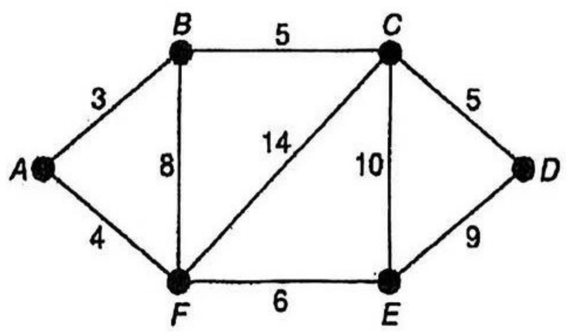
兰州大学 2024-2025 学年第二学期

《图论》期中测验试题(2024-4-29)

1. Show that if a directed graph D has no directed cycles, then D has a source (a vertex with in-degree 0) and a sink (a vertex with out-degree 0) .
2. Let T_1 and T_2 be spanning trees of a connected graph G.
   (1) If e is any edge of T_1, show that there exists an edge f of T_2 such that the graph $(T_1-\{e\})\cup\{f\}$ is also a spanning tree.
   (2) Deduce that T_1 can be "transformed" into T_2 by replacing the edges of T_1 one at a time by edges of T_2 in such a way that a spanning tree is obtained at each stage.
3. Prove Ore's theorem: If G is a simple graph with n (≥ 3) vertices, and if $\deg(u)+\deg(v)\geq n$ for each pair of non-adjacent vertices u and v, then G is Hamiltonian.
4. Prove that, if S is any set of edges of a connected graph G with an even number of edges in common with each cutset of G, then S can be split into edge-disjoint cycles.
5. Show that the Petersen graph is non-planar.
6. Let G be a connected plane graph. Prove that G is bipartite if and only if its dual G^* is Eulerian.
7. Use the shortest path algorithm (Dijkstra's algorithm) to find shortest paths from A to all other vertices with the corresponding distances in the following weighted graph.



华英学院 22级 王一鑫 80 Midterm for Graph Theory

1. proof. 4

Suppose ~~not, then D doesn't have a source and a sink~~.
~~That is, for $\forall v \in V(D)$. $indeg(v)>0$, $outdeg(v)>0$.~~
~~If D has a directed cycles, then every vetex in the cycle~~
~~has the~~ same indegree and outdegree ~~so every vertex of~~
~~Now every vertex in D has different indegree and outdegree.~~
~~Suppose now $u, v \in V(D)$ has~~ have different ~~indegree and outdegree~~
For a directed cycles, choose two adjacent vertices ~~which~~ v w
s.t. $indeg(v) = outdeg(v) = indeg(w) = outdeg(w)$ = change the direction between them will lead to a non-directed cycles, and v can become a source and a sink.

2. proof: 16

(1) If e is any edge of T_1, then $T_1 - \{e\}$ is disconnected, by the property of spanning trees. Suppose $T_1 - \{e\} = U \cup V$, U and V are connected. Since T_2 is also a spanning tree, then there exists an edge f of T_2 s.t. f connects U and V. then $(T_1 - \{e\}) \cup \{f\}$ is a connected graph with no cycles. thus $(T_1 - \{e\}) \cup \{f\}$ is also a spanning tree.

$E(T_1) \setminus E(T_2)$

subtrees

(2). Susppose $|E(T_1) \setminus E(T_2)| = k$. if $k=0$. Then $T_1 = T_2$.
if $k \neq 0$. Choose an edge e of T_1. and replace it with an edge f of T_2. This process holds since (1).
Then $|E(T_1) \setminus E(T_2)| = k-1$ after doing this.
We can replace the edges of T_1 one at a time by edges of T_2 in such a way that a spanning tree is obtained at each stage.

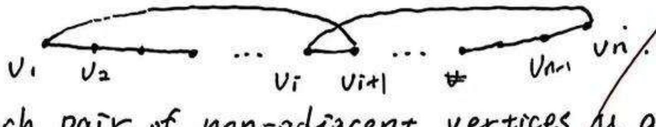
Induction

3. proof: 12

~~We show this theorem by induction on the edges of the simple graph G.~~
~~For $n=3$ it holds obviously.~~
~~Now suppose $n=k-1$ the conclusion holds.~~
Suppose not. WLOG, we may assume G is non-Hamiltonian which can be Hamiltonian after adding an edge. The for $v_1, v_2, \cdots, v_n$. we have a ~~trait~~ shown as follows

H-path

要考虑边



Since for each pair of non-adjacent vertices u and v, $deg(u) + deg(v) \geq n$
The $deg(v_1) + deg(v_n) \geq n$. There must have a vertex v_i and its adjacent vertex v_{i+1}, s.t. v_1 and v_{i+1} are adjacent, v_i and v_n are adjacent. otherewise, $deg(v_1) + deg(v_n) < n$. which is a contradiction to our hyphothesis. Now $v_1 \to v_2 \to \cdots \to v_i \to v_n \to v_{n-1} \to \cdots \to v_{i+1} \to v_1$ is a Hamiltonian cycle, which is a contradiction. Thus G is Hamiltonian.

4. proof. S can be split into edge-disjoint cycles iff S is ~~an~~ Eulerian, iff ~~every~~ each vertex in S have even degree. each component of G[S]

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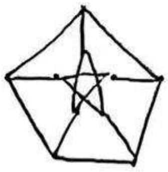
Let G' be the graph obtained from G by removing all edges of G that are not in S. Then now it suffices to show that ~~G~~ each vertex in G' has even degree.

Assume ~~there~~ exists a vertex v of G' that has odd degree.

Since all ~~vertices~~ edges incident with v in G forms a cutset. and by hypothesis. S have an even number of edges in common with the cutset. and the common edges are exactly the edges incident with v. which is odd a contradiction.

or Union of some disjoint cutsets.

5. proof. The Peterse graph



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contains ~~A~~ a subgraph contractiable to $K_{3,3}$. • and ∘ in the graph.

By Kuratowski's theorem, Petersen graph is non-plana

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6. proof G is bipartite iff every cycle in G has even lengths. and thus each cutset of G^* has an even number of edges. then each vertex in G^* has even degree. By Euler's theorem we know that G^* is Eulerian ✓ Coversely. ~~若~~. following in the same way. G^* is Eulerian $\Rightarrow G^{**}$ is biparite. and G is a connected plane graph. $G^{**} = G$.

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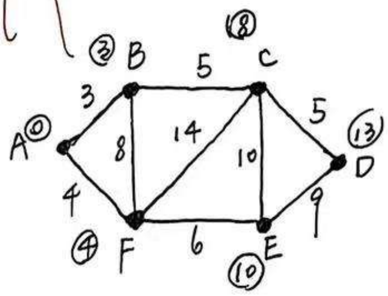
7. 

By Dijkstra's algorithm, we know that we can labelled each vertex. ~~like~~ as the graph shown

A	to B	A→B	3
	to C	A→B→C	8
	to D	A→B→C→D	13
	to E	A→F→E	10
	to F	A→F	4.

要有讨论过程.