Homework 4

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Problem 1. (Exercise 3.7)

Find the fundamental sets of cycles and cutsets of the graph in Fig**??** associated with the spanning tree shown.

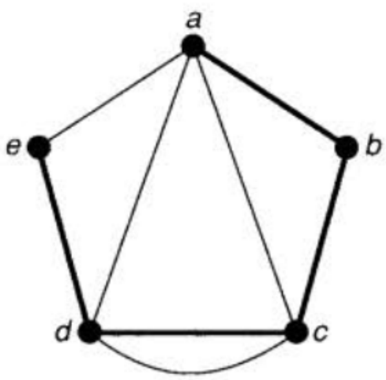

图 1: Figure For Problem 1

Solution.

1. **Cycles**: By definition, we can find *abcdea*, *abcda*, *abca* *cdc*.

2. **Cutsets**: $\{ab, ae, ad, ac\}$; $\{bc, ae, ad, ac\}$; $\{cd, ae, ad, cd\}$; $\{ed, ae, ad\}$.

$\square$

PROBLEM 2. (Exercise 3.10)

If G is a connected graph, a **centre** of G is a vertex v with the property that the maximum of the distances between v and the other vertices of G is as small as possible. By successively removing all the end-vertices, prove that every tree has either one centre or two adjacent centres. Give an example of a tree of each type with seven vertices.

SOLUTION.

We will follow the hint by removing all the end-vertices. Denote $M(v) = \max\{\mathrm{d}(v, w) | w \in V(G)\}$. Notive that the maximum distance $M(v)$ from a given vertex v to any other vertex w occurs only when w is a end-vertex.

First, let G be a tree with n vertices ($n \geq 2$). Then G must have at least two end-vertices. Delete all end-vertices from G, then the resulting graph G' is still a tree.

After removing end-vertices, $E(v)$ in G' is just one less than $E(v)$ in G'. Again, delete end-vertices from G' so that the resulting G'' is still a tree with the same centers.

Note that all vertices that G had as centers will still remain centers in $G' \to G'' \to G''' \ldots$. Continue this process until the remaining tree has either one vertex or one edge.

So at the end, if one vertex is there this implies tree G has one center. If one edge is there then tree G has two centers which are adjacent.

One example with seven vertices are shown as Fig**??**.

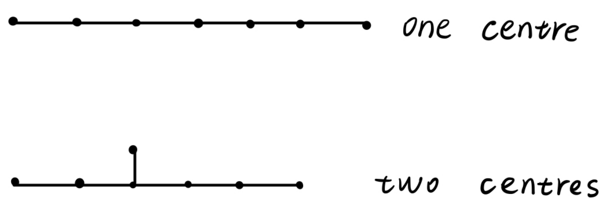

图 2: Example

□

PROBLEM 3. (Exercise 3.11)

Let T_1 and T_2 be spanning trees of a connected graph G.

(i) If e is any edge of T_1, show that there exists an edge f of T_2 such that the graph $(T_1 - \{e\}) \cup \{f\}$ (obtained from T_1 on replacing e by f) is also a spanning tree.

(ii) Deduce that T_1 can be 'transformed' into T_2 by replacing the edges of T_1 one at a time by edges of T_2 in such a way that a spanning tree is obtained at each stage. (This result will be needed in Chapter 7.)

SOLUTION.

(i) Let $e \in E(T_1)$. Removing e from T_1 disconnects it into two components, A and B. Since T_2 is a spanning tree, it contains a unique edge f connecting A and B. Adding f to $T_1 - \{e\}$ reconnects A and B resulting in a spanning tree $(T_1 - e) \cup f$.

(ii) Let $k = |E(T_1) \setminus E(T_2)|$. If $k = 0$, then $T_1 = T_2$. Otherwise, choose

$e \in E(T_1) \setminus E(T_2)$. By (i), replace e with some $f \in E(T_2)$ to get a new spanning tree T_1' with $|E(T_1') \setminus E(T_2)| = k-1$. Repeat this process inductively until all edges of T_1 are replaced by edges of T_2, maintaining a spanning tree at each step.

□

PROBLEM 4. (Exercise 3.12)

Verify directly that there are exactly 125 labelled trees on five vertices.

SOLUTION.

For unlabelled trees on five vertices, there are three cases sohwn in Fig??.

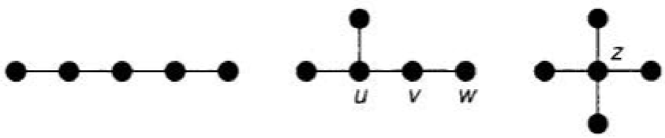

图 3: unlabelled Trees

For the firs tree, there are $5!/2 = 60$ labelled trees. For the second one, there are $5 \times 4 \times 3$ labelled trees, depending on the different choices of u, v and w. For the last one, there are 5 labelled trees depending on z.

Therefore, there are exactly 125 labelled trees on five vertices.

□

PROBLEM 5.

Show that, for each value of n, the graph associated with the alcohol $C_nH_{2n+1}OH$ is a tree (the oxygen vertex has degree 2). Draw the tree corresponding to the molecule C_2H_5OH.

SOLUTION.

The graph is a connected graph. We can calculate the number of vertices and edges:

1. **Vertices**: $n + (2n + 1) + 1 + 1 = 3n + 3$.
2. **Edges**: $\frac{1}{2}(4n + (2n + 1) + 2 + 1) = 3n + 2$.

This is directly from the property of $C_nH_{2n+1}OH$, and is therefore a tree, by Theorem 3.1(iii). The tree corresponding is shown in Fig **??**.

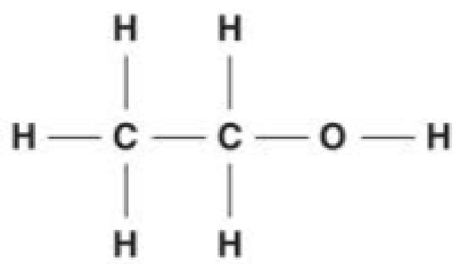

图 4: Tree of Alcohol

□

PROBLEM 6. (Exercise 3.16)

In the first proof of Cayley's theorem, find the labelled tree that corresponds to the sequence $(7, 6, 5, 4, 3, 2, 1)$.

Solution.

Follow the construction in the proof of Cayley's theorem, we find the desired labelled tree in Fig ??

图 5: Labelled Tree

□

> Problem 7.
>
> Show that every tree with maximum degree k has at least k leaves.

Solution.

The case $k = 1$ is trivial. Assume that $k \geq 2$. Let u be a vertex of degree k - So u is not a leaf.

We have

$$\sum_{v \in V} \deg v = 2|E|$$

in every graph. Also,

$$|E| = |V| - 1$$

in every tree. Thus

$$\sum_{v \in V} \deg v = 2|V| - 2$$

Define L to be the set of leaves of the graph. The degree of every non-leaf vertex is at least 2, so it follows

$$\sum_{v \in V} \deg v = \deg u + \sum_{v \in L} \deg v + \sum_{v \in V \setminus (L \cup \{u\})} \deg v \geq k + |L| + 2(|V| - |L| - 1)$$

Thus

$$2|V| - 2 \geq k + 2|V| - |L| - 2$$

and it follows that $|L| \geq k$.

$\square$