Homework 3

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PROBLEM 1.

For given $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, let

$$C_s(x) = \{y = (y_1, \cdots, y_n) : |x_i - y_i| < s/2, i = 1, \cdots, n\}$$

be the open cube of side length s centered at x. Show that the collection

$$\mathcal{B} = \{C_s(x) : x \in \mathbb{R}^n \text{ and } s > 0\}$$

is a basis for the Euclidean topology on $\mathbb{R}^n$.

SOLUTION.

Recall the definition of the basis for the Euclidean topology on $\mathbb{R}^n$.

(i) Every element of $\mathcal{B}$ is an open subset of the Euclidean topology.

(ii) Every open subset of the Euclidean topology is the union of some collection of elements of $\mathcal{B}$.

First we will show (i).

Let $C_s(x) \in \mathcal{B}$. For any $y \in C_s(x)$, we show there exists an open Euclidean ball $B(y, \varepsilon) \subseteq C_s(x)$.

By definition of $C_s(x)$, for each coordinate $i (i = 1, 2, \cdots, n)$, $|y_i - x_i| < \frac{s}{2}$. Then the distance from y to the boundary of $C_s(x)$ in the i-th coordinate is $d_i = \frac{s}{2} - |y_i - x_i| > 0$.

Let $\varepsilon = \min_{1 \le i \le n} d_i$. For any $z \in B(y, \varepsilon)$, by the Euclidean norm:

$$|z_i - y_i| \le d(z, y) < \varepsilon \le d_i \quad \forall i = 1, 2, \cdots, n.$$

Thus,

$$|z_i - x_i| \le |z_i - y_i| + |y_i - x_i| < d_i + |y_i - x_i| = \frac{s}{2}.$$

Hence, $z \in C_s(x)$, so $B(y, \varepsilon) \subseteq C_s(x)$. Therefore, $C_s(x)$ is open in the Euclidean topology.

Now, it suffices to show (ii).

Let $U \subseteq \mathbb{R}^n$ be open in the Euclidean topology. For every $x \in U$, there exists $r > 0$ such that the open ball $B(x, r) \subseteq U$. We construct a cube $C_s(x) \subseteq B(x, r)$ as follows:

Choose $s = \frac{2r}{\sqrt{n}}$. For any $y \in C_s(x)$, the Euclidean distance satisfies:

$$d(y, x) \le \sqrt{n} \cdot \max_{1 \le i \le n} |y_i - x_i| < \sqrt{n} \cdot \frac{s}{2} = \sqrt{n} \cdot \frac{r}{\sqrt{n}} = r.$$

Thus, $C_s(x) \subseteq B(x, r) \subseteq U$.

Since every $x \in U$ has such a cube $C_s(x) \subseteq U$, the set U is the union of all such cubes:

$$U = \bigcup_{x \in U} C_{s_x}(x), \quad \text{where } s_x = \frac{2r_x}{\sqrt{n}} \text{ and } B(x, r_x) \subseteq U.$$

The result follows.

$$\boxed{\mathcal{B} \text{ is a basis for the Euclidean topology on } \mathbb{R}^n.}$$

$\square$

PROBLEM 2.

A map $f : X \to Y$ is **open**, if for any open subset $U \subset X$, the image $f(U)$ is open in Y. Let $f : X \to Y$ be a continuous open map. Show that if X is second countable, then $f(X)$ is also second countable.

SOLUTION.

Since X is second countable, let $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$ be a countable basis for X.

Define $\mathcal{C} = \{f(B_n) : B_n \in \mathcal{B}\}$. Each $f(B_n)$ is open in Y by the definition of open map f. Under the subspace topology on $f(X)$, every $f(B_n)$ is also open in $f(X)$. Moreover, $\mathcal{C}$ is obviously countable. This follows directly from the countability of $\mathcal{B}$.

It suffices to show that $\mathcal{C}$ forms a basis for $f(X)$. Let $V \subseteq f(X)$ be open in $f(X)$. Then $V = W \cap f(X)$ for some open set $W \subseteq Y$. For any $y \in V$, there exists $x \in X$ such that $f(x) = y$. Since f is continuous, $f^{-1}(W)$ is open in X. By the basis criterion of $\mathcal{B}$, there exists $B \in \mathcal{B}$ such that:

$$x \in B \subseteq f^{-1}(W).$$

Applying f, we get:

$$y = f(x) \in f(B) \subseteq f(f^{-1}(W)) = W \cap f(X) = V.$$

Thus, $\mathcal{C}$ satisfies the basis criterion for $f(X)$.

$\mathcal{C}$ is a countable basis for $f(X)$, so $f(X)$ is second countable.

□

Problem 3.

Let $f, g : X \to Y$ be continuous. Assume that Y is Hausdorff. Show that $\{x : f(x) = g(x)\}$ is closed in X.

Solution.

Denote $C = \{x \mid f(x) = g(x)\}$.

Suppose C is not closed. Then there exists $x_1 \notin C$ where x_1 is a limit point of C. Thus $f(x_1) \neq g(x_1)$ in Y. Since Y is Hausdorff, we can find disjoint open sets U and V containing $f(x_1)$ and $g(x_1)$ respectively. Moreover $f^{-1}(U)$ and $g^{-1}(V)$ are open sets in X since f and g are continuous and both sets contain the point x_1.

Now consider the set $A = f^{-1}(U) \cap g^{-1}(V)$. This set is open and must also contain the point x_1. Since x_1 is a limit point of C, the set A must contain some point $z \in C$, and $f(z) = g(z)$ by definition. Since $z \in A$, we have $f(z) \in U$ and $g(z) \in V$. This implies that $f(z) = g(z) \in U \cap V$, $U \cap V \neq \emptyset$. A contradiction since U and V were chosen to be disjoint.

So C must contain all its limit points and thus C is closed.

$\square$

Problem 4.

Show that every manifold has a basis of coordinate balls.

Solution.

Let U be an open set in M. Let $x \in U$. By definition there exists some open neighborhood C of x. and there is a homeomorphism φ to some open neighborhood of the origin of $\mathbb{R}^n$. Let B denote the component of $U \cap C$ which contains x. Then $\varphi|_B$ is a homeomorphism onto its image, which is again a connected open neighborhood of the origin of $\mathbb{R}^n$.

By possibly postcomposing with a translation, we may WLOG assume that $\varphi(x) = 0$. Now take a small open ball S centered at 0 in $\mathbb{R}^n$ such that $S \subseteq \varphi(B)$. Its preimage $\varphi^{-1}(S)$ is homeomorphic to $\mathbb{R}^n$ via $\varphi^{-1} \circ \psi$, where ψ is a homeomorphism between an ε-ball and all of $\mathbb{R}^n$, so $\varphi^{-1}(S)$ together with the restriction of φ_C is a coordinate chart. Finally we have $x \in \varphi^{-1}(S) \subseteq U$.

Therefore, for every open set U of M and for every point $x \in U$, we may find a coordinate chart S which has $x \in S \subseteq U$, so coordinate neighborhoods form a basis for the topology of M.

$\square$

PROBLEM 5.

Let X be a topological space satisfying the T_1-axiom. Then each one-point subset is closed in X. The topological space X is said to be **regular** if each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively. Prove that every manifold is regular and hence metrizable.

SOLUTION.

For any manifold, it's Hausdorff thus satisfying the T_1-axiom.

Lemma. X is regular iff for any $x \in X$ and any open neighborhood U, there exists an open neighborhood V such that $\overline{V} \subseteq U$.

Proof. $\Rightarrow$ Suppose X is a regular space, for $x \in X$, let U be an open neighborhood of x, then U^C is a closed set which does not contain x. Thus there exist disjoint open neighborhoods U_1 and V_1 containing x and U^C. Then $U_1 \subseteq V_1^C$.

Therefore

$$\overline{U_1} \subseteq \overline{V_1^C} = V_1^C \subseteq U$$

So $\overline{U}_1 \subseteq U$.

$\Leftarrow$ Suppose $x \in X$ and A, which is a closed set and does not contain x. Then A^C is an open neighborhood of x. Since there is an open neighborhood U such that $\overline{U} \subseteq A^C$, denote $V = \overline{U}^C$, then $A \subseteq V$. Thus V is an open neighborhood of A, and $U \cap V = \emptyset$.

Let M be a manifold. We aim to show that M is regular, i.e., for every point $x \in M$ and every open neighborhood U of x, there exists an open set V such that $x \in V \subseteq \overline{V} \subseteq U$.

Since M is locally Euclidean, there exists an open neighborhood $W \subseteq M$ containing x and a homeomorphism $\phi : W \to \phi(W) \subseteq \mathbb{R}^n$.

Define $U' = U \cap W$. Then U' is an open neighborhood of x contained in W. Under the homeomorphism ϕ, the image $\phi(U') \subseteq \mathbb{R}^n$ is open.

Since $\mathbb{R}^n$ is regular, there exists an open set $V' \subseteq \mathbb{R}^n$ such that:

$$\phi(x) \in V' \subseteq \overline{V'} \subseteq \phi(U').$$

Let $V = \phi^{-1}(V')$. Then V is an open set in M satisfying:

$$x \in V \subseteq \phi^{-1}(\overline{V'}) \subseteq \phi^{-1}(\phi(U')) = U' \subseteq U.$$

Since homeomorphisms preserve closures, we have $\overline{V} = \phi^{-1}(\overline{V'})$. Therefore:

$$\overline{V} \subseteq U' \subseteq U.$$

Hence, every manifold is regular. By Urysohn's metrization theorem it's metrizable.

$\square$