Homework 8

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Problem 1.

For each of the following surface presentations, compute the Euler characteristic and determine which of our standard surfaces it represents.

(a) $\langle a, b, c \mid abacb^{-1}c^{-1} \rangle$

(b) $\langle a, b, c \mid abca^{-1}b^{-1}c^{-1} \rangle$

Solution.

(a) $\langle a, b, c \mid abacb^{-1}c^{-1} \rangle$

The relator represents the boundary of a polygon with 6 edges. Each edge label appears once in forward and once in inverse form, indicating that we are gluing pairs of edges. So:

$$F = 1, \quad E = \frac{6}{2} = 3, \quad V = 1$$
$$\chi = V - E + F = 1 - 3 + 1 = -1$$

Since the Euler characteristic is odd, this must be a non-orientable surface. For non-orientable surfaces:

$$\chi = 2 - k \Rightarrow k = 3$$

Thus, the surface is the connected sum of 3 real projective planes $\mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$.

(b) $\langle a, b, c \mid abca^{-1}b^{-1}c^{-1} \rangle$

This relator also gives a 6-gon with edge pairings. The calculation is the same:

$$F = 1, \quad E = 3, \quad V = 2 \Rightarrow \chi = 0$$

An orientable surface with $\chi = 0$ corresponds to the torus $\mathbb{T}^2$(orientable genus 1).

□

Problem 2.

Let X be a topological space. Show that for any points $p, q \in X$, path homotopy is an equivalence relation on the set of all paths in X from p to q.

Solution.

Let X be a topological space and let $p, q \in X$. Consider the set of all paths from p to q, i.e., continuous maps $f : [0, 1] \to X$ with $f(0) = p$, $f(1) = q$.

We define an equivalence relation called path homotopy. Two paths f, g from p to q are path homotopic, denoted $f \simeq g$, if there exists a continuous map $H : [0,1] \times [0,1] \to X$ such that:

$$H(s,0) = f(s), \quad H(s,1) = g(s), \quad H(0,t) = p, \quad H(1,t) = q$$

for all $s, t \in [0,1]$.

We verify that this defines an equivalence relation.

(1) Reflexivity: Let f be a path from p to q. Define $H(s,t) = f(s)$. Then H is constant in t and clearly satisfies the conditions above, hence $f \simeq f$.

(2) Symmetry: If $f \simeq g$ via H, define $K(s,t) = H(s, 1-t)$. Then K is continuous and satisfies:

$$K(s,0) = H(s,1) = g(s), \quad K(s,1) = H(s,0) = f(s)$$

with the same endpoints $K(0,t) = p$, $K(1,t) = q$. Hence $g \simeq f$.

(3) Transitivity: Suppose $f \simeq g$ via H and $g \simeq h$ via G. Define

$$F(s,t) = \begin{cases} H(s,2t), & 0 \le t \le \frac{1}{2} \\ G(s,2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

This function is continuous by the pasting lemma and gives a homotopy from f to h, so $f \simeq h$.

$\square$

PROBLEM 3.

Let X be a path-connected topological space.

(a) Let $f, g : I \to X$ be two paths from p to q. Show that $f \sim g$ if and only if $f \cdot \overline{g} \sim c_p$.

(b) Show that X is simply connected if and only if any two paths in X with the same initial and terminal points are path-homotopic.

SOLUTION.

(a) ($\Rightarrow$) Suppose $f \sim g$. I'll show $f \sim g \Rightarrow f\overline{g} \sim c_p$. There is a homotopy $H(s,t)$ satisfies

$$H(s,0) = f(s) \quad H(s,1) = g(s) \quad H(0,t) = p \quad H(1,t) = q$$

Explicitly, we define our new homotopy as

$$K(s,t) = \begin{cases} f(2s) & \text{for } s \in [0, \frac{1-t}{2}] \\ H(1-t, \frac{1}{t}(s - \frac{1-t}{2})) & \text{for } s \in [\frac{1-t}{2}, \frac{1+t}{2}] \\ g(2-2s) & \text{for } s \in [\frac{1+t}{2}, 1] \end{cases}$$

Then

$$K(s,0) = f \cdot \bar{g}(s) \quad K(s,1) = p \quad K(0,t) = p \quad K(1,t) = p$$

This gives a path-homotopy from $f \cdot \overline{g}$ to c_p.

($\Leftarrow$) To prove that when $f \sim g$, we can construct a homotopy from f to g. We define the homotopy $H(s,t)$ as follows:

$$H(s,t) = \begin{cases} K(2s,t) & \text{for } s \in [0, 1/2), \\ g(2s-1) & \text{for } s \in [1/2, 1]. \end{cases}$$

where $K(s,t)$ is a known homotopy from $f \cdot \overline{g}$ to c_p.

When $s = 0$, $H(0,t) = K(0,t) = p$. When $s = 1$, $H(1,t) = g(2-1) = g(1) = q$.

At $s = 1/2$, $H(1/2, t) = K(1, t) = p$, and $g(2 \times 1/2 - 1) = g(0) = p$, ensuring continuity.

When $t = 0$, $H(s, 0)$ for $s \in [0, 1/2]$ is $f \cdot \bar{g}(2s)$, and for $s \in [1/2, 1]$, it is $g(2s - 1)$, thus $H(s, 0) = f \cdot g \cdot \bar{g} \sim f$. When $t = 1$, $H(s, 1) = p$, hence $c_p \cdot g \sim g$.

Therefore, $H(s, t)$ is a homotopy from f to g, proving that $f \sim g$.

(b) ($\Rightarrow$) Assume X is simply connected. Let $f, g : I \to X$ be two paths from p to q. By part (a), $f \sim g$ if and only if the loop $f \cdot \overline{g}$ (the concatenation of f and the reverse of g) is path-homotopic to c_p. Since X is simply connected, every loop based at p is homotopic to c_p. Thus, $f \cdot \overline{g} \sim c_p$, and by part (a), $f \sim g$. Hence, all paths with the same endpoints are path-homotopic.

($\Leftarrow$) Conversely, suppose any two paths in X with the same endpoints are path-homotopic. Let $h : I \to X$ be any loop based at p. By assumption, h is path-homotopic to the constant path c_p, since both are paths from p to p. Thus, $\pi_1(X, p)$ contains only the trivial element. Since X is path-connected and $\pi_1(X, p)$ is trivial for all p, X is simply connected.

$\square$

PROBLEM 4.

Suppose $f, g : S^n \to S^n$ are continuous maps such that $f(x) \neq -g(x)$ for any $x \in S^n$. Prove that f and g are homotopic.

SOLUTION.

We construct an explicit homotopy. For each $x \in S^n$ and $t \in [0, 1]$, define the linear interpolation:

$$H(x,t) = \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}$$

Suppose there exists some $x \in S^n$ and $t \in [0,1]$ such that $(1-t)f(x) + tg(x) = 0$. Then:

$$(1-t)f(x) = -tg(x)$$

Taking norms gives:

$$(1-t)\|f(x)\| = t\|g(x)\|$$

Since $f(x)$ and $g(x)$ are unit vectors, $(1-t) = t$, which implies $t = \frac{1}{2}$. Substituting back yields $f(x) = -g(x)$, contradicting the given condition $f(x) \neq -g(x)$. Thus, the denominator never vanishes.

The numerator and denominator are continuous, and the denominator is non-zero. Hence, H is continuous.

Moreover, when $t = 0$, $H(x,0) = f(x)$. When $t = 1$, $H(x,1) = g(x)$.

Therefore, H defines a homotopy from f to g, proving that $f \simeq g$.

□

PROBLEM 5. Suppose X is a topological space, and g is any path in X from p to q. Let

$$\Phi_g : \pi_1(X,p) \to \pi_1(X,q)$$

denote the group isomorphism defined in Theorem 7.13.

(a) Show that if h is another path in X starting at q, then

$$\Phi_{g \cdot h} = \Phi_h \circ \Phi_g.$$

(b) Suppose $\psi : X \to Y$ is continuous, and show that the following diagram commutes:

$$\begin{tikzcd}
\pi_1(X,p) \arrow[r, "\psi_*"] \arrow[d, "\Phi_g"'] & \pi_1(Y,\psi(p)) \arrow[d, "\Phi_{\psi\circ g}"] \\
\pi_1(X,q) \arrow[r, "\psi_*"'] & \pi_1(Y,\psi(q))
\end{tikzcd}$$

SOLUTION.

(a) Let $[f] \in \pi_1(X,p)$, then

$$\begin{aligned}
\Phi_h \circ \Phi_g[f] &= \Phi_h[\bar{g}] \cdot [f] \cdot [g] = [\bar{h}] \cdot [\bar{g}] \cdot [f] \cdot [g] \cdot [h] \\
&= [\overline{g \cdot h}] \cdot [f] \cdot [g \cdot h] = \Phi_{g \cdot h}([f])
\end{aligned}$$

(b) We need to verify the commutative diagram, i.e., $\psi_* \circ \Phi_g = \Phi_{\psi\circ g} \circ \psi_*$. For any loop $[\alpha] \in \pi_1(X,p)$, we calculate the results of both sides of the diagram:

First, we calculate the left side:

$$\psi_*(\Phi_g([f])) = \psi_*([\bar{g} \cdot f \cdot g]) = [\psi \circ (\bar{g} \cdot f \cdot g)].$$

It is equal to

$$[(\psi \circ \bar{g}) \cdot (\psi \circ f) \cdot (\psi \circ g)]$$

Based on the definition of $\Phi_{\psi\circ g}$, we calculate the right side:

$$\Phi_{\psi\circ g}(\psi_*([f])) = \Phi_{\psi\circ g}([\psi \circ f]) = \left[(\overline{\psi \circ g}) \cdot (\psi \circ f) \cdot (\psi \circ g)\right].$$

The results of both sides are identical, thus the diagram commutes.

$\square$