Homework 2

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Problem 1.

Suppose $f : X \to Y$ is a bijective continuous map. Show that the following are equivalent.

(a) f is a homeomorphism.

(b) f is open.

(c) f is closed.

Solution. We show the equivalence by proving (a) $\implies$ (b), (b) $\implies$ (c), and (c) $\implies$ (a).

Step 1. (a) $\implies$ (b) : If f is a homeomorphism, $f^{-1} : Y \to X$ is continuous. For any open $U \subseteq X$, since f^{-1} preserves openness, $f(U) = (f^{-1})^{-1}(U)$ is open in Y .

Step 2. (b) $\implies$ (c): Assume f is open. For any closed $C \subseteq X$, then $X \setminus C$ is open. Then $f(X \setminus C) = Y \setminus f(C)$ is open, so $f(C)$ is closed.

Step 3. (c) $\Rightarrow$ (a): Assume f is closed. For closed $D \subseteq Y$, $f^{-1}(D)$ is closed in X by continuity. As f is bijective and closed, $D = f(C)$ for some closed $C \subseteq X$, so f^{-1} preserves closedness and is continuous. Thus, f is a homeomorphism.

Hence, all conditions are equivalent. □

PROBLEM 2.

Prove the following statements.

(a) Every homeomorphism is a local homeomorphism.

(b) Every local homeomorphism is continuous and open.

SOLUTION.

(a) Let $f : X \to Y$ be a homeomorphism. By definition, f is bijective, continuous, and has a continuous inverse f^{-1}. For every $x \in X$, the entire space X is an open neighborhood of x. The restriction $f|_X : X \to Y$ is itself a homeomorphism because f is globally a homeomorphism. Thus, f satisfies the local homeomorphism condition at every point, making it a local homeomorphism.

(b) Let $f : X \to Y$ be a local homeomorphism.

Continuity: Let $V \subseteq Y$ be open. For any $x \in f^{-1}(V)$, there exists an open neighborhood $U_x \subseteq X$ of x such that $f|_{U_x} : U_x \to f(U_x)$ is a homeomorphism. Since $f(U_x)$ is open in Y, the set $f(U_x) \cap V$ is open, and $f|_{U_x}^{-1}(f(U_x) \cap V)$ is open in U_x, hence in X. Thus, $f^{-1}(V)$ is a union of open sets

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} f|_{U_x}^{-1}(f(U_x) \cap V)$$

therefore open. This proves f is continuous.

Openness: Let $U \subseteq X$ be open. For $y \in f(U)$, choose $x \in U$ with $f(x) = y$. There exists an open neighborhood $W_x \subseteq X$ of x such that $f|_{W_x} : W_x \to f(W_x)$ is a homeomorphism onto an open set. Since $U \cap W_x$ is open in W_x, $f(U \cap W_x)$ is open in Y. Hence, $f(U) = \bigcup_{x \in U} f(U \cap W_x)$ is open. This proves f is open.

$\square$

Problem 3.

Prove that any two spaces in the following are homeomorphic:

(a) the subspace $\mathbb{R}^2 \setminus \{(0,0)\}$ of $\mathbb{R}^2$;

(b) the subspace $\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ of $\mathbb{R}^3$;

(c) the subspace $\{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1\}$ of $\mathbb{R}^3$.

Solution.

Step 1. Homeomorphism between (a) and (b).

Define $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \text{Cylinder}$ by:

$$f(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, \ln \sqrt{x^2+y^2} \right).$$

This maps the plane in (a) to the cylinder in (b). The inverse is:

$$f^{-1}(a,b,z) = (e^z a, e^z b),$$

which is continuous. Thus, f is a homeomorphism.

Step 2. Homeomorphism between (b) and (c).

Parametrize the cylinder as $(\cos v, \sin v, z)$ and define $g : \text{Cylinder} \to \text{Hyperboloid}$ by:

$$g(\cos v, \sin v, z) = (\cosh z \cos v, \cosh z \sin v, \sinh z).$$

This satisfies $x^2 + y^2 - z^2 = \cosh^2 z - \sinh^2 z = 1$. The inverse is:

$$g^{-1}(x, y, z) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, \sinh^{-1} z \right),$$

which is continuous. Hence, g is a homeomorphism.

$\square$

Problem 4.

Suppose that X is a topological space, and for every $p \in X$ there exists a continuous function $f : X \to \mathbb{R}$ such that $f^{-1}(0) = \{p\}$. Show that X is Hausdorff.

Solution. To prove X is Hausdorff, we show that for any two distinct points $p, q \in X$, there exist disjoint open neighborhoods U and V containing p and q, respectively.

By hypothesis, for each $p \in X$, there is a continuous function $f_p : X \to \mathbb{R}$ with $f_p^{-1}(0) = \{p\}$. For distinct points p and q, consider f_p. Since $q \neq p$, we have $f_p(q) \neq 0$. Let $a = f_p(q)$, so $a \neq 0$.

Choose two disjoint open intervals in $\mathbb{R}$:

$$I_1 = \left(-\frac{|a|}{2}, \frac{|a|}{2} \right), \quad I_2 = \begin{cases} \left(\frac{a}{2}, \frac{3a}{2} \right) & \text{if } a > 0, \\ \left(\frac{3a}{2}, \frac{a}{2} \right) & \text{if } a < 0. \end{cases}$$

These intervals are disjoint because I_1 contains 0, while I_2 is centered at a with radius $|a|/2$.

Let $U = f_p^{-1}(I_1)$ and $V = f_p^{-1}(I_2)$. Since f_p is continuous, U and V are open in X. Then $p \in U$ because $f_p(p) = 0 \in I_1$. $q \in V$ because $f_p(q) = a \in I_2$. $U \cap V = \emptyset$, as $I_1 \cap I_2 = \emptyset$.

Thus, X is Hausdorff by definition. $\square$

Problem 5.

Let X and Y be topological spaces.

(a) Suppose $f : X \to Y$ is continuous and $p_n \to p$ in X. Show that $f(p_n) \to f(p)$ in Y.

(b) Prove that if X is first countable, the converse is true: if $f : X \to Y$ is a map that $p_n \to p$ in X implies $f(p_n) \to f(p)$ in Y, then f is continuous.

Solution.

(a) Let $f : X \to Y$ be continuous, and suppose $p_n \to p$ in X. We show $f(p_n) \to f(p)$ in Y.

Let V be any open neighborhood of $f(p)$ in Y. By continuity, there exists an open neighborhood U of p in X such that $f(U) \subseteq V$. Since $p_n \to p$, there exists $N \in \mathbb{N}$ such that $p_n \in U$ for all $n \geq N$. Therefore, $f(p_n) \in f(U) \subseteq V$ for all $n \geq N$, proving $f(p_n) \to f(p)$.

(b) Assume X is first countable, and $f : X \to Y$ satisfies that $p_n \to p \implies f(p_n) \to f(p)$. We prove f is continuous.

Suppose f is not continuous at some $p \in X$. Then there exists an open neighborhood V of $f(p)$ such that for every neighborhood U of p, $f(U) \nsubseteq V$. Let $\{U_n\}$ be a countable nested neighborhood basis at p. For each n, choose $p_n \in U_n$ with $f(p_n) \notin V$. The sequence $\{p_n\}$ converges to p, but $f(p_n) \notin V$ for all n, contradicting $f(p_n) \to f(p)$. Hence, f must be continuous.

□