Homework 4

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PROBLEM 1.

(a) Every subspace of a Hausdorff space is Hausdorff.

(b) Every subspace of a second countable space is second countable.

SOLUTION.

(a) Let X be a Hausdorff space, and let Y be a subspace of X. Let x_1 and x_2 be elements of Y such that $x_1 \neq x_2$. Since X is Hausdorff, there exist disjoint neighborhoods U_1 and U_2 in X of x_1 and x_2, respectively. Hence a set containing x_1 in Y is $V_1 = U_1 \cap Y$, which is open in Y by definition of the subspace topology on Y. Thus V_1 is a neighborhood of x_1 in Y. Similarly, a set containing x_2 in Y is $V_2 = U_2 \cap Y$, which is open in Y by the definition of the subspace topology on Y. Thus V_2 is a neighborhood of x_2 in Y. Now since $V_1 \subset U_1$ and $V_2 \subset U_2$, and U_1 and U_2 are disjoint, it follows that V_1 and V_2 are disjoint. Thus, Y is Hausdorff.

(b) Let $(X, \mathcal{T})$ be second countable, and let $(A, \mathcal{T}_A)$ be a subspace. Since $(X, \mathcal{T})$ is second countable, let $\mathcal{B}$ be a countable basis for $\mathcal{T}$.

Now consider $\mathcal{B}' = \{B \cap A \mid B \in \mathcal{B}\}$. Then $\mathcal{B}'$ is a countable basis for $\mathcal{T}_A$. The subspace $(A, \mathcal{T}_A)$ is second countable.

$\square$

PROBLEM 2.

Let X be a topological space, and let $A \subset X$. Prove that if X is metrizable, then for any $x \in \bar{A}$, there is a sequence of points of A converging to x.

SOLUTION.

Step 1: We show X is first countable first. Let $(X, \mathcal{T})$ be a metrizable topological space and let $x \in X$. Since the space is metrizable, there is a metric d that induces $\mathcal{T}$.

We aim to show that

$$B_x = \left\{B_{1/n}(x) \mid n \in \mathbb{N}\right\}$$

is a countable neighborhood basis of x.

Note first that B_x is countable because the natural numbers are countable. Moreover, x is clearly a member of each set in B_x. And since d induces $\mathcal{T}$, the sets in B_x are open.

Since d induces $\mathcal{T}$, any open set U with $x \in U$ is a union of open balls with the metric d. One of these open balls contains x, so we have $B_r(x) \subset U$ for some radius r. Pick some number m such that $mr > 1$. Then

$$B_{1/m}(x) \subset B_r(x) \subset U.$$

Step 2: For each n we pick $x_n \in \left(\bigcap_{i=1}^{n} B_{1/i}\right) \cap A$. Now if U is any open neighbourhood of x, some $B_{1/N} \subset U$ and then all x_n for $n \geq N$ are in U. So $x_n \to x$, and we have the required sequence.

$\square$

PROBLEM 3.

Prove that a surjective topological embedding is a homeomorphism.

SOLUTION.

Suppose $f : X \to Y$ is a surjective topological embedding, so $f : X \to f(X)$ is a homeomorphism, but $f(X) = Y$ since f is surjective, so $f : X \to Y$ is a homeomorphism.

$\square$

PROBLEM 4.

Let X be a topological space. The **diagonal** of $X \times X$ is the subset $\Delta = \{(x, x) : x \in X\} \subseteq X \times X$. Show that X is Hausdorff if and only if Δ is closed in $X \times X$.

SOLUTION.

$\Leftarrow$ Suppose first that Δ is closed in $X \times X$. To show that X is Hausdorff, we must show that if x and y are any two points of X, then there are open sets U and V in X such that $x \in U, y \in V$, and $U \cap V = \emptyset$.

For $p = (x, y) \in X \times X$. Since $x \neq y$, $p \notin \Delta$. This means that p is in the open set $(X \times X) \setminus \Delta$. Thus, there must be an open set W in the product topology such that $p \in W \subset (X \times X) \setminus \Delta$. Consider the basis in the product topology are sets of the form $U \times V$, where U and V are open in X, so let $p \in U \times V \subseteq W$ for such $U, V \subset X$. Since $x \neq y$, $x \in U$ and $y \in V$, we have two disjoint neighborhoods U and V. Thus X is Hausdorff.

$\Rightarrow$ Now suppose that X is Hausdorff. To show that Δ is closed in $X \times X$, we need only show that $(X \times X) \setminus \Delta$ is open.

Take any point $p \in (X \times X) \setminus \Delta$. Since X is Hausdorff, for $x \neq y \in X$, there are disjoint open neighborhoods U and V containing x and y respectively. Let $W = U \times V$, then W is an open neighborhood of p, and $W \subseteq (X \times X) \setminus \Delta$. Thus $(X \times X) \setminus \Delta$ is open.

$\square$

PROBLEM 5.

Show that real projective space $\mathbb{P}^n$ is an n-manifold. [Hint: consider the subsets $U_i \subseteq \mathbb{R}^{n+1}$ where $x_i = 1$.]

SOLUTION.

By definition,

$$\mathbb{RP}^n = \left(\mathbb{R}^{n+1} \setminus \{0\}\right) / \sim,$$

where $x \sim y \Leftrightarrow \exists \lambda \in \mathbb{R} \setminus \{0\}$ such that $x = \lambda y$.

The topology on $\mathbb{RP}^n$ is, by definition, the quotient topology induced by the canonical projection

$$\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n \quad \text{where} \quad (x_0, \ldots, x_n) \mapsto [x_0, \ldots, x_n]$$

where $[x_0, \ldots, x_n] \in \mathbb{RP}^n$ denotes the equivalence class of $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$. This makes π a quotient map.

Second Countablity: Second countability simply follows from second countability of $\mathbb{R}^{n+1} \setminus \{0\}$.

Locally Euclidean: To show that $\mathbb{RP}^n$ is locally Euclidean, we need to exhibit a cover for $\mathbb{RP}^n$ by coordinate charts. For each $0 \leq i \leq n$, define

$$U_i \subset \mathbb{R}^{n+1} \setminus \{0\} \quad \text{by} \quad U_i = \left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\} : x_i = 1\right\}.$$

U_i is an open subset of $\mathbb{R}^{n+1} \setminus \{0\}$. Define $V_i \subset \mathbb{RP}^n$ to be $\pi(U_i)$. Then, V_i is an open subset of $\mathbb{RP}^n$ and $\pi_i = \pi|_{U_i}$ is also a quotient map. The sets $V_i, 0 \leq i \leq n$, form an open cover of $\mathbb{RP}^n$.

For each $0 \leq i \leq n$, define the map

$$f_i : V_i \to \mathbb{R}^n \quad \text{by} \quad f_i[x_0, \ldots, x_n] = (x_0, x_{i-1}, x_{i+1}, \ldots, x_n)\,.$$

The map $g_i = f_i \circ \pi_i : U_i \to \mathbb{R}^n$ is given by

$$g_i(x_0, \ldots, x_n) = (x_0, x_{i-1}, \ldots, x_n)\,.$$

Since g_i is continuous, by the characteristic property of quotient maps, f_i is also continuous.

By our definition of f_i ,it is bijective. Moreover, for each $0 \leq i \leq n$, consider the map

$$h_i : \mathbb{R}^n \to \mathbb{R}^{n+1} \backslash \{0\} \quad \text{given by} \quad h_i(u_1, \ldots, u_n) = (u_1, \ldots, u_i, 1, u_{i+1}, \ldots, u_n).$$

Then, h_i is continuous and its image is contained in V_i. Since $\pi_i \circ h_i = \varphi_i^{-1}$. So, φ_i^{-1} is continuous.

Hence, φ_i is a homeomorphism for each $0 \leq i \leq n$. Now we have shown that $\mathbb{RP}^n$ is locally Euclidean.

Hausdorff:

To show that $\mathbb{RP}^n$ is Hausdorff, choose $\bar{x}$ and $\bar{y}$, two distinct points in $\mathbb{RP}^n$.

If there exists $0 \leq i \leq n$ such that both points lie in V_i, then $f_i(\bar{x})$ and $f_i(\bar{y})$ are two distinct points in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is Hausdorff, there exists a pair of disjoint open sets A and B with $f_i(\bar{x}) \in A$ and $f_i(\bar{y}) \in B$. Hence, $f_i^{-1}(A)$ and $f_i^{-1}(B)$ are disjoint open subsets of V_i (and hence of $\mathbb{RP}^n$) such that $\bar{x} \in f_i^{-1}(A)$ and $\bar{y} \in f_i^{-1}(B)$.

On the other hand, suppose there is no $i, 0 \leq i < n$, such that $\bar{x}$ and $\bar{y}$ both lie in V_i. Let $(x_0, \ldots, x_n)$ and $(y_0, \ldots, y_n)$ be representatives of $\bar{x}$ and $\bar{y}$, respectively. There exists $i \neq j, 0 \leq i, j \leq n$, such that

$$x_i \neq 0, y_j \neq 0, \quad \text{and} \quad x_j = 0, y_i = 0.$$

Fix the representatives so that $x_i = 1 = y_j$. WLOG, let $i < j$. Choose $0 < \varepsilon < 1$. The sets

$$A = \{[a_0, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n] : |a_k - x_k| < \varepsilon, k \neq i\} \subset V_i$$
$$B = \{[b_0, \dots, b_{j-1}, 1, b_{j+1}, \dots, b_n] : |b_k - y_k| < \varepsilon, k \neq j\} \subset V_j$$

are open sets containing $\bar{x}$ and $\bar{y}$, respectively. This is because $f_i(A)$ is an open rectangle in $\mathbb{R}^n$ centered on $f_i(\bar{x})$ having side length 2ε, and similarly $f_j(B)$ is an open rectangle in $\mathbb{R}^n$ centered on $f_j(\bar{y})$ having side length 2ε. They are disjoint because if

$$[a_0, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n] = [b_0, \dots, b_{j-1}, 1, b_{j+1}, \dots, b_n],$$

then we must have $a_j \neq 0, b_i \neq 0$, and $a_j b_i = 1$. But, $|a_j| < 1$ and $|b_i| < 1$, so this is not possible.

Hence, $\mathbb{RP}^n$ is Hausdorff, and so $\mathbb{RP}^n$ is an n-manifold.

$\square$