兰州大学
LANZHOU UNIVERSITY

习题二.

1. 解: $P=\frac{3}{10}\times\frac{3}{9}\times\frac{1}{8}\times\frac{2}{7}\times\frac{2}{6}\times\frac{2}{5}\times\frac{1}{4}\times\frac{1}{3}\times\frac{1}{2}=\frac{1}{50400}$.

3. 解: $P(A)=\frac{\#(A)}{\#(\Omega)}=\frac{C_a^0C_b^2C_\alpha^2+C_a^1C_b^1C_{\alpha+1}^2+C_a^2C_b^0C_{\alpha+2}^2}{C_{a+b}^2\,C_{\alpha+\beta+2}^2}$.

4. 证明: 设一个家庭中有n个小孩为事件An. 有k个男孩为事件Bk.

$P(A_n)=\alpha p^n$. $P(B_k|A_n)=\cancel{\alpha p^n}\,C_n^k(\frac{1}{2})^n$

由全概率公式. $P(B_k)=\sum_{n=k}^{\infty}P(A_n)P(B_k|A_n)=\sum_{n=k}^{\infty}\alpha p^n\cdot C_n^k(\frac{1}{2})^n$.

$\cancel{=\alpha\sum_{i=0}^{\infty}(\frac{p}{2})^{k+i}\cdot C_{k+i}^k=\alpha\cdot(\frac{p}{2})^k\cdot\sum_{i=0}^{\infty}(\frac{p}{2})^i\cdot C_{k+i}^{i}}$.

令 $I_k=\sum_{n=k}^{\infty}C_n^k(\frac{p}{2})^n$. 注意到:

$$I_{k+1}-\frac{p}{2-p}I_k=\sum_{n=k+1}^{\infty}C_n^{k+1}(\frac{p}{2})^n-\frac{p}{2-p}\sum_{n=k}^{\infty}C_n^k(\frac{p}{2})^n$$

$$=\sum_{n=k+1}^{\infty}[C_n^{k+1}-\frac{p}{2-p}C_n^k)(\frac{p}{2})^n-\frac{p}{2-p}\cdot(\frac{p}{2})^k$$

$$=\sum_{n=k+1}^{\infty}[C_n^{k+1}-\frac{p}{2-p}(C_{n+1}^{k+1}-C_n^{k+1})](\frac{p}{2})^n-\frac{p}{2-p}\cdot(\frac{p}{2})^k$$

$$=\sum_{n=k+1}^{\infty}[\frac{p}{2-p}C_n^{k+1}(\frac{p}{2})^{n-1}-\frac{p}{2-p}C_{n+1}^{k+1}(\frac{p}{2})^n]-\frac{p}{2-p}\cdot(\frac{p}{2})^k$$

$$=\frac{p}{2-p}\cdot(\frac{p}{2})^k-\frac{p}{2-p}(k+2)(\frac{p}{2})^{k+1}+\frac{p}{2-p}(k+2)(\frac{p}{2})^{k+1}\cdots-\frac{p}{2-p}\cdot(\frac{p}{2})^k$$

$$=0.$$

$\{I_k\}$是一个公比$\frac{p}{2-p}$的等比数列 $I_0=\sum_{n=0}^{\infty}C_n^0(\frac{p}{2})^n=\sum_{n=0}^{\infty}(\frac{p}{2})^n=\frac{1}{1-\frac{p}{2}}=\frac{2}{2-p}$

$I_k=I_0\cdot(\frac{p}{2-p})^k=\frac{2p^k}{(2-p)^{k+1}}$ $P(B_k)=\frac{2\alpha p^k}{(2-p)^{k+1}}$.

5、解：(1). 令A为事件"家庭中至少有一个男孩". B为事件"家庭中至少有两个男孩"

$$P(AB)=P(B)=\sum_{k=2}^{\infty}\frac{2\alpha p^k}{(2-p)^{k+1}}=\frac{2\alpha}{2-p}\cdot\frac{p^2}{(2-p)^2}\cdot\frac{1}{1-\frac{p}{2-p}}=\frac{\alpha p^2}{(2-p)^2(1-p)}$$

$$P(A)=\sum_{k=1}^{\infty}\frac{2\alpha p^k}{(2-p)^{k+1}}=\frac{2\alpha}{2-p}\cdot\frac{p}{2-p}\cdot\frac{1}{1-\frac{p}{2-p}}=\frac{\alpha p}{(2-p)(1-p)}.$$

$$P(B|A)=\frac{P(AB)}{P(A)}=\frac{P(B)}{P(A)}=\frac{p}{2-p}$$

(2). 令C为事件"家中没有女孩". D为事件"家中正好有一个男孩".

$$P(CD)=P(D)=\cancel{\frac{2\alpha p}{(2-p)^2}}\ \alpha p\times\frac{1}{2}=\frac{1}{2}\alpha p.$$

$$P(C)=1-\frac{\alpha p}{1-p}+\sum_{n=1}^{\infty}\alpha p^n\left(\frac{1}{2}\right)^n=1-\frac{\alpha p}{1-p}+\frac{\frac{\alpha p}{2}}{1-\frac{p}{2}}=\frac{p^2-\alpha p-3p+2}{(1-p)(2-p)}$$

$$P(D|C)=\frac{P(CD)}{P(C)}=\frac{\alpha p(1-p)(2-p)}{2(p^2-\alpha p-3p+2)}$$

16、证明： 三口之家.

$P(A)=P\{3个男孩\}+P\{2个男孩1个女孩\}$

$=\frac{1}{8}+3\times\frac{1}{8}=\frac{1}{2}.$

$P(B)=P\{1男2女\}+P\{2男1女\}=\frac{3}{8}+\frac{3}{8}=\frac{3}{4}.$

$P(AB)=\frac{3}{8}=P(A)P(B)$，故A·B独立

四口之家：

$P(A)=P\{4男\}+P\{3男1女\}$

$=\frac{1}{16}+4\times\frac{1}{16}=\frac{5}{16}$

$P(B)=P\{1女3男\}+P\{2女2男\}+P\{3女1男\}$

$=\frac{4}{16}+\frac{6}{16}+\frac{4}{16}=\frac{7}{8}$

$P(AB)=\{1女3男\}=\frac{4}{16}\neq P(A)P(B)$ 故A·B不独立

19. 证明："⇒"、不妨设 $\hat{A}_1,\cdots,\hat{A}_n$ 中前m个为 $\bar{A}_i$，后n-m个为 A_i.

当 $m=0$、$P(A_1A_2\cdots A_n)=P(A_1)P(A_2)\cdots P(A_n)$.

$m=1$、
$$P(\bar{A}_1A_2\cdots A_n)=P(A_2\cdots A_n)-P(A_1A_2\cdots A_n)$$
$$=P(A_2)\cdots P(A_n)-P(A_1)P(A_2)\cdots P(A_n)$$
$$=P(\bar{A}_1)P(A_2)\cdots P(A_n).$$

设 $m=k$ 时 $P(\bar{A}_1\bar{A}_2\cdots\bar{A}_kA_{k+1}\cdots A_n)=P(\bar{A}_1)\cdots P(\bar{A}_k)P(A_{k+1})\cdots P(A_n)$.

则 $m=k+1$ 时
$$P(\bar{A}_1\bar{A}_2\cdots\bar{A}_{k+1}A_{k+2}\cdots A_n)=P(\bar{A}_1\bar{A}_2\cdots\bar{A}_kA_{k+2}\cdots A_n)-P(\bar{A}_1\bar{A}_2\cdots\bar{A}_kA_{k+1}\cdots A_n)$$
$$\cancel{=P(A_1A_2\cdots A}$$
$$=P(\bar{A}_1)P(\bar{A}_2)\cdots P(\bar{A}_k)P(A_{k+2})\cdots P(A_n)-P(\bar{A}_1)\cdots P(\bar{A}_k)P(A_{k+1})\cdots P(A_n)$$
$$=P(\bar{A}_1)P(\bar{A}_2)\cdots P(\bar{A}_{k+1})P(A_{k+2})\cdots P(A_n).$$

从而有 2^n 个等式成立.

"⇐". $P(A_1A_2\cdots A_n)=P(A_1)P(A_2)\cdots P(A_n)$,
$$P(A_1\cdots A_{i-1}A_{i+1}\cdots A_n)=P(A_1\cdots A_{i-1}A_iA_{i+1}\cdots A_n)-P(A_1\cdots A_{i-1}\bar{A}_iA_{i+1}\cdots A_n).$$
$$=P(A_1)\cdots P(A_{i-1})P(A_i)P(A_{i+1})\cdots P(A_n)+$$
$$P(A_1)\cdots P(A_{i-1})P(\bar{A}_i)P(A_{i+1})\cdots P(A_n).$$
$$=P(A_1)\cdots P(A_{i-1})P(A_{i+1})\cdots P(A_n).$$

由归纳推理可知它能满足独立性.

数学萃英班
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概率论作业

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33. 解：设第n回时出正面的概率为p_n. 由题 $p_1=c$.

有递推关系式 $p_n = p\,p_{n-1} + (1-p)(1-p_{n-1}) = (2p-1)p_{n-1} + 1-p$.

注意到上式可化为 $p_n - \frac{1}{2} = (2p-1)(p_{n-1} - \frac{1}{2})$.

故 $p_n - \frac{1}{2} = (p_1 - \frac{1}{2})(2p-1)^{n-1}$，即 $p_n = (c-\frac{1}{2})(2p-1)^{n-1} + \frac{1}{2}$

故第n回时出正面的概率为 $(c-\frac{1}{2})(2p-1)^{n-1} + \frac{1}{2}$.

$0<p<1 \quad -1<2p-1<1$，$\lim\limits_{n\to\infty} (c-\frac{1}{2})(2p-1)^{n-1} + \frac{1}{2} = \frac{1}{2}$.

34. 解：$p_{n+1} = \frac{1}{4}q_n$，$q_{n+1} = p_n + \frac{1}{2}q_n + r_n$，$r_{n+1} = \frac{1}{4}q_n$.

由题 $p_0 = 0$，$q_0 = 1$，$r_0 = 0$.

$p_1 = \frac{1}{4} \quad q_1 = \frac{1}{2} \quad r_1 = \frac{1}{4}$.

注意到 $p_n = r_n$，

$q_{n+1} = 2p_n + \frac{1}{2}q_n = \frac{1}{2}q_{n-1} + \frac{1}{2}q_n$.

$q_{n+1} + \frac{1}{2}q_n = q_n + \frac{1}{2}q_{n-1} = \cdots = q_1 + \frac{1}{2}q_0 = 1$.

$q_{n+1} = -\frac{1}{2}q_n + 1$.

$q_{n+1} - \frac{2}{3} = -\frac{1}{2}(q_n - \frac{2}{3})$

$q_{n+1} = \frac{1}{3}\times(-\frac{1}{2})^{n+1} + \frac{2}{3} = \frac{2}{3}[1-(-\frac{1}{2})^{n+2}]$.

$p_{n+1} = r_{n+1} = \frac{1}{2}(1-q_{n+1}) = \frac{1}{6} + \frac{1}{3}\times(-\frac{1}{2})^{n+2}$

$\lim\limits_{n\to\infty} q_{n+1} = \lim\limits_{n\to\infty} \frac{2}{3}[1-(-\frac{1}{2})^{n+2}] = \frac{2}{3}$.

$\lim\limits_{n\to\infty} p_{n+1} = \lim\limits_{n\to\infty} r_{n+1} = \lim\limits_{n\to\infty} [\frac{1}{6} + \frac{1}{3}\times(-\frac{1}{2})^{n+2}] = \frac{1}{6}$.

37. 解：~~作出~~负南大正确~~决策~~意见的概率服从二项分布 $B(7, 0.6)$.

至少有4个人负南大正确意见才可作出正确决策

$$p = \sum_{k=4}^{7} C_7^k (0.6)^k (0.4)^{7-k} = 35\times0.6^4\times0.4^3 + 21\times0.6^5\times0.4^2 + 7\times0.6^6\times0.4 + 0.6^7$$

$$= 0.710208.$$

39. 解：设销售量 x，进货量 m.　$x \sim p(7)$.

$P(x \le m) \ge 0.999$. 即 $P(x > m) < 0.001$,

$\sum_{k=m+1}^{\infty} P(x=k) = \sum_{k=m+1}^{\infty} \frac{7^k}{k!} e^{-7} < 0.001$ 经查表 $m=16$ 时满足条件

40. 解：设应装 $100+k$ 只.

$$p = \sum_{i=0}^{k} C_{100+k}^{i}\, 0.015^i \cdot 0.985^{100+k-i} \ge 80\%.$$

本题中 $b(100+k, 0.015) \to \frac{1.5^i}{i!} e^{-1.5}$ 于是

$\lambda = (100+k)(0.015) \approx 1.5$.

$\sum_{i=0}^{k} \frac{1.5^i}{i!} e^{-1.5} \ge 80\%$. 经查表. $k=2$

于是应装 102 只.

41. 解：(1) 每试管2毫升中所含细菌数服从泊松分布 $P(2)$.

故 5只试管中都有细菌的概率为 $(1-\frac{2^0}{0!}e^{-2})^5 = (1-e^{-2})^5 \approx 0.48$

(2) $p = \sum_{i=3}^{5} C_5^i (1-e^{-2})^i e^{-2(5-i)} = 0.9801$

44. 解：设一分钟内的车流量服从 $p(\lambda)$. 于是

$$e^{-\lambda}=0.2 \qquad \lambda=\ln 5.$$

则两分钟内车流量服从 $p(2\lambda)$.

$$p=1-e^{-2\ln 5}-2\ln 5\, e^{-2\ln 5}=0.8312$$

47. 解：由于基数大，可视作100件中次品数服从二项分布．假设合格率99%．则抽出至少2件次品的概率为

$$p=1-(0.99)^{100}-100\times 0.01\times(0.99)^{99}$$

$$=1-e^{-1}-e^{-1}=0.2642$$

概率不可忽略．因此不能断言．