多元统计分析 - HW2.

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3-1. 证明：由A为n阶对称幂等矩阵知 $A^2-A=0$

故A只有特征值0和1. 此外 由 $\mathrm{rank}(A)=r$ 知

A有r重特征值1与n-r重特征值0.

于是存在正交矩阵T, s.t. $T'AT=\begin{pmatrix} I_r & 0 \\ 0 & O_{n-r}\end{pmatrix}$

令 $X=TY$. $\cancel{X'AX=Y'T'ATY=}$ $Y=T^{-1}X=T'X$.

$D(Y)=T'D(X)T=T'(\sigma^2 I_n)T=\sigma^2 I_n$

$E(Y)=T'E(X)=T'\mu$

因此 $Y\sim N_n(T'\mu,\sigma^2 I_n)$. 故 $\frac{Y}{\sigma}\sim N_n(\frac{T'}{\sigma}\mu, I_n)$

而 $\frac{1}{\sigma^2}X'AX=\frac{1}{\sigma^2}(TY)'A(TY)=\frac{1}{\sigma^2}Y'\begin{pmatrix} I_r & 0 \\ 0 & O_{n-r}\end{pmatrix}Y$

记 $B=\begin{pmatrix} I_r & 0 \\ 0 & O_{n-r}\end{pmatrix}$. 有分解 $B=B'B$

故 $\frac{1}{\sigma^2}X'AX=\frac{1}{\sigma^2}Y'BY=(\frac{1}{\sigma}Y')B'B(\frac{1}{\sigma}Y)=(\frac{1}{\sigma}BY)'(\frac{1}{\sigma}BY)$

而 $\frac{1}{\sigma}BY\sim N_r(\frac{BT'}{\sigma}\mu, I_r)$. 由非中心参数定义知

$\delta=(\frac{BT'}{\sigma}\mu)'(\frac{BT'}{\sigma}\mu)=\frac{1}{\sigma^2}\mu'TB'BT'\mu=\frac{1}{\sigma^2}\mu'A\mu$

3-3. 证明：因 $\Sigma>0$. 则有 $\mathrm{rank}(\Sigma)=p$. $\Sigma=\Sigma^{\frac{1}{2}}\cdot\Sigma^{\frac{1}{2}}$.

令 $Y=\Sigma^{-\frac{1}{2}}(X-\mu)\sim N_p(0, I_p)$.

$\xi=(X-\mu)'A(X-\mu)=Y'\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}Y:=Y'CY$.

$\eta=(X-\mu)'B(X-\mu)=Y'\Sigma^{\frac{1}{2}}B\Sigma^{\frac{1}{2}}Y:=Y'DY$

由3-2知, $Y'CY$ 与 $Y'DY$ 相互独立 $\Leftrightarrow CD=0$

$\Sigma^{\frac{1}{2}}A\Sigma^{\frac{1}{2}}\Sigma^{\frac{1}{2}}B\Sigma^{\frac{1}{2}}=0 \Leftrightarrow \Sigma A\Sigma B\Sigma=0$.

A+

3-4. Wishart分布性质4.

将 $X_{(\alpha)}$ 割分成两部分. $X_{(\alpha)} = \begin{pmatrix} X_{(\alpha)}^{(1)} \\ \cdots \\ X_{(\alpha)}^{(2)} \end{pmatrix} \begin{matrix} r \\ p-r \end{matrix} \sim N_p(0, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix})$

显然 $X_{(\alpha)}^{(1)} \sim N_r(0, \Sigma_{11})$ $X_{(\alpha)}^{(2)} \sim N_{p-r}(0, \Sigma_{22})$

$$W = \sum_{\alpha=1}^{n} X_{(\alpha)} X_{(\alpha)}' = \sum_{\alpha=1}^{n} \begin{pmatrix} X_{(\alpha)}^{(1)} \\ X_{(\alpha)}^{(2)} \end{pmatrix} (X_{(\alpha)}^{(1)'}, X_{(\alpha)}^{(2)'}) = \sum_{\alpha=1}^{n} \begin{pmatrix} X_{(\alpha)}^{(1)} X_{(\alpha)}^{(1)'} & X_{(\alpha)}^{(1)} X_{(\alpha)}^{(2)'} \\ X_{(\alpha)}^{(2)} X_{(\alpha)}^{(1)'} & X_{(\alpha)}^{(2)} X_{(\alpha)}^{(2)'} \end{pmatrix}$$

这说明 $W_{11} = \sum_{\alpha=1}^{n} X_{(\alpha)}^{(1)} X_{(\alpha)}^{(1)'}$ $X_{(\alpha)}^{(1)} \sim N_r(0, \Sigma_{11})$

$W_{22} = \sum_{\alpha=1}^{n} X_{(\alpha)}^{(2)} X_{(\alpha)}^{(2)'}$ $X_{(\alpha)}^{(2)} \sim N_{p-r}(0, \Sigma_{22})$

由定义可知 $W_{11} \sim W_r(n, \Sigma_{11})$. $W_{22} \sim W_{p-r}(n, \Sigma_{22})$.

又由 $\Sigma_{12} = 0$ 知 $X_{(\alpha)}^{(1)}$ 与 $X_{(\alpha)}^{(2)}$ 相互独立. 进一步. $W_{11}$ 只与 $X_{(\alpha)}^{(1)}$ 有关.

$W_{22}$ 只与 $X_{(\alpha)}^{(2)}$ 有关. 因此 $W_{11}$, $W_{22}$ 相互独立.

Hotelling $T^2$ 分布性质5.

$Y_{(\alpha)} = CX_{(\alpha)} + d$ $(\alpha = 1, 2, \cdots, n)$.

故 $Y_{(\alpha)} \sim N_p(C\mu, C\Sigma C')$ $(\alpha = 1, 2, \cdots, n)$

$\bar{Y} = C\bar{X} + d$.

$$A_y = \sum_{\alpha=1}^{n} (Y_{(\alpha)} - \bar{Y})(Y_{(\alpha)} - \bar{Y})'$$

$$= \sum_{\alpha=1}^{n} (CX_{(\alpha)} - C\bar{X})(CX_{(\alpha)} - C\bar{X})'$$

$$= C \sum_{\alpha=1}^{n} (X_{(\alpha)} - \bar{X})(X_{(\alpha)} - \bar{X})' C'$$

故 $T_y^2 = n(n-1)(\bar{Y} - \mu_y)' A_y^{-1} (\bar{Y} - \mu_y)$

$$= n(n-1)(\bar{X} - \mu)' C' [C A_x C']^{-1} C(\bar{X} - \mu)$$

$$= n(n-1)(\bar{X} - \mu)' A_x^{-1} (\bar{X} - \mu) = T_x^2$$

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3-5. 解: $f_{X_i}(X_i) = (2\pi)^{-\frac{p}{2}}|\Sigma|^{-\frac{1}{2}}\exp\{-\frac{1}{2}(X_i-\mu)'\Sigma^{-1}(X_i-\mu)\}$

$$L(\mu) = \prod_{i=1}^{n} f_{X_i}(X_i) = (2\pi)^{-\frac{np}{2}}|\Sigma|^{-\frac{n}{2}}\exp\{-\frac{1}{2}\sum_{i=1}^{n}(X_i-\mu)'\Sigma^{-1}(X_i-\mu)\}$$

$$\ln L(\mu) = -\frac{np}{2}\ln(2\pi)-\frac{n}{2}\ln|\Sigma|-\frac{1}{2}\sum_{i=1}^{n}(X_i-\mu)'\Sigma^{-1}(X_i-\mu).$$

$\mu$的极大似然估计为$\bar{X}$.

$$\lambda = \frac{\max_{\mu=\mu_0} L(\mu,\Sigma_0)}{\max_{\mu\neq\mu_0} L(\mu,\Sigma_0)}$$

$$= \frac{|2\pi\Sigma_0|^{-\frac{n}{2}}\exp\{-\frac{1}{2}\sum_{i=1}^{n}(X_i-\mu_0)'\Sigma_0^{-1}(X_i-\mu_0)\}}{|2\pi\Sigma_0|^{-\frac{n}{2}}\exp\{-\frac{1}{2}\sum_{i=1}^{n}(X_i-\bar{X})'\Sigma_0^{-1}(X_i-\bar{X})\}}.$$

$$= \frac{\exp\{-\frac{1}{2}\sum_{i=1}^{n}tr[(X_i-\mu_0)'\Sigma_0^{-1}(X_i-\mu_0)]\}}{\exp\{-\frac{1}{2}\sum_{i=1}^{n}tr[(X_i-\bar{X})'\Sigma_0^{-1}(X_i-\bar{X})]\}}$$

$$= \frac{\exp\{-\frac{1}{2}\sum_{i=1}^{n}tr[\Sigma_0^{-1}(X_i-\mu_0)(X_i-\mu_0)']}{\exp\{-\frac{1}{2}\sum_{i=1}^{n}tr[\Sigma_0^{-1}(X_i-\bar{X})(X_i-\bar{X})']}$$

$$= \exp\{-\frac{n}{2}tr[(\bar{X}-\mu_0)'\Sigma_0^{-1}(\bar{X}-\mu_0)]\}$$

$$= \exp\{-\frac{n}{2}(\bar{X}-\mu_0)'\Sigma_0^{-1}(\bar{X}-\mu_0)\}$$

因此 $\ln\lambda = -\frac{n}{2}(\bar{X}-\mu_0)'\Sigma_0^{-1}(\bar{X}-\mu_0)$. $-2\ln\lambda \sim \chi^2(p)$

3-7. 解: 令 $Y_{(\alpha)} = CX_{(\alpha)}$. 则 $Y_{(\alpha)} \sim N_k(C\mu, C\Sigma C')$.

即需要给出 $H_0: C\mu = r \Leftrightarrow H_0: \mu_y = r$ 的检验.

取检验统计量 $F = \frac{n-k}{(n-1)k}T^2 \overset{H_0\text{下}}{\sim} F(k, n-k)$.

其中 $T^2 = (n-1)[\sqrt{n}(\bar{Y}-C\mu)]'A_Y^{-1}[\sqrt{n}(\bar{Y}-C\mu)]$

$= (n-1)n(C\bar{X}-r)'[CAC']^{-1}(C\bar{X}-r)$

$A = \sum_{i=1}^{n}(X_{(i)}-\bar{X})(X_{(i)}-\bar{X})'$. 取 $k=p-1$, $r=0$ 得

检验$H_0$的似然比统计量及分布。

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3-9. 解：
$$\lambda=\frac{\max_{\mu}L(\mu,\Sigma_0)}{\max_{\mu,\Sigma}L(\mu,\Sigma)}$$
$$=\frac{\max L(\bar{X},\Sigma_0)}{\max L(\bar{X},\Sigma)}$$
$$=\frac{|2\pi\Sigma_0|^{-\frac{n}{2}}\exp\{-\frac{1}{2}\sum_{i=1}^{n}(X_i-\bar{X})'\Sigma_0^{-1}(X_i-\bar{X})\}}{|2\pi\cdot\frac{1}{n}A|^{-\frac{n}{2}}\exp\{-\frac{1}{2}\sum_{i=1}^{n}(X_i-\bar{X})'(\frac{1}{n}A)^{-1}(X_i-\bar{X})\}}.$$
$$=\frac{|\Sigma_0|^{-\frac{n}{2}}etr\{-\frac{1}{2}\Sigma_0^{-1}A)}{(\frac{n}{e})^{\frac{np}{2}}|A|^{-\frac{n}{2}}}$$
$$=(\frac{e}{n})^{\frac{np}{2}}etr(-\frac{1}{2}\Sigma_0^{-1}A)|\Sigma_0^{-1}A|^{\frac{n}{2}}$$
由定理3.2.1 $-2\ln\lambda\sim\chi^2(\frac{p(p+1)}{2})$.

3-10. 解：
$$\lambda=\frac{\max_{\mu^{(1)}=\mu^{(2)}=\mu,\Sigma}L(\mu,\Sigma)}{\max_{\mu^{(1)},\mu^{(2)},\Sigma>0}L_1(\mu^{(1)},\Sigma)\cdot L_2(\mu^{(2)},\Sigma)}$$
$$=\frac{L(\bar{X},\frac{T}{n})}{L(\bar{X}^{(1)},\bar{X}^{(2)},\frac{A_1+A_2}{n})}$$
其中 $A_1=\sum_{i=1}^{n_1}(X_{(i)}-\bar{X})(X_{(i)}-\bar{X})'$ $A_2=\sum_{i=1}^{n_2}(X_{(i)}-\bar{X})(X_{(i)}-\bar{X})'$.

代入有
$$\lambda=\frac{|A|^{\frac{n}{2}}}{|T|^{\frac{n}{2}}}=(\frac{|A|}{|A+B|})^{\frac{n}{2}}.$$
$$T=A+B=A+\sum_{i=1}^{2}n_i(\bar{X}^{(i)}-\bar{X})(\bar{X}^{(i)}-\bar{X})'=A+\frac{n_1n_2}{n}(\bar{X}^{(1)}-\bar{X}^{(2)})(\bar{X}^{(1)}-\bar{X}^{(2)})'$$
由行列式的打洞原理知
$$|T|=|A+\frac{n_1n_2}{n}(\bar{X}^{(1)}-\bar{X}^{(2)})(\bar{X}^{(1)}-\bar{X}^{(2)})'|$$
$$=\begin{vmatrix}A & -\sqrt{\frac{n_1n_2}{n}}(\bar{X}^{(1)}-\bar{X}^{(2)})\\ \sqrt{\frac{n_1n_2}{n}}(\bar{X}^{(1)}-\bar{X}^{(2)})' & 1\end{vmatrix}=|A|\cdot|1+\frac{n_1n_2}{n}(\bar{X}^{(1)}-\bar{X}^{(2)})'A^{-1}(\bar{X}^{(1)}-\bar{X}^{(2)})|$$
$$\frac{|A|}{|T|}=\frac{1}{1+\frac{n_1n_2}{n}(\bar{X}^{(1)}-\bar{X}^{(2)})'A^{-1}(\bar{X}^{(1)}-\bar{X}^{(2)})}=\frac{1}{1+\frac{1}{n-2}T^2},\quad T^2\sim T^2(p,n-2).$$
这是由于 $\sqrt{\frac{n_1n_2}{n}}(\bar{X}^{(1)}-\bar{X}^{(2)})\overset{H_0}{\sim}N_p(0,\Sigma)$. $A=A_1+A_2\sim W_p(n-2,\Sigma)$.