Homework 6

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PROBLEM 1. **Exercise 4.4.**

Prove that a topological space X is disconnected if and only there exists a nonconstant continuous function from X to the discrete space $\{0, 1\}$.

SOLUTION.

$\Rightarrow$ If X is disconnected then there exist non-empty open sets $U, V \subset X$ such that $U \cap V = \emptyset$ and $X = U \cup V$. Let f be equal to 0 on U and 1 on V. Such f is what we need.

$\Leftarrow$ Suppose we have such a map f and let U the preimage of 0 and V the preimage of 1. Since $\{0, 1\}$ has the discrete topology the singletons are open hence U, V are open sets in X. Then $U \cap V = f^{-1}(\{0\} \cap \{1\}) = \emptyset$. Moreover $X = f^{-1}(\{0, 1\}) = U \cup V$. Therefore X is disconnected.

$\square$

PROBLEM 2. 4-1.

Show that for $n > 1$, $\mathbb{R}^n$ is not homeomorphic to any open subset of $\mathbb{R}$. [Hint: if $U \subseteq \mathbb{R}$ is open and $x \in U$, then $U \setminus \{x\}$ is not connected.]

SOLUTION.

Suppose for contradiction that there exists a homeomorphism $f : \mathbb{R}^n \to U$, where $n > 1$ and $U \subseteq \mathbb{R}$ is an open subset.

Choose any point $x \in \mathbb{R}^n$. Since homeomorphisms preserve topological properties $\mathbb{R}^n \setminus \{x\}$ is connected and $U \setminus \{f(x)\}$ must then also be connected.

However for any open $U \subseteq \mathbb{R}$ and any $f(x) \in U$, the space $U \setminus \{f(x)\}$ is not connected. This creates a contradiction.

$\square$

PROBLEM 3. 4-4.

Show that the following topological spaces are not manifolds:

(a) the union of the x-axis and the y-axis in $\mathbb{R}^2$

(b) the conical surface $C \subseteq \mathbb{R}^3$ defined by

$$C = \{(x, y, z) : z^2 = x^2 + y^2\}$$

SOLUTION.

(a) Suppose X were a n-manifold($n \geq 2$). Then there would be a nbhd U of the origin in X that is homeomorphic to $\mathbb{R}^n$. Then we also have that U with the origin removed is homeomorphic to $\mathbb{R}^n$ with one point removed. But this can't be since U without the origin is not connected, whereas $\mathbb{R}^n$ with one point removed is connected.

   Suppose it were a 1-manifold and V is a nbhd of the origin which is homeomorphic to $\mathbb{R}$. Then removing the origin gives us 4 components in V and 2 components in $\mathbb{R}$. So U is also not 1-manifold.

(b) Similar to (a).

□

PROBLEM 4. 4-5.

Let $M = \mathbb{S}^1 \times \mathbb{R}$, and let $A = \mathbb{S}^1 \times \{0\}$. Show that the space M/A obtained by collapsing A to a point is homeomorphic to the space C of Problem 4-4(b), and thus is Hausdorff and second countable but not locally Euclidean.

SOLUTION. Let

$$\begin{tikzcd} M \arrow[d, "g"'] \arrow[dr, "f\circ g"] & \\ M/A \arrow[r, "f"] & C \end{tikzcd}$$

Then f is continuous since g and $f \circ g$ are continuous.

For $\forall(\alpha, \beta, \gamma) \in C$, we have:

$$f\left(\left[\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}, \frac{\beta}{\sqrt{\alpha^2+\beta^2}}, \gamma\right]\right) = (\alpha, \beta, \gamma)$$

Thus f is surjective. Now for $[x], [y] \in M/A$, If $f([x]) = f([y]) = (\alpha, \beta, \gamma)$,

- $\gamma = 0$, then $C_x = C_y = 0 \Rightarrow [x] = [y]$.
- $\gamma \neq 0$, then $[x] = [(\frac{\alpha}{\gamma}), (\frac{\beta}{\gamma}), \gamma] = [y] \Rightarrow f$is injective.

Therefore f is homeomorphic.

Since M is Hausdorff and second countable, then M/A is Hausdorff and second countable.

According to problem 4-4, C is not a manifold, and M/A is homeomorphic to C, then M/A is not locally Euclidean.

□

Problem 5. 4-11.

Let X be a topological space, and let CX be the cone on X (see Example 3.53).

(a) Show that CX is path-connected.

(b) Show that CX is locally connected if and only if X is, and locally path-connected if and only if X is.

Solution.

(a) Let $CX = (X \times I) \setminus (X \times \{0\})$ and let $(x_1, y_1), (x_2, y_2) \in CX$, $f_1, f_2 \in [0,1] \to CX$ such that

$$f_1(y) = (x_1, y_1(1-y)), \quad f_2(y) = (x_2, y_2 y).$$

then f_1 and f_2 are paths. Consider

$$f_3(y) = \begin{cases} f_1(2y), & \text{if } y \in [0, \frac{1}{2}] \\ f_2(2y), & \text{if } y \in (\frac{1}{2}, 1) \end{cases}$$

Since $f_1(1) = (x_1, 0) = (x_2, 0) = f_2(1)$ in CX, it is continuous in CX. Moreover f_3 connectes $(x_1, y_1), (x_2, y_2) \in CX$, so CX is path-connected.

(b) $\Rightarrow$ Suppose CX is locally (path-) connected. Then for each $p \in CX$, there exists a neighborhood U_p of p, which is (path-) connected. Note that $X \times \{1\} \cong X$, so CX is locally (path-) connected. Then we can put a projection $\varphi : CX \to X$, such that $\varphi(U_p)$ is a (path-) connected neighborhood of $\varphi(p)$, so α is (locally path-) connected.

$\Leftarrow$ Suppose X is locally (path-) connected. For each $q \in X$, there exists a neighborhood V_q of q which is (path-) connected. $\Rightarrow$ $(V_q \times I) \setminus (V_q \times \{0\})$ is path-connected. Hence, CX is locally (path-) connected.

$\square$