Homework 7

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2025 年 4 月 10 日

Problem 1. (Exercise 4.5)

(i) For which values of k is the k-cube Q_k planar?

(ii) For which values of r, s, and t is the complete tripartite graph $K_{r,s,t}$ planar?

Solution.

(i) $k \geq 3$. This is because when $k = 4$, Q_k contains $K_{3,3}$. Moreover, Q_k can be constructed by two Q_{k-1}, so Q_k is not planar ($k \geq 4$).

(ii) Without loss of generality assume $r \geq s \geq t$.

If $r \geq 3$, and $s + t \geq 3$, then the graph is not planar (It contains $K_{3,3}$).

In the cases $r \leq 2$ or $s + t \leq 2$, it is planar.

□

Problem 2. (Exercise 4.7)

Prove that the Petersen graph is non-planar

(i) by removing the two 'horizontal' edges and using Theorem 4.2;

(ii) by using Theorem 4.3.

Solution.

(i) By removing the two 'horizontal' edges we find that Petersen graph contains $K_{3,3}$, shown in Fig 1.

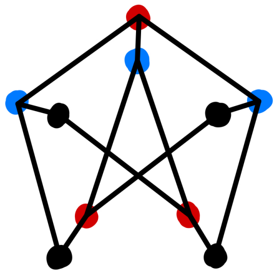

图 1: $K_{3,3}$

By Theorem 4.2, Petersen graph is not planar.

(ii) Petersen graph is contractible to K_5 as Fig 2. Apply Theorem 4.3.

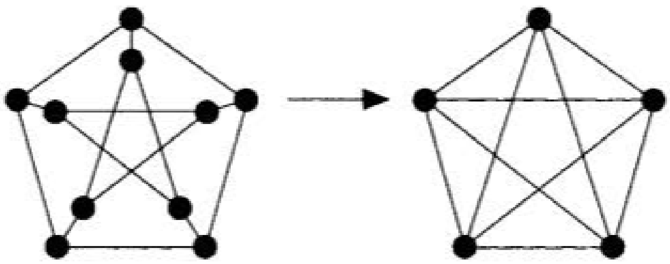

图 2: K_5

□

Problem 3. (Exercise 4.9)

If two homeomorphic graphs have n_i vertices and m_i edges ($i = 1, 2$), show that

$$m_1 - n_1 = m_2 - n_2.$$

Solution.

Any graph is homeomorphic to a graph with no vertices of degree 2. In the process of removing a vertex of degree 2, we will lose one vertex and one edge, so the difference $m - n$ is unchanged. Therefore, it suffices to show that homeomorphic graphs with no vertices of degree 2 are isomorphic (and so have the same number of vertices and the same number of edges). This is because, in the absence of vertices of degree 2, a homeomorphism between two graphs must take vertices to vertices and edges to edges.

□

Problem 4. (Exercise 4.10)

A planar graph G is outerplanar if G can be drawn in the plane so that all of its vertices lie on the exterior boundary.

(i) Show that K_4 and $K_{2,3}$ are not outerplanar.

(ii) Deduce that, if G is an outerplanar graph, then G contains no subgraph homeomorphic or contractible to K_4 or $K_{2,3}$.

(The converse result also holds, yielding a Kuratowski-type criterion for a graph to be outerplanar.)

Solution.

(i) Let G' be obtained from G by adding an extra vertex to G adjacent to every other vertex. Then G' is planar if and only if G is outerplanar.

By Theorem 4.1 we know that K_5 and $K_{3,3}$ are non-planar, thus K_4 and $K_{2,3}$ are not outerplanar.

(ii) Kuratowski's theorem shows that if G' is planar, then G' contains no subgraph homeomorphic or contractible to K_5 or $K_{3,3}$.By the argument in (i), G contains no subgraph homeomorphic or contractible to K_4 or $K_{2,3}$.

$\square$

PROBLEM 5. (Exercise 4.11)

By placing its vertices at the points $(1, 1^2, 1^3)$, $(2, 2^2, 2^3)$, $(3, 3^2, 3^3)$, $\cdots$, prove that any simple graph can be drawn without crossings in Euclidean three-dimensional space so that each edge is represented by a straight line.

SOLUTION.

If there are two crossing edges with endpoints on that curve, then the endpoints are four coplanar points on the curve. Call the points (t_i, t_i^2, t_i^3), $i = 1, 2, 3, 4$, where t_1, t_2, t_3, t_4 are distinct real numbers, and suppose they all lie on a plane $A + Bx + Cy + Dz = 0$ where A, B, C, D are not all zero. Then A, B, C, D satisfy the equation

$$\begin{bmatrix} 1 & t_1 & t_1^2 & t_1^3 \\ 1 & t_2 & t_2^2 & t_2^3 \\ 1 & t_3 & t_3^2 & t_3^3 \\ 1 & t_4 & t_4^2 & t_4^3 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

However, since the coefficient matrix is a nonsingular Vandermonde matrix, the equation has no nontrivial solution. Thus any simple graph can be drawn without crossings in Euclidean three-dimensional space so that each edge is represented by a straight line.

□

PROBLEM 6. (Exercise 4.15)

(i) Use Euler's formula to prove that, if G is a connected planar graph of girth 5 with n vertices and m edges, then $m \leq \frac{5}{3}(n-2)$. Deduce that the Petersen graph is non-planar.

(ii) Obtain an inequality, generalizing that in part (i), for connected planar graphs of girth r.

SOLUTION.

(i) Since G has girth 5, we have $5f \leq 2m$. Combining this with Euler's formula $n - m + f = 2$ gives the required inequality.

$$5(2-n+m) \leq 2m \Leftrightarrow m \leq \frac{5}{3}(n-2)$$

If the Petersen graph were planar, then this inequality would be $15 \leq \frac{40}{3}$, which is false. Thus, the Petersen graph is non-planar.

(ii) If G has girth r, then $rf \leq 2m$. Combining this with Euler's formula gives the inequality

$$r(2-n+m) \leq 2m \Leftrightarrow m \leq \frac{r(n-2)}{r-2}.$$

□

PROBLEM 7. (Exercise 4.16)

Let G be a polyhedron (or polyhedral graph), each of whose faces is bounded by a pentagon or a hexagon.

(i) Use Euler's formula to show that G must have at least 12 pentagonal faces.

(ii) Prove, in addition, that if G is such a polyhedron with exactly three faces meeting at each vertex (such as a football), then G has exactly 12 pentagonal faces.

SOLUTION.

(i) By the property of polyhedron, we have $n \leq \frac{2}{3}m$, so that $-n \geq -\frac{2}{3}m$ and so, by Euler's formula, we have:

$$f = 2 + m - n \geq 2 + m - \frac{2}{3}m = 2 + \frac{1}{3}m.$$

Now if we have have p pentagonal faces and h hexagonal faces, then there are $p + h$ faces, so $p + h = f$. and $5p + 6h = 2m$

$$p + h \geq 2 + \frac{1}{6}(5p + 6h) \Leftrightarrow p \geq 12$$

(ii) Now $v = \frac{2}{3}e$, we can change the inequality to equality.

□

PROBLEM 8. (Exercise 4.17)

Let G be a simple plane graph with fewer than 12 faces, in which each vertex has degree at least 3.

(i) Use Euler's formula to prove that G has a face bounded by at most four edges.

(ii) Give an example to show that the result of part (i) is false if G has 12 faces.

SOLUTION.

(i) By problem 7 we have

$$f \geq= 2 + \frac{1}{3}m.$$

If not, then G has faces bounded by at least 5 edges, that is $5f \leq 2m$.

$$10 + \frac{5}{3}m \leq 2m \Rightarrow m \geq 30$$

Thus, $f \geq 2 + \frac{1}{3}m \geq 12$, which is a contradiction since G is a simple plane graph with fewer than 12 faces.

(ii) The situation in Problem 7 (ii).

□

PROBLEM 9. (Exercise 4.18)

(i) Let G be a simple connected cubic plane graph, and let C_k be the number of k-sided faces. By counting the number of vertices and edges of G, prove that

$$3C_3 + 2C_4 + C_5 - C_7 - 2C_8 - 3C_9 - \cdots = 12.$$

(ii) Use this result to deduce the result of Exercise 4.16(ii).

(iii) Deduce also that G has at least one face bounded by at most five edges.

Solution.

(i) If G has n vertices, m edges, and f faces, then

$$\begin{aligned} f &= C_3 + C_4 + C_5 + C_6 + \dots, \\ 2m &= 3C_3 + 4C_4 + 5C_5 + 6C_6 + \dots, \\ 3n &= 3C_3 + 4C_4 + 5C_5 + 6C_6 + \dots. \end{aligned}$$

The last equality is from the definition of cubic graph. Substituting these expressions for f, m, and n into Euler's formula yields the result.

(ii) Since $C_3 = C_4 = C_7 = C_8 = \cdots = 0$, we deduce that $C_5 = 12$.

(iii) If G has no face bounded by at most five edges, then $C_3 = C_4 = C_5 = 0$, and the left-hand side is negative; this is a contradiction.

$\square$