Homework 7

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Problem 1.

A topological space is **totally disconnected** if all of its components are singletons.

(1) The subset $\mathbb{Q}$ of all rational numbers is a subspace of one-dimensional Euclidean space $\mathbb{R}^1$. Prove that $\mathbb{Q}$ is totally disconnected.

(2) Is $\mathbb{Q}$ discrete?

(3) Prove that the Cantor set is totally disconnected.

Solution.

(1) if $C \subseteq \mathbb{Q}$ and $q_1 < q_2$ exist in C, we can find an irrational s in between and then $\{(-\infty, s) \cap \mathbb{Q}, (s, \infty) \cap \mathbb{Q}\}$ disconnects $\mathbb{Q}$ and C. So any subset of $\mathbb{Q}$ with two or more points is disconnected.

(2) In the subspace topology, a singleton $\{q\}$ in $\mathbb{Q}$ would need to be the intersection of an open set in $\mathbb{R}$ with $\mathbb{Q}$.

Any open set in $\mathbb{R}$ containing q must contain an interval around q, which includes infinitely many rational numbers. Hence, singletons in $\mathbb{Q}$ are not open. Since $\mathbb{Q}$ has no isolated points, it is not discrete.

(3) Consider two distinct points x and y in the Cantor set C. Their ternary expansions differ at some position n.

At the n-th stage of the Cantor set construction, the interval containing x and y is split, and the middle third is removed, placing x and y in different intervals.

These intervals are clopen in C, allowing x and y to be separated by clopen sets. Thus, the connected component of x cannot contain y, showing all components are singletons. Hence, C is totally disconnected.

$\square$

PROBLEM 2. (Exercise 4.18)

Let M_1 and M_2 be n-manifolds. For $i = 1, 2$, let $B_i \subseteq M_i$ be regular coordinate balls, and let $M_i' = M_i \setminus B_i$. Choose a homeomorphism $f : \partial M_2' \to \partial M_1'$ (such a homeomorphism exists by Problem 4-17). Let $M_1 \# M_2$ (called a **connected sum of M_1 and M_2**) be the adjunction space $M_1' \cup_f M_2'$ (Fig. 1).

(a) Show that $M_1 \# M_2$ is an n-manifold (without boundary).

(b) Show that if M_1 and M_2 are connected and $n > 1$, then $M_1 \# M_2$ is connected.

(c) Show that if M_1 and M_2 are compact, then $M_1 \# M_2$ is compact.

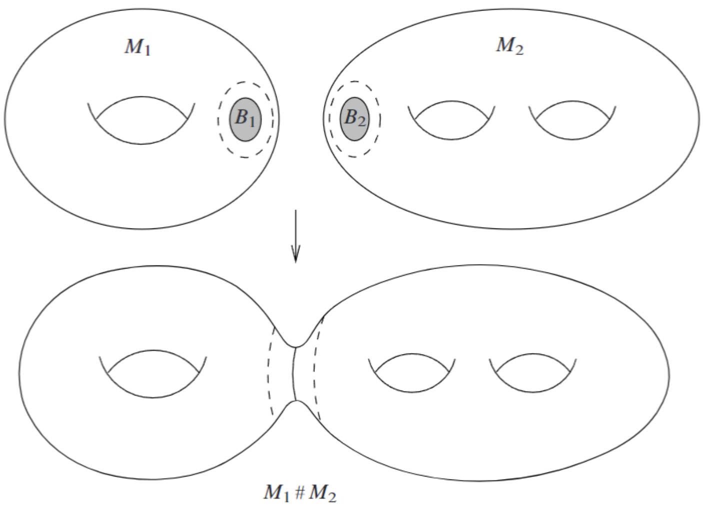

图 1: Connected Sum

SOLUTION.

(a) For points in the interior of M_1' or M_2', neighborhoods remain homeomorphic to $\mathbb{R}^n$.

For points on the glued boundary, consider collar neighborhoods around $\partial M_1'$ and $\partial M_2'$. These collars, homeomorphic to $S^{n-1} \times [0, 1]$ and $S^{n-1} \times (-1, 0]$, merge to form $S^{n-1} \times (-1, 1)$. Using the radial extension of the homeomorphism f, this merged neighborhood is homeomorphic to $\mathbb{R}^n$. Thus, every point in $M_1 \# M_2$ has a neighborhood homeomorphic to $\mathbb{R}^n$, making it an n-manifold without boundary.

(b) A theorem from the book furnishes topological embeddings

$$e_i : M_i' \to M_1 \# M_2 \quad \text{such that:}$$

$$e_1(M_1') \cup e_2(M_2') = M_1 \# M_2$$

$$e_1(M_1') \cap e_2(M_2') = e_1(\partial M_1') = e_2(\partial M_2')$$

Since the M_i' have nonempty boundary, $e_1(M_1') \cap e_2(M_2') \neq \emptyset$. Now, $e_i(M_i')$ being a topological embedding, it is connected if and only if M_i' is connected, which is true if M_i is connected.

So assuming that M_1 and M_2 are connected, this shows that $M_1 \# M_2$ is the union of nonempty connected sets with nonempty intersection, which implies that it is connected.

(c) M_1' and M_2' are compact as they are closed subsets of compact manifolds. The adjunction space $M_1' \cup_f M_2'$ is a quotient of the compact space $M_1' \sqcup M_2'$, hence compact.

□

PROBLEM 3. (Exercise 6.1)

Show that a connected sum of one or more projective planes contains a subspace that is homeomorphic to the Möbius band.

SOLUTION. By Fig 2 from Wikipedia we know that the right diagram is the subset of the left diagram.

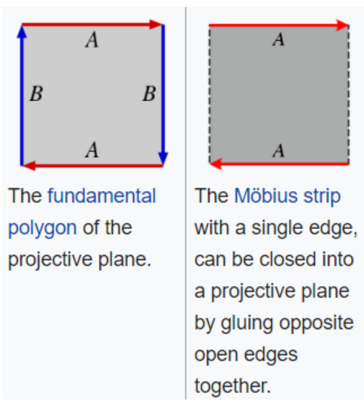

图 2: Projective planes vs Möbius band

□

PROBLEM 4. (Exercise 6.2)

Note that both a disk and a Möbius band are manifolds with boundary, and both boundaries are homeomorphic to $\mathbb{S}^1$. Show that it is possible to obtain a space homeomorphic to a projective plane by attaching a disk to a Möbius band along their boundaries.

SOLUTION.

To see that P^2 minus a disk is a Mobius band, see Fig 3. In the upper left is P^2, drawn as a fundamental polygon with sides identified. In the upper right, we've removed a disk. The boundary of the now-missing disk is drawn at the lower left as a dashed line. In the lower right we obtain the required Möbius band.

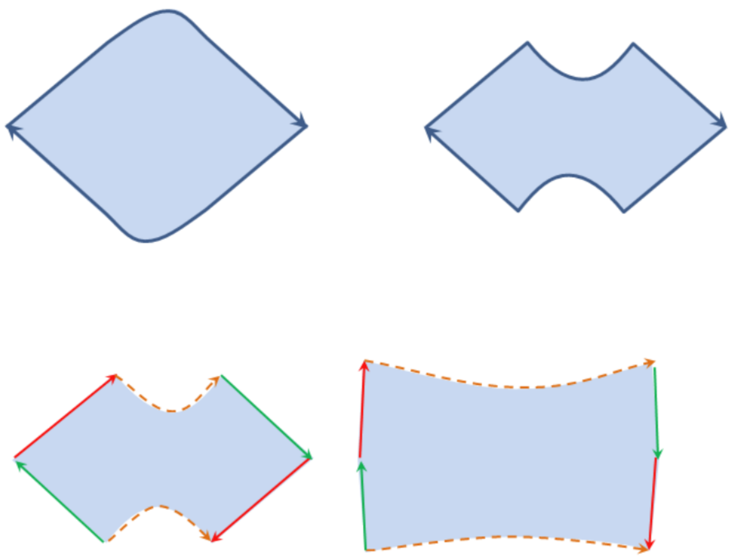

图 3: Construction

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Problem 5. (Exercise 6.3)

Show that the Klein bottle is homeomorphic to a quotient obtained by attaching two Möbius bands together along their boundaries.

Solution. See Fig 4.

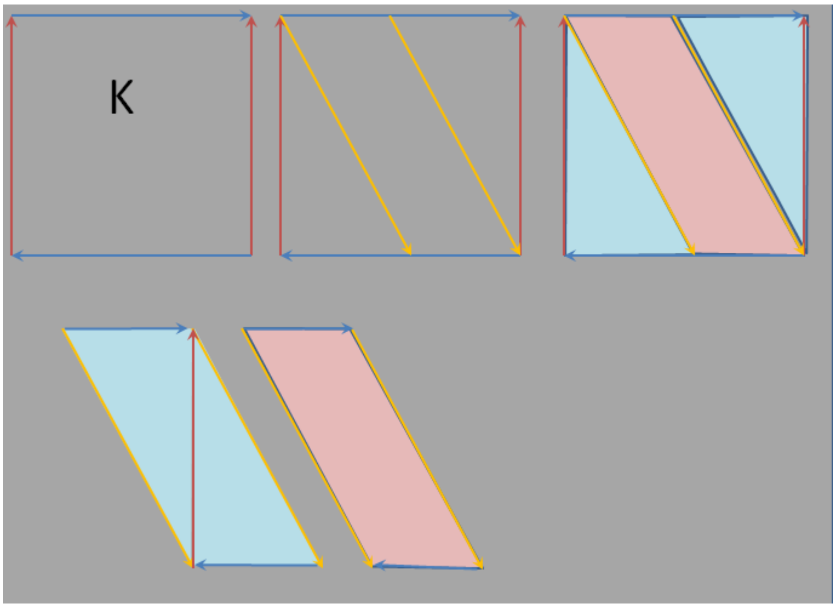

图 4: Construction

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