

Homework 3

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PROBLEM 1. (Exercise 2.18)

- (a) A connected graph is Eulerian if and only if its set of edges can be split up into edge-disjoint cycles
- (b) A connected graph is semi-Eulerian if and only if it has exactly two vertices of odd degree.

SOLUTION.

- (a) We prove by induction on the number of edges. Assume G has $n > 0$ (since the case $n = 0$ is trivial) edges and the statement holds for all graphs with fewer than n edges.

If G has more than one connected component, the claim holds for each (smaller) component, hence for all of G . Hence we now assume that G is connected.

\Rightarrow : Consider an Eulerian trail $v_0v_1v_2\ldots v_n$ with $v_0 = v_n$. Consider all (i, j) with $0 \leq i < j \leq n$ and $v_i = v_j$. Such pairs exist because $v_0 = v_n$ (and $n > 0$). Select (i, j) among these pairs such that $j - i$ is minimized.

Then $v_i v_{i+1} \dots v_j$ is a simple cycle γ and $v_0 \dots v_i v_{j+1} \dots v_n$ is an Eulerian trail of the graph $G' := G - \gamma$. By induction, G' can be split up into edge-disjoint cycles. Together with γ we obtain a partitioning of G into edge-disjoint cycles.

\Leftarrow : Assume G can be split up into edge-disjoint cycles. Since $n > 0$, there must be at least one cycle in this partitioning. If there is exactly one cycle, it is automatically an Eulerian trail and we are done. we remove from G the edges of C to form a new (possibly disconnected) graph H with fewer edges than G , and H can also be split up into edge-disjoint cycles. By the induction hypothesis, each component of H has an Eulerian trail. But each component of H has at least one vertex in common with C , by connectedness.

It follows that we can obtain the required Eulerian trail of G by tracing the edges of C until a non-isolated vertex of H is reached, tracing the Eulerian trail of the component of H that contains that vertex, and then continuing along the edges of C until we reach a vertex belonging to another component of H , and so on. The whole process terminates when we return to the initial vertex. See Figure 1.

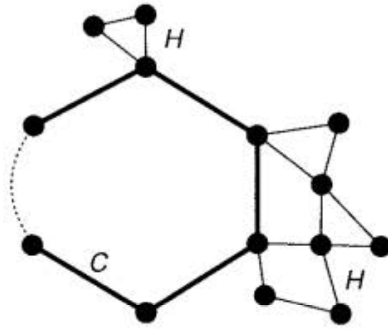


图 1: Schema

(b) \Rightarrow Suppose that P is a non-closed trail that includes every edge of G .

Whenever P passes through a vertex, there is a contribution of 2 towards the degree of that vertex, except the original and the end points. Since each edge occurs exactly once in P , the degree of each vertex must be a sum of $2s$, except two vertices of odd degree.

\Leftarrow : The proof is by induction on the number of edges of G . Suppose that G has two vertices of odd degree. Since G is connected, each vertex has degree at least 2 and so G contains a cycle C .

we remove from G the edges of C to form a new (possibly disconnected) graph H with fewer edges than G , and H has exactly two vertices of odd degree. By the induction hypothesis, H is semi-Eulerian, and each component of H has at least one vertex in common with C , by connectedness.

It follows that we can obtain the required non-closed Eulerian trail of G by starting from one of the vertex of odd degree, and tracing the edges of H until a non-isolated vertex of C is reached, tracing the edges of C that incident with that vertex until we reach a vertex belonging to another component of H , and so on. Finally we end at another vertex of odd degree in H .

□

PROBLEM 2. (Exercise 2.19)

Let G be a connected graph with $k(> 0)$ vertices of odd degree.

- (i) Show that the minimum number of trails, that have no edges in common, is $\frac{1}{2}k$.

- (ii) How many continuous pen-strokes are needed to draw the diagram in Fig2 without repeating any line?

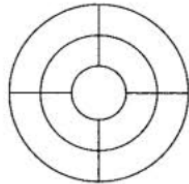


图 2: Figure for problem 2

SOLUTION.

- (i) At least $\frac{1}{2}k$ trails are needed, so as to use up all k vertices of odd degree.

If we now add $\frac{1}{2}k$ edges to G and join these vertices in pairs, then we obtain an Eulerian graph G' , since the degree of each vertex of G' is even. We then obtain the require $\frac{1}{2}k$ trails by writing down an Eulerian trail for G' and then omitting the added edges.

- (ii) Since the graph has 8 vertices of odd degree, then there are at least 4 trails which have no edges in common. That is, 4 continuous pen-strokes are needed to draw the diagram.

□

PROBLEM 3. (Exercise 2.21)

An Eulerian graph is **randomly traceable** from a vertex v if, whenever we start from v and traverse the graph in an arbitrary way never using any edge twice, we eventually obtain an Eulerian trail.

- (i) Show that the graph in Fig3 is randomly traceable.

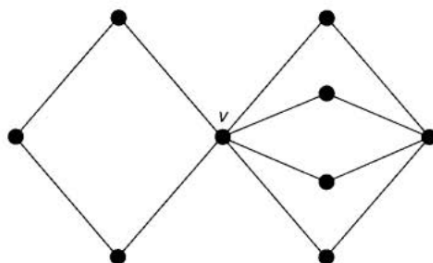


图 3: Figure for problem 3

- (ii) Give an example of an Eulerian graph that is not randomly traceable.
- (iii) Why might a randomly traceable graph be suitable for the layout of an exhibition.

SOLUTION.

- (i) The edges of G can be split into three cycles in different ways, but all have the common vertex v . For example, one of the partitioning is shown as Fig 4.

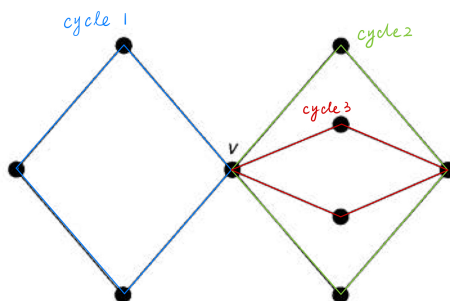


图 4: Example of partitioning

Starting from v and traverse the graph in an arbitrary way will trace the three cycles separately, and we eventually obtain an Eulerian trail never using any edge twice. Thus the graph is randomly traceable.

- (ii) Change the vertex v to be the leftmost one. Shown as Fig 5.

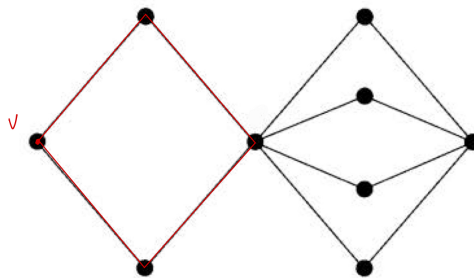


图 5: conterexample

- (iii) A randomly traceable graph ensures that visitors can navigate the exhibition in any direction without retracing their steps, making sure they can enjoy the whole exhibition without going down the same way.

□

PROBLEM 4. (Exercise 2.24)

Let D be the digraph whose vertices are the pairs of integers 11, 12, 13, 21, 22, 23, 31, 32, 33, and whose arcs join ij to kl if and only if $j = k$. Find an Eulerian trail in D and use it to obtain a circular arrangement of nine 1s, nine 2s, and nine 3s in which each of the 27 possible triples $(111, 233, \dots)$ occurs exactly once. (Problems of this kind arise in communication theory.)

SOLUTION. Actually, we can start from 11 and finally return to 11. Every vertex of D has indegree and outdegree 3, it may have many Euler trails. I found one of these Euler trails is

$$\begin{aligned} &11 \rightarrow 12 \rightarrow 21 \rightarrow 12 \rightarrow 22 \rightarrow 22 \rightarrow 21 \rightarrow 11 \rightarrow 13 \rightarrow \\ &33 \rightarrow 31 \rightarrow 13 \rightarrow 31 \rightarrow 12 \rightarrow 23 \rightarrow 33 \rightarrow 33 \rightarrow 32 \rightarrow \\ &21 \rightarrow 13 \rightarrow 32 \rightarrow 23 \rightarrow 32 \rightarrow 22 \rightarrow 23 \rightarrow 31 \rightarrow 11 \end{aligned}$$

Then we can use it to obtain a circular arrangement like

$$112122211331312333213232231$$

Since it has 27 digits, it has 27 three-digit substrings, like the 212 that starts at position 3. That substring arises from the edge going from vertex 21 to vertex 12; we used that edge only once, so there can be only one 212 substring. Thus, each of the 27 possible three-digit substrings must be represented exactly once in our string. \square

PROBLEM 5. (Exercise 2.26)

- (a) Find an Eulerian trail in the infinite triangular lattice S .
- (b) Verify that S satisfies the conditions of Theorem 2.14. i.e., G is a countable connected graph which is Eulerian.

SOLUTION.

1. We draw the Eulerian trail in the infinite triangular lattice S by the following figures.

The number in the edge shows the order we connect the lines. As shown

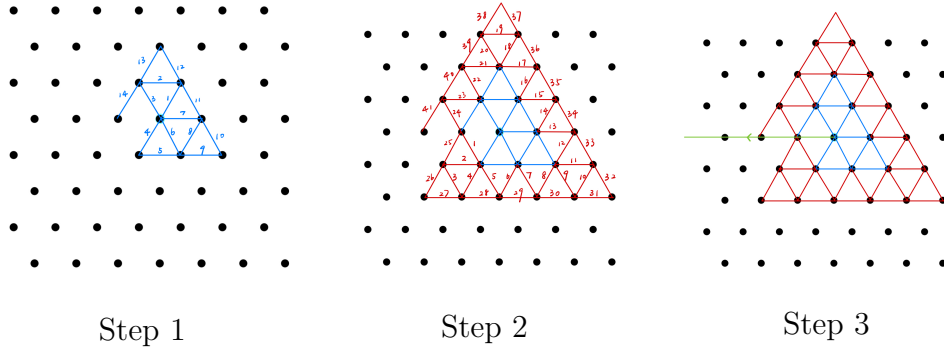


图 6: Construction of Euler trail

in Fig 6, there is a two-way infinite trail that includes every edge of S , where the green one is one trail, and the other can be extended to an infinite trail in the same way as the figure.

2. We have shown that S is Eulerian, then it suffices to show that S is a countable connected graph.

Since S is Eulerian, each two vertices can be connected by a path. Thus S is a connected locally countable infinite graph since every vertex in S has degree 6. Apply Theorem 1.4, S is a countable graph, the result follows.

□

PROBLEM 6. (Exercise 2.29(v))

For which values of k is the k -cube Q_k Hamiltonian?

SOLUTION. We show Q_k is Hamiltonian for all $k \geq 2$ by induction.

For $k = 0, 1$, there is no Hamiltonian cycle. For $k = 2$, Q_2 actually is C_4 , which is a Hamiltonian cycle, so the statement is true for $n = 2$.

Now, suppose Q_k has a Hamiltonian circuit C . We can construct Q_{k+1} from taking two copies of Q_k , then adding 0 to the end of the vertex bit-strings in one copy of Q_k , and adding 1 in the other. Take $k = 2$ as an example in Figure 7.

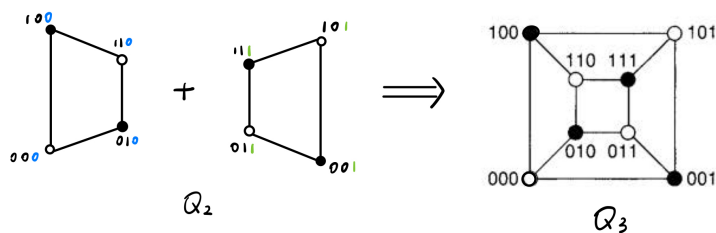


图 7: Example

In the first copy, take the Hamiltonian cycle C , and delete the final edge to form a Hamiltonian path H and label the start and end vertices v_1 and v_2 respectively. Then, join v_2 to w_2 , the corresponding vertex in the second copy of Q_k , and reverse the path H to get another Hamiltonian path in the second copy of Q_k that ends at w_1 . As w_1 is adjacent to v_1 , join these two vertices with a final edge that thus completes the Hamiltonian cycle. Hence, Q_{k+1} has a Hamiltonian cycle whenever Q_k does, so by induction Q_k has a Hamiltonian circuit for all $k \geq 2$.

□

PROBLEM 7. (Exercise 2.34)

- (i) Let G be a graph with n vertices and $\frac{1}{2}(n-1)(n-2) + 2$ edges. Use Ore's theorem to prove that G is Hamiltonian.
- (ii) Find a non-Hamiltonian graph with n vertices and $\frac{1}{2}(n-1)(n-2) + 1$ edges.

SOLUTION.

- (i) Let G be a simple graph with n vertices and $m = \frac{1}{2}(n-1)(n-2) + 2$ edges. Let u, v be two non-adjacent vertices of G , and let v_1, \dots, v_{n-2} be the other $n-2$ vertices of G .

Consider the subgraph H of G induced by the vertices v_1, \dots, v_{n-2} . This is the subgraph containing all edges of G with both endpoints from the set $\{v_1, \dots, v_{n-2}\}$.

Since the number of edges in H is at most $\binom{n-2}{2} = \frac{1}{2}(n-2)(n-3)$, there are at least

$$\frac{1}{2}(n-1)(n-2) - \frac{1}{2}(n-2)(n-3) + 2 = n$$

edges in G , each with exactly one endpoint u or v (since u, v are not adjacent). In other words, $\deg(u) + \deg(v) \geq n$. Ore's theorem now proves G is hamiltonian.

- (ii) The required non-Hamiltonian graph is constructed by taking a complete graph K_{n-1} and adding a new vertex connected by a single edge to one of the vertices in K_{n-1} .

we obtain a graph with $\frac{1}{2}(n-1)(n-2) + 1$ edges. This new vertex has degree 1, making it impossible to form a Hamiltonian cycle because a Hamiltonian cycle would require entering and exiting this vertex, which is not possible with only one edge.

□

PROBLEM 8. (Exercise 2.53)

Let G be a Hamiltonian graph and let S be any set of k vertices in G . Prove that the graph $G - S$ has at most k components.

SOLUTION. Let C be a Hamilton cycle in G . Fix any subset S of V .

We delete the vertices of S from C . After the first vertex is deleted, C is still connected, but has become a path. When any of the subsequent $k - 1$ vertices is deleted, it either breaks some path into two shorter paths, increasing the number of connected components by one or removes a vertex at an end of some path, leaving the number of connected components unchanged, or reducing it by one if this component was a N_0 . So $C \setminus S$ has at most $1 + (k - 1) = k$ connected components.

Notice that if two vertices u and v are in the same connected component of $C \setminus S$, then they will also be in the same connected component of $G \setminus S$. This is because adding edges can only connect things more fully, reducing the number of connected components. For example, if there is a $u - v$ walk in C , then any pair of consecutive vertices in that walk is adjacent in C so is also adjacent in G . Therefore the same walk is a $u - v$ walk in G . This tells us that the number of connected components of $G \setminus S$ is at most the number of connected components of $C \setminus S$, which we have shown to be at most k .

□