Homework 9

萃英学院　　2022 级　　王一鑫

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Problem 1. (Exercise 5.2)

Find the chromatic number of each graph in Fig 1.

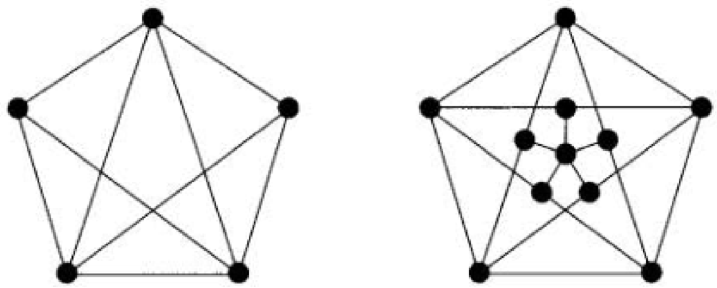


图 1: Figure for Problem 1.

Solution. Apply Theorem 5.2. The largest vertex-degree of the left graph is 4 and the right one is 5. Thus the left graph is 4-colourable and the right one is 5-colourable.

Moreover, the left graph is not 3-colourable and the right one is not 4-colourable. Thus the chromatic number of the left graph is 4 and the right one is 5.

□

Problem 2. (Exercise 5.6)

A lecture timetable is to be drawn up. Since some students wish to attend several lectures, certain lectures must not coincide, as shown by asterisks in the following table. How many periods are needed to timetable all seven lectures?

	a	b	c	d	e	f	g
a	-	*	*	*	-	-	*
b	*	-	*	*	*	-	*
c	*	*	-	*	-	*	-
d	*	*	*	-	-	*	-
e	-	*	-	-	-	*	-
f	-	-	*	*	-	-	*
g	*	*	-	-	-	*	-

Solution.

To answer this, we draw the graph whose vertices correspond to the seven lectures, with two vertices adjacent whenever te corresponding lectures are to be kept apart, see Fig 2. Now we colour the vertices, then the colours correspond to the periods needed, thus 4 periods are needed.

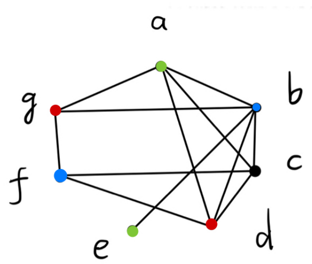


图 2: Corresponding graph.

□

Problem 3. (Exercise 5.7)

Let G be a simple graph with n vertices, which is regular of degree d. By considering the number of vertices that can be assigned the same colour, prove that $\chi(G) \geq \frac{n}{n-d}$.

Solution.

If c_i is the number of vertices coloured i, for $1 \leq i \leq \chi(G)$, then $c_i \leq n-d$.

Thus,

$$n \leq c_1 + c_2 + \cdots + c_{\chi(G)} \leq \chi(G) \times (n-d),$$

and so

$$\chi(G) \geq \frac{n}{n-d}.$$

□

Problem 4. (Exercise 5.8)

Let G be a simple planar graph containing no triangles.

(i) Using Euler's formula, show that G contains a vertex of degree at most 3.

(ii) Use induction to deduce that G is 4-colourable.
(In fact, it can be proved that G is 3-colourable.)

Solution.

(i) Without loss of generality we can assume that G is connected and has at least three vertices. If each vertex has degree at least 4, then, with the above notation, we have

$$4n \leq 2m, \text{ so } 2n \leq m.$$

It follows immediately from Corollary 4.8(ii) that

$$2n \leq 2n - 4,$$

which is a contradiction.

(ii) We prove the theorem by induction on the number of vertices. Suppose then that G is a simple planar graph with n vertices, and that all simple planar graphs with $n-1$ vertices are 4-colourable. By (i), G contains a vertex v of degree at most 3. If we delete v and its incident edges, then the graph that remains has $n-1$ vertices and is thus 4-colourable. A 4-colouring of G is then obtained by colouring v with a colour different from the (at most three) vertices adjacent to v.

□

Problem 5. (Exercise 5.11)

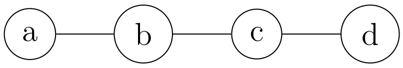
(i) Find the chromatic polynomials of the six connected simple graphs on four vertices.

(ii) Verify that each of the polynomials in part (i) has the form

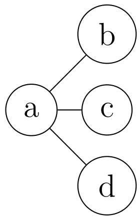
$$k^4 - mk^3 + ak^2 - bk,$$

where m is the number of edges, and a and b are positive constants.

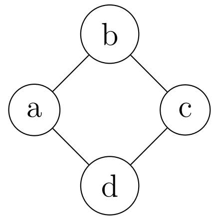
Solution. All 6 graphs are shown in Fig 3.



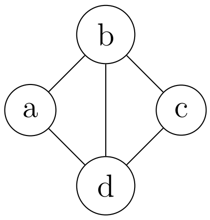
(a) Path P_4



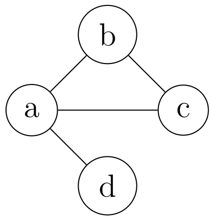
(b) Star $K_{1,3}$



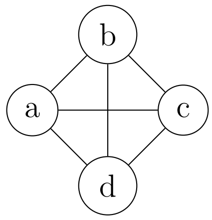
(c) Cycle C_4



(d) Diamond



(e) Kite graph



(f) Complete graph K_4

图 3: All six connected simple graphs on four vertices

(i)

- **1. Path graph P_4:**

$$P_{P_4}(k) = k(k-1)^3$$

- **2. Star graph $K_{1,3}$:**

$$P_{K_{1,3}}(k) = k(k-1)^3$$

- **3. Cycle graph C_4:**

$$P_{C_4}(k) = k(k-1)^2(k-2)$$

- **4. Diamond graph (i.e., K_4 minus one edge):**

$$P_{\text{diamond}}(k) = k(k-1)(k-2)^2$$

- **5. Kite graph:**

$$P_{\text{kite}}(k) = k(k-1)^2(k-2)$$

- **6. Complete graph K_4:**

$$P_{K_4}(k) = k(k-1)(k-2)(k-3)$$

(ii)

- **1. Path graph P_4:**

$$P_{P_4}(k) = k(k-1)^3 = k^4 - 3k^3 + 3k^2 - k$$

- **2. Star graph $K_{1,3}$:**

$$P_{K_{1,3}}(k) = k(k-1)^3 = k^4 - 3k^3 + 3k^2 - k$$

- **3. Cycle graph C_4:**

$$P_{C_4}(k) = k(k-1)^2(k-2) = k^4 - 4k^3 + 6k^2 - 3k$$

- **4. Diamond graph:**

$$P_{\text{diamond}}(k) = k(k-1)^2(k-2) = k^4 - 5k^3 + 10k^2 - 8k$$

- **5. Kite graph:**

$$P_{\text{kite}}(k) = k(k-1)(k-2)^2 = k^4 - 4k^3 + 5k^2 - 2k$$

- **6. Complete graph K_4:**

$$P_{K_4}(k) = k(k-1)(k-2)(k-3) = k^4 - 6k^3 + 11k^2 - 6k$$

So the polynomials in part (i) has the form

$$k^4 - mk^3 + ak^2 - bk,$$

where m is the number of edges, and a and b are positive constants.

□

Problem 6. (Exercise 5.13)

(i) Prove that the chromatic polynomial of $K_{2,s}$ is

$$k(k-1)^s + k(k-1)(k-2)^s.$$

(ii) Prove that the chromatic polynomial of C_n is

$$(k-1)^n + (-1)^n(k-1).$$

Solution.

(i) Let the two parts of $K_{2,s}$ be:

- Set $A = \{a_1, a_2\}$
- Set $B = \{b_1, b_2, \ldots, b_s\}$

We compute the number of proper k-colorings of $K_{2,s}$ by separating into two cases:

Case 1: a_1 and a_2 are assigned the same color.

There are:

- k choices for the common color of a_1 and a_2;
- Each vertex in B must then avoid this color: $(k-1)^s$ total choices.

So this case contributes:

$$k(k-1)^s.$$

Case 2: a_1 and a_2 are assigned different colors.

There are:

- $k(k-1)$ ways to assign different colors to a_1 and a_2;
- Each vertex in B must avoid both of these colors: $(k-2)^s$ total choices.

So this case contributes:

$$k(k-1)(k-2)^s.$$

Therefore,

$$P_{K_{2,s}}(k) = k(k-1)^s + k(k-1)(k-2)^s.$$

(ii) Apply Theorem 5.6.

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k),$$

Choose any edge e of C_n. Then:

- $C_n - e$ becomes a path graph P_n.
- C_n/e becomes another cycle, but with $n-1$ vertices, i.e., C_{n-1}.

Thus, applying the deletion-contraction formula to C_n, we have the recurrence:

$$P_{C_n}(k) = P_{P_n}(k) - P_{C_{n-1}}(k).$$

We already know the chromatic polynomial of a path graph P_n is:

$$P_{P_n}(k) = k(k-1)^{n-1}.$$

Therefore, the recurrence becomes:

$$P_{C_n}(k) = k(k-1)^{n-1} - P_{C_{n-1}}(k).$$

and

$$P_{C_n}(k) = (k-1)^n + (-1)^n(k-1).$$

satisfies the relation.

□

PROBLEM 7. (Exercise 5.14)

Prove that, if G is a disconnected simple graph, then its chromatic polynomial $P_G(k)$ is the product of the chromatic polynomials of its components. What can you say about the degree of the lowest non-vanishing term?

SOLUTION.

If G is disconnected, then its components $G_1, G_2, \ldots, G_r$ are vertex-disjoint and have no edges between them. This means that the colorings on different components are completely independent, a coloring on one component has no impact on the valid colorings of the others.

Therefore, to obtain a proper coloring of the entire graph G, we simply choose a proper coloring for each component G_i independently. The total number of proper colorings of G is the product of the number of proper colorings of each G_i. Thus,

$$P_G(k) = P_{G_1}(k) \cdot P_{G_2}(k) \cdots P_{G_r}(k) = \prod_{i=1}^{r} P_{G_i}(k).$$

Each chromatic polynomial $P_{G_i}(k)$ has lowest non-zero term of degree 1. Thus, for G with r connected components, the lowest non-zero term of the product polynomial $P_G(k)$ has degree r, since the product of r polynomials each having lowest degree term k will yield a term of k^r as the lowest-degree nonzero term.

$\square$

PROBLEM 8. (Exercise 5.16)

(i) Use the results of Exercises 5.14 and 5.15 to prove that, if

$$P_G(k) = k(k-1)^n,$$

then G is a tree on n vertices.

(ii) Find three graphs with chromatic polynomial

$$k^5 - 4k^4 + 6k^3 - 4k^2 + k.$$

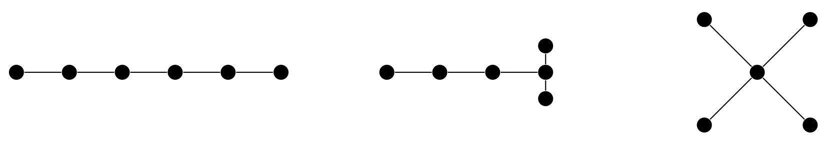
SOLUTION.

(i) Since

$$k(k-1)^{n-1} = k^n - (n-1)k^{n-1} + \cdots + (-1)^{n-1}k,$$

and G has n vertices, $n-1$ edges and one component, it follows from Theorem 3.1(iii) that G is a tree on n vertices.

(ii) Since $P_G(k) = k(k-1)^4$, G must be a tree on five vertices:



$\square$