Homework 4

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Problem 1. **The completion principle for Banach spaces**

Two normed spaces X and Y over $\mathbb{K}$ are called **normisomorphic** iff there exists a linear bijective operator $j : X \to Y$ such that j is isometric, i.e.,

$$\|j(u)\| = \|u\| \quad \text{for all } u \in X.$$

Let D be a normed space over $\mathbb{K}$. The Banach space X over $\mathbb{K}$ is called a **completion** of D iff the set D is dense in X and the X-norm coincides with the D-norm on D.

a. **Uniqueness of completion** Show that two completions X and Y of D are normisomorphic.

b. **Existence of a completion** Show that there exists a Banach space X over $\mathbb{K}$ that is a completion of D.

Solution.

a. Let $D \subseteq X$ and $D \subseteq Y$. We define the operator $j : D \subseteq X \to Y$ by $j(u) = u$. Then we have $\|j(u)\| = \|u\|$ on D. By the extension principle in

Section 3.6, there exists a unique extension $j : X \to Y$ with $\|j(u)\| = \|u\|$ on X.

It suffices to show that j is a surjective. In order to prove that $j(X) = Y$, notice that X and Y are all Banach spaces. Let (u_n) be a sequence in D with $u_n \to v$ in Y as $n \to \infty$. Then (u_n) is a Cauchy sequence in X and Y. Thus, we obtain $u_n \to u$ in X as $n \to \infty$, i.e., $j(u) = v$.

b. Two Cauchy sequences (u_n) and (v_n) in D are called equivalent iff

$$\|u_n - v_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Let X be the set of the corresponding equivalence classes $u = [(u_n)]$. We define operations by

$$[(u_n)] + [(v_n)] := [(u_n + v_n)], \quad \alpha[(u_n)] := [(\alpha u_n)].$$

It is easy to show that these operations are independent of the choice of the representatives.

1. Suppose $[(u_n)] = [(u'_n)]$ and $[(v_n)] = [(v'_n)]$, i.e.,

$$\|u_n - u'_n\| \to 0 \quad \text{and} \quad \|v_n - v'_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Then by the triangle inequality:

$$\|(u_n+v_n)-(u'_n+v'_n)\| = \|(u_n-u'_n)+(v_n-v'_n)\| \leq \|u_n-u'_n\|+\|v_n-v'_n\|.$$

Hence, $\|(u_n + v_n) - (u'_n + v'_n)\| \to 0$, so $[(u_n + v_n)] = [(u'_n + v'_n)]$.

2. Suppose $[(u_n)] = [(u'_n)]$, i.e., $\|u_n - u'_n\| \to 0$. Then:

$$\|(\alpha u_n) - (\alpha u'_n)\| = |\alpha| \cdot \|u_n - u'_n\|.$$

Since $\|u_n - u'_n\| \to 0$, it follows that $\|(\alpha u_n) - (\alpha u'_n)\| \to 0$, so $[(\alpha u_n)] = [(\alpha u'_n)]$.

This way X becomes a linear space.

Furthermore, we define

$$\|u\| := \|[(u_n)]\| = \lim_{n\to\infty} \|u_n\|. \tag{1}$$

Since $|\|u_n\| - \|u_m\|| \leq \|u_n - u_m\|$, this limit exists. Moreover, it follows from

$$|\|u_n\| - \|v_n\|| \leq \|u_n - v_n\|$$

that the limit in (1) is independent of the choice of the representative (u_n) of u. It follows easily from (1) that X is a normed space.

Let $w \in D$. Then the constant sequence (w) is a Cauchy sequence and hence $[(w)]$ lies in X. We identify w with the equivalence class $[(w)]$. This way D becomes a subset of X, i.e., $D \subseteq X$.

Each Cauchy sequence (u_n) in D converges to $u = [(u_n)]$ in X. This follows from $u - u_m = [(u_n - u_m)]$ for fixed m. Then we have

$$\|u - u_m\| = \lim_{n\to\infty} \|u_n - u_m\| < \varepsilon$$

Hence D is *dense* in X.

Finally, we show that X is a Banach space. To this end, let (w_n) be a Cauchy sequence in X. We choose a sequence (u_n) in D with $\|u_n - w_n\| < 1/n$ for all n. Hence (u_n) is a Cauchy sequence, this is from

$$\|u_n - u_m\| \leq \|u_n - w_n\| + \|w_n - w_m\| + \|u_m - w_m\|$$

and $u_n \to u$ in X as $n \to \infty$. This implies $w_n \to u$ as $n \to \infty$.

$\square$

Problem 2. **Separation of convex sets**

Let A and B be nonempty convex sets in the real normed space X. Show that

(i) A and B can be separated by a closed hyperplane provided

$$B \cap \mathrm{int} A = \varnothing \quad \text{and} \quad \mathrm{int} A \neq \varnothing.$$

(ii) A and B can be strictly separated by a closed hyperplane provided

$$A \cap B = \varnothing \quad \text{and both } A \text{ and } B \text{ are open.}$$

(iii) A and B can be strictly separated by a closed hyperplane provided

$$A \cap B = \varnothing, \quad A \text{ is closed, and } B \text{ is compact.}$$

Solution.

(i) Suppose first that

$$B \cap A = \varnothing \quad \text{and} \quad \mathrm{int} A \neq \varnothing.$$

Set

$$K := A - B$$

then K is a nonempty convex sets and $\mathrm{int} K \neq \varnothing$. Besides, $0 \notin K$. In fact, suppose there exists $x_1 \in A$ and $x_2 \in B$, such that $x_1 - x_2 = 0$, then

$$x_1 = x_2 \in A \cap B$$

which contradicts with $A \cap B = \varnothing$.

Applying **Hanh-Banach** theorem we can find a hyperplane which seperates K and 0.

$$f(x) \leq r \quad \text{for all } x \in K \quad f(0) \geq r$$

Thus $f(x) \leq 0$ for all $x \in K$, then there exist $y \in A$ and $z \in B$ such that $f(y-z) \leq 0$, which implies $f(y) \leq f(z)$.

Then we can find $s \in \mathbb{R}$ such that

$$\sup_{y \in A} f(y) \leq s \leq \inf_{z \in B} f(z)$$

So A and B can be separated by a closed hyperplane.

Now we show that the condition can be weakened to

$$B \cap \text{int}A = \varnothing \quad \text{and} \quad \text{int}A \neq \varnothing.$$

Since int$A$ is a nonempty convex set which has interior point. By the conclusion we have given

$$f(x) \leq s \quad \text{for all } x \in \text{int}A \quad f(x) \geq s \quad \text{for all } x \in B$$

By the continuity of f, we have

$$f(x) \leq s \quad \text{for all } x \in \overline{\text{int}A} = \overline{A}$$

and further

$$f(x) \leq s \quad \text{for all } x \in A$$

This is the statement.

(ii) The set $K = A - B$ is open and convex and does not contain 0. Hence there exists a continuous linear form $f \neq 0$ on X such that $f(K) < 0$. If $\alpha = \sup\{f(a) : a \in A\}$, then α is finite and

$$f(A) \leq \alpha, \quad f(B) \geq \alpha.$$

However, since $f \neq 0$, f is an open map of X onto $\mathbb{R}$. Hence from $f(A) \leq \alpha$ and $f(B) \geq \alpha$ follows indeed $f(A) < \alpha$ and $f(B) > \alpha$.

(iii) There exists an open convex neighbourhood U of 0 in X such that

$$(A + U) \cap (B + U) = \emptyset.$$

Since $A + U$ and $B + U$ are open and f is an open map of X onto $\mathbb{R}$. Applying (ii) to the sets $A + U$ and $B + U$, we deduce the existence of a continuous linear form $f \neq 0$ on X and a real number α such that

$$f(A + U) < \alpha, \quad f(B + U) > \alpha.$$

whence the assertion.

□

PROBLEM 3. **Extension of linear positive functionals (the Krein theorem)**

Suppose that X is a real ordered normed space in the sense of Section 1.19 of AMS Vol.108 with the order cone X_+ and that L is a linear subspace of X such that

$$L \cap \mathrm{int} X_+ \neq \varnothing.$$

Let $F : L \to \mathbb{R}$ be a linear functional such that

$$F(u) \geq 0 \quad \text{for all } u \in L \text{ with } u \geq 0.$$

Show that F can be extended to a linear continuous functional $f : X \to \mathbb{R}$ such that $f(u) \geq 0$ for all $u \in X$ with $u \geq 0$.

SOLUTION. We first show a similar result.

> Suppose that X is a real ordered normed space L is a linear subspace of X such that each $x \in X$ corresponds at least one element $m \in L$ for which $x \leq m$. If F is a positive linear functional on L, then F can be extended to a positive linear functional on X.

Proof. We define p on X by

$$p(x) = \inf\{F(y) : y \in L, y \geq x\}.$$

Our hypothesis on L entails that to each $x \in X$ corresponds m and m' in L such that $m' \leq x \leq m$. It follows that p is bounded. In fact,

$$F(m') \leq p(x) \leq F(m).$$

Obviously p is a sublinear function on X. If x is in L, it is evident that $p(x) \leq F(x)$. On the other hand, if x and y are in L and $y \geq x$, then $F(y) \geq F(x)$, this is from

$$F(y-x) = F(y) - F(x) \geq 0 \text{ with } y - x \geq 0$$

so $p(x) \geq F(x)$ for x in L. Thus in fact $F(x) = p(x)$ for x in L.

Applying **Hanh-Banach** theorem, we deduce that there exists a linear form f on X that extends f and that satisfies $f(x) \leq p(x)$ for x in X. Then, if $x \geq 0$,

$$f(-x) \leq p(-x) = \inf\{F(y) : y \in L, y + x \geq 0\} \leq 0 \quad \text{and so} \quad f(x) \geq 0.$$

Thus f is a positive linear form on X.

Proof of Problem 3. Suppose

$$x_0 \in L \cap \mathrm{int} X_+.$$

Let U be any balanced neighbourhood of 0 in X such that $x_0 + U \subseteq X_+$. If x belongs to X and $\lambda > 0$ is such that $x \in \lambda U$, then $-x/\lambda \in U$ and so

$$x_0 - x/\lambda \in X_+; \quad \text{that is,} \quad x \leq \lambda x_0 = m.$$

The hypothesis of the previous statement is fulfilled and the construction used in its proof shows that $p(x) \leq \lambda F(x_0)$ if $x \in \lambda U$ and $\lambda > 0$. We can extend F into a positive linear form f on X such that $f(x) \leq p(x)$ for all x in X. Hence $f(x) \leq \lambda F(x_0)$ if $x \in \lambda U$, $\lambda > 0$. This entails that

$$|f(x)| \leq \lambda |F(x_0)| \quad \text{if} \quad x \in \lambda U, \ \lambda > 0.$$

Thus f is continuous on X. □

PROBLEM 4. **Density and duality**

Let X and Y be Banach spaces over $\mathbb{K}$ such that the embedding

$$X \hookrightarrow Y \tag{2}$$

is continuous, and X is dense in Y. Show that the following are met:

(i) The embedding $Y^* \hookrightarrow X^*$ is continuous.

(ii) If X is reflexive, then Y^* is dense in X^*.

SOLUTION.

(i) It follows from (2) that

$$\|j(x)\|_Y \leq \text{const}\|x\|_X \quad \text{for all} \quad x \in X.$$

where $j : X \to Y$ is linear, continuous and injective.

Let $y \in Y^*$ be given. Then, for all $x \in X$,

$$|y^*(j(x))| \leq \|y^*\|\|j(x)\| \leq \text{const}\|y^*\|\|x\|. \tag{3}$$

Let $\bar{y}^* : X \to \mathbb{K}$ denote the restriction of the functional $y^* : Y \to \mathbb{K}$ to the subset X of Y. By (3), we get $\bar{y}^* \in X^*$, where

$$\bar{y}^*(x) = y^*(j(x)) \quad \text{for all} \quad x \in X, \tag{4}$$

and

$$\|\bar{y}^*\| \leq \text{const}\|y^*\| \quad \text{for all} \quad y^* \in Y^*. \tag{5}$$

We want to show that $\bar{y}^* = 0$ implies $y^* = 0$. In fact, let $\bar{y}^* = 0$. Since X is *dense* in Y, it follows from (4) that $y^*(j(x)) = 0$ for all $j(x) \in Y$, and hence $y^* = 0$.

Set $p(y^*) = \bar{y}^*$. Then, it follows from our considerations above that the operator

$$p : Y^* \to X^*$$

is *injective* and continuous. Therefore, we have $Y^* \hookrightarrow X^*$. Moreover, it follows from (4) and (5) that

$$y^*(x) = y^*(j(x)) \quad \text{for all} \quad y^* \in Y^*, \ x \in X, \tag{6}$$

and

$$\|y^*\| \leq \text{const}\|y^*\| \quad \text{for all} \quad y^* \in Y^*.$$

(ii) If the assertion is not true, then the closure of Y^* in the Banach space X^* is a *proper* closed linear subspace of X^*. By the *Hahn-Banach theorem*, there exists a functional $f \in (X^*)^*$ such that

$$f(x^*) = 0 \quad \text{for all} \quad x^* \in Y^* \tag{7}$$

and $f \neq 0$. Since X is *reflexive*, there exists an $x \in X$ such that

$$f(x^*) = x^*(x) \quad \text{for all} \quad x^* \in X^*.$$

By (6) and (7),

$$x^*(j(x)) = 0 \quad \text{for all} \quad x^* \in Y^*.$$

Thus $j(x) = 0$ and $x = 0$. Furthermore, $f = 0$, which contradicts with $f \neq 0$.

$\square$