Homework 3

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Problem 1. ***The Parseval equation.***

Let $(u_n)_{n\geq 1}$ be an orthonormal system in the separable Hilbert space X over $\mathbb{K}$. Show that (u_n) is complete iff

$$\sum_{n\geq 1} |\langle u_n, u\rangle|^2 = \|u\|^2 \quad \text{for all } u \in X.$$

Solution.

"$\Rightarrow$" Recall that u_n is complete if and only if, for all $u \in X$,

$$u = \lim_{m\to\infty} \sum_{n=1}^{m} \langle u_n, u\rangle u_n$$

$$\|u\|^2 = \langle u, u\rangle = \lim_{m\to\infty} \left\langle \sum_{n=1}^{m} \langle u_n, u\rangle u_n, \sum_{k=1}^{m} \langle u_k, u\rangle u_k \right\rangle.$$

Since $\langle u_n, u_k\rangle = \delta_{nk}$,

$$\|u\|^2 = \lim_{m\to\infty} \sum_{n=1}^{m} |\langle u_n, u\rangle|^2 = \sum_{n=1}^{\infty} |\langle u_n, u\rangle|^2.$$

"$\Leftarrow$" By Parseval's equation,

$$\|u - S_m\|^2 = \sum_{n=1}^{\infty} \left| \langle u_n, u - \sum_{k=1}^{m} \langle u_k, u\rangle u_k \right|^2.$$

$$= \sum_{n=1}^{\infty} \left| \langle u_n, u \rangle - \sum_{k=1}^{m} \langle u_k, u \rangle \langle u_n, u_k \rangle \right|^2 .$$

$$= \sum_{n=1}^{\infty} |\langle u_n, u \rangle - \langle u_n, u \rangle|^2 \to 0 \quad \text{as } m \to \infty.$$

Thus, $u = \lim_{m\to\infty} S_m$, and (u_n) is complete.

$\square$

PROBLEM 2. ***A fundamental completeness theorem.***

Let $-\infty \le a < b \le \infty$. We are given a measurable function $f : (a, b) \to \mathbb{K}$ (e.g., f is continuous) such that

$$|f(x)| \le Ce^{-\alpha|x|} \quad \text{for all } x \in \mathbb{R} \text{ and fixed } \alpha > 0 \text{ and } C > 0.$$

Show that the linear hull of the system $\{x^n f(x)\}_{n=0,1,\ldots}$ is dense in the Hilbert space $L_2^{\mathbb{K}}(a, b)$.

SOLUTION. We want to show that span$\{x^n f(x)\}$ is dense in $L_2^{\mathbb{K}}(a, b)$. According to **Corollary 3** in **Section 3.3**, we have to show that if $u \in L_2^{\mathbb{K}}(a, b)$ and

$$\langle v_n, u \rangle \equiv \int_a^b x^n f(x) u(x) dx = 0 \quad \text{for all } n = 0, 1, \ldots, \tag{1}$$

then $u = 0$. To this end, let $M = \{k \in \mathbb{R} : |k| < \alpha - 1\}$ and set

$$g(k) := \int_a^b f(x) u(x) e^{-ikx} dx \quad \text{for all } k \in M.$$

Formally,

$$g^{(n)}(k) = \int_a^b f(x) u(x) (-ix)^n e^{-ikx} dx \quad \text{for all } k \in M,\ n = 0, 1, 2, \ldots . \tag{2}$$

For all $x \in \mathbb{R}$ and $k \in M$, we get

$$\begin{aligned} \left|f(x)u(x)(-ix)^n e^{-ikx}\right| &\leq Ce^{-\alpha|x|}|x|^n e^{|kx|}|u(x)| \\ &\leq Ce^{-|x|}|x|^n|u(x)| \\ &\leq \text{Const(n)}|u(x)| \end{aligned} \tag{3}$$

Since u is an element of $L_2^{\mathbb{K}}(a,b)$,

$$\int_a^b |u(x)|\, dx < \infty.$$

Thus, the majorant condition (3) justifies formula (2). Consequently, the function g is analytic on the strip M. By (1) and (2),

$$g^{(n)}(0) = 0 \quad \text{for all } n = 0, 1, \ldots.$$

Hence $g(k) = 0$ for all $k \in M$. By lemma, this implies $u(x) = 0$ for almost all $x \in [a,b]$.

$\square$

Problem 3. ***The completeness of the system of the Laguerre functions.***

$$x^n e^{-\frac{x}{2}}, \quad n = 0, 1, \ldots, \quad x \in \mathbb{R},$$

the Schmidt orthogonalization method yields a system of functions

$$L_n(x)e^{-\frac{x}{2}}, \quad n = 0, 1, \ldots, \quad x \in \mathbb{R}.$$

Show that the following are true:

(i) System forms a complete orthonormal system in $L_2^{\mathbb{K}}(0,\infty)$.

(ii) Explicitly,

$$L_n(x) := \frac{(-1)^n}{n!} e^x \frac{d^n}{dx^n}\left(e^{-x}x^n\right), \quad n = 0, 1, \ldots, \quad x \in \mathbb{R}.$$

Solution.

(i) Since

$$f(x) = e^{-\frac{x}{2}} \leq Ce^{-\alpha|x|} \quad \text{for all } x \in (0, \infty)$$

where $C = 1$ and $\alpha = -\frac{1}{2}$.

Thus, by problem 2, the linear hull of the system $\{x^n e^{-\frac{x}{2}}\}_{n=0,1,\cdots}$ is dense in the Hilbert space $L_2^{\mathbb{K}}(0, \infty)$. Moreover, the assertion follows from **Proposition** 2 in **Section** 3.3.

(ii) Since

$$\lim_{x\to\infty} e^{-x}x^n = 0$$

The derivatives of $e^{-x}x^n \to 0$ as $x \to \infty$.

Recall that

$$L_n(x) := \frac{(-1)^n}{n!}e^x \frac{d^n}{dx^n}\left(e^{-x}x^n\right), \quad n = 0, 1, \ldots, \quad x \in \mathbb{R}.$$

Integrating by parts yields that

$$\begin{aligned}
\int_0^\infty e^{-x}x^m L_n(x) &= \int_0^\infty x^m \frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^n\right) \\
&= -m\frac{(-1)^n}{n!}\int_0^\infty x^{m-1}\frac{d^{n-1}}{dx^{n-1}}\left(e^{-x}x^n\right) \\
&= m!\frac{(-1)^{m+n}}{n!}\int_0^\infty \frac{d^{n-m}}{dx^{n-m}}\left(e^{-x}x^n\right) \\
&= \begin{cases} 0 & m < n \\ n! & m = n \end{cases}
\end{aligned}$$

Since L_m is a polynomial of degree m, let $u_n(x) := L_n(x)e^{-\frac{x}{2}}$, we have

$$\langle u_n, u_m \rangle = \int_0^\infty e^{-x}L_m(x)L_n(x) = 0$$

for all $m < n$.

If $m = n$, we have

$$\langle u_n, u_n \rangle = \int_0^\infty e^{-x} L_n^2(x) = \frac{1}{n!} \int_0^\infty e^{-x} x^n L_n(x) = 1$$

$\square$

PROBLEM 4. ***The nonhomogeneous stationary Schrödinger equation.***

Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous function that vanishes outside a compact interval. Set

$$v(x) := \int_{-\infty}^\infty \frac{ie^{ip|x-y|}}{2p} f(y)\, dy.$$

Show that, for each $p \in \mathbb{R}$ with $p \neq 0$, the function v is a C^2-solution of

$$-v'' - p^2 v = f \quad \text{on } \mathbb{R}.$$

SOLUTION. Recall that if

$$F(x) = \int_{\varphi(x)}^{\psi(x)} f(x, y)\, dy$$

then

$$F'(x) = \int_{\varphi(x)}^{\psi(x)} \frac{\partial f(x, y)}{\partial x}\, dy + f(x, \psi(x))\psi'(x) - f(x, \varphi(x))\varphi'(x)$$

Since

$$v(x) := \int_{-\infty}^\infty \frac{ie^{ip|x-y|}}{2p} f(y)\, dy = \int_{-\infty}^x \frac{ie^{ip(x-y)}}{2p} f(y)\, dy + \int_x^\infty \frac{ie^{ip(y-x)}}{2p} f(y)\, dy$$

We compute

$$v'(x) = \int_{-\infty}^x \frac{-e^{ip(x-y)}}{2} f(y)\, dy + \frac{i}{2p} f(x) + \int_x^\infty \frac{e^{ip(y-x)}}{2} f(y)\, dy - \frac{i}{2p} f(x)$$

and

$$
\begin{aligned}
v''(x) &= \int_{-\infty}^{x} \frac{-ipe^{ip(x-y)}}{2} f(y)\,dy - \frac{1}{2}f(x) + \int_{x}^{\infty} \frac{-ipe^{ip(y-x)}}{2} f(y)\,dy - \frac{1}{2}f(x) \\
&= \int_{-\infty}^{\infty} \frac{-ipe^{ip|x-y|}}{2} f(y)\,dy - f(x)
\end{aligned}
$$

Thus

$$
\begin{aligned}
-v'' - p^2 v &= \int_{-\infty}^{\infty} \frac{ipe^{ip|x-y|}}{2} f(y)\,dy + f(x) - p^2 \int_{-\infty}^{\infty} \frac{ie^{ip|x-y|}}{2p} f(y)\,dy \\
&= f
\end{aligned}
$$

□

PROBLEM 5. ***Graph closed operators.***

Let $A : D(A) \subseteq X \to X$ be a linear operator on the Hilbert space X over $\mathbb{K}$ such that $D(A)$ is dense in X. The set

$$G(A) := \{(u, Au) : u \in D(A)\}$$

is called the graph of A. The operator A is called graph closed iff $G(A)$ is closed in $X \times X$, i.e.,

$$u_n \to u \quad \text{and} \quad Au_n \to v \quad \text{in } X \text{ as } n \to \infty$$

imply $Au = v$. The linear operator $B : D(B) \subseteq X \to X$ is called the closure of A iff $A \subseteq B$ and

$$\overline{G(A)} = G(B).$$

We write $\overline{A}$ instead of B. Show the following:

(i) The adjoint operator A^* is graph closed.

(ii) The closure $\overline{A}$ exists iff it follows from $u_n \in D(A)$ for all n along with

$$Au_n \to v \quad \text{and} \quad u_n \to 0 \quad \text{as } n \to \infty$$

that $v = 0$.

(iii) If there exists a linear graph closed operator $C : D(C) \subseteq X \to X$ such that $A \subseteq C$, then the closure $\overline{A}$ exists and

$$\overline{A} \subseteq C.$$

Hence the closure $\overline{A}$ is the smallest graph closed extension of A. In particular, $\overline{A}$ is uniquely determined by A.

(iv) If A is symmetric, then the closure $\overline{A}$ exists and is symmetric.

(v) If $\overline{A}$ exists, then $(\overline{A})^* = A^*$.

(vi) If A is self-adjoint, then $\overline{A} = A$.

(vii) The operator A is graph closed iff $D(A)$ is a Hilbert space over $\mathbb{K}$ equipped with the inner product

$$\langle u, v \rangle_A := \langle u, v \rangle + \langle Au, Av \rangle.$$

Solution.

(i) Recall that $v \in D(A^*)$ iff there exists $A^*v \in X$ such that

$$\langle Au, v \rangle = \langle u, A^*v \rangle$$

Suppose now that $v_n \in D(A^*)$, then

$$\langle Au, v_n \rangle = \langle u, A^*v_n \rangle$$

If $v_n \to v$ and $A^* v_n \to w$, we know

$$\langle Au, v\rangle = \lim_{n\to\infty} \langle Au, v_n\rangle = \lim_{n\to\infty} \langle u, A^* v_n\rangle = \langle u, w\rangle$$

This implies $A^* v = w$, so the adjoint operator A^* is graph closed.

(ii) $\Rightarrow$ If the closure $\overline{A}$ exists, then $\overline{G(A)} = G(\overline{A})$. For all $u_n \in D(A)$, $(u_n, Au_n) \in G(A)$. Since $Au_n \to v$ and $u_n \to 0$ as $n \to \infty$, $(0, v) \in \overline{G(A)} = G(\overline{A})$. Therefore, $v = \overline{A}0 = 0$.

$\Leftarrow$ To construct the closure $\overline{A}$, we suppose $(u_n, Au_n) \to (u, v)$ in $X \times X$. Since A is densely defined, $u_n \to u$ implies $u \in \overline{D(A)} = X$. To ensure v is uniquely determined by u, assume another sequence $\{u'_n\} \subseteq D(A)$ with $u'_n \to u$ and $Au'_n \to v'$. Then $w_n = u_n - u'_n \to 0$ and $Aw_n = Au_n - Au'_n \to v - v'$. By the given condition, $v - v' = 0$, hence $v = v'$. Thus $\overline{A}$ is well-defined.

Next, we show that $\overline{A}$ is closed. Suppose $u_n \to u$ and $\overline{A}u_n \to v$. Since $(u_n, \overline{A}u_n) \in G(\overline{A}) = \overline{G(A)}$, the limit $(u, v) \in \overline{G(A)}$, implying $v = \overline{A}u$. Thus, $\overline{A}$ is closed, thus the closure $\overline{A}$ exists.

(iii) Suppose that $u_n \in D(A) \subseteq D(C)$. Since C is a linear graph closed operator

$$u_n \to u \quad \text{and} \quad Cu_n \to v \quad \text{in } X \text{ as } n \to \infty$$

imply $Cu = v$. Thus $G(\overline{A}) = \overline{G(A)} \subseteq G(C)$.

(iv) Suppose now $u_n \in D(A)$ for all n along with

$$Au_n \to v \quad \text{and} \quad u_n \to 0 \quad \text{as } n \to \infty$$

Since A is symmetric, then

$$v^2 = \langle v, v\rangle = \lim_{n\to\infty} \langle v, Au_n\rangle = \lim_{n\to\infty} \langle u_n, Av\rangle = 0$$

then $v = 0$. By (ii) we show the closure $\overline{A}$ exists.

Moreover, for $u_n \to u$ and $v_n \to v$,

$$\langle \overline{A}u, v\rangle = \lim_{n\to\infty} \langle Au_n, v_n\rangle = \lim_{n\to\infty} \langle u_n, Av_n\rangle = \langle u, \overline{A}v\rangle$$

Thus $\overline{A}$ is symmetric.

(v) **Step 1**: We show $A^* \subseteq (\overline{A})^*$. Since $A \subseteq \overline{A}$, for all $u \in D(A)$ and $v \in D(A^*)$, we have

$$\langle Au, v\rangle = \langle u, A^*v\rangle = \langle \overline{A}u, v\rangle = \langle u, (\overline{A})^*v\rangle.$$

By the definition of A^*, this implies $v \in D((\overline{A})^*)$ and $A^*v = (\overline{A})^*v$. Hence, $A^* \subseteq (\overline{A})^*$.

Step 2: We show $(\overline{A})^* \subseteq A^*$. Since $A \subseteq \overline{A}$, for all $u \in D(A)$ and $v \in D((\overline{A})^*)$, we have

$$\langle \overline{A}u, v\rangle = \langle Au, v\rangle = \langle u, A^*v\rangle = \langle u, (\overline{A})^*v\rangle.$$

By the definition of $(\overline{A})^*$, this implies $v \in D(A^*)$ and $A^*v = (\overline{A})^*v$. Hence, $(\overline{A})^* \subseteq A^*$.

Combining the two inclusions, we conclude that

$$(\overline{A})^* = A^*.$$

(vi) Since A is self-adjoint, by (i) we know that $A = A^*$ is graph closed, that is

$$\overline{G(A)} = G(A)$$

then $A = \overline{A}$.

(vii) $\Rightarrow$ If A is graph closed, i.e.,

$$u_n \to u \quad \text{and} \quad Au_n \to v \quad \text{in } X \text{ as } n \to \infty$$

imply $Au = v$. Suppose (u_n) is a Cauchy sequence in D(A), that is

$$\|u_n - u_m\|_A = \langle u_n - u_m, u_n - u_m\rangle_A^{\frac{1}{2}} < \varepsilon \quad \text{for all } n, m > N_\varepsilon$$

where

$$\langle u_n - u_m, u_n - u_m\rangle_A = \|u_n - u_m\|^2 + \|Au_n - Au_m\|^2$$

Thus $\|\cdot\| \leq \|\cdot\|_A$, so $\|u_n - u_m\|_A < \varepsilon$ implies $\|u_n - u_m\| < \varepsilon$ and $\|Au_n - Au_m\| < \varepsilon$. Since X is a Hilbert space with $\|\cdot\|$, we have $u_n \to u$ and $Au_n \to v$ as $n \to \infty$. Moreover, the graph closed operator A shows that $Au = v$. This implies

$$\|u_n - u\|_A \to 0 \quad \text{as } n \to \infty$$

and $u \in D(A)$. So $D(A)$ is a Hilbert space over $\mathbb{K}$ equipped with the inner product

$$\langle u, v\rangle_A := \langle u, v\rangle + \langle Au, Av\rangle.$$

$\Leftarrow$ If $D(A)$ is a Hilbert space, we aim to show that A is graph closed. Let $(u_n) \subseteq D(A)$ be a sequence such that $u_n \to u$ in X and $Au_n \to v$ in X. We need to prove that $Au = v$.

Since $(D(A), \|\cdot\|_A)$ is a Hilbert space, it is complete with respect to the norm $\|\cdot\|_A$. Observe that

$$\|u_n - u\|_A^2 = \|u_n - u\|^2 + \|Au_n - Au\|^2.$$

Given $u_n \to u$ in $D(A)$, we have $\|u_n - u\|_A \to 0$ and $\|u_n - u\| \to 0$, thus

$$\|Au_n - Au\| \to 0$$

Additionally, since $Au_n \to v$ in X, it follows that $\|Au_n - v\| \to 0$. Combining these results,

$$\|Au - v\| \leq \|Au - Au_n\| + \|Au_n - v\| \to 0 \quad \text{as } n \to \infty.$$

This shows that $Au = v$, so A is graph closed.

$\square$

PROBLEM 6. ***The Kato perturbation theorem.***
Let $A : D(A) \subseteq X \to X$ be a linear self-adjoint operator on the complex Hilbert space X, and let $B : D(B) \subseteq X \to X$ be a linear symmetric operator such that $D(A) \subseteq D(B)$ and

$$\|Bu\| \leq a\|Au\| + b\|u\| \quad \text{for all } u \in D(A) \tag{4}$$

where a and b are fixed real numbers with $0 \leq a < 1$ and $b \geq 0$. Show that $A + B$ is self-adjoint.

SOLUTION. Let $\alpha \in \mathbb{R}$ with $\alpha \neq 0$. Since $i\alpha \in \rho(A)$, the operator $(A - i\alpha I)^{-1} : X \to X$ is linear and continuous. We shall show ahead that

$$\left\|B(A - i\alpha I)^{-1}\right\| < 1 \quad \text{for all} \quad \alpha \in \mathbb{R} : |\alpha| \geq \alpha_0, \tag{5}$$

provided α_0 is sufficiently large. Since

$$(A + B - i\alpha I)(A - i\alpha I)^{-1} = I + B(A - i\alpha I)^{-1},$$

Applying the conclusion of Neumann series, it follows from (5) that

$$R(A + B - i\alpha I) = X \quad \text{for all} \quad \alpha \in \mathbb{R} : |\alpha| \geq \alpha_0$$

Thus, by Problem 5.5(i) in p416, $A + B$ is self-adjoint.

Proof of (5)**.** By Problem 5.4(i),

$$\left\|(A-i\alpha I)^{-1}u\right\| \le |\alpha|^{-1}\|u\| \quad \text{for all} \quad u \in X.$$

Furthermore,

$$\begin{aligned}\|Av\|^2 + |\alpha|^2\|v\|^2 &= (Av - i\alpha v \mid Av - i\alpha v) \\ &= \|Av - i\alpha v\|^2 \quad \text{for all} \quad v \in D(A).\end{aligned}$$

Letting $v := (A-i\alpha I)^{-1}u$, this implies

$$\|A(A-i\alpha I)^{-1}u\|^2 \le \|u\|^2 \quad \text{for all} \quad u \in X.$$

Thus, it follows from (4) that

$$\begin{aligned}\|B(A-i\alpha I)^{-1}u\| &\le a\|A(A-i\alpha I)^{-1}u\| + b\|(A-i\alpha I)^{-1}u\| \\ &\le \left(a + b|\alpha|^{-1}\right)\|u\| \quad \text{for all} \quad u \in X.\end{aligned}$$

This yields (5). □

PROBLEM 7. ***A classical inequality.***

Show that, for all $u \in C_0^\infty(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} \left(u_\xi^2 + u_\eta^2 + u_\zeta^2\right) dx \ge \int_{\mathbb{R}^3} \frac{u^2}{4r^2} dx,$$

where $x = (\xi, \eta, \zeta)$.

SOLUTION. Set $v = r^{\frac{1}{2}}u$. Since $v = \sqrt{\xi^2+\eta^2+\zeta^2}$. We have

$$\begin{aligned}u_\xi^2 + u_\eta^2 + u_\zeta^2 &= |\nabla u|^2 = |\nabla r^{-\frac{1}{2}} v|^2 \\ &= |-\frac{1}{2}r^{-\frac{3}{2}}v\nabla r + r^{-\frac{1}{2}}\nabla v|^2 \\ &= r^{-1}(v_\xi^2 + v_\eta^2 + v_\zeta^2) - \frac{1}{2}r^{-2}(v^2)_r + (4r^3)^{-1}v^2\end{aligned}$$

Since $u \in C_0^\infty(\mathbb{R}^3)$, for sufficiently large R,

$$\int_{\mathbb{R}} r^{-2}(v^2)_r = \int_0^\pi \int_0^{2\pi} \sin\theta \int_0^R (v^2)_r \, dr \, d\theta \, d\varphi = 0$$

then

$$u_\xi^2 + u_\eta^2 + u_\zeta^2 = r^{-1}(v_\xi^2 + v_\eta^2 + v_\zeta^2) + (4r^3)^{-1}v^2 \geq (4r^3)^{-1}v^2 = \frac{u^2}{4r^2}$$

thus

$$\int_{\mathbb{R}^3} \left(u_\xi^2 + u_\eta^2 + u_\zeta^2\right) dx \geq \int_{\mathbb{R}^3} \frac{u^2}{4r^2} dx,$$

□