \delta > 0$, such that$|x - z| < \delta$ and $x, z \in [a, b]$, we have $|K(x, y) - K(z, y)| < \varepsilon$. Then

$$|(Au)(x) - (Au)(z)| \leq \int_a^b |K(x, y) - K(z, y)||u(y)|\mathrm{d}y \leq (b-a)\|u\|\varepsilon$$

By Arzel$\grave{a}$-Ascoli theorem, A is a compact operator. Obviously A is monotone increasing. Further,

$$Au_0 = 1 \geq u_0$$

and

$$Av_0 = 2\int_a^b K(x,y)\mathrm{d}y + 1 \leq 2(b-a)d + 1 \leq 2 = v_0$$

By Theorem 1.E, we have the conclusion.

PROBLEM 15. Let $\alpha \in \mathbb{R}$ and $f \in \mathbb{X}$ be given. Then, the nonlinear integral equation

$$u(x) = \alpha \int_a^b \sin u(x)\mathrm{d}x + f(x)$$

has a solution $u \in \mathbb{X}$.

SOLUTION. Define the operator

$$(Au)(x) := \alpha \int_a^b \sin u(x)\mathrm{d}x + f(x)$$

We first show that $A : \mathbb{X} \to \mathbb{X}$ is a compact operator. Continuity is obviously, it suffices to show that $A(\mathbb{X})$ is bounded and equicontinuous.

For all $u \in \mathbb{X}$, we have

$$\|Au\| \leq \alpha(b-a) + \underset{a \leq x \leq b}{max} f(x) := r$$

Since $f \in \mathbb{X}$, it's bounded in $[a,b]$. For $u, f \in \mathbb{X}$, u and f are uniformly continuous. For each $\varepsilon > 0$, there is a $\delta > 0$ such that $|x-y| < \delta$ imply $|u(x) - u(y)| < \frac{\varepsilon}{2\alpha(b-a)}$ and $|f(x) - f(y)| < \frac{\varepsilon}{2}$.

$$|(Au)(x) - (Au)(y)| \leq |\alpha| \int_a^b \underset{a \leq x \leq b}{max} |cosu(z)||u(x) - u(y)|\mathrm{d}x + |f(x) - f(y)| < \varepsilon$$

By Arzelà-Ascoli theorem, A is a compact operator. Further we have priori estimate

$$\|u\| = |t|\|\alpha \int_a^b \sin u(x)\mathrm{d}x + f(x)\| = |t|r$$

Apply Leray-Schaude Principle we have the conclusion.

PROBLEM 16. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Then, the system

$$\xi = 10^{27} + \sin f(\xi, \eta) \quad \eta = \cos f(\xi, \eta)$$

has a solution $(\xi, \eta) \in \mathbb{R}^2$

SOLUTION. Define operators

$$A\xi = 10^{27} + \sin f(\xi, \eta) \quad B\eta = \cos f(\xi, \eta)$$

By Arzelà-Ascoli theorem, A and b are compact operators. Details are similar to the previous problem. Further, we have the priori estimate

$$\|\xi\| = |t|\|10^{27} + \sin f(\xi, \eta)\| \leq (10^{27} + 1)|t|$$

and

$$\|\eta\| = |t|\| \cos f(\xi, \eta)\| \leq |t|$$

Apply Leray-Schaude Principle we have the conclusion.

PROBLEM 17. Let $\sigma(A)$ denote the spectrum of the linear operator $A : \mathbb{X} \to \mathbb{X}$. Show that $\sigma(A) = 2$ provided $\mathbb{X} := \mathbb{C}$ and $Au = 2u$.

SOLUTION. Consider whether $(\lambda I - A)^{-1}$ exists. Since $Au = 2u$

$$(\lambda I - A)u = (\lambda - 2)u$$

If $\lambda = 2$, $(\lambda I - A)^{-1}$ doesn't exist. If $\lambda \neq 2$, $(\lambda I - A)^{-1} = \frac{I}{\lambda - 2}$. By definition, $\sigma(A) = 2$.

Problem 18. The spectral radius. Let $A : \mathbb{X} \to \mathbb{X}$ be a linear continuous operator on the complex Banach space $\mathbb{X}$. Define the spectral radius $r(A)$ of A through

$$r(A) := \underset{\lambda \in \sigma(A)}{sup} |\lambda|$$

Show that $r(A) = \lim_{n \to \infty} \|A^n\|^{\frac{1}{n}}$

Solution. For any $\varepsilon > 0$, let's define the two following matrics

$$A_{\pm} = \frac{A}{r(A) \pm \varepsilon}$$

Thus

$$r(A_{\pm}) = \frac{r(A)}{r(A) \pm \varepsilon}$$

and

$$r(A_+) < 1 < r(A_-)$$

Then

$$\lim_{n \to \infty} A_+^k = 0$$

This shows the existence of $N_+ \in \mathbb{N}$ such that, for all $n \geq N_+$

$$\|A_+^n\| < 1$$

Therefore

$$\|A^n\|^{\frac{1}{n}} < r(A) + \varepsilon$$

Similarly, the theorem on power sequences implies that $\|A_-^n\|$ is not bounded and that there exists $N_- \in \mathbb{N}$ such that, for all n N_-

$$\|A_-^n\| > 1$$

Therefore

$$\|A^n\|^{\frac{1}{n}} > r(A) - \varepsilon$$

Let $N = max\{N_+, N_-\}$, for $\forall \varepsilon$, $\exists N \in \mathbb{N}$, $\forall n \geq N$

$$r(A) - \varepsilon < \|A^n\|^{\frac{1}{n}} < r(A) + \varepsilon$$

That is, $r(A) = \lim\limits_{n\to\infty} \|A^n\|^{\frac{1}{n}}$.

Problem 19. **Volterra integral operator**. Let $\mathbb{X} := C[a,b]_{\mathbb{C}}$, where $-\infty < a < b < \infty$. Define the operator $A : \mathbb{X} \to \mathbb{X}$ through

$$(Au)(x) := \int_a^x K(x,y)u(y)\mathrm{d}y \quad \text{for all } x \in [a,b]$$

where $K : [a,b] \times [a,b] \to \mathbb{C}$ is continuous. Then, $r(A) = 0$, and hence $\sigma(A) = \{0\}$.

Solution. Set

$$c = \max_{a \leq x,y \leq b} |K(x,y)|$$

Then

$$|(Au)(x)| = \int_a^x K(x,y)u(y)\mathrm{d}y \leq c\|u\|(x-a)$$

and

$$|(A^2u)(x)| = \int_a^x K(x,y)(Au)(y)\mathrm{d}y \leq c^2\|u\| \int_a^x (y-a)\mathrm{d}y = \frac{c^2\|u\|(x-a)^2}{2!}$$

Continuing in this way, we have

$$|(A^nu)(x)| \leq \frac{c^n\|u\|(x-a)^n}{n!}$$

and therefore

$$\|A^nu\| = \max_{a \leq x \leq b} |(A^nu)(x)| = \frac{c^n\|u\|(b-a)^n}{n!}$$

Thus we find the radius $r(A) = \lim\limits_{n\to\infty} \|A^n\|^{\frac{1}{n}} = 0$, and hence $\sigma(A) = \{0\}$.

PROBLEM 20. **Fredholm integral operator**.Let $\mathbb{X} := C[a,b]_{\mathbb{C}}$, where $-\infty < a < b < \infty$. Define the operator $A : \mathbb{X} \to \mathbb{X}$ through

$$(Au)(x) := \int_a^b K(x,y)u(y)\mathrm{d}y \quad \text{for all } x \in [a,b]$$

where $K : [a,b] \times [a,b] \to \mathbb{C}$ is continuous. Show that

$$r(A) \leq (b-a) \underset{x,y\in[a,b]}{max} |K(x,y)|$$

SOLUTION. We have shown that

$$\|A\| \leq \underset{\|u\|\leq 1}{sup}\ c\|u\|(b-a) = (b-a) \underset{x,y\in[a,b]}{max} |K(x,y)|$$

Combining with

$$r(A) \leq \|A\|$$

We have the conclusion.

PROBLEM 21. **The Banach space $l_\infty^{\mathbb{K}}$**. Let $\mathbb{K}^\infty$ denote the space of all sequences $(u_n)_{n\geq 1}$, where $u_n \in \mathbb{K}$ for all $n \in \mathbb{N}$. Moreover, let $l_\infty^{\mathbb{K}}$ denote the set of all $(u_n) \in \mathbb{K}^\infty$ such that

$$\|(u_n)\|_\infty := \underset{n\geq 1}{sup} |u_n| < \infty$$

Define

$$\alpha(u_n) + \beta(v_n) = (\alpha u_n + \beta v_n) \quad \text{for all} \ \ \alpha, \beta \in \mathbb{K}$$

Show that $\mathbb{K}^\infty$ is an infinite-dimensional linear space over $\mathbb{K}$.

SOLUTION. Since $\mathbb{K}$ is a linear space, It suffices to show that $\alpha(u_n)+\beta(v_n) \in$

$\mathbb{K}^\infty$. We have

$$\alpha(u_n) + \beta(v_n) = (\alpha u_n + \beta v_n) \quad \text{for all} \quad \alpha, \beta \in \mathbb{K}$$

Since $\alpha u_n + \beta v_n \in \mathbb{K}$, $\alpha(u_n) + \beta(v_n) \in \mathbb{K}^\infty$.

Now we choose $(e_{1n}), e_{2n}, \cdots e_{Nn}$, and $e_{kn} = \{0, \cdots, 0, 1, 0 \cdots\}$ such that 1 arise in the k-th place. Thus for each $N = 1, 2, \cdots$,

$$\alpha_1(e_{1n}) + \alpha_2(e_{2n}) + \cdots \alpha_N(e_{Nn}) = 0$$

always implies $\alpha_1 = \alpha_2 = \cdots = \alpha_N = 0$, so $\mathbb{K}^\infty$ is an infinite-dimensional linear space over $\mathbb{K}$.

Problem 22. $l_\infty^{\mathbb{K}}$ is an infinite-dimensional Banach space over $\mathbb{K}$ with respect to the norm $\|\cdot\|_\infty$.

Solution. First we show that $l_\infty^{\mathbb{K}}$ is linear. For $\forall (u_n), (v_n) \in l_\infty^{\mathbb{K}}$ and $\alpha, \beta \in \mathbb{K}$, we have

$$\|\alpha(u_n)+\beta(v_n)\| = \|(\alpha u_n+\beta v_n)\| = \sup_{n\geq 1} |\alpha u_n+\beta v_n| \leq \alpha \sup_{n\geq 1} |u_n|+\beta \sup_{n\geq 1} |v_n| < \infty$$

Thus $\alpha(u_n) + \beta(v_n) \in l_\infty^{\mathbb{K}}$.

Then we show each Cauchy sequences is convergent. Choose a Cauchy sequence $(u_n^{(k)})$ in $l_\infty^{\mathbb{K}}$, which means for $\forall \varepsilon > 0$, there exists $N > 0$, such that $\forall k_1, k_2 \geq N$, we have

$$\|(u_n^{(k_1)} - u_n^{(k_2)})\| = \sup_{n\geq 1} |u_n^{(k_1)} - u_n^{(k_2)}| < \varepsilon$$

That is, for each $u_n \in \mathbb{K}$, $u_n^{(k)}$ is a Cauchy sequence, and by applying the traditional Cauchy convergent criterion, $u_n^{(k)}$ converges to u_n^*, then

$$\|(u_n^{(k)} - u_n^*)\| = \sup_{n\geq 1} |u_n^{(k)} - u_n^{(*)}| < \varepsilon$$

$l_\infty^{\mathbb{K}}$ is an infinite-dimensional Banach space over $\mathbb{K}$ with respect to the norm $\|\cdot\|_\infty$.

Classical function spaces on $[a,b]$. Let $-\infty < a < b < \infty$. Show that the following function spaces are Banach spaces.

PROBLEM 23. Let $B[a,b]$ denote the set of all bounded functions $u : [a,b] \to \mathbb{R}$ and set

$$\|u\| := \sup_{a\le x\le b} |u(x)|$$

SOLUTION. First we show $\|\cdot\|$ is a norm.

- $\|u\| = \sup_{a\le x\le b} |u(x)| = 0 \Leftrightarrow u \equiv 0$.
- For $\alpha \in \mathbb{R}$, $\|\alpha u\| = \sup_{a\le x\le b} |\alpha u(x)| = \alpha \sup_{a\le x\le b} |u(x)| = \|\alpha u\|$
- For $\forall u, v \in [a,b]$

  $$\|u+v\| = \sup_{a\le x\le b} |u(x)+v(x)| \le \sup_{a\le x\le b} |u(x)| + \sup_{a\le x\le b} |v(x)| = \|u\| + \|v\|$$

  Then we show each Cauchy sequences is convergent. Choose a Cauchy sequence (u_n) in $B[a,b]$, i.e.

  $$\|u_n - u_m\| = \sup_{a\le x\le b} |u_n(x) - u_m(x)| < \varepsilon \quad \text{for all } n, m \ge N$$

  This implies the pointwise convergence

  $$u_n(x) \to u(x) \quad \text{for all } x \in [a,b]$$

  Letting $m \to \infty$, we obtain

  $$\sup_{a\le x\le b} |u_n(x) - u(x)| \le \varepsilon$$

Thus the convergence is uniform on the interval $[a, b]$. Further

$$\|u\| = \|u - u_n + u_n\| \leq \|u - u_n\| + \|u_n\| < \varepsilon + \|u_n\|$$

Since u_n is bounded, $u \in B[a, b]$, and

$$u_n \to u \text{ in } B[a, b] \text{ as } n \to \infty$$

Problem 24. For $0 < \alpha \leq 1$, let $C^{0,\alpha}[a, b]$ denote the set of all the so-called Hölder continuous function $u : [a, b] \to \mathbb{R}$, i.e., by definition,

$$|u(x) - u(y)| \leq \text{const}|x - y|^{\alpha} \quad \text{for all } x, y \in [a, b]$$

Let

$$H_{\alpha}(u) := \sup \frac{|u(x) - u(y)|}{|x - y|^{\alpha}}$$

where the supremum is taken over all $x, y \in [a, b]$ with $x \neq y$. In particular

$$|u(x) - u(y)| \leq H_{\alpha}(u)|x - y|^{\alpha} \quad \text{for all } x, y \in [a, b]$$

Set

$$\|u\| := \underset{a \leq x \leq b}{max} |u(x)| + H_{\alpha}(u)$$

Solution. It's easy to show $\| \cdot \|$ is a norm as before. Choose a Cauchy sequence (u_n) in $C^{0,\alpha}$, i.e.

$$\|u_n - u_m\| = \underset{a \leq x \leq b}{max} |u_n(x) - u_m(x)| + H_{\alpha}(u_n - u_m) < \varepsilon \quad \text{for all } n, m \geq N$$

Obviously $H_{\alpha}(u_n - u_m) \geq 0$, it implies

$$\underset{a \leq x \leq b}{max} |u_n(x) - u_m(x)| < \varepsilon$$

The pointwise convergence holds

$$u_n(x) \to u(x) \quad \text{for all } x \in [a,b]$$

Letting $m \to \infty$, we obtain

$$\|u_n - u\| = \underset{a \le x \le b}{max} |u_n(x) - u(x)| + H_\alpha(u_n - u) \le \varepsilon$$

Thus the convergence is uniform on the interval $[a,b]$. Further

$$\|u\| = \|u - u_n + u_n\| \le \|u - u_n\| + \|u_n\| \le \varepsilon + \|u_n\|$$

u_n is Hölder continuous, which implies $\|u_n\| < \infty$. Thus u is also Hölder continuous.

Problem 25. Let $C^k[a,b]$ with $k = 1, 2, \cdots$ denote the set of all continuous functions $u : [a,b] \to \mathbb{R}$ that have continuous derivatives on $[a,b]$ up to the order k. Set

$$\|u\| := \sum_{j=0}^{k} \underset{a \le x \le b}{max} |u^{(j)}(x)|$$

where u^j denotes the jth derivative.

Solution. It's easy to show $\| \cdot \|$ is a norm as before. Choose a Cauchy sequence (u_n) in $C^k[a,b]$, i.e.

$$\|u_n - u_m\| = \sum_{j=0}^{k} \underset{a \le x \le b}{max} |u_n^{(j)}(x) - u_m^{(j)}(x)| < \varepsilon \quad \text{for all } n, m \ge N$$

$$\underset{a \le x \le b}{max} |u_n^{(j)}(x) - u_m^{(j)}(x)| < \varepsilon \quad \text{for all } j = 1, 2, \cdots, k$$

It implies the pointwise convergence

$$u_n^{(j)}(x) \to u^{(j)}(x) \quad \text{for all } x \in [a,b] \text{ and } j = 1, 2, \cdots, k$$

Letting $m \to \infty$, we obtain

$$\|u_n - u\| = \sum_{j=0}^{k} \underset{a \le x \le b}{max} |u_n^{(j)}(x) - u^{(j)}(x)| \le \varepsilon$$

Thus the convergence is uniform on the interval $[a,b]$. Further for $\varepsilon > 0$, $|x-y| < \delta$, $j = 1, 2, \cdots, k$

$$|u^{(j)}(x) - u^{(j)}(y)| \le |u^{(j)}(x) - u_n^{(j)}(x)| + |u_n^{(j)}(x) - u_n^{(j)}(y)| + |u_n^{(j)}(y) - u^{(j)}(y)| < \varepsilon$$

We obtain $u \in C^k[a,b]$.

Problem 26. For $0 < \alpha \le 1$ and $k = 1, 2, \cdots$, let $C^{k,\alpha}[a,b]$ denote the set of all functions $u \in C^k[a,b]$ with $u^{(k)} \in C^{0,\alpha}[a,b]$. Set

$$\|u\| := \sum_{j=0}^{k} \underset{a \le x \le b}{max} |u^{(j)}(x)| + H_\alpha(u^{(k)})$$

Solution. Combining previous two problems.

Problem 27. Let $C[a,b]_{\mathbb{C}}$ denote the set of all complex continuous functions $u : [a,b] \to \mathbb{C}$. Define

$$\|u\| := \underset{a \le x \le b}{max} |u(x)|$$

Solution. Apply the condition of $C[a,b]$ to real and imaginary parts respectively.

PROBLEM 28. **Density**. Let D be a dense subset of the normed space X over $\mathbb{K}$. Show that

$$\langle u^*, u \rangle = 0 \quad \text{for all } u \in D \text{ and fixed } u^* \in X$$

implies $u^* = 0$.

SOLUTION. Let $v \in X$ be given. Since D is dense in X, there exists a sequence (u_n) in D such that $u_n \to v$ as $n \to \infty$. The functional u^* is continuous and $u^*(u_n) = 0$ for all n. Hence

$$\langle u^*, v \rangle = \lim_{n \to \infty} \langle u^*, u_n \rangle = 0 \quad \text{for all } v \in X$$

Therefore, $u^* = 0$.