习题2.

6、证明：令 $g(p)=\sum_{i=m}^{n} C_n^i p^i(1-p)^{n-i} - m C_n^m \int_0^p t^{m-1}(1-t)^{n-m}dt$.

$$g'(p)=\sum_{i=m}^{n}\left[iC_n^i p^{i-1}(1-p)^{n-i} - (n-i)C_n^{n-i} p^i (1-p)^{n-i-1}\right] - mC_n^m p^{m-1}(1-p)^{n-m}$$

$$=\sum_{i=m}^{n}\left[nC_{n-1}^{i-1}p^{i-1}(1-p)^{n-i}\right] - \sum_{i=m}^{n} nC_{n-1}^{n-i-1}p^i(1-p)^{n-i-1}\Big] - mC_n^m p^{m-1}(1-p)^{n-m}$$

$$=\sum_{i=m-1}^{n-1}\left[nC_{n-1}^i p^i(1-p)^{n-i-1}\right] - \sum_{i=m}^{n-1}\left[nC_{n-1}^i p^i (1-p)^{n-i-1}\right] - mC_n^m p^{m-1}(1-p)^{n-m}$$

$$= nC_{n-1}^{m-1}p^{m-1}(1-p)^{n-m} - mC_n^m p^{m-1}(1-p)^{n-m}$$

$$=(nC_{n-1}^{m-1}-mC_n^m)p^{m-1}(1-p)^{n-m}=0$$

又 $g(0)=0$.

故 $g(p)$ 为常数. 且为0. 令 $p=F(x)$. 有.

$\cancel{\sum_{i=m}^n C_n^i p^i(1-p)^{n-i}}$

$$\sum_{i=m}^n C_n^i (F(x))^i[1-F(x)]^{n-i} = mC_n^m\int_0^{F(x)} t^{m-1}(1-t)^{n-m}dt.$$

7解: $P(a<x_1<\cdots<x_n<b) = P(a< x_i \le X_{(i)} < x_i+dx_i <b)$ $i=1,2,\cdots,n$

$$=\int\cdots\int_{a<x_1<\cdots<x_n<b} n!\, f(x_1)dx_1\cdot f(x_2)dx_2\cdots f(x_n)dx_n$$

这里的 $n!$ 源于 $X_{(i)}$ 为次序统计量.

又 $X_1,\cdots,X_n$ iid $\sim f$, $P(a<x_1<\cdots<x_n<b)=[F(b)-F(a)]^n$.

即 $\int\cdots\int_{a<x_1<\cdots<x_n<b} f(x_1)\cdots f(x_n)dx_1\cdots dx_n = \frac{1}{n!}[F(b)-F(a)]^n$.

$(X_{(1)},\cdots,X_{(n)})$ 的 $\sim n!\, f(x_1)\cdots f(x_n)\, I_{(x_1\le x_2\cdots\le x_n)}$

故 $f_m(x)=\int\cdots\int_{x_1\le\cdots\le x\le\cdots\le x_n} n!\, f(x_1)\cdots \widehat{f(x)}\cdots f(x_n)\, dx_1\cdots dx_{m-1}dx_{m+1}\cdots dx_n$

$\cancel{= f(x_m)}$

$$= n! \int \cdots \int_{x_1<\cdots<x_{m-1}<x} \cancel{n!}\, f(x_1)\cdots f(x_{m-1}) f(x)\, dx_1 \cdots dx_{m-1} \cdot$$

$$\int \cdots \int_{x<x_{m+1}<\cdots<x_n} f(x_{m+1})\cdots f(x_n)\, dx_1 \cdots dx_{m\,n}$$

由已推恒等式,

$$上式 = n!\ \frac{1}{(m-1)!}[F(x)-F(-\infty)]^{m-1} \frac{1}{(n-m)!}[F(+\infty)-F(x)]^{n-m} f(x)$$

$$= \frac{n!}{(m-1)!\,(n-m)!}(F(x))^{m-1}(1-F(x))^{n-m} f(x).$$

即为 $X_{(m)}$ 的密度函数.

8. 解: (1) $P(X_{(n)} \geq 0.99) = 1 - P(X_{(n)} < 0.99)$

$$= 1 - \cancel{[F(x)]^n}\, [F(0.99)]^n$$

$$= 1 - 0.99^n \geq 0.95.$$

即 $n \geq 298.073$. n 至少为 299.

(2) $f_{1n}(x,y) = n(n-1) f(x) f(y) [F(y)-F(x)]^{n-2} I_{(x<y)}$

由于 x, y iid $\sim U(0,1)$.

$$f_{1n}(x,y) = n(n-1)(y-x)^{n-2} I_{(x<y)}$$

令 $\begin{cases} y-x = v \\ x = u. \end{cases}$ 则 $(u,v) \sim f(u,v)$

而 $f(u,v) = f(x,y) \cdot \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = f(x,y) = f(u, u+v)$.

其中, $\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}$ 为变元变换的 Jacobi 行列式.

$$f(u,v) = n(n-1) \cdot v^{n-2} I_{(0<u<1-v)}$$

$$f_{R_n}(v) = \int_0^{1-v} n(n-1) v^{n-2}\, du = n(n-1) v^{n-2}(1-v) \quad 0<v<1.$$

这就是极差 R_n 的密度函数

(3) 证明: 由 $Z_n = 2n(1-R_n)$, $R_n = 1-\frac{Z_n}{2n}$.

$\cancel{f_{Z_n}(z) = f_{R_n}(1-\frac{z}{2n})\cdot|}$

$$f_{Z_n}(z) = f_{R_n}\left(1-\frac{z}{2n}\right)\cdot\left|\frac{\partial(1-\frac{z}{2n})}{\partial z}\right|$$

$$= \frac{1}{2n}\cdot n(n-1)\cdot\left(1-\frac{z}{2n}\right)^{n-2}\left(\frac{z}{2n}\right).$$

$$\lim_{n\to\infty} f_{Z_n}(z) = \lim_{n\to\infty}\frac{1}{2n}\cdot n(n-1)\cdot\left(1-\frac{z}{2n}\right)^{n-2}\left(\frac{z}{2n}\right)$$

$$= \lim_{n\to\infty}\frac{n(n-1)}{4n^2}z\cdot e^{(n-2)\ln(1-\frac{z}{2n})} = \frac{z}{4}\cdot e^{\lim\limits_{n\to\infty}\frac{\ln(1-\frac{z}{2n})}{\frac{1}{n-2}}}$$

$$= \frac{z}{4}\cdot e^{-\frac{z}{2}} = \frac{(\frac{1}{2})^2}{\Gamma(2)}z^{2-1}\cdot e^{-\frac{1}{2}z} \sim \Gamma(2,\tfrac{1}{2}).$$

即极限分布服从 χ^2_4.

9、解: (1) 经验分布函数为

$$F_n(x) = \begin{cases} 0 & x \le -1 \\ 0.1 & -1 < x \le -0.7 \\ 0.2 & -0.7 < x \le -0.3 \\ 0.3 & -0.3 < x \le -0.1 \\ 0.4 & -0.1 < x \le 0 \\ 0.5 & 0 < x \le 0.15 \\ 0.6 & 0.15 < x \le 0.2 \\ 0.7 & 0.2 < x \le 0.25 \\ 0.8 & 0.25 < x \le 1 \\ 0.9 & 1 < x \le 2 \\ 1 & 2 < x. \end{cases}$$

(2) $f_6(x) = \frac{10!}{5!\,1!\,4!}[F(x)]^5[1-F(x)]^4 \cdot f(x)$

$\cancel{= 1260x^5(1-x)^4}$

$$E\{F(X_{(6)})\} = \int_{-\infty}^{+\infty} F(x)\cdot f_6(x)\,dx$$

$$= 1260\int_0^1 [F(x)]^6[1-F(x)]^4 \cdot dF(x)$$

$$= 1260\cdot B(7,5) = 1260\cdot\frac{\Gamma(7)\Gamma(5)}{\Gamma(12)} = \frac{6}{11}$$

$$E\{F^2(X_{(6)})\} = \int_{-\infty}^{+\infty} F^2(x)\cdot f_6(x)\,dx$$

$$= 1260\cdot B(8,5) = \frac{7}{22}$$

$$D\{F(X_{(6)})\} = E\{F^2(X_{(6)})\} - E\{F(X_{(6)})\}^2 = \frac{5}{242}$$

(3) $\cancel{F_6(0.2) = \int_{-\infty}^{0.2} f_6(x)\,dx}$

$F_6(X_{(6)}$ [illegible] $) = P(X_{(6)} < 0.2) = \sum_{i=6}^{10} C_{10}^{i}(F(x))^i(1-F(x))^{10-i}$

$x = 0.2 \quad F(x) = 0.5793, \quad F_6(X_{(6)}) = \sum_{i=6}^{10} C_{10}^{i}(0.5793)^i(0.4207)^{10-i}$

$= 0.5804.$

10. 解： $F_Y(x) = P(Y < x) = P(\text{至少有一个} X_i < x, i=1,2,\cdots,n)$

$$= 1-(1-F(x))^n$$

$$= \begin{cases} 1-e^{-(\frac{x}{\beta})^{\alpha}\cdot n} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

故Y仍服从威布尔分布，且形状参数为α，刻度参数为$\beta\cdot n^{-\frac{1}{\alpha}}$.

24. (1) $\Gamma(p,\alpha)$ 的分布函数为

$$p(x)=\frac{\alpha^p}{\Gamma(p)}\cdot x^{p-1}\cdot e^{-\alpha x},\quad x>0.$$

因此 $$\varphi(t)=\int_0^{+\infty}\frac{\alpha^p}{\Gamma(p)}\cdot x^{p-1}\cdot e^{(it-\alpha)x}\,dx.$$

$$=\int_0^{+\infty}\frac{1}{\Gamma(p)}\cdot\left(\frac{\alpha}{\alpha-it}\right)^p\cdot\left[(\alpha-it)x\right]^{p-1}\cdot e^{-(\alpha-it)x}\,d(\alpha-it)x$$

$$=\left(\frac{\alpha}{\alpha-it}\right)^p\cdot\frac{1}{\Gamma(p)}\cdot\int_0^{+\infty}\left[(\alpha-it)x\right]^{p-1}\cdot e^{-(\alpha-it)x}\,d(\alpha-it)x$$

$$=\left(\frac{\alpha}{\alpha-it}\right)^p\cdot\frac{\Gamma(p)}{\Gamma(p)}=\left(\frac{\alpha}{\alpha-it}\right)^p.$$

(2) $$E(X)=\int_0^{+\infty}xp(x)\,dx=\int_0^{+\infty}\frac{\alpha^p}{\Gamma(p)}\cdot x^p\cdot e^{-\alpha x}\,dx.$$

$$=\frac{1}{\alpha}\int_0^{+\infty}\frac{1}{\Gamma(p)}\cdot(\alpha x)^p\cdot e^{-\alpha x}\,d(\alpha x)$$

$$=\frac{1}{\alpha}\cdot\frac{\Gamma(p+1)}{\Gamma(p)}=\frac{p}{\alpha}.$$

$$E(X^2)=\int_0^{+\infty}x^2p(x)\,dx=\frac{1}{\alpha^2}\cdot\frac{1}{\Gamma(p)}\cdot\int_0^{+\infty}(\alpha x)^{p+1}e^{-\alpha x}\,d(\alpha x)$$

$$=\frac{1}{\alpha^2}\cdot\frac{\Gamma(p+2)}{\Gamma(p)}=\frac{(p+1)p}{\alpha^2}.$$

$$D(X)=E(X^2)-E^2(X)=\frac{(p+1)p-p^2}{\alpha^2}=\frac{p}{\alpha^2}.$$

(3) 令 $T=\sum_{i=1}^n X_i$, 由 $X_1,\cdots,X_n$ 相互独立.

$$\varphi(t)=E(e^{itT})=E(e^{it\sum_{i=1}^n X_i})=\prod_{i=1}^n E(e^{itX_i})=\prod_{i=1}^n\left(\frac{\alpha}{\alpha-it}\right)^{p_i}$$

$$=\left(\frac{\alpha}{\alpha-it}\right)^{\sum_{i=1}^n p_i}=\left(\frac{\alpha}{\alpha-it}\right)^p.$$

则 $\sum_{i=1}^n X_i\sim\Gamma(p,\alpha)$.

(4) $\Gamma(\frac{n}{2},\frac{1}{2})$ 的特征函数为

$$\varphi(t)=\left(\frac{\frac{1}{2}}{\frac{1}{2}-it}\right)^{\frac{n}{2}}=(1-2it)^{-\frac{n}{2}}.$$

由特征函数的唯一性, $\Gamma(\frac{n}{2},\frac{1}{2})$ 是 χ_n^2 分布.

26. 证明：设 $Y_{(i)}=\frac{1}{\lambda}X_{(i)}$，$i=1,2,\cdots,n$，故

$Y_{(1)},Y_{(2)},\cdots,Y_{(n)}\overset{i.i.d.}{\sim}Exp(1)$.

$\frac{T}{\lambda}=\sum_{i=1}^{r}\frac{X_{(i)}}{\lambda}+(n-r)\frac{X_{(r)}}{\lambda}=\sum_{i=1}^{r}Y_{(i)}+(n-r)Y_{(r)}$.

作变换

$$\begin{cases}Z_1=n\cancel{X}Y_{(1)}\\Z_2=(n-1)(\cancel{X}Y_{(2)}-\cancel{X}Y_{(1)})\\\cdots\\Z_r=(n-r+1)(\cancel{X}Y_{(r)}-\cancel{X}Y_{(r-1)})\end{cases}，即\begin{cases}Y_{(1)}=\frac{Z_1}{n}\\Y_{(2)}=\frac{Z_1}{n}+\frac{Z_2}{n-1}\\\vdots\\Y_{(r)}=\frac{Z_1}{n}+\cdots+\frac{Z_r}{n-r+1}\end{cases}$$

变换的Jacobi行列式为 $|J|=\cancel{\frac{n!}{(n-r)!}}\ \frac{(n-r)!}{n!}$

由次序统计量的密度函数公式，有

$$f(y_1,y_2,\cdots,y_r)=A_n^r\cdot e^{-\sum_{i=1}^{r}y_i}\cdot e^{-y_r\cdot(n-r)}=\frac{n!}{(n-r)!}\cdot e^{-[\sum_{i=1}^{r}y_i+(n-r)y_r]}$$

$0\le x_1<x_2<\cdots<x_r$

注意到 $\sum_{i=1}^{r}y_i+(n-r)y_r=n\cdot\frac{Z_1}{n}+(n-1)\cdot\frac{Z_2}{n-1}+\cdots+(n-r+1)\cdot\frac{Z_r}{n-r+1}$

$=Z_1+Z_2+\cdots+Z_r$.

$$f(z_1,z_2,\cdots,z_r)=|J|\cdot A_n^r\cdot e^{-\sum_{i=1}^{r}z_i}=e^{-\sum_{i=1}^{r}z_i}=e^{-z_1}e^{-z_2}\cdots e^{-z_r},$$

$z_1,z_2,\cdots,z_r>0$

则 $Z_1,Z_2,\cdots,Z_r\overset{i.i.d.}{\sim}Exp(1)$.

故 $2\cdot\frac{T}{\lambda}=2\cdot1\cdot\sum_{i=1}^{r}Z_i\sim\chi^2_{2r}$.

27. 证明：由26结论，令 $Y_i=\frac{X_{(i)}-\mu}{\sigma}$，于是 $Y_i\sim Exp(1)$

$Z_i=(n-i+1)(Y_i-Y_{i-1})=\frac{n-i+1}{\sigma}(X_{(i)}-X_{(i-1)})$,

$2Z_i\sim\chi^2_2$.

28. 证明: $p(x_1) = \frac{\lambda^{\alpha_1}}{\Gamma(\alpha_1)} x_1^{\alpha_1 - 1} e^{-\lambda x_1}$

$p(x_2) = \frac{\lambda^{\alpha_2}}{\Gamma(\alpha_2)} x_2^{\alpha_2 - 1} e^{-\lambda x_2}$. 由 $X_1 \cdot X_2$ 独立, 联合密度函数为

$p(x_1, x_2) = p(x_1) p(x_2) = \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x_1^{\alpha_1-1} x_2^{\alpha_2-1} \cdot e^{-\lambda(x_1+x_2)}$.

令 $\begin{cases} y_1 = x_1 + x_2 \\ y_2 = \frac{x_1}{x_1+x_2}, \end{cases}$ 则 $\begin{cases} x_1 = y_1 y_2 \\ x_2 = y_1 - y_1 y_2, \end{cases}$

$|J| = \left|\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)}\right| = y_1$

$p(y_1, y_2) = y_1 p(y_1 y_2) p(y_1 - y_1 y_2) = y_1 \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} (y_1 y_2)^{\alpha_1-1} (y_1 - y_1 y_2)^{\alpha_2-1} \cdot e^{-\lambda y_1}$

$= \cancel{\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} y_2^{\alpha_1-1}} \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2)} \cdot \cancel{x} y_1^{\alpha_1+\alpha_2-1} \cdot e^{-\lambda y_1} \cdot \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} y_2^{\alpha_1-1}(1-y_2)^{\alpha_2}$.

则 $y_1 \sim \Gamma(\alpha_1+\alpha_2, \lambda)$. $y_2 \sim Be(\alpha_1, \alpha_2)$. $y_1 \cdot y_2$ 独立.

11. 解: $X_{n+1} \sim N(a, \sigma^2)$. $\bar{X} \sim N(a, \frac{\sigma^2}{n})$.

故 $X_{n+1} - \bar{X} \sim N(0, \frac{n+1}{n}\sigma^2)$.

$\sqrt{\frac{n}{n+1}} \frac{X_{n+1} - \bar{X}}{\sigma} \sim N(0,1)$.

$\frac{nS_n^2}{\sigma^2} \sim \chi^2_{n-1}$, 故

$\sqrt{\frac{n}{n+1}} \frac{X_{n+1} - \bar{X}}{\sigma} \Big/ \sqrt{\frac{nS_n^2}{(n-1)\sigma^2}} = \frac{X_{n+1} - \bar{X}}{S_n} \cdot \sqrt{\frac{n-1}{n+1}} \sim t_{n-1}$

12. 解: $\bar{X} \sim N(\mu_1, \frac{\sigma^2}{m})$. $\bar{Y} \sim N(\mu_2, \frac{\sigma^2}{n})$.

$\alpha(\bar{X} - \mu_1) + \beta(\bar{Y} - \mu_2) \sim N(0, (\frac{\alpha^2}{m} + \frac{\beta^2}{n})\sigma^2)$.

$[\alpha(\bar{X} - \mu_1) + \beta(\bar{Y} - \mu_2)] \Big/ \sqrt{(\frac{\alpha^2}{m} + \frac{\beta^2}{n})\sigma^2}$

$\frac{mS_{1m}^2}{\sigma^2} \sim \chi_{(m-1)}$ $\frac{nS_{2n}^2}{\sigma^2} \sim \chi_{(n-1)}$.

故 $\frac{mS_{1m}^2 + nS_{2n}^2}{\sigma^2} \sim \chi_{(m+n-2)}$.

$$T=\frac{\alpha(\bar{X}-\mu_1)+\beta(\bar{Y}-\mu_2)}{\sqrt{\frac{mS_{1m}^2+nS_{2n}^2}{n+m-2}\left(\frac{\alpha^2}{m}+\frac{\beta^2}{n}\right)}}\sim t_{(n+m-2)}$$

13. 解：作正交变换 $Y=AX$. 其中正交矩阵 A 为

$$A=\begin{pmatrix}\frac{1}{\sqrt{n}}, & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}}\\ \sqrt{\frac{n-1}{n}}, & -\sqrt{\frac{1}{n(n-1)}} & \cdots & -\sqrt{\frac{1}{n(n-1)}}\\ c_{31} & c_{32} & \cdots & c_{3n}\\ \vdots & \vdots & & \vdots\\ c_{n1} & c_{n2} & \cdots & c_{nn}\end{pmatrix}$$

$Y_1=\sqrt{n}\bar{X}$，$Y_2=\sqrt{\frac{n}{n-1}}(X_1-\bar{X})$，$\sum_{i=1}^n X_i^2=\sum_{i=1}^n Y_i^2$

$(n-1)S^2=\sum_{i=1}^n(X_i-\bar{X})^2=\sum_{i=1}^n X_i^2-n\bar{X}^2=\sum_{i=1}^n Y_i^2-Y_1^2=\sum_{i=2}^n Y_i^2$.

而 $Y_i=\sum_{j=1}^n c_{ij}X_j$，$i=2,3,\cdots$ 均值为 $a\sum_{j=1}^n c_{ij}=a\sqrt{n}\sum_{j=1}^n\frac{1}{\sqrt{n}}c_{ij}=0$ (正交矩阵性质)

即 $Y_2,Y_3,\cdots,Y_n\sim N(0,\sigma^2)$. $\frac{(n-1)}{\sigma^2}S^2=\sum_{i=2}^n\left(\frac{Y_i}{\sigma}\right)^2\sim\chi^2_{(n-1)}$.

故 $T=Y_2\Big/\sqrt{(Y_3^2+\cdots+Y_n^2)/n-2}\sim t_{(n-2)}$

$\xi=\sqrt{\frac{n-1}{n}}Y_2\Big/\sqrt{\frac{1}{n-1}\sum_{i=2}^n Y_i^2}=\frac{n-1}{\sqrt{n}}\cdot\frac{Y_2}{\sqrt{Y_2^2+\cdots+Y_n^2}}=\frac{n-1}{\sqrt{n}}\cdot\frac{T}{\sqrt{n-2+T^2}}=g(T)$

$T\sim t_{(n-2)}$. 密度函数为 $f_T(t)=\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}-1\right)\sqrt{(n-2)\pi}}\left(1+\frac{t^2}{n-2}\right)^{-\frac{n-1}{2}}$

ξ 的密度函数为 $f_\xi(x)=\cancel{\frac{n-1}{\sqrt{n}}\cdot\frac{\quad}{n-2}}\,f_T(g^{-1}(x))\cdot g^{-1\prime}$

14. 解. 令 $Y_i=\frac{X_i}{\sigma_i}\sim N(0,1)$

$$\xi=\sum_{i=1}^n\frac{(X_i-Z)^2}{\sigma_i^2}=\sum_{i=1}^n\left(\frac{X_i}{\sigma_i}\right)^2-2\sum_{i=1}^n\frac{X_i}{\sigma_i^2}\cdot\sum_{i=1}^n\frac{X_i}{\sigma_i^2}\Big/\sum_{i=1}^n\frac{1}{\sigma_i^2}+\left(\sum_{i=1}^n\frac{X_i}{\sigma_i^2}\Big/\sum_{i=1}^n\frac{1}{\sigma_i^2}\right)^2\cdot\sum_{i=1}^n\frac{1}{\sigma_i^2}$$

$$=\sum_{i=1}^n Y_i^2-\left(\sum_{i=1}^n\frac{Y_i}{\sigma_i}\right)^2\Big/\sum_{i=1}^n\frac{1}{\sigma_i^2}$$

作正交变换 $Z=AY$.

$$A=\begin{pmatrix}\frac{1}{\sigma_1\left(\sum_{i=1}^n\frac{1}{\sigma_i^2}\right)^{\frac12}}, & \cdots, & \frac{1}{\sigma_n\left(\sum_{i=1}^n\frac{1}{\sigma_i^2}\right)^{\frac12}}\\ a_{21}, & \cdots, & a_{2n}\\ \vdots & & \\ a_{n1}, & \cdots, & a_{nn}\end{pmatrix}.$$

即 $Z_1=\sum_{i=1}^{n}\frac{Y_i}{\sigma_i}/(\sum_{i=1}^{n}\frac{1}{\sigma_i^2})^{1/2}$，且 $\sum_{i=1}^{n}Y_i^2=\sum_{i=1}^{n}Z_i^2$.

$\xi=\sum_{i=1}^{n}Z_i^2-Z_1^2=\sum_{i=2}^{n}Z_i^2$.

而 $Z_i\overset{i.i.d}{\sim}N(0,1)$ $i=2,3,\cdots,n$.

$\xi\sim\chi^2_{(n-1)}$

16. 解：$n\bar{X}=\sum_{i=1}^{n}X_i$，由于 $X_i\overset{i.i.d.}{\sim}\chi^2_m$，

$n\bar{x}\sim\chi^2_{nm}$，令 $Y=\frac{n\bar{X}}{n}=\bar{X}$.

$n\bar{x}$ 的密度函数为 $\frac{1}{2^{\frac{nm}{2}}\Gamma(\frac{nm}{2})}x^{\frac{nm}{2}-1}e^{-\frac{1}{2}x}$.

$\bar{X}\sim\frac{n}{2^{\frac{nm}{2}}\Gamma(\frac{nm}{2})}\cancel{x}(ny)^{\frac{nm}{2}-1}e^{-\frac{ny}{2}}$.

35. 解：由 $X_1,X_2,\cdots,X_n\overset{i.i.d.}{\sim}U(0,\theta)$.

~~EX_i~~ $EX_i=\frac{0+\theta}{2}=\frac{\theta}{2}$. $DX_i=\frac{\theta^2}{12}$.

则 $E\bar{X}=\frac{\theta}{2}$，$D\bar{X}=\frac{\theta^2}{12}$. 由中心极限定理.

$$\frac{\bar{X}-E\bar{X}}{\sqrt{D\bar{X}/n}}=\frac{\bar{X}-\frac{\theta}{2}}{\sqrt{\frac{\theta^2}{12n}}}\overset{\mathcal{L}}{\sim}N(0,1).$$

$\bar{X}$ 的渐近分布为 $N(\frac{\theta}{2},\frac{\theta^2}{12n})$

36. 证明：$X_1,X_2,\cdots,X_n\overset{i.i.d.}{\sim}P(\lambda)$.

$E\bar{X}=\lambda$，$D\bar{X}=\lambda$. 由中心极限定理.

$$\frac{\bar{X}-\lambda}{\sqrt{\lambda/n}}\xrightarrow{\mathcal{L}}N(0,1)$$

由辛钦大数定理. $\bar{X}\xrightarrow{P}\lambda$. 即 $\frac{\bar{X}}{\lambda}\xrightarrow{P}1$

由 Slutsky 引理. $\frac{\bar{X}-\lambda}{\sqrt{\bar{X}/n}}=\frac{\bar{X}-\lambda}{\sqrt{\lambda/n}}/\sqrt{\frac{\bar{X}}{\lambda}}\xrightarrow{\mathcal{L}}N(0,1)$

37. 证明: $\bar{X} \sim N(\mu, \frac{\sigma_1^2}{m})$ $\bar{Y} \sim N(\mu, \frac{\sigma_2^2}{n})$.

$\bar{X}-\bar{Y} \sim N(0, \frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n})$

由中心极限定理

$\frac{\bar{X}-\bar{Y}}{\sqrt{\frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n}}} \sim \xrightarrow{\mathcal{L}} N(0,1)$

由辛钦大数定理. $S_X^2 \xrightarrow{P} \sigma_1^2$ $S_Y^2 \xrightarrow{P} \sigma_2^2$.

即 $\sqrt{\frac{\frac{S_X^2}{m}+\frac{S_Y^2}{n}}{\frac{\sigma_1^2}{m}+\frac{\sigma_2^2}{n}}} \xrightarrow{P} 1$.

由 Slutsky 引理. $\frac{\bar{X}-\bar{Y}}{\sqrt{\frac{S_X^2}{m}+\frac{S_Y^2}{n}}} \xrightarrow{\mathcal{L}} N(0,1)$.

38. 解: 单参数指数族 $X \sim P(\lambda)$. 则

$f(\lambda, x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!} = e^{-\lambda} e^{x \ln \lambda} \cdot \frac{1}{x!}$.

多参数指数族. $X \sim \Gamma(\alpha, \lambda)$.

$f(\alpha, \lambda, x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \cdot e^{-\lambda x} I_{(0,\infty)}(x)$.

$X = (X_1, X_2, \cdots, X_n) \sim \frac{\lambda^{n\alpha}}{\Gamma^n(\alpha)} \prod_{i=1}^n x_i^{\alpha-1} \cdot e^{-\lambda \sum_{i=1}^n x_i} \prod_{i=1}^n I_{(0,\infty)}(x_i)$.

$= \frac{\lambda^{n\alpha}}{\Gamma^n(\alpha)} \exp\{-\lambda \sum_{i=1}^n x_i + (\alpha-1)\sum_{i=1}^n \ln x_i\} \prod_{i=1}^n I_{(0,\infty)}(x_i)$

39. 解: $X \sim NB(r, p)$. $f(x, \theta) = \binom{x+r-1}{r-1} \theta^r (1-\theta)^x = \theta^r \exp\{x \ln(1-\theta)\} \binom{x+r-1}{r-1}$

$0<\theta<1$

$x = 0, 1, 2, \cdots$

令 $\varphi = \ln(1-\theta)$ $\theta = 1-e^\varphi$

$f(x, \varphi) = (1-e^\varphi)^r \exp\{x\varphi\} \binom{x+r-1}{r-1}$. 为指数族的自然形式.

自然参数空间 $\Theta^* = \{\varphi: -\infty < \varphi < 0\}$.

$X \sim Exp(\lambda)$. $f(x, \lambda) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$ 已成为自然形式

自然参数空间 $\Theta^* = \{\lambda: \lambda > 0\}$

40. 证明: 由概率密度函数的[illegible]性质

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1.\ 即\quad C(\theta) = \frac{1}{\int_{-\infty}^{+\infty} \exp\{\sum_{j=1}^{k}\theta_j T_j(x)\} h(x)\,dx}$$

$$-\frac{\partial \log C(\theta)}{\partial \theta_j} = -\frac{\partial \log C}{\partial C}\cdot\frac{\partial C(\theta)}{\partial \theta_j} = -\frac{1}{C(\theta)}\frac{\partial C(\theta)}{\partial \theta_j}$$

上式代入化简有 $-\int_{-\infty}^{+\infty} \exp\{\sum_{j=1}^{k}\theta_j T_j(x)\} h(x)\,dx \cdot$

$$-\left(\frac{1}{\int_{-\infty}^{+\infty} \exp\{\sum_{j=1}^{k}\theta_j T_j(x)\} h(x)\,dx}\right)^2 \cdot \int_{-\infty}^{+\infty} T_j(x) \exp\{\sum_{j=1}^{k}\theta_j T_j(x)\} h(x)\,dx$$

$$= \int_{-\infty}^{+\infty} T_j(x)\cdot C(\theta)\exp\{\sum_{j=1}^{k}\theta_j T_j(x)\} h(x)\,dx = E_\theta(T_j(x)).$$

$$Cov(T_j(x), T_s(x)) = E_\theta(T_j(x) - E_\theta(T_j(x)))(T_s(x) - E_\theta(T_s(x))).$$

$$= E_\theta(T_j(x)T_s(x)) - E_\theta(T_j(x))E_\theta(T_s(x)).$$

$$-\frac{\partial^2 \log C(\theta)}{\partial\theta_j\,\partial\theta_s} = \frac{\partial}{\partial\theta_s}\left[\int_{-\infty}^{+\infty} T_j(x)\cdot C(\theta)\exp\{\sum_{j=1}^{k}\theta_j T_j(x)\} h(x)\,dx\right]$$

$$= \frac{\partial}{\partial\theta_s}C(\theta)\int_{-\infty}^{+\infty} T_j(x)\exp\{\sum_{j=1}^{k}\theta_j T_j(x)\} h(x)\,dx +$$

$$C(\theta)\frac{\partial}{\partial\theta_s}\left[\int_{-\infty}^{+\infty} T_j(x)\exp\{\sum_{j=1}^{k}\theta_j T_j(x)\} h(x)\,dx\right.$$

$$= -\int_{-\infty}^{+\infty} T_j(x)\,C(\theta)\exp\{\sum_{j=1}^{k}\theta_j T_j(x)\} h(x)\,dx\,\cdot$$

$$\int_{-\infty}^{+\infty} T_s(x)\,C(\theta)\exp\{\sum_{j=1}^{k}\theta_j T_j(x)\} h(x)\,dx$$

$$+ \int_{-\infty}^{+\infty} T_j(x)T_s(x)\,C(\theta)\exp\{\sum_{j=1}^{k}\theta_j T_j(x)\} h(x)\,dx.$$

$$= E_\theta(T_j(x), T_s(x)) - E_\theta(T_j(x))E_\theta(T_s(x)).$$

42. 证明: $X = (X_1, X_2, \cdots, X_n) \sim \prod_{i=1}^{n}\frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = e^{-n\lambda}\cdot e^{\ln\lambda\cdot\sum_{i=1}^{n}x_i}\frac{1}{\prod_{i=1}^{n}x_i!}$

$$P(X_1 = x_1, \cdots, X_n = x_n \mid T = t) = \frac{P(X_1 = x_1, \cdots, X_n = x_n, T = t)}{P(T = t)}$$

$$= \frac{\prod_{i=1}^{n-1}\frac{\lambda^{x_i}}{x_i!}\cdot\frac{\lambda^{t-\sum_{i=1}^{n-1}x_i}}{(t-\sum_{i=1}^{n-1}x_i)!}\cdot e^{-n\lambda}}{\sum_{k_1+\cdots+k_n=t}\frac{\lambda^t}{k_1!\cdots k_n!}\cdot e^{-n\lambda}}$$ 与 λ 无关.

$$g(T(x), \lambda) = e^{-n\lambda+\ln\lambda\cdot\sum_{i=1}^{n}x_i}\quad h(x) = \frac{1}{\prod_{i=1}^{n}x_i!}$$

由因子分解定理, ~~$C(\lambda) = e^{-n\lambda}$, $T(x) = \sum_{i=1}^{n}x_i$, $Q(\lambda) = \ln\lambda$~~

$T(x)$ 为充分统计量 ~~$h(x)$~~

43. 证明: (1) $X=(X_1,\cdots,X_n)\overset{i.i.d.}{\sim} g(p)$

$P(X_1=x_1,X_2=x_2,\cdots,X_n=x_n|T=t)=\dfrac{P(X_1=x_1,\cdots,X_n=x_n,T=t)}{P(T=t)}$

$=\dfrac{\prod_{i=1}^{n-1}p(1-p)^{x_i-1}\cdot p(1-p)^{t-\sum_{i=1}^{n-1}x_i-1}}{\sum_{x_1+\cdots+x_n=t}\prod_{i=1}^{n}p(1-p)^{x_i-1}}=\dfrac{p^n(1-p)^{t-n}}{\binom{t-1}{n-1}p^n(1-p)^{t-n}}=\dfrac{1}{\binom{t-1}{n-1}}$ 与 p 无关.

(2) $(X_1,\cdots,X_n)\sim p^n(1-p)^{\sum_{i=1}^n x_i}=p^n(1-p)^{t(x)-n}=g(t(x),p)h(x)$. $h(x)\equiv 1$

由因子分解定理知 $T(x)$ 为充分统计量.

45. $X_1,X_2,\cdots,X_n\overset{i.i.d.}{\sim}Exp(\lambda)$, 可知联合密度函数

$f(x_1,\cdots,x_n)=\lambda^n\cdot exp\{-\lambda\sum_{i=1}^n x_i\}\prod_{i=1}^n I_{(0,\infty)}(x_i)$

$=\lambda^n\cdot exp\{-\lambda n\bar{x}\}\prod_{i=1}^n I_{(0,\infty)}(x_i)$.

$=g(\bar{x},\lambda)\cdot h(x)$.

其中 $g(\bar{x},\lambda)=\lambda^n exp\{-\lambda n\bar{x}\}$ $h(x)=\prod_{i=1}^n I_{(0,\infty)}(x_i)$. 由因子分解定理可得 $\bar{x}$ 为充分统计量.

46. $X_1,\cdots,X_n\overset{i.i.d.}{\sim}N(\theta,\theta^2)$. 可知联合密度函数

$f(x_1,\cdots,x_n)=(2\pi\theta^2)^{-n/2}exp\{-\frac{1}{2\theta^2}\sum_{i=1}^n(x_i-\theta)^2\}$

$=(2\pi\theta^2)^{-n/2}exp\{-\frac{1}{2\theta^2}(\sum_{i=1}^n x_i^2-2\theta\sum_{i=1}^n x_i+n\theta^2\}$

$=(2\pi\theta^2)^{-n/2}\cdot exp\{-\frac{n}{2}\}exp\{-\frac{1}{2\theta^2}\sum_{i=1}^n x_i^2+\frac{1}{\theta}\sum_{i=1}^n x_i\}$

$=g(t(x),\theta)h(x)$. $h(x)\equiv 1$

由因子分解定理, 充分统计量为 $(\sum_{i=1}^n x_i^2,\sum_{i=1}^n x_i)$, 与 $\bar{x}$ 不一一对应.

故 $\bar{x}$ 并非充分统计量.

47. 解：由样本独立，计算联合密度有

$$f(x_1,\cdots,x_m;y_1,\cdots,y_n)=(2\pi\sigma^2)^{-m/2}\exp\{-\tfrac{1}{2\sigma^2}\sum_{i=1}^m(x_i-a)^2\}(2\pi\sigma^2)^{-n/2}\exp\{-\tfrac{1}{2\sigma^2}\sum_{j=1}^n(\cancel{x}y_j-b)^2\}$$

$$=(2\pi\sigma^2)^{-(m+n)/2}\exp\{-\tfrac{1}{2\sigma^2}(ma^2+nb^2)\}\exp\{-\tfrac{1}{2\sigma^2}(\sum_{i=1}^m x_i^2+\sum_{j=1}^n y_j^2-2a\sum_{i=1}^m x_i-2b\sum_{j=1}^n y_j)\}$$

$$=(2\pi\sigma^2)^{-(m+n)/2}\exp\{-\tfrac{1}{2\sigma^2}(ma^2+nb^2)\}\exp\{-\tfrac{1}{2\sigma^2}(\sum_{i=1}^m x_i^2+\sum_{j=1}^n y_j^2-2am\bar{x}-2bn\bar{y})\}$$

由因子分解定理，统计量 $(\sum_{i=1}^m x_i^2+\sum_{j=1}^n y_j^2,\ \bar{x},\ \bar{y})$ 为充分统计量，进而与之对应的统计量 $(\bar{X},\bar{Y},S^2)$ 也为充分统计量

$f=C^*(\varphi)\exp\{\varphi_1T_1+\varphi_2T_2+\varphi_3T_3\}h(x)$，其中 $\varphi_1=-\frac{1}{2\sigma^2}$，$\varphi_2=\frac{am}{\sigma^2}$，$\varphi_3=\frac{bn}{\sigma^2}$.

自然参数空间为 $\Theta^*=\{(\varphi_1,\varphi_2,\varphi_3),\ -\infty<\varphi_1<0,\ -\infty<\varphi_2<\infty,\ -\infty<\varphi_3<\infty\}$

显然有内点，故 $(\sum_{i=1}^m x_i^2+\sum_{j=1}^n y_j^2,\ \bar{x},\ \bar{y})$ 为完全统计量，进而 $(\bar{X},\bar{Y},S^2)$ 为完全统计量.

48. 联合密度函数为 $f(x_1,\cdots,x_n)=(2\theta)^{-n}\exp\{-\frac{1}{\theta}\sum_{i=1}^n|x_i|\}=g(T(x),\theta)h(x)$

$h(x)\equiv1$. 由因子分解定理 $T=\sum_{i=1}^n|x_i|$ 是 θ 的充分统计量.

又自然参数空间 $\Theta^*=\{\varphi:-\infty<\varphi<0\}$ 显然有内点，故 T 为完全统计量

49. 证明：取合~~分布~~密度函数 $f(x_1,\cdots,x_n)=\exp\{-\sum_{i=1}^n(x_i-\theta)\}\prod_{i=1}^n I_{(\theta,\infty)}(x_i)$

$$=\exp\{n\theta\}\cancel{\prod_{i=1}^n I_{(\theta,\infty)}(x_{(1)})}I_{(x_{(1)}>\theta)}\cdot\exp\{-\sum_{i=1}^n x_i\}$$

$$=g(x_{(1)},\theta)\cdot h(x)$$

$h(x)=\exp\{-\sum_{i=1}^n x_i\}$. 由因子分解定理，$X_{(1)}$ 为充分统计量.

~~$f(x_1,\cdots,x_n)=C^*$~~ $X_{(1)}$ 的密度函数为 $g(t)=\begin{cases}ne^{-n(t-\theta)} & t>\theta\\ 0 & t\le\theta\end{cases}$

$\forall\varphi$. $E_\theta\varphi(T(X)) = \int_\theta^{+\infty}\varphi(t)\cdot ne^{-n(t-\theta)}dt = 0$. $\int_\theta^{+\infty}\varphi(t)\cdot e^{-nt}dt=0$

两边求导有 $\varphi(\theta)\cdot e^{-n\theta}=0$ $\forall\theta\in(-\infty,+\infty)$.

从而 $\varphi(t)=0$, 故 由定义 $X_{(1)}$ 为完全充分统计量.

50. 证明: 密度函数为 $f(x_1,\cdots,x_n)=\frac{1}{\theta^n}\prod_{i=1}^{n}I_{(-\frac{\theta}{2},\frac{\theta}{2})}(x_i)=\frac{1}{\theta^n}I_{(-\frac{\theta}{2}<x_{(1)}<x_{(n)}<\frac{\theta}{2})}$

$=g((x_{(1)},x_{(n)}),\theta)h(x)$, $h(x)\equiv1$

故 $(X_{(1)},X_{(n)})$ 为充分统计量.

取 $T(X)=\frac{X_{(n)}}{X_{(1)}}$, 令 $Y_i=\frac{X_i}{\theta}\sim U(-\frac{1}{2},\frac{1}{2})$.

$\frac{Y_{(n)}}{Y_{(1)}}=\frac{X_{(n)}}{X_{(1)}}$ 与 θ 无关.

令 $\frac{X_{(n)}}{X_{(1)}}=z$. 存在常数 a,b $P(z\leq a)=P(z>b)$

定义 $\varphi(z)=\begin{cases}-1 & z<a\\ 1 & z>b\\ 0 & \text{otherwise.}\end{cases}$ 则 $E_\theta\varphi(z)=0$, 但 $\varphi(z)\neq0$

从而 $(X_{(1)},X_{(n)})$ 并非完全统计量.

51. 证明: 密度函数为 $f(x_1,\cdots,x_n)=\frac{1}{\theta^n}\prod_{i=1}^{n}I_{(\theta,2\theta)}(x_i)=\frac{1}{\theta^n}I_{(\theta<x_{(1)}<x_{(n)}<2\theta)}$

由因子分解定理知 $(X_{(1)},X_{(n)})$ 是充分统计量.

取 $T(X)=\frac{X_{(n)}}{X_{(1)}}$ 令 $Y_i=\frac{X_i}{\theta}\sim U(1,2)$. $\frac{Y_{(n)}}{Y_{(1)}}=\frac{X_{(n)}}{X_{(1)}}$ 与 θ 无关.

令 $\frac{X_{(n)}}{X_{(1)}}=z$, 存在常数 a,b, 使得 $P(z<a)=P(z>b)>0$.

定义 $\varphi(z)=\begin{cases}-1 & z<a\\ 1 & z>b\\ 0 & \text{otherwise.}\end{cases}$ 则 $E_\theta(\varphi(z))=0$ 但 $\varphi(z)\neq0$

从而 $(X_{(1)},X_{(n)})$ 是充分统计量但非完全统计量.

52、证明：(1) $X_1,\cdots,X_n$ 的密度函数为

$$f(x_1,\cdots,x_n)=\lambda^{-n}\exp\{-\frac{1}{\lambda}\sum_{i=1}^{n}x_i+\frac{n}{\lambda}\mu\}\prod_{i=1}^{n}I_{(x_i>\mu)}$$

$$=\lambda^{-n}\exp\{-\frac{1}{\lambda}\sum_{i=1}^{n}x_{(i)}+\frac{n}{\lambda}\mu\}I_{(x_{(1)}>\mu)}$$

由因子分解定理知，$(X_{(1)},\sum_{i=1}^{n}X_{(i)})$ 是 (λ,μ) 的充分统计量.

(2) 已证 $X_{(1)}$ 的充分性，下证完全性. $X_{(1)}$ 的分布密度函数为

$$f(t)=\begin{cases}\frac{n}{\lambda}[\exp\{-\frac{t-\mu}{\lambda}\}]^n & x>\mu\\ 0 & \text{otherwise.}\end{cases}$$

$\forall\varphi$，$E_\theta(\varphi(X_{(1)}))=\int_{\mu}^{+\infty}\varphi(t)\cdot\frac{n}{\lambda}\exp\{-\frac{n}{\lambda}(t-\mu)\}dt=0.$

即 $\int_{\mu}^{+\infty}\varphi(t)\cdot\exp\{-\frac{n}{\lambda}t\}dt=0$ 求导有

$\varphi(\mu)\cdot\exp\{-\frac{n}{\lambda}\mu\}=0$ 从而 $\varphi=0$，一切 $\theta\in\Theta$.

则 $X_{(1)}$ 是完全统计量. 自然为有界完全统计量.

$\sum_{i=1}^{n}(X_i-X_{(1)})=\sum_{i=1}^{n}(X_{(i)}-X_{(1)})$.

令 $Y_i=X_i-\mu$ $i=1,2,\cdots,n$. $Y_i\sim\lambda^{-1}\exp\{-\frac{x}{\lambda}\}$ 与 μ 无关.

$X_{(i)}-X_{(1)}=Y_{(i)}-Y_{(1)}$ 与 μ 无关. 进而 $\sum_{i=1}^{n}(X_{(i)}-X_{(1)})$ 与 μ 无关.

由Basu定理. $X_{(1)}$ 与 $\sum_{i=1}^{n}(X_i-X_{(1)})$ 独立.

2. 解： $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ $\bar{Y} \sim N(\mu, \frac{\sigma^2}{n})$

由于 $\bar{X}, \bar{Y}$ 相互独立， $\bar{X}-\bar{Y} \sim (0, \frac{2}{n}\sigma^2)$.

$P(|\bar{X}-\bar{Y}|>\sigma) = P(\frac{|\bar{X}-\bar{Y}|}{\sigma}\cdot\sqrt{\frac{n}{2}} > \sqrt{\frac{n}{2}}) = 2(1-\phi(\sqrt{\frac{n}{2}})) \approx 0.01$

从而 $n \approx 13$

4. 解： $S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2 = \frac{1}{n}(\sum_{i=1}^{n}X_i^2 - n\bar{X}^2)$

由于两点分布中， $\sum_{i=1}^{n}X_i^2 = \sum_{i=1}^{n}X_i$，故 $S_n^2 = \frac{1}{n}(n\bar{X}-n\bar{X}^2) = \bar{X}(1-\bar{X})$.

$\sum_{i=1}^{n}X_i \sim B(n,p)$. $P(\bar{X}=\frac{k}{n}) = C_n^k p^k(1-p)^{n-k}$.

$P(S_n^2 = \frac{k}{n} - \frac{k^2}{n^2}) = P(\bar{X} = \frac{k}{n}, \frac{n-k}{n})$

$$= \begin{cases} C_n^k[p^k(1-p)^{n-k} + p^{n-k}(1-p)^k] & k=0,1,\cdots,[\frac{n}{2}], n\text{为奇} \\ C_n^k[p^k(1-p)^{n-k} + p^{n-k}(1-p)^k] & k=0,1,\cdots,\frac{n}{2}-1 \\ C_n^{\frac{n}{2}} p^{\frac{n}{2}}(1-p)^{\frac{n}{2}} & \end{cases} \quad n\text{为偶}$$

5. 证明： 令 $u = \frac{x_1}{x_2}$, $v = \sqrt{x_1^2+x_2^2}$. 反解得

$$x_1 = \frac{uv}{\sqrt{1+u^2}}, \quad x_2 = \frac{v}{\sqrt{1+u^2}}, \quad (x_2>0)$$

或

$$x_1 = \frac{-uv}{\sqrt{1+u^2}}, \quad x_2 = \frac{-v}{\sqrt{1+u^2}} \quad (x_2<0)$$

$$\begin{vmatrix} \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} \\ \frac{\partial v}{\partial x_1} & \frac{\partial v}{\partial x_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{x_2} & -\frac{x_1}{x_2^2} \\ \frac{x_1}{\sqrt{x_1^2+x_2^2}} & \frac{x_2}{\sqrt{x_1^2+x_2^2}} \end{vmatrix} = \frac{1+u^2}{v} = J^{-1}, \text{故 } J = \frac{v}{1+u^2}$$

$$f(x_1,x_2) = (2\pi\sigma^2)^{-\frac{2}{2}}\exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^{2}(x_i)^2\} = \frac{1}{2\pi\sigma^2}\exp\{-\frac{1}{2\sigma^2}(x_1^2+x_2^2)\}$$

$$f(u,v) = \cancel{\frac{1}{2\pi\sigma^2}\exp\{-\frac{1}{2\sigma^2}} f(x_1,x_2)|J|(x_2>0) + f(x_1+x_2)|J|(x_2<0)$$

$$= \frac{1}{\pi\sigma^2}\exp\{-\frac{1}{2\sigma^2}v^2\}\cdot\frac{v}{1+u^2}.$$ 从而 $\frac{x_1}{x_2}$ 和 $\sqrt{x_1^2+x_2^2}$ 相互独

$$= f_V(v)\cdot f_U(u)$$

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固定 σ，有

53. 证明：$f(x_1,\cdots,x_n)=(2\pi\sigma^2)^{-n/2}\exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^n(x_i-a)^2\}$

$=(2\pi\sigma^2)^{-n/2}\exp\{-\frac{n}{2\sigma^2}a^2\}\exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2+\frac{a}{\sigma^2}\sum_{i=1}^n x_i\}$

$=g(\sum_{i=1}^n x_i,a)h(x)$，$h(x)\equiv 1$

由因子分解定理，$\sum_{i=1}^n x_i$ 为 a 的充分统计量。

又 $f(x_1,\cdots,x_n)$ 的自然形式，$\varphi=\frac{a}{\sigma^2}$，参数空间 $\Theta^*:\{\varphi:-\infty<\varphi<\infty\}$

可知其有内点。$\sum_{i=1}^n x_i$ 为 a 的完全统计量，

于是 $\bar{x}=\frac{1}{n}\sum_{i=1}^n x_i$ 为 a 的充分完全统计量。

令 $Y_i=x_i-a$，$Y_i\sim N(0,\sigma^2)$。

$X_{(n)}-X_{(1)}=Y_{(n)}-Y_{(1)}$ 与 a 无关。由Basu定理，$\bar{X}$ 与 $X_{(n)}-X_{(1)}$ 独立

54. 证明：由独立性。$f(x_1,\cdots,x_n;y_1,\cdots,y_n)$ ~~$=(2\pi\sigma^2)^{-n}\exp\{-\frac{1}{2\sigma^2}$~~

$=(2\pi\sigma_1^2)^{-\frac{n}{2}}\exp\{-\frac{1}{2\sigma_1^2}\sum_{i=1}^n(x_i-a)^2\}(2\pi\sigma_2^2)^{-\frac{n}{2}}\exp\{-\frac{1}{2\sigma_2^2}\sum_{i=1}^n(y_i-b)^2\}$

$=(2\pi\sigma_1\sigma_2)^{-n}\exp\{-\frac{n}{2\sigma_1^2}a^2-\frac{n}{2\sigma_2^2}b^2\}\exp\{-\frac{1}{2\sigma_1^2}\sum_{i=1}^n x_i^2-\frac{1}{2\sigma_2^2}\sum_{i=1}^n y_i^2+\frac{a}{\sigma_1^2}\sum_{i=1}^n x_i+\frac{b}{\sigma_2^2}\sum_{i=1}^n y_i\}$

由因子分解定理，$(\sum_{i=1}^n x_i^2,\sum_{i=1}^n y_i^2,\sum_{i=1}^n x_i,\sum_{i=1}^n y_i)$ 为 (σ_1,σ_2,a,b) 的充分统计量。

化为自然形式，$\varphi_1=-\frac{1}{2\sigma_1^2}$，$\varphi_2=-\frac{1}{2\sigma_2^2}$，$\varphi_3=\frac{a}{\sigma_1^2}$，$\varphi_4=\frac{b}{\sigma_2^2}$。

自然参数空间为 $\Theta^*=\{(\varphi_1,\varphi_2,\varphi_3,\varphi_4):\varphi_1<0,\varphi_2<0,-\infty<\varphi_3<+\infty,-\infty<\varphi_4<+\infty\}$

作为 $\mathbb{R}^4$ 的子集有内点，于是为完全统计量，进而充分

$T(X,Y)=(\bar{X},\bar{Y},Q_1^2,Q_2^2)$ 为 (σ_1,σ_2,a,b) 的完全统计量。

$\bar{X}\sim(a,\frac{\sigma_1^2}{n})$，$X_i-\bar{X}\sim(0,\frac{n+1}{n}\sigma_1^2)$，$\frac{X_i-\bar{X}}{\sqrt{\frac{n+1}{n}}\sigma}\sim(0,1)$。

$\frac{Q_1^2}{\sigma_1^2}\sim\chi^2_{n-1}$。令 $T_i=\frac{X_i-\bar{X}}{\sqrt{\frac{n+1}{n}}\sigma}\Big/\sqrt{\frac{Q_1^2}{\sigma_1^2}/n-1}=\frac{X_i-\bar{X}}{\sqrt{Q_1^2}}\cdot\sqrt{\frac{n(n-1)}{n+1}}\sim t_{n-1}$。

同理 $T_i'=\frac{Y_i-\bar{Y}}{\sqrt{Q_2^2}}\cdot\sqrt{\frac{n(n-1)}{n+1}}\sim t_{n-1}$。$r(X,Y)=\frac{n+1}{n(n-1)}\sum_{i=1}^n T_i\cdot T_i'$ 与 (σ_1,σ_2,a,b) 无关

由Basu定理，$T(X,Y)$ 与 $r(X,Y)$ 独立。