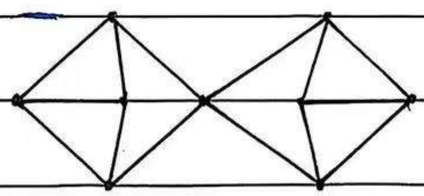
2.5. <proof>. Suppose ~~be~~ a simple graph and its complement can both be disconnected, then there exist two vertices such that there ~~is~~ has not a path connecting them, but $G \cup \bar{G}$ is a complete graph. a contradiction.

2.7 <proof>

(i) Consider a vertex with degree exactly k. the edges incident to v are $e_1, e_2, \cdots, e_k$, connecting v to other vertices. Removing all this k edges will disconnect G.

This set of k edges is a cutset, demonstrating that the edge connectivity $\lambda(G) \le k$.

(ii).



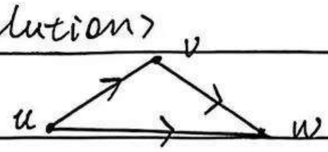
$\kappa(G) = 1 \quad \lambda(G) = 2 \quad \delta(G) = 3$

2.8. (ii) <proof>

$\Leftarrow$ If G has a cut vertex v, then any two distinct vertices are still joined by at least ~~two~~ one path~~s~~, then G is connected by definition. a contradiction.

$\Rightarrow$ Assume there exist two vertices u and v with only one internally disjoint path. Any other path between u and v must share an internal vertex w. Removing w would disconnect u from v, a contradiction.

2.12. <Solution>

(i) 

(ii) We deduce by induction.

For G of n vertices.

① $n=3$ trivial. as (i)

② $n=k-1$. holds. then for $n=k$.

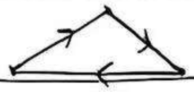
Take a vertex s, which beats all other teams.

Remove s, the remaining graph is a transitive tournament of size $k-1$, by hypothesis, it can be ranked.

Since the transitivity guarantee that s can be ranked in G, we ~~pro~~ have shown the result.

(iii). By (ii), we note that there ~~is~~ are no directed paths from the lowest ranked vertex to any ~~o~~ other vertex. then G is not strongly directed.

2.13: proof:

(i) 

(ii) $\Rightarrow$ Suppose G is not strongly connected.

then there ~~is no a directed path exists~~

exist vertices u and v, s.t. there is not a directed path from u to v.

Denote $R(u)$ as the set of vertices that u can't reach. then $R(u) \neq \phi$.

$u \in G \setminus R(u)$, we claim that each arc joing a vertex of $G \setminus R(u)$ and a vertex of $R(u)$ is directed from $G \setminus R(u)$ to $R(u)$. Otherwise, there is a directed path from u to the vertex in $G \setminus R(u)$. a contradiction.

then G is not irreducible. so G is strongly connected.

$\Leftarrow$ Suppose G is strongly connected bout not irreduciable

Theren we can split the set of vertices of G into two disjoint sets V_1 and V_2 so that each arc joining a vertex of V_1 and a vertex of V_2 is directed from V_1 to V_2. let $u \in V_1$ and $v \in V_2$, we note that there is not a direeted path from v to u, a contradiction.

2.14 <Solution>.

The infinite star obtained by joining the origin to infinitely many points on the unit circle is an counterexample.

2.44 <proof>

We prove by induction on k.

$k=1$. trivial

Suppose it's true for $k=n-1$ then we have $2(n-1)$ vertices with at most $(n-1)^2$ edges.

Suppose there is an edge between two vertices u, v we added. there there are at most $2(n-1)$ edges added. because if we add more, there must be a vertex w which both conneeted u and v, thus create a triangle. then we have $(k-1)^2+2(k-1)+1=k^2$ edges at most.

An example is $K_{k,k}$ which can reach the upper bound

Problem: Let G be a connected graph. Then $d(x,z) \le d(x,y)+d(y,z)$ for any triple of vertices x, y, z of G.

<proof> Let P be the shortest walk from x to y. Q from y to z. Combine P. Q, this is a walk from x to z. If $d(x,z)>d(x,y)+d(y,z)$ we can walk from x to z as $P \cup Q$, then it's a contradiction to the definition of $d(\cdot,\cdot)$. so $d(x,z) \le d(x,y)+d(y,z)$