Homework 5

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PROBLEM 1. Let $A \subseteq \mathbb{R}$ be the set of integers, and let X be the quotient space $\mathbb{R}/A$ obtained by collapsing A to a point as in Example 3.52. (We are not using the notation $\mathbb{R}/\mathbb{Z}$ for this space because that has a different meaning, described in Example 3.92.)

(a) Show that X is homeomorphic to a wedge sum of countably infinitely many circles. [Hint: express both spaces as quotients of a disjoint union of intervals.]

(b) Show that the equivalence class A does not have a countable neighborhood-basis in X, so X is not first or second countable.

SOLUTION.

(a) The wedge sum of countably infinitely many circles is the space $B = \bigvee_{n\in A} S^1_n = \coprod_{n\in A} S^1_n/P$, where P is the set of distinguished points. Since S^1 is the quotient space $I/(0 \sim 1)$, we can express B as $B = \coprod_{n\in A} I_n/\sim$, where $\sim$ is the relation $p_i \sim p_j$ for all i, j and $0_i \sim 1_i$ for all i. With this one defines the quotient map $q : \coprod_{n\in A} I_n \to B$ by sending elements to their equivalence classes.

Let $\mathbb{R} = \coprod_{n\in A} I_n/(1_n \sim 0_{n+1})$. Then we find a quotient map $r : \prod_{n\in A} I_n \to \mathbb{R}/A$ that makes the same identifications as q to prove that the two spaces are homeomorphic by the uniqueness of the quotient space.

(b) Let $X = \mathbb{R} \setminus A$ be the quotient space obtained by collapsing the set of integers $A \subseteq \mathbb{R}$ to a single point.

The space X is homeomorphic to a wedge sum of countably infinitely many circles $\bigvee_{n\in A} S^1_n$, where each circle corresponds to the interval $[n, n+1]$ with its endpoints identified to the common basepoint A.

In the quotient topology, a neighborhood of A in X corresponds to an open set in $\mathbb{R}$ containing all integers $n \in \mathbb{Z}$, with each n surrounded by an open interval $(n - \varepsilon_n, n + \varepsilon_n)$ for some $\varepsilon_n > 0$. These intervals are "collapsed" to form loops around A in X.

Suppose for contradiction that $\{U_k\}_{k\in\mathbb{N}}$ is a countable neighborhood basis at A. For each U_k, there exists a corresponding open set in $\mathbb{R}$ containing all integers $n \in \mathbb{Z}$, with intervals $\left(n - \varepsilon_n^{(k)}, n + \varepsilon_n^{(k)}\right)$ for some $\varepsilon_n^{(k)} > 0$.

For each integer $n \in \mathbb{Z}$, define $\varepsilon = \frac{1}{2}\varepsilon_n^{(n)}$. Construct an open set $V \subseteq \mathbb{R}$ by taking:

$$V = \bigcup_{n\in\mathbb{Z}} (n - \varepsilon, n + \varepsilon).$$

This V maps to a neighborhood U of A in X.

By construction, U does not contain any U_k, contradicting the assumption that $\{U_k\}$ is a neighborhood basis.

$\square$

Problem 2.

Let G be a topological group and let $H \subseteq G$ be a subgroup. Show that its closure $\overline{H}$ is also a subgroup.

Solution.

By hypothesis, let the product with inversion

$$f : G \times G \to G \quad \text{defined by} \quad f(x, y) = xy^{-1}$$

is continuous.

Therefore, $f^{-1}(\overline{H})$ is closed. Now, notice that $H \times H \subseteq f^{-1}(\overline{H})$. So, taking closures,

$$\overline{H \times H} \subseteq f^{-1}(\overline{H}).$$

Now, we just have to show that $\overline{H} \times \overline{H} \subseteq \overline{H \times H}$, to conclude that

$$f(\overline{H} \times \overline{H}) \subseteq \overline{H}.$$

In fact, For any topological spaces X, Y and subsets $A \subseteq X$, $B \subseteq Y$:

$$\overline{A} \times \overline{B} \subseteq \overline{A \times B}$$

Let $(x, y) \in \overline{A} \times \overline{B}$, W be any open neighborhood of (x, y) in $X \times Y$. By the definition of product topology, there exists a basic open set $U_0 \times V_0 \subseteq W$ where U_0 is open in X containing x and V_0 is open in Y containing y

From the closure properties:

$$U_0 \cap A \neq \emptyset \quad (\text{since } x \in \overline{A})$$
$$V_0 \cap B \neq \emptyset \quad (\text{since } y \in \overline{B})$$

Therefore:

$$(U_0 \times V_0) \cap (A \times B) = (U_0 \cap A) \times (V_0 \cap B) \neq \emptyset$$

This implies $W \cap (A \times B) \neq \emptyset$. Since W was an arbitrary neighborhood of (x, y), it follows that $(x, y) \in \overline{A \times B}$.

Take $A = B = H$, we obtain the result.

$\square$

PROBLEM 3.

Suppose Γ is a normal subgroup of the topological group G. Show that the quotient group G/Γ is a topological group with the quotient topology. [Hint: it might be helpful to use Problems 3-5 and 3-22.]

SOLUTION. Since G is a topological group, then the usual inversion and group multiplication operations on G,

$$i : G \to G \quad \text{given by} \quad i(g) = g^{-1}$$

and

$$m : G \times G \to G \quad \text{given by} \quad m(g, h) = gh,$$

are continuous. We know from elementary abstract algebra that since Γ is normal, the quotient group G/Γ is itself a group under the operations

$$\bar{i} : G/\Gamma \to G/\Gamma \quad \text{given by} \quad \bar{i}(g\Gamma) = g^{-1}\Gamma$$

and

$$\bar{m} : G/\Gamma \times G/\Gamma \to G/\Gamma \quad \text{given by} \quad \bar{m}(g\Gamma, h\Gamma) = (gh)\Gamma.$$

To show that G/Γ is a topological group, it remains to show that these operations are continuous with respect to the quotient topology on G/Γ. Now, let $\pi : G \to G/\Gamma$ be the standard projection map $\pi(g) = g\Gamma$ for $g \in G$.

First, the **inversion map**. Let

$$\psi = \pi \circ i : G \to G/\Gamma \quad \text{be given by} \quad \psi(g) = g^{-1}\Gamma$$

which is clearly continuous as the composition of two continuous maps. Then it's easy to see that ψ is constant on the fibers of π:

$$\pi(x) = \pi(y) \quad \Longleftrightarrow \quad x\Gamma = y\Gamma \quad \Longleftrightarrow \quad x = \gamma y \text{ for some } \gamma \in \Gamma$$

hence

$$\psi(x) = x^{-1}\Gamma = (\gamma y)^{-1}\Gamma = y^{-1}\gamma^{-1}\Gamma = y^{-1}\Gamma = \psi(y).$$

Thus, ψ passes to the quotient to give rise to a (unique) continuous map $\overline{\psi} : G/\Gamma \to G/\Gamma$ such that $\overline{\psi} \circ \pi = \pi \circ i = \psi$; it is obvious that $\bar{\psi} = \bar{i}$, the inversion map of the group G/Γ, so inversion is continuous.

Next, the **multiplication map**. Let

$$\varphi = \pi \circ m : G \times G \to G/\Gamma$$

which is continuous as the composition of continuous maps. Define $\pi \times \pi : G \times G \to G/\Gamma \times G/\Gamma$ as the product of "two copies" of the quotient map on G.

First we will show that the map $p = \pi \times \pi : G \times G \to (G/\Gamma) \times (G/\Gamma)$ is a quotient map. It is obviously surjective and continuous, as the product of two surjective and continuous maps. The quotient map $\pi : G \to G/\Gamma$ is open (from Problem 3-22(a)) and so p is open as the finite product of open maps (from Problem 3-5), and so p is a quotient map.

Then φ is constant on the fibers of $\pi \times \pi$:

$$\pi \times \pi(g, h) = \pi \times \pi(g', h') \quad \Longrightarrow \quad g = \gamma g',\ h = h'\gamma \text{ for some } \gamma, \gamma' \in \Gamma.$$

Then

$$\psi(g, h) = gh\Gamma = \gamma g' h' \gamma \Gamma = g' h' \Gamma = \psi(g', h')$$

Since Γ is normal. Thus, φ passes to the quotient and induces a (unique) continuous map $\overline{\varphi} : G/\Gamma \times G/\Gamma \to G/\Gamma$ such that $\overline{\varphi} \circ (\pi \times \pi) = \pi \circ m = \varphi$.

Then clearly $\overline{\psi} = \overline{m}$, the multiplication map of the group G/Γ, and so this multiplication is continuous.

Thus, G/Γ is a topological group since both inversion and the group multiplication operation are continuous (with respect to the quotient topology).

$\square$

> PROBLEM 4.
>
> Consider the action of $O(n)$ on $\mathbb{R}^n$ by matrix multiplication as in Example 3.88(b). Prove that the quotient space is homeomorphic to $[0, \infty)$. [Hint: consider the function $f : \mathbb{R}^n \to [0, \infty)$ given by $f(x) = |x|$.]

SOLUTION. Recall that we have the action given as follows

$$a : O(n) \times \mathbb{R}^n \to \mathbb{R}^n; \quad (M, x) \mapsto Mx$$

But this map can be rewritten with

$$a_M : \mathbb{R}^n \to \mathbb{R}^n; \quad x \mapsto Mx$$

Then with this action we can define the orbits as follows:

$$O(n) \cdot x = \{a_M(x) : x \in \mathbb{R}^n\}$$

on which we have the equivalence relation

$$yRx \iff y \in a_M(x)$$

Now we can define the quotient space as follows:

$$Q := \mathbb{R}^n / O(n) = \{[x] : x \in \mathbb{R}^n\}$$

Now let us use the following function:

$$\varphi : Q \to \mathbb{R}_{\geq 0}; \quad [x] \mapsto \|x\|$$

I claim that this function is a homeomorphism. To do so let us check injectivity. Let $[x], [y] \in Q$ s.t. $\varphi([x]) = \varphi([y]) \iff \|x\| = \|y\|$. This is equivalent to say $\|Mx\| = \|My\|$ since M is orthogonal, from where follows that $[x] = [y]$. For surjectivity, let $r \in [0, \infty)$. Then take $R = [(r, 0_2, 0_3, \ldots, 0_n)]$, then $\varphi(R) = r$. This shows that φ is bijective. Furthermore, this function is clearly continuous since it is polynomial in each component. To finish this proof, I claim that the inverse of φ is given by

$$\varphi^{-1} = \pi : \mathbb{R}_{\geq 0} \to Q; \quad x \mapsto [x]$$

Indeed

$$\varphi^{-1}(\varphi([x])) = \varphi^{-1}(\|x\|) = [x]$$

and

$$\varphi(\varphi^{-1}(x)) = \varphi([x]) = \|x\| = x$$

But since we have endowed Q with the quotient topology we know that π is continuous. Thus we have shown that it is homeomorphic.

$\square$

Problem 5.

Suppose X is a connected topological space, and $\sim$ is an equivalence relation on X such that every equivalence class is open. Show that there is exactly one equivalence class, namely X itself.

Solution.

For each $x \in X$, let $U_x = \{y \in X \mid x \sim y\}$. Let $x_0 \in X$ be given. If $U_{x_0} = X$, we are done. Suppose otherwise. Let

$$V = \bigcup_{x \in U_{x_0}^c} U_x.$$

Then $U_{x_0} \cup V = X$ and $U_{x_0} \cap V = \emptyset$. U_{x_0} is open since it is an equivalence class. V is open since it is a union of equivalence classes, each of which is open.

Therefore, X is disconnected. This is a contradiction, so $U_{x_0} = X$.

$\square$