多元统计分析 - HW1

王一鑫

兰州大学 LANZHOU UNIVERSITY

2-1. 2-2. 2-4 ~~2-6~~ 2-8 2-13 2-14 2-15

2-1 解: $Y = AX + d. \Rightarrow Y \sim N(A\mu + d, A\Sigma A')$.

其中 $A = \begin{pmatrix} 0.5 & -1 & 0.5 \\ -0.5 & 0 & -0.5 \end{pmatrix}$ $\mu = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ $d = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\Sigma = 2I_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

计算得 $A\mu + d = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $A\Sigma A' = \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$.

A+

2-2 证明: (1) 令 $Y_1 = X_1 + X_2 = (1, 1)\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

$Y_2 = X_1 - X_2 = (1, -1)\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

Y_1, Y_2 均为正态随机变量. 又

$\mathrm{cov}(Y_1, Y_2) = (1,1)\Sigma\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \sigma^2(1+\rho, 1+\rho)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$.

即 X_1+X_2 与 X_1-X_2 相互独立.

(2) $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} X_1 + X_2 \\ X_1 - X_2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

~~2-4~~ 令 $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. $Y = AX \sim (A\mu, A\Sigma A')$

$A\mu = \begin{pmatrix} \mu_1 + \mu_2 \\ \mu_1 - \mu_2 \end{pmatrix}$ $A\Sigma A' = \sigma^2\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \sigma^2\begin{pmatrix} 2(1+\rho) & 0 \\ 0 & 2(1-\rho) \end{pmatrix}$

则 $X_1 + X_2 \sim N(\mu_1 + \mu_2, 2\sigma^2(1+\rho))$

$X_1 - X_2 \sim N(\mu_1 - \mu_2, 2\sigma^2(1-\rho))$

2-4. 解: (1) 1° $(X_1, X_2 | X_3)$. $\mu = (\mu_1, \mu_2, \vdots\, \mu_3)'$ $\Sigma = \left(\begin{array}{cc|c} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \hline \rho & \rho & 1 \end{array}\right)$

$\mu_{1\cdot 2} = (\mu_1, \mu_2)' + \begin{pmatrix} \rho \\ \rho \end{pmatrix} \cdot 1 \cdot (X_3 - \mu_3) = \begin{pmatrix} \mu_1 + \rho(X_3 - \mu_3) \\ \mu_2 + \rho(X_3 - \mu_3) \end{pmatrix}$

$\Sigma_{11\cdot 2} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} - \begin{pmatrix} \rho \\ \rho \end{pmatrix} \cdot 1 \cdot (\rho\ \rho) = \begin{pmatrix} 1-\rho^2 & \rho-\rho^2 \\ \rho-\rho^2 & 1-\rho^2 \end{pmatrix}$.

$(X_1, X_2 | X_3) \sim N(\mu_{1\cdot 2}, \Sigma_{11\cdot 2})$

2°. $(X_1 | X_2, X_3)$ $\mu = (\mu_1 \vdots, \mu_2, \mu_3)$ $\Sigma = \left(\begin{array}{c|cc} 1 & \rho & \rho \\ \hline \rho & 1 & \rho \\ \rho & \rho & 1 \end{array}\right)$

$$\mu_{1\cdot 2} = (\mu_1) + (\rho, \rho)\frac{1}{1-\rho^2}\begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}\begin{pmatrix} x_2-\mu_2 \\ x_3-\mu_3 \end{pmatrix}$$

$$= \mu_1 + \frac{\rho}{1+\rho}[(x_2-\mu_2)+(x_3-\mu_3)].$$

$$\Sigma_{11\cdot 2} = \Sigma_{11} - \Sigma_{12}\cdot\Sigma_{22}^{-1}\Sigma_{21} = 1-(\rho,\rho)\frac{1}{1-\rho^2}\begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}\begin{pmatrix} \rho \\ \rho \end{pmatrix} = 1-\frac{2\rho^2}{1+\rho}.$$

$$(X_1 \mid X_2, X_3) \sim N(\mu_{1\cdot 2}, \Sigma_{11\cdot 2})$$

(2) 由(1)中求出的条件协方差矩阵知为 $\rho-\rho^2$.

2-6 解: (1) $3X_1 - 2X_2 + X_3 = (3,-2,1)\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$.

令 $A=(3,-2,1)$. $3X_1-2X_2+X_3 = AX \sim N(A\mu, A\Sigma A')$

$A\mu = 13$ $A\Sigma A' = (3,-2,1)\begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = (2,-1,1)\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = 9.$

$3X_1-2X_2+X_3 \sim N(13, 9)$

(2) $\mathrm{cov}\left(X_3, X_3 - (a_1, a_2)\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}\right)$

$= \cancel{\mathrm{cov}\left(X_3, (-a_1, -a_2, 1)\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}\right)} = D(X_3) - a_1\mathrm{cov}(X_3, X_1) - a_2(X_3, X_2)$

$= 2 - a_1 - 2a_2 = 0 \Rightarrow a_1 + 2a_2 = 2$ 即 $(a_1, a_2)'$ 满足该条件即可。

2-8. 证明: 首先 $\begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}X + \begin{pmatrix} d \\ c \end{pmatrix}$. 故 Y. Z 联合~~正态~~分布是多元正态分布.

Y. Z 独立 $\Leftrightarrow \mathrm{cov}(Y, Z) = 0 \Leftrightarrow \mathrm{cov}(AX+d, BX+c) = 0 \Leftrightarrow A\Sigma B' = O_{m\times k}$

2-13 证明：

(1) $\Sigma = E[(X-\mu)(X-\mu)']$

$= E[XX' - X\mu' - \mu X' + \mu\mu']$

$= E[XX'] - E[X]\mu' - \mu E[X'] + \mu\mu'$

$= E[XX'] - \mu\mu'$.

于是 $E[XX'] = \Sigma + \mu\mu'$

(2) 由(1)知

$E[X'AX] = E[tr(X'AX)] = \cancel{tr(E[X'AX])}\ E[tr(AXX')]$

$= tr(E[AXX']) = tr(AE[XX'])$

$= tr(A\Sigma + A\mu\mu')$

$= tr(\Sigma A) + \mu' A\mu$

(3) $E[X'AX] = tr(\sigma^2 I_p(I_p - \frac{1}{p}1_p 1_p')) + a1_p'(I_p - \frac{1}{p}1_p 1_p')a1_p$

$= tr[\sigma^2 I_p - \frac{\sigma^2}{p}1_p 1_p'] + a^2(1_p' 1_p - \frac{1}{p}1_p' 1_p 1_p' 1_p)$

$= p\sigma^2 - \sigma^2 + a^2(p-p) = \sigma^2(p-1)$.

2-14. 证明：p元正态分布的概率密度函数为

$f(x) = (2\pi)^{-\frac{p}{2}}|\Sigma|^{-\frac{1}{2}}\exp\{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\}$.

$\ln L(\mu,\Sigma) = -\frac{np}{2}\ln 2\pi - \frac{n}{2}\ln|\Sigma| - \frac{1}{2}tr[\Sigma^{-1}\sum_{i=1}^{n}(X_{(i)}-\mu)(X_{(i)}-\mu)']$

$\frac{\partial \ln L(\mu,\Sigma)}{\partial\mu} = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial tr[(X_{(i)}-\mu)'\Sigma^{-1}(X_{(i)}-\mu)]}{\partial(X_{(i)}-\mu)} = -\frac{1}{2}\sum_{i=1}^{n}(\Sigma^{-1}+(\Sigma^{-1})')(X_{(i)}-\mu) = 0$

由于 Σ^{-1} 正定. $\hat{\mu} = \bar{X}$

又 $\ln L(\mu,\Sigma) = -\frac{np}{2}\ln(2\pi) - \frac{n}{2}\ln|\Sigma| - \frac{1}{2}\sum_{i=1}^{n}(X_{(i)}-\mu)'\Sigma^{-1}(X_{(i)}-\mu)$

$= -\frac{np}{2}\ln(2\pi) - \frac{n}{2}\ln|\Sigma| - \frac{1}{2}\sum_{i=1}^{n}tr[(X_{(i)}-\mu)'\Sigma^{-1}(X_{(i)}-\mu)]$

$= -\frac{np}{2}\ln(2\pi) - \frac{n}{2}\ln|\Sigma| - \frac{1}{2}\sum_{i=1}^{n}tr((X_{(i)}-\mu)(X_{(i)}-\mu)'\Sigma^{-1})$

由矩阵微商相关结论知 $\frac{\partial \ln|X|}{\partial |X|} = (X^{-1})'$ $\frac{\partial tr(AX)}{\partial X} = A'$

令 $U = \Sigma^{-1}$. $\frac{\partial \ln L(\mu,\Sigma)}{\partial U} = \frac{n}{2}\frac{\partial \ln|U|}{\partial U} - \frac{1}{2}\sum_{i=1}^{n}\frac{\partial tr((X_{(i)}-\mu)(X_{(i)}-\mu)'U)}{\partial U}$

$= \frac{n}{2}(U^{-1})' - \frac{1}{2}\sum_{i=1}^{n}(X_{(i)}-\mu)'(X_{(i)}-\mu)$

$= \frac{n}{2}\Sigma - \frac{A'}{2}\sum_{i=1}^{n}(X_{(i)}-\bar{X})'(X_{(i)}-\bar{X}) = 0.$

$\hat{\Sigma} = \frac{A}{n}$

2-15 同 2-14. 有 $\frac{n}{2}\Sigma - \frac{1}{2}\sum_{i=1}^{n}(X_{(i)}-\mu_0)'(X_{(i)}-\mu_0) = 0$. 即

$\hat{\Sigma}_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(X_{(i)}-\mu_0)'(X_{(i)}-\mu_0).$