An Introduction to Partial Differential Equations

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Contents

Chapter 1

Preliminaries

1.1 Notations

(i) $\mathbb{R}^n = n$-dimensional real Euclidean space, $\mathbb{R} = \mathbb{R}^1$.
$S^{n-1} = \partial B(0,1) = (n-1)$-dimensional unit sphere in $\mathbb{R}^n$.

(ii) $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) = i$th standard coordinate vector.

(iii) A typical point in $\mathbb{R}^n$ is $x = (x_1, \ldots, x_n)$.
We will also, depending upon the context, regard x as a row or as a column vector.

(iv) $\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$ = open upper half-space.
$\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$.

(v) A point in $\mathbb{R}^{n+1}$ will often be denoted as $(x,t) = (x_1, \ldots, x_n, t)$, and we usually interpret $t = x_{n+1}$ = time.
A point $x \in \mathbb{R}^n$ will sometimes be written $x = (x', x_n)$ for $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$.

(vi) U, V, and W usually denote open subsets of $\mathbb{R}^n$. We write

$$V \subset\subset U$$

if $V \subset \overline{V} \subset U$ and $\overline{V}$ is compact, and say V is *compactly contained in* U.

(vii) ∂U = boundary of U, $\overline{U} = U \cup \partial U$ = closure of U.

(viii) $U_T = U \times (0, T]$.

(ix) $\Gamma_T = \partial U_T - U_T$ = parabolic boundary of U_T.

(x) $B(x,r) = \{y \in \mathbb{R}^n \mid |x-y| \leq r\}$ = closed ball in $\mathbb{R}^n$ with center x and radius $r > 0$.

(xi) $\alpha(n)$ = volume of unit ball $B(0,1)$ in $\mathbb{R}^n$:

$$\alpha(n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

$n\alpha(n) =$ surface area of unit sphere $\partial B(0,1)$ in $\mathbb{R}^n$.

Example.

$$\int_{\mathbb{R}} e^{-x^2}\,dx = \sqrt{\pi}.$$

$$\int_{\mathbb{R}^n} e^{-|x|^2}\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e^{-x_1^2-x_2^2-\cdots-x_n^2}\,dx_1dx_2\ldots dx_n$$
$$= \pi^{n/2}.$$

Applying Polar Coordinate,

$$\begin{aligned}\int_0^\infty r^{n-1}\int_{\partial B(0,1)} e^{-r^2}\,d\sigma\,dr &= \int_{\partial B(0,1)} d\sigma \int_0^\infty r^{n-1}e^{-r^2}\,dr \\ &= \omega_{n-1}\int_0^\infty r^{n-1}e^{-r^2}\,dr \\ &= \omega_{n-1}\cdot\frac{1}{2}\Gamma\left(\frac{n}{2}\right).\end{aligned}$$

$$\begin{aligned}\alpha(n) = |B(0,1)| &= \int_{B(0,1)} 1\,dx \\ &= \int_0^1\int_{\partial B(0,1)} r^{n-1}\,d\sigma\,dr \\ &= \omega_{n-1}\int_0^1 r^{n-1}\,dr \\ &= \frac{1}{n}\omega_{n-1}\end{aligned}$$

1.2 Calculus

Theorem 1.2.1: Gauss-Green Theorem

(i) Suppose $u \in C^1(\overline{U})$. Then

$$\int_U u_{x_i}\,dx = \int_{\partial U} u\nu^i\,dS \quad (i = 1,\ldots,n).$$

(ii) We have

$$\int_U \operatorname{div}\mathbf{u}\,dx = \int_{\partial U}\mathbf{u}\cdot\nu\,dS$$

for each vector field $\mathbf{u} \in C^1(\overline{U};\mathbb{R}^n)$.

Assertion (ii), also called the *Divergence Theorem*, follows from (i) applied to each component of $\mathbf{u} = (u^1,\ldots,u^n)$.

Theorem 1.2.2: Integration by parts formula

Let $u, v \in C^1(\overline{U})$. Then

$$\int_U u_{x_i} v\, dx = -\int_U u v_{x_i}\, dx + \int_{\partial U} u v \nu^i\, dS \quad (i = 1, \dots, n).$$

Proof. Apply Gauss-Green Theorem to uv. □

Theorem 1.2.3: Green's formulas

Let $u, v \in C^2(\overline{U})$. Then

(i)
$$\int_U \Delta u\, dx = \int_{\partial U} \frac{\partial u}{\partial \nu}\, dS,$$

(ii)
$$\int_U Dv \cdot Du\, dx = -\int_U u \Delta v\, dx + \int_{\partial U} \frac{\partial v}{\partial \nu} u\, dS,$$

(iii)
$$\int_U u\Delta v - v\Delta u\, dx = \int_{\partial U} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\, dS.$$

Proof. Using the *Integration by parts formula* with u_{x_i} in place of u and $v \equiv 1$, we see

$$\int_U u_{x_i x_i}\, dx = \int_{\partial U} u_{x_i} \nu^i\, dS.$$

Summing over $i = 1, \dots, n$ establishes (i).

To derive (ii), we employ the *Integration by parts formula* with v_{x_i} replacing v. Write (ii) with u and v interchanged and then subtract to obtain (iii). □

Remark.

> $$Du \cdot \nu = \frac{\partial u}{\partial \nu}$$

Theorem 1.2.4: Polar coordinates

(i) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and summable. Then

$$\int_{\mathbb{R}^n} f\,dx = \int_0^\infty \left(\int_{\partial B(x_0,r)} f\,dS \right) dr$$

for each point $x_0 \in \mathbb{R}^n$.

(ii) In particular,

$$\frac{d}{dr}\left(\int_{B(x_0,r)} f\,dx \right) = \int_{\partial B(x_0,r)} f\,dS$$

for each $r > 0$.

1.3 Measure Theory

Theorem 1.3.1: Dominated Convergence Theorem

Assume the functions $\{f_k\}_{k=1}^\infty$ are integrable and

$$f_k \to f \quad \text{a.e.}$$

Suppose also

$$|f_k| \le g \quad \text{a.e.},$$

for some summable function g. Then

$$\int_{\mathbb{R}^n} f_k\,dx \to \int_{\mathbb{R}^n} f\,dx.$$

Proof. See notes for Real Analysis. □

Theorem 1.3.2: Lebesgue's Differentiation Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally summable.

(i) Then for a.e. point $x_0 \in \mathbb{R}^n$,

$$\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} f\,dx \to f(x_0) \quad \text{as } r \to 0.$$

(ii) In fact, for a.e. point $x_0 \in \mathbb{R}^n$,

$$\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} |f(x) - f(x_0)|\,dx \to 0 \quad \text{as } r \to 0.$$

Chapter 2

Four Important Linear Partial Differential Equations

2.1 Laplace Equation

Among the most important of all partial differential equations are undoubtedly *Laplace's equation*

$$\Delta u = 0$$

and *Poisson's equation*

$$-\Delta u = f.$$

First of all we study the properties of **harmonic** functions.

Definition 2.1.1: Harmonic Function

A C^2 function u satisfying $\Delta u = 0$ is called a *harmonic function*.

2.1.1 Mean-Value Formulas

Theorem 2.1.2: Mean-value formulas for Laplace's equation

If $u \in C^2(U)$ is harmonic, then

$$u(x) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u \, d\sigma = \frac{1}{|B(x,r)|} \int_{B(x,r)} u \, dy$$

for each ball $B(x,r) \subset U$.

Proof. Set

$$\varphi(r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y) \, d\sigma(y),$$

where

$$|\partial B(x,r)| = n\alpha(n)r^{n-1}, \quad \text{and } \alpha(n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}.$$

By a change of variables $y = x + rz$, we have

$$\varphi(r) = \frac{r^{n-1}}{n\alpha(n)r^{n-1}} \int_{\partial B(0,1)} u(x+rz)\, d\sigma(z).$$

Then,

$$\varphi'(r) = \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} \nabla u(x+rz) \cdot z\, d\sigma(z).$$

By the divergence theorem:

$$\int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}\, d\sigma = \int_{B(x,r)} \Delta u\, dy.$$

We have

$$\begin{aligned}
\varphi'(r) &= \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} \nabla u(x+rz) \cdot \frac{y-x}{r}\, d\sigma(z) \\
&= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} \nabla u(y) \cdot \nu\, d\sigma(y) \\
&= \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} \frac{\partial u(y)}{\partial \nu}\, d\sigma(y) \\
&= \frac{1}{n\alpha(n)r^{n-1}} \int_{B(x,r)} \Delta u(y)\, dy \\
&= \frac{r}{n} \cdot \frac{1}{|B(x,r)|} \int_{B(x,r)} \Delta u(y)\, dy
\end{aligned}$$

Since u is harmonic ($\Delta u = 0$), we conclude that $\varphi'(r) = 0$, which implies that $\varphi(r)$ is constant.

By the Lebesgue Differentiation Theorem

$$\varphi(r) = \lim_{t\to 0} \frac{1}{|\partial B(x,t)|} \int_{\partial B(x,t)} u\, d\sigma = u(x)$$

By employing polar coordinates, we have

$$\int_{B(x,r)} u\, dy = \int_0^r \int_{\partial B(x,s)} u\, d\sigma\, ds$$

and

$$\int_{B(x,r)} u\, dy = n\alpha(n) \int_0^r s^{n-1} u(x)\, ds.$$

Finally,

$$u(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} u\, dy.$$

□

Theorem 2.1.3: Converse to mean-value property

If $u \in C^2(U)$ satisfies

$$u(x) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u \, dS$$

for each ball $B(x,r) \subset U$, then u is harmonic.

Proof. Suppose not. WLOG, there exists a ball $B(x,r) \subset U$, such that $\Delta u > 0$ in $B(x,r)$.

Define

$$\varphi(r) = \frac{1}{|B(x,r)|} \int_{\partial B(x,r)} u \, d\sigma.$$

Then

$$\varphi'(r) = \frac{r}{n} \cdot \frac{1}{|B(x,r)|} \int_{B(x,r)} \Delta u(y) \, dy > 0.$$

However, since $\varphi(r)$ is a constant, $\varphi'(r) = 0$, which is a contradiction (as $\Delta u > 0$). Therefore, u must be harmonic. □

2.1.2 Maximum Principle

Theorem 2.1.4: Strong maximum principle

Suppose $u \in C^2(U) \cap C(\overline{U})$ is harmonic within U.

(i) Then

$$\max_{\overline{U}} u = \max_{\partial U} u.$$

(ii) Furthermore, if U is connected and there exists a point $x_0 \in U$ such that

$$u(x_0) = \max_{\overline{U}} u,$$

then u is constant within U.

Assertion (i) is the maximum principle for Laplace's equation and (ii) is the strong maximum principle. Replacing u by $-u$, we recover also similar assertions with "min" replacing "max".

Proof. Suppose $\exists x_0 \in U$ such that $u(x_0) = \max_{\overline{U}} u := M$.

Then for $0 < r < \delta(x_0) = \text{dist}(x_0, \partial U)$, the mean value property implies that

$$u(x_0) = \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} u(x) \, dx \leq M.$$

Since $u(x_0) = M$, equality holds iff $u(x) = M$ for all $x \in B(x_0, r)$.

Repeating this argument, we see that $u(y) = M$ for all $y \in B(x_0, r)$.

Hence the set $\{x \in U \mid u(x) = M\}$ is both open and relatively closed in U, and thus equals U if U is connected. □

Before we learn the weak maximum principle, we first introduce the following definition:

Definition 2.1.5: Subharmonic and Superharmonic Functions

Let u be a C^2 function in U. Then u is a subharmonic (superharmonic) function in U if $\Delta u \geq 0$ ($\Delta u \leq 0$).

Subharmonic and superharmonic functions both have maximum principle, we only show one of it.

Theorem 2.1.6: Maximum Principle for Subharmonic Functions

Let U be a bounded domain in $\mathbb{R}^n$ and $u \in C^2(U) \cap C(\overline{U})$ be subharmonic in U. Then u attains its maximum in $\overline{U}$, i.e.,

$$\max_{\overline{U}} u = \max_{\partial U} u.$$

Proof. 1. First, consider $\Delta u > 0$ in U. If u has a local maximum at a point $x_0 \in U$, then the Hessian matrix $\nabla^2 u(x_0)$ is negative semi-definite. Thus,

$$\Delta u(x_0) = \operatorname{tr}(\nabla^2 u(x_0)) \leq 0,$$

which is a contradiction.

2. Now consider $\Delta u \geq 0$. To handle this, for any $\varepsilon > 0$, define

$$u_\varepsilon(x) = u(x) + \varepsilon |x|^2.$$

Then,

$$\Delta u_\varepsilon(x) = \Delta u(x) + \Delta(\varepsilon |x|^2) = \Delta u(x) + 2n\varepsilon > 0.$$

By Step 1, we have

$$\max_{\overline{U}} u_\varepsilon = \max_{\partial U} u_\varepsilon.$$

Observe that

$$\max_{\overline{U}} u \leq \max_{\overline{U}} u_\varepsilon = \max_{\partial U} u_\varepsilon \leq \max_{\partial U} u + \max_{\partial U} (\varepsilon |x|^2).$$

Taking $\varepsilon \to 0$, we conclude

$$\max_{\overline{U}} u = \max_{\partial U} u.$$

□

Theorem 2.1.7: Uniqueness

Let $g \in C(\partial U)$, $f \in C(U)$. Then there exists at most one solution $u \in C^2(U) \cap C(\overline{U})$ of the boundary-value problem

$$\begin{cases} -\Delta u = f & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases}$$

Proof. If u and $\tilde{u}$ are both solutions for the boundary-value problem , apply maximum principle to the harmonic functions $w := \pm(u - \tilde{u})$. □

Remark.

If U is unbounded, the conclusion fails.

2.1.3 Regularity

Our main result in this section is that if $u \in C^2$ is harmonic, then necessarily $u \in C^\infty$.

Let $\Omega \subset \mathbb{R}^n$ be open. For $\varepsilon > 0$, denote

$$\Omega_\varepsilon = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > \varepsilon\}.$$

Definition 2.1.8: Mollifier

Define the mollifier as:

$$\eta(x) = \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right), & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1. \end{cases}$$

where $\eta \in C^\infty(\mathbb{R}^n)$ and satisfies $\int_{\mathbb{R}^n} \eta(x)\, dx = 1$.
Let

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right).$$

This is called the *standard mollifier.*

By definition, we know that η_ε has support, and $spt(\eta_\varepsilon) \subset B(0, \varepsilon)$.

Definition 2.1.9: Mollification

For a function f, define its mollification as:

$$f^\varepsilon = \eta_\varepsilon * f = \int_\Omega \eta_\varepsilon(x-y) f(y)\, dy = \int_{B(0,\varepsilon)} \eta_\varepsilon(y) f(x-y)\, dy.$$

Theorem 2.1.10: Properties of molifiers

$f^\varepsilon \in C^\infty(\Omega_\varepsilon)$.

Proof. We can rewrite:

$$f^\varepsilon(x) = \int_\Omega \eta_\varepsilon(y) f(x-y)\, dy = \frac{1}{\varepsilon^n} \int_\Omega \eta\left(\frac{y}{\varepsilon}\right) f(x-y)\, dy.$$

Change variables with $z = \frac{y}{\varepsilon}$, $dy = \varepsilon^n dz$:

$$f^\varepsilon(x) = \int_{\mathbb{R}^n} \eta(z) f(x - \varepsilon z)\, dz.$$

Since $\eta \in C^\infty$, we conclude $f^\varepsilon \in C^\infty$. In fact, fix $x \in U_\varepsilon$, $i \in \{1, \ldots, n\}$, and h so small that $x + he_i \in U_\varepsilon$. Then:

$$\frac{f^\varepsilon(x+he_i) - f^\varepsilon(x)}{h} = \frac{1}{\varepsilon^n} \int_\Omega \frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\varepsilon}\right) - \eta\left(\frac{x-y}{\varepsilon}\right)\right] f(y)\, dy$$

$$= \frac{1}{\varepsilon^n} \int_V \frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\varepsilon}\right) - \eta\left(\frac{x-y}{\varepsilon}\right)\right] f(y)\, dy,$$

for some open set $V \subset\subset U$. As

$$\frac{1}{h}\left[\eta\left(\frac{x+he_i-y}{\varepsilon}\right) - \eta\left(\frac{x-y}{\varepsilon}\right)\right] \to \frac{1}{\varepsilon}\eta_{x_i}\left(\frac{x-y}{\varepsilon}\right)$$

uniformly on V, the partial derivative $f^\varepsilon_{x_i}(x)$ exists and equals

$$f^\varepsilon_{x_i}(x) = \int_\Omega \eta_{\varepsilon, x_i}(x-y) f(y)\, dy.$$

A similar argument shows that $D^\alpha f^\varepsilon(x)$ exists, and

$$D^\alpha f^\varepsilon(x) = \int_\Omega D^\alpha \eta_\varepsilon(x-y) f(y)\, dy, \quad (x \in U_\varepsilon),$$

for each multiindex α.

□

Theorem 2.1.11: Smoothness

Suppose $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω. Then $u \in C^\infty(\Omega)$.

Proof. Let η be the standard mollifier. Define

$$u^\varepsilon(x) = \eta_\varepsilon * u = \int_\Omega \eta_\varepsilon(x-y) u(y)\, dy = \frac{1}{\varepsilon^n} \int_{B(x,\varepsilon)} \eta\left(\frac{x-y}{\varepsilon}\right) u(y)\, dy.$$

Using polar coordinates and the mean value property:

$$
\begin{aligned}
u^\varepsilon(x) &= \frac{1}{\varepsilon^n}\int_0^\varepsilon \int_{\partial B(x,r)} \eta\left(\frac{r}{\varepsilon}\right) u(y)\, d\sigma dr \\
&= \frac{1}{\varepsilon^n}\int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) \frac{n\alpha(n)r^{n-1}}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} u(y)\, d\sigma dr \\
&= \frac{1}{\varepsilon^n}\int_0^\varepsilon \eta\left(\frac{r}{\varepsilon}\right) n\alpha(n) r^{n-1} u(x)\, dr \\
&= u(x)\frac{1}{\varepsilon^n}\int_0^\varepsilon \int_{\partial B(x,r)} \eta\left(\frac{r}{\varepsilon}\right)\, d\sigma dr \\
&= u(x)\int_{B(x,\varepsilon)} \frac{1}{\varepsilon^n}\eta\left(\frac{r}{\varepsilon}\right)\, dr \\
&= u(x) \in C^\infty(\Omega_\varepsilon)
\end{aligned}
$$

□

The stronger conclusion is that

Theorem 2.1.12: Analyticity

Assume u is harmonic in Ω, then u is analytic in Ω

.

2.1.4 Interior Estimate

Theorem 2.1.13: Estimates on Derivatives (Theorem 7)

Assume u is harmonic in U. Then:

$$|D^\alpha u(x_0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))}$$

for each ball $B(x_0, r) \subset U$ and each multiindex α of order $|\alpha| = k$.
Here:

$$C_0 = \frac{1}{\alpha(n)}, \quad C_k = \frac{(2^{n+1}nk)^k}{\alpha(n)}, \quad (k = 1, 2, \dots),$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$.

Proof. We argue by induction on k.

1. For $k = 0$:

$$\text{LHS} = u(x_0), \quad \text{RHS} = \frac{1}{\alpha(n)r^n}\int_{B(x_0,r)} u.$$

Since u is harmonic, by the mean value property:

$$|u(x_0)| = \frac{1}{\alpha(n)r^n}\int_{B(x_0,r)} |u| \leq \frac{1}{\alpha(n)r^n}\int_{B(x_0,r)} |u|.$$

2. For $k = 1$:

$$C_1 = \frac{2^n n}{\alpha(n)}.$$

Observe that $\frac{\partial u}{\partial x_i}$ is still harmonic, since $\Delta \frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i}(\Delta u) = 0$.

By the mean value property:

$$\frac{\partial u}{\partial x_i}(x_0) = \frac{1}{\alpha(n) r^n} \int_{B(x_0,r)} \frac{\partial u}{\partial x_i}.$$

Using the divergence theorem:

$$\frac{\partial u}{\partial x_i}(x_0) = \frac{1}{\alpha(n) r^n} \int_{\partial B(x_0,r)} u\, \nu_i,$$

where ν_i is the i-th component of the outward unit normal. Then:

$$\left| \frac{\partial u}{\partial x_i}(x_0) \right| \leq \frac{2^n}{\alpha(n) r^{n+1}} \|u\|_{L^1(B(x_0,r))}.$$

Combining these inequalities, we deduce:

$$|\nabla u(x_0)| \leq \frac{2^n n}{\alpha(n) r^{n+1}} \|u\|_{L^1(B(x_0,r))}.$$

3. Inductive Step: Assume the estimate holds for $k-1$. Then for $|\alpha| = k$, apply similar arguments using the derivatives of u, the divergence theorem, and scaling properties to obtain:

$$|D^\alpha u(x_0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x_0,r))}.$$

Thus, the result follows by induction. □

Chapter 3

Sobolev Spaces

3.1 Hölder Spaces

We first discuss the simpler Hölder Spaces. Recall the definition of Hölder continuity:

Definition 3.1.1: Hölder Continuity

Suppose functions u satisfying

$$|u(x) - u(y)| \leq C|x-y|^\gamma \quad (x, y \in U),$$

for some $0 < \gamma \leq 1$ and a constant C. Such a function is said to be Hölder continuous with exponent γ.

Then we give the definition of Hölder norms.

Definition 3.1.2: Hölder Norms

(i) If $u : U \to \mathbb{R}$ is bounded and continuous, we write

$$\|u\|_{C(\overline{U})} := \sup_{x \in U} |u(x)|.$$

(ii) The γ^{th}-Hölder seminorm of $u : U \to \mathbb{R}$ is

$$[u]_{C^{0,\gamma}(\overline{U})} := \sup_{x,y \in U, x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\gamma}.$$

The γ^{th}-Hölder norm is defined as

$$\|u\|_{C^{0,\gamma}(\overline{U})} := \|u\|_{C(\overline{U})} + [u]_{C^{0,\gamma}(\overline{U})}.$$

The general Hölder space $C^{k,\gamma}$ is defined based on it.

Definition 3.1.3: Hölder Space

The Hölder space $C^{k,\gamma}(\overline{U})$ consists of all functions $u \in C^k(\overline{U})$ for which the norm

$$\|u\|_{C^{k,\gamma}(\overline{U})} := \sum_{|\alpha|\leq k} \|D^\alpha u\|_{C(\overline{U})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\overline{U})}$$

is finite.

So the space $C^{k,\gamma}(\overline{U})$ consists of those functions u that are k-times continuously differentiable and whose k^{th}-partial derivatives are bounded and Hölder continuous with exponent γ.

Remark.

> We may write C^α instead of $C^{0,\alpha}$ for brevity.

3.2 Sobolev Spaces

The Hölder spaces are unfortunately not often suitable settings for elementary PDE theory, as we usually cannot make good enough analytic estimates to demonstrate that the solutions we construct actually belong to such spaces. What are needed rather are some other kinds of spaces, containing **less smooth** functions. In practice we must strike a balance, by designing spaces comprising functions which have some, but not too great, smoothness properties.

Motivation for Definition of Weak Derivative: Assume we are given a function $u \in C^1(U)$. Then if $\varphi \in C_c^\infty(U)$, we see from the integration by parts formula that

$$\int_U u\varphi_{x_i}\,dx = -\int_U u_{x_i}\varphi\,dx \quad (i = 1, \dots, n).$$

There are no boundary terms, since φ has compact support in U and thus vanishes near ∂U.

More generally now, if k is a positive integer, $u \in C^k(U)$, and $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multiindex of order $|\alpha| = \alpha_1 + \cdots + \alpha_n = k$, then

$$\int_U uD^\alpha\varphi\,dx = (-1)^{|\alpha|}\int_U D^\alpha u\varphi\,dx.$$

This equality holds since

$$D^\alpha\varphi = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}\varphi.$$

Definition 3.2.1: Weak Partial Derivative

Suppose $u, v \in L^1_{\text{loc}}(U)$ and α is a multiindex. We say that $v \in L^1_{\text{loc}}(U)$ is the α-weak partial derivative of u, written

$$D^\alpha u = v,$$

provided

$$\int_U u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_U v\varphi \, dx$$

for all test functions $\varphi \in C_c^\infty(U)$.

In the sense of weak derivative, we introduce the definition of Sobolev spaces.

Definition 3.2.2: Sobolev Space

The Sobolev space $W^{k,p}(U)$ consists of all locally summable functions $u : U \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.

Remark.

> If $p = 2$, we usually write
>
> $$H^k(U) = W^{k,2}(U) \quad (k = 0, 1, \dots).$$
>
> The letter H is used, since $H^k(U)$ is a Hilbert space. Note that $H^0(U) = L^2(U)$.

Definition 3.2.3: Norm in Sobolev Space

If $u \in W^{k,p}(U)$, we define its norm to be

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha|\leq k} \int_U |D^\alpha u|^p \, dx\right)^{1/p}, & \text{if } 1 \leq p < \infty, \\ \sum_{|\alpha|\leq k} \operatorname{ess\,sup}_U |D^\alpha u|, & \text{if } p = \infty. \end{cases}$$

We denote $W_0^{k,p}(U)$ the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$ and denote $H_0^k := W_0^{k,2}$.

3.3 Sobolev Embedding Theorems

In this section we will know some important conclusions of Sobolev Embedding theorems without proof.

Definition 3.3.1: Embedding

Let X and Y be Banach spaces, $X \subseteq Y$. We say that X is compactly embedded in Y, written $X \hookrightarrow Y$, provided:

(i) $\|u\|_Y \leq C\|u\|_X$ for all $u \in X$, for some constant C,

(ii) Each bounded sequence in X is precompact in Y.

The Sobolev Embedding Theorem is considered in three situations, and the main results are showing as follows:

Theorem 3.3.2: Sobolev Embedding

(1) $W^{1,p} \hookrightarrow L^q$, where $q = \frac{np}{n-p}$ for $1 \leq p < n$ (Gagliardo-Nirenberg-Sobolev inequality).

(2) $W^{1,p} \hookrightarrow C^\alpha$, where $\alpha = 1 - \frac{n}{p}$ for $p > n$ (Morrey's inequality).

(3) $W^{1,p} \hookrightarrow$ BMO (Poincaré Inequality) for $p = n$, where BMO stands for Bounded Mean Oscillation

$$\|u\|_{\text{BMO}} := \sup_{x \in Q} \frac{1}{|Q|} \int_Q |u - \frac{1}{|Q|} \int_Q u| < \infty.$$

Remark.

> $W_{1,p} \hookrightarrow L^\infty$ is not true. The conterexample is
>
> $$u(x) = \ln(\ln(1 + \frac{1}{|x|}))$$

Chapter 4

Weak Solutions: Part I

4.1 Guide

The goal of this chapter is to discuss regularity results for weak solutions to elliptic equations

$$\mathscr{L}u + cu = f \tag{*}$$

where $\mathscr{L} := -\mathrm{div}(A(x)\nabla) = -D_j(a_{ij}(x)D_i)$

Definition 4.1.1: Weak Solution

We say $u \in H^1(\Omega)$ is a weak solution of $(*)$ if

$$\int_\Omega (a_{ij}D_iuD_j\varphi + cu\varphi) = \int_\Omega f\varphi,$$

where $\varphi \in H_0^1(\Omega)$

This is motivated by integrating by parts. In fact

$$\mathscr{L}u\varphi + cu\varphi = f\varphi, \quad \int_\Omega \mathscr{L}u\varphi = -\int_\Omega D_j(a_{ij}D_iu)\varphi = \int_\Omega a_{ij}D_iuD_j\varphi.$$

The definition makes sense provided the following assumptions.

(1) (Ellipticity) The leading coefficients $a_{ij} \in L^\infty$ are *uniformly elliptic*. That is, for some positive constant λ, there holds

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for any } x \in \Omega \text{ and } \xi \in \mathbb{R}^n.$$

(2) The coefficient $c \in L^{\frac{n}{2}}(\Omega)$ and the nonhomogeneous term $f \in L^{\frac{2n}{n+2}}(\Omega)$.

Assumptions (2) is from Sobolev Embedding Theorems.

Take $\varphi = u$, and the right-hand side becomes $\int_\Omega fu$.
Using Sobolev embedding $H^1 \hookrightarrow W^{1,2} \hookrightarrow L^{\frac{2n}{n-2}}$, we have $u \in L^{\frac{2n}{n-2}}$.

By Hölder's inequality:

$$\int_\Omega cu^2 \leq \|u\|^2_{L^{\frac{2n}{n-2}}} \|c\|_{L^{\frac{n}{2}}},$$

and

$$\int_\Omega fu \leq \|u\|_{L^{\frac{2n}{n-2}}} \|f\|_{L^{\frac{2n}{n+2}}}.$$

Our main result is to show that the weak solution $u \in H^1$ can lead to $u \in C^\alpha$ if we have better assumptions on coefficients a_{ij} and c and the nonhomogeneous term f.

4.2 Growth of Local Integrals

This section will provide some general knowledge of **Campanato** and **BMO** spaces.

NOTATIONS

- $B_R(x_0)$ denotes the ball in $\mathbb{R}^n$ of radius R centered at x_0.
- Let Ω be a bounded connected open set in $\mathbb{R}^N$ and let $u \in L^1(\Omega)$. For any ball $B_r(x_0) \subset \Omega$, define
$$u_{x_0,r} := \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u$$

Theorem 4.2.1: Campanato Sapce $\hookrightarrow$ Hölder Space

Suppose $u \in L^2(\Omega)$ satisfies

$$\int_{B_r(x)} |u - u_{x,r}|^2 \leq M^2 r^{n+2\alpha} \quad \text{for any } B_r(x) \subset \Omega,$$

for some $\alpha \in (0,1)$. Then $u \in C^\alpha(\Omega)$, and for any $\Omega' \subset\subset \Omega$, there holds

$$\sup_{\Omega'} |u| + \sup_{\substack{x,y\in\Omega' \\ x\neq y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq c\{M + \|u\|_{L^2(\Omega)}\},$$

where $c = c(n, \alpha, \Omega, \Omega') > 0$.

Proof. Denote $R_0 = \text{dist}(\Omega', \partial\Omega)$. For any $x_0 \in \Omega'$ and $0 < r_1 < r_2 \leq R_0$, we have

$$|u_{x_0,r_1} - u_{x_0,r_2}|^2 \leq 2\left(|u(x) - u_{x_0,r_1}|^2 + |u(x) - u_{x_0,r_2}|^2\right).$$

Integrating with respect to x in $B_{r_1}(x_0)$ shows that

$$\begin{aligned}\int_{B_{r_1}(x_0)} |u_{x_0,r_1} - u_{x_0,r_2}|^2 &= |u_{x_0,r_1} - u_{x_0,r_2}|^2 \frac{\omega_n r_1^n}{n} \\ &\leq \int_{B_{r_1}(x_0)} \text{RHS},\end{aligned}$$

Thus, we have:

$$|u_{x_0,r_1} - u_{x_0,r_2}|^2 \leq \frac{n}{\omega_n r_1^n} \left\{ \int_{B_{r_1}(x_0)} |u - u_{x_0,r_1}|^2 + \int_{B_{r_2}(x_0)} |u - u_{x_0,r_2}|^2 \right\}.$$

Using the assumption $\int_{B_r(x_0)} |u - u_{x_0,r}|^2 \leq M^2 r^{n+2\alpha}$, we obtain:

$$\begin{aligned}|u_{x_0,r_1} - u_{x_0,r_2}|^2 &\leq \frac{n}{\omega_n r_1^n} \left\{ M^2 r_1^{n+2\alpha} + M^2 r_2^{n+2\alpha} \right\}. \\ &:= c(n) M^2 r_1^{-n} \left\{ r_1^{n+2\alpha} + r_2^{n+2\alpha} \right\}.\end{aligned}$$

For any $R \leq R_0$, with $r_1 = R/2^{i+1}$ and $r_2 = R/2^i$, we obtain:

$$|u_{x_0,2^{-(i+1)}R} - u_{x_0,2^{-i}R}| \leq c(n) 2^{-(i+1)\alpha} M R^\alpha.$$

Thus, for $h < k$,

$$|u_{x_0,2^{-h}R} - u_{x_0,2^{-k}R}| \leq \frac{c(n)}{2^{(h+1)\alpha}} M R^\alpha \sum_{i=0}^{k-h-1} \frac{1}{2^{i\alpha}} \leq \frac{c(n,\alpha)}{2^{h\alpha}} M R^\alpha.$$

This shows that $\{u_{x_0,2^{-i}R}\} \subset \mathbb{R}$ is a Cauchy sequence, and hence it converges. Its limit $\hat{u}(x_0)$ is independent of the choice of R, as the estimate can be applied with $r_1 = 2^{-i}R$ and $r_2 = 2^{-i}\bar{R}$.

Let:

$$\hat{u}(x_0) = \lim_{r \to 0} u_{x_0,r},$$

and for any $0 < r \leq R_0$,

$$|u_{x_0,r} - \hat{u}(x_0)| \leq c(n,\alpha) M r^\alpha. \tag{*}$$

Recall the Lebesgue theorem, $\{u_{x,r}\}$ converges to u in $L^1(\Omega)$ as $r \to 0^+$, so $u = \hat{u}$ almost everywhere. Moreover, the uniform convergence implies u is continuous. We further deduce:

$$|u(x)| \leq CMR^\alpha + |u_{x,R}|,$$

for $x \in \Omega'$, $R \leq R_0$, hence u is bounded with the estimate:

$$\sup_{x \in \Omega'} |u(x)| \leq c\{M R_0^\alpha + \|u\|_{L^2(\Omega')}\}.$$

To prove u is Hölder continuous, let $x, y \in \Omega'$ with $R = |x-y| < R_0/2$. Then:

$$|u(x) - u(y)| \le |u(x) - u_{x,2R}| + |u(y) - u_{y,2R}| + |u_{x,2R} - u_{y,2R}|.$$

The first two terms are estimated using (*), while for the last term:

$$|u_{x,2R} - u_{y,2R}| \le |u_{x,2R} - u(\zeta)| + |u_{y,2R} - u(\zeta)|.$$

Integrating over ζ in $B_{2R}(x) \cap B_{2R}(y)$, we get:

$$|u_{x,2R} - u_{y,2R}|^2 \le \frac{2}{|B_{2R}(x)|} \left\{ \int_{B_{2R}(x)} |u - u_{x,2R}|^2 + \int_{B_{2R}(y)} |u - u_{y,2R}|^2 \right\} \le c(n,\alpha) M^2 R^{2\alpha}.$$

Thus:

$$|u(x) - u(y)| \le c(n,\alpha) M |x-y|^\alpha.$$

For $|x-y| > R_0/2$, we have:

$$|u(x) - u(y)| \le 2 \sup_{\Omega'} |u| \le c \left\{ M + \frac{1}{R_0^\alpha} \|u\|_{L^2(\Omega)} \right\} |x-y|^\alpha.$$

This concludes the proof.

□

A special case of the theorem is the following result due to Morrey

Corollary 4.2.2

Suppose $u \in H^1_{\text{loc}}(\Omega)$ satisfies

$$\int_{B_r(x)} |Du|^2 \le M^2 r^{n-2+2\alpha} \quad \text{for any } B_r(x) \subset \Omega,$$

for some $\alpha \in (0,1)$. Then $u \in C^\alpha(\Omega)$, and for any $\Omega' \subset\subset \Omega$, there holds

$$\sup_{\Omega'} |u| + \sup_{\substack{x,y \in \Omega' \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \le c\{M + \|u\|_{L^2(\Omega)}\},$$

where $c = c(n, \alpha, \Omega, \Omega') > 0$.

Proof. By the Poincaré inequality, we obtain

$$\int_{B_r(x)} |u - u_{x,r}|^2 \le c(n) r^2 \int_{B_r(x)} |Du|^2 \le c(n) M^2 r^{n+2\alpha}.$$

By applying Theorem 4.2.1, we have the result. □

The following lemma is needed in the next section.

Lemma 4.2.3

Suppose $u \in H^1(\Omega)$ satisfies

$$\int_{B_r(x_0)} |Du|^2 \leq Mr^\mu \quad \text{for any } B_r(x_0) \subset \Omega,$$

for some $\mu \in [0, n)$. Then for any $\Omega' \subset\subset \Omega$, there holds for any $B_r(x_0) \subset \Omega$ with $x_0 \in \Omega'$:

$$\int_{B_r(x_0)} |u|^2 \leq c(n, \lambda, \mu, \Omega, \Omega') \left\{ M + \int_\Omega u^2 \right\} r^\lambda,$$

where $\lambda = \mu + 2$ if $\mu < n - 2$, and λ is any number in $[0, n)$ if $n - 2 \leq \mu < n$.

Proof. As before, denote $R_0 = \mathrm{dist}(\Omega', \partial\Omega)$. For any $x_0 \in \Omega'$ and $0 < r \leq R_0$, the Poincaré inequality yields

$$\int_{B_r(x_0)} |u - u_{x_0,r}|^2 \leq cr^2 \int_{B_r(x_0)} |Du|^2 \, dx \leq c(n)Mr^{\mu+2}.$$

This implies that

$$\int_{B_r(x_0)} |u - u_{x_0,r}|^2 \leq c(n)Mr^\lambda,$$

For any $0 < \rho < r \leq R_0$, we have

$$\int_{B_\rho(x_0)} |u|^2 \leq 2 \int_{B_\rho(x_0)} |u_{x_0,r}|^2 + 2 \int_{B_\rho(x_0)} |u - u_{x_0,r}|^2.$$

Using the above inequalities:

$$\begin{aligned} \int_{B_\rho(x_0)} |u|^2 &\leq c(n)\rho^n |u_{x_0,r}|^2 + 2 \int_{B_r(x_0)} |u - u_{x_0,r}|^2 \\ &\leq c(n) \left\{ \left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |u|^2 + Mr^\lambda \right\} \end{aligned}$$

where we used

$$|u_{x_0,r}|^2 \leq \frac{c(n)}{r^n} \int_{B_r(x_0)} |u|^2.$$

Hence, the function $\varphi(r) = \int_{B_r(x_0)} |u|^2$ satisfies the inequality

$$\varphi(\rho) \leq c(n) \left\{ \left(\frac{\rho}{r}\right)^n \varphi(r) + Mr^\lambda \right\} \quad \text{for any } 0 < \rho < r \leq R_0,$$

for some $\lambda \in (0, n)$. If we replace Mr^λ on the right-hand side with $M\rho^\lambda$, the result follows. Specifically, we obtain:

$$\int_{B_\rho(x_0)} |u|^2 \leq c \left\{ \left(\frac{\rho}{r}\right)^\lambda \int_{B_r(x_0)} |u|^2 + M\rho^\lambda \right\}.$$

Choose $r = R_0$. This implies

$$\int_{B_\rho(x_0)} |u|^2 \leq c\rho^\lambda \left(\int_\Omega |u|^2 + M \right) \quad \text{for any } \rho \leq R_0.$$

□

Remark.

> In this chapter, the constant c may not be the same since we don't care about constant.

For this purpose, we need the following technical lemma

Lemma 4.2.4

Let $\varphi(t)$ be a nonnegative and nondecreasing function on $[0, R]$. Suppose that

$$\varphi(\rho) \leq A\left[\left(\frac{\rho}{r}\right)^\alpha + \varepsilon\right]\varphi(r) + Br^\beta,$$

for any $0 < \rho \leq r \leq R$, with A, B, α, β nonnegative constants and $\beta < \alpha$. Then for any $\gamma \in (\beta, \alpha)$, there exists a constant $\varepsilon_0 = \varepsilon_0(A, \alpha, \beta, \gamma)$ such that if $\varepsilon < \varepsilon_0$, we have for all $0 < \rho \leq r \leq R$:

$$\varphi(\rho) \leq c\left\{\left(\frac{\rho}{r}\right)^\gamma \varphi(r) + B\rho^\beta\right\},$$

where c is a positive constant depending on A, α, β, γ. In particular, we have for any $0 < r \leq R$:

$$\varphi(r) \leq c\left\{\frac{\varphi(R)}{R^\gamma} r^\gamma + Br^\beta\right\}.$$

Proof. For $\tau \in (0, 1)$ and $r < R$, we have:

$$\varphi(\tau r) \leq A\tau^\alpha \left[1 + \varepsilon\tau^{-\alpha}\right]\varphi(r) + Br^\beta.$$

Choose $\tau < 1$ in such a way that $2A\tau^\alpha = \tau^\gamma$ and assume $\varepsilon_0\tau^{-\alpha} < 1$. Then we get for every $r < R$:

$$\varphi(\tau r) \leq \tau^\gamma \varphi(r) + Br^\beta.$$

Therefore, for all integers $k > 0$:

$$\begin{aligned}
\varphi(\tau^{k+1} r) \leq \tau^\gamma \varphi(\tau^k r) + B\tau^{k\beta} r^\beta &\leq \tau^{(k+1)\gamma}\varphi(r) + B\tau^{k\beta} r^\beta \sum_{j=0}^{k} \tau^{j(\gamma-\beta)} \\
&\leq \tau^{(k+1)\gamma}\varphi(r) + \frac{B\tau^{k\beta} r^\beta}{1 - \tau^{\gamma-\beta}}
\end{aligned}$$

By choosing k such that $\tau^{k+2}r < \rho \le \tau^{k+1}r$, the last inequality gives:

$$\varphi(\rho) \le \frac{1}{\tau^\gamma}\left(\frac{\rho}{r}\right)^\gamma \varphi(r) + \frac{B\rho^\beta}{\tau^{2\beta}(1-\tau^{\gamma-\beta})}.$$

□

4.3 Hölder Continuity of Solutions

In this section we will prove Hölder regularity for solutions. The basic idea is to **freeze** the leading coefficients and then to **compare solutions with harmonic functions**. The regularity of solutions depends on how close solutions are to harmonic functions. Hence we need some regularity assumption on the leading coefficients.

Suppose $a_{ij} \in L^\infty(B_1)$ is **uniformly elliptic** in $B_1 = B_1(0)$, that is,

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \text{for any } x \in B_1,\ \xi \in \mathbb{R}^n.$$

In the following we assume that a_{ij} is at least **continuous**. We assume that $u \in H^1(B_1)$ satisfies

$$\int_{B_1} a_{ij}D_iuD_j\varphi + cu\varphi = \int_{B_1} f\varphi \quad \text{for any } \varphi \in H_0^1(B_1). \tag{*}$$

Before we prove the Hölder estimates for solutions, we first show some necessary lemmas.

Lemma 4.3.1: Basic Estimates for Harmonic Functions

Suppose $\{a_{ij}\}$ is a constant positive definite matrix with

$$\lambda|\xi|^2 \le a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^n$$

for some $0 < \lambda \le \Lambda$. Suppose $w \in H^1(B_r(x_0))$ is a weak solution of

$$a_{ij}D_{ij}w = 0 \quad \text{in } B_r(x_0).$$

Then for any $0 < \rho \le r$, there hold

$$\int_{B_\rho(x_0)} |Dw|^2 \le c\left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |Dw|^2,$$

$$\int_{B_\rho(x_0)} |Dw - (Dw)_{x_0,\rho}|^2 \le c\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |Dw - (Dw)_{x_0,r}|^2,$$

where $c = c(\lambda, \Lambda)$.

Proof. To show this, we may apply another lemma

Lemma 4.3.2

Suppose $\{a_{ij}\}$ is a constant positive definite matrix with

$$\lambda|\xi|^2 \le a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^n$$

for some constants $0 < \lambda \le \Lambda$. Suppose $u \in C^1(B_1)$ satisfies

$$\int_{B_1} a_{ij} D_i u D_j \varphi = 0 \quad \text{for any } \varphi \in C_0^1(B_1).$$

Then for any $0 < \rho \le r$, there hold

$$\int_{B_\rho} |u|^2 \le c\left(\frac{\rho}{r}\right)^n \int_{B_r} |u|^2, \tag{1.9}$$

$$\int_{B_\rho} |u - u_\rho|^2 \le c\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |u - u_r|^2, \tag{1.10}$$

where $c = c(\lambda, \Lambda)$ is a positive constant and u_r denotes the average of u in B_r.

□