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Problem 1.

proof:

(a) ① Since $\phi \in \mathcal{T}_1$. $X \setminus X = \phi$ is finite. $X, \phi \in \mathcal{T}_1$.

~~(b) ①~~ ② For $\{A_\lambda\}_{\lambda\in\Lambda} \in \mathcal{T}_1$, WLOG, suppose for $\forall \lambda \in \Lambda$ $A_\lambda \neq \phi$. Then $X \setminus A_\lambda$ is finite. $\bigcup_{\lambda\in\Lambda} A_\lambda \subseteq X$

Apply De Morgan's Law, $X \setminus \bigcup_{\lambda\in\Lambda} A_\lambda = \bigcap_{\lambda\in\Lambda} X \setminus A_\lambda$ is finite

thus $\bigcup_{\lambda\in\Lambda} A_\lambda$ ~~is~~ $\in \mathcal{T}_1$

③. For $A, B \in \mathcal{T}_1$. WLOG, suppose $A, B \neq \phi$. $A \cap B \subseteq A \subseteq X$.

Then $X \setminus A$, $X \setminus B$ ~~is~~ are finite.

$|X \setminus (A \cap B)| = |(X \setminus A) \cup (X \setminus B)| \leq |X \setminus A| + |X \setminus B| < \infty$.

Thus $X \setminus A \cap B$ is finite. $A \cap B \in \mathcal{T}_1$.

Therefore, $\mathcal{T}_1$ is a topology on X. □

(b) ① $X \setminus X = \phi$ is countable. $\phi, X \in \mathcal{T}_2$.

② For $A, B \in \mathcal{T}_2$. WLOG, suppose $A, B \neq \phi$. $A \cap B \subseteq A \subseteq X$.

$X \setminus A$, $X \setminus B$ are countable. $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$ is also countable. $A \cap B$ ~~is~~ $\in \mathcal{T}_2$.

③ For $\{A_\lambda\}_{\lambda\in\Lambda} \in \mathcal{T}_2$. WLOG, suppose for $\forall \lambda \in \Lambda$, $A_\lambda \neq \phi$. $X \setminus A_\lambda$ is countable. $\bigcup_{\lambda\in\Lambda} A_\lambda \subseteq X$.

$X \setminus \bigcup_{\lambda\in\Lambda} A_\lambda = \bigcap_{\lambda\in\Lambda} (X \setminus A_\lambda)$ is countable. $\bigcup_{\lambda\in\Lambda} A_\lambda \in \mathcal{T}_2$

Therefore, $\mathcal{T}_2$ is a topology on X. □

(c) ① Since $p \in X$. $\phi, X \in \mathcal{T}_3$

② For $\phi \neq A, B \in \mathcal{T}_3$. For $p \in X$. $p \in A$, $p \in B$. thus $p \in A \cap B$

$A \cap B \in \mathcal{T}_3$

③ For $\phi \neq \{A_\lambda\}_{\lambda \in \Lambda} \in \mathcal{T}_3$. For $p \in X$. $p \in A_\lambda$, $\lambda \in \Lambda$.

then $p \in \bigcup_{\lambda \in \Lambda} A_\lambda$. $\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathcal{T}_3$

Thus, $\mathcal{T}_3$ is a topology on X. □

2. <proof>. To prove that the metrics d and d' generate the same topology on M, we need to show that every open set in (M, d) is open in (M, d') and vice versa.

Step 1. Open sets in (M, d) are open in (M, d')

Let U be an open set in (M, d). Then for $\forall x \in U$.

$\exists \varepsilon > 0$. s.t. $B_x^{(d)}(\varepsilon) \subset U$.

Now consider $B_x^{(d')}(c\varepsilon)$ in (M, d'). If $y \in B_x^{(d')}(c\varepsilon)$,

then $d'(x, y) = c \cdot d(x, y) < c\varepsilon$. thus $d(x, y) < \varepsilon$.

$y \in B_x^{(d)}(\varepsilon) \subseteq U$. so $B_x^{(d')}(c\varepsilon) \subseteq U$. showing U is open in (M, d')

Step 2. Open sets in (M, d') are open in (M, d).

Let V be an open set in (M, d') Then for $\forall x \in V$.

$\exists \delta > 0$ s.t. $B_x^{(d')}(\delta) \subset V$

For $B_x^{(d)}(\delta/c)$ in (M, d). $y \in B_x^{(d)}(\delta/c)$. $d(x, y) = \frac{1}{c} d'(x, y) < \delta/c$

$d'(x, y) < \delta$. $y \in B_x^{(d')}(\delta) \subseteq V$, $B_x^{(d)}(\delta/c) \subseteq V$. V is open in (M, d)

Thus the topologies generated by d and d' are the same

Problem 3

<proof>. (a).

Step 1. $\overline{X\setminus B} \subseteq X\setminus \text{Int}(B)$.

Let $x \in \overline{X\setminus B}$. then every neighborhood contains a point of $X\setminus B$. If $x\in \text{Int}(B)$. then there exists a neighborhood contained in B, which has no points in $X\setminus B$. this is a contradiction. so $\overline{X\setminus B} \subseteq X\setminus \text{Int}(B)$

Step 2 $X\setminus \text{Int}(B) \subseteq \overline{X\setminus B}$.

0 Let $x\in X\setminus \text{Int}(B)$. then $x \notin \text{Int}(B)$. so there is no neighborhood contained in B. which implies every neighborhood contains a point of $X\setminus B$. $x\in\overline{X\setminus B}$.

Therefore $\overline{X\setminus B} = \text{Int}(B)$.

(b) Apply the conclusion in (a).

$$X\setminus \text{Int}(X\setminus B) = \overline{X\setminus(X\setminus B)} = \bar{B}$$

thus $X\setminus \bar{B} = X\setminus(X\setminus \text{Int}(X\setminus B)) = \text{Int}(X\setminus B)$. □

10 Problem 4.

<proof> Lemma: $\bar{A}$ contains all limit points of A

<proof>. Suppose not. then there exist a limit point $x\in A$. s.t. $x\notin\bar{A}$.

Since $\bar{A}$ is closed. then $x\in X\setminus\bar{A}$ is open so there exists a neighborhood contained in $X \setminus \bar{A}$. which is a contradiction with the definition of the limit point. $\square$

$\Rightarrow$ For a closed subset $A \subseteq X$. $A = \bar{A}$ contains all its limit points

$\Leftarrow$ If A contains all limit points, then denote $A' = \{x \mid x \text{ is a limit point of } A \subseteq X\}$. $A' \subseteq A$. $A \cup A' = A$. but $A \subseteq A \cup A'$. therefore $A = A \cup A' = \bar{A}$ which is closed $\square$

<proof>

5 (a) $\Rightarrow$ For an arbitrary $A \subseteq X$. $\overline{f(A)}$ is closed in Y. Since f is continuous. $f^{-1}(\overline{f(A)})$ is closed. $f(A) \subseteq \overline{f(A)}$, then $A \subseteq f^{-1}(\overline{f(A)})$. Since RHS is a closed set. $\bar{A} \subseteq f^{-1}(\overline{f(A)})$. $f(\bar{A}) \subseteq \overline{f(A)}$.

10

$\Leftarrow$ For any closed subset $B \subseteq Y$. let $C = f^{-1}(B) \subseteq X$. Since $f(\bar{C}) \subseteq \overline{f(C)} = \bar{B} = B$. $\bar{C} \subseteq f^{-1}(B) = C$. thus $C = \bar{C}$ is closed. f is continuous.

(b) $\Rightarrow$ f is continuous. For any $B \subseteq Y$. $\operatorname{Int} B$ is open, thus $f^{-1}(\operatorname{Int} B)$ is open. since $\operatorname{Int} B \subseteq B$. $f^{-1}(\operatorname{Int}(B)) \subseteq f^{-1}(B)$

Since $\operatorname{Int}(f^{-1}(B))$ is the largest open set contained in $f^{-1}(B)$

$f^{-1}(\operatorname{Int}(B)) \subseteq \operatorname{Int}(f^{-1}(B))$.

$\Leftarrow$ $f^{-1}(\mathrm{Int}\, B) \subseteq \cancel{f}\, \mathrm{Int}(f^{-1}(B))$ for all $B \subseteq Y$. Consider any open set $U \subseteq Y$. $\mathrm{Int}\, U = U$. $f^{-1}(U) \subseteq \mathrm{Int}\, f^{-1}(U)$. Since $\mathrm{Int}\, f^{-1}(U) \subseteq f^{-1}(U)$, $f^{-1}(U) = \mathrm{Int}\, f^{-1}(U)$ is open. □

50