Homework 9

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PROBLEM 1. 7-6

For any path-connected space X and any base point $p \in X$, show that the map sending a loop to its circle representative induces a bijection between the set of conjugacy classes of elements of $\pi_1(X, p)$ and $[S^1, X]$ (the set of free homotopy classes of continuous maps from S^1 to X).

SOLUTION.

Define a map

$$\varphi : \pi_1(X, p)/\mathrm{conj} \longrightarrow [S^1, X]$$

by sending the conjugacy class of an element $[g] \in \pi_1(X, p)$ (represented by a loop $g : [0, 1] \to X$ with $g(0) = g(1) = p$) to the free homotopy class $[\tilde{g}]$ of the corresponding map $\tilde{g} : S^1 \to X$, obtained by identifying the endpoints of the loop.

We claim that φ is a bijection. We verify this by showing that φ is well-defined, injective, and surjective.

Well-definedness. Suppose that $[g]$ and $[g']$ are conjugate in $\pi_1(X, p)$, i.e., there exists a loop $h \in \pi_1(X, p)$ such that

$$[g'] = [hgh^{-1}].$$

Then the map $\tilde{g'}$ is homotopic to $\tilde{g}$, since pre- and post-composing with the path h and its inverse results in a loop freely homotopic to g. Thus $[\tilde{g}] = [\tilde{g'}]$, and φ is well-defined on conjugacy classes.

Injectivity. Suppose $\varphi([g]) = \varphi([g'])$, i.e., the maps $\tilde{g}$ and $\tilde{g'}$ are freely homotopic. This means there exists a homotopy

$$H : S^1 \times [0,1] \to X$$

such that $H(s,0) = \tilde{g}(s)$, $H(s,1) = \tilde{g'}(s)$. At each time t, $H(\cdot,t)$ is a loop in X, so the endpoints of g and g' move continuously under the homotopy.

Let $h : [0,1] \to X$ be the path defined by $h(t) = H(0,t) = H(1,t)$. Then h is a path from p to p, and we have

$$g' \simeq hgh^{-1}$$

as loops based at p, which implies that $[g'] = [hgh^{-1}]$ in $\pi_1(X,p)$, i.e., $[g]$ and $[g']$ are conjugate. Thus φ is injective.

Surjectivity. Let $f : S^1 \to X$ be a continuous map. Since X is path-connected, there exists a point $a \in X$ such that $f(1) = a$. Choose a path $\gamma : [0,1] \to X$ from p to a, i.e., $\gamma(0) = p$, $\gamma(1) = a$.

Define a new map $g : S^1 \to X$ by

$$g = \gamma^{-1} \cdot f \cdot \gamma,$$

where the composition denotes the concatenation of the path γ^{-1} with f and then with γ. Then g is a loop based at p, and the map $\tilde{g}$ is freely homotopic to f. Hence $\varphi([g]) = [f]$, and φ is surjective.

Therefore, φ is a well-defined bijection between the set of conjugacy classes of $\pi_1(X,p)$ and the set $[S^1, X]$ of free homotopy classes of maps from S^1 to X.

$\square$

Problem 2. 7-8

Prove that a retract of a Hausdorff space is a closed subset.

Solution.

Let $A \subseteq X$ be a retract of the topological space X, and suppose X is Hausdorff.

Let $r : X \to A$ be a retraction, i.e., a continuous map such that $r(a) = a$ for all $a \in A$. Let $x \in X \setminus A$, and set $a = r(x) \in A$. Since X is Hausdorff and $x \neq a$, there exist disjoint open neighborhoods U of x and V of a. Then consider the open set $r^{-1}(V \cap A) \cap U \subseteq X$. We claim this is an open neighborhood of x disjoint from A.

To see why it is disjoint from A, suppose for contradiction that there exists $y \in A \cap (r^{-1}(V \cap A) \cap U)$. Then:

$$y \in r^{-1}(V \cap A) \cap U \quad \Rightarrow \quad y \in U \text{ and } r(y) \in V \cap A.$$

However, since $y \in A$ and r acts as the identity on A, it follows that $r(y) = y$. Therefore, $y \in U \cap V$, contradicting the fact that $U \cap V = \emptyset$.

Hence, no such $y \in A$ exists in the set $r^{-1}(V \cap A) \cap U$, so this open neighborhood of x is entirely contained in $X \setminus A$. Since such a neighborhood exists for every $x \in X \setminus A$, the complement $X \setminus A$ is open, and thus A is closed.

□

Problem 3. 7-10

Let X and Y be topological spaces. Show that if either X or Y is contractible, then every continuous map from X to Y is homotopic to a constant map.

SOLUTION. We consider two cases.

(1) **Case 1: X is contractible.**

By definition, there exists a point $x_0 \in X$ and a continuous map $H : X \times [0,1] \to X$ such that

$$H(x,0) = x \quad \text{and} \quad H(x,1) = x_0 \quad \text{for all } x \in X.$$

Let $f : X \to Y$ be any continuous map. Define the homotopy $F : X \times [0,1] \to Y$ by

$$F(x,t) := f(H(x,t)).$$

Then:

$$F(x,0) = f(H(x,0)) = f(x), \quad F(x,1) = f(H(x,1)) = f(x_0) \quad \text{for all } x \in X.$$

Hence, $f \simeq c$, where $c(x) := f(x_0)$ is the constant map. Thus, f is homotopic to a constant map.

(2) **Case 2: Y is contractible.**

Then there exists a point $y_0 \in Y$ and a continuous map $G : Y \times [0,1] \to Y$ such that

$$G(y,0) = y \quad \text{and} \quad G(y,1) = y_0 \quad \text{for all } y \in Y.$$

Let $f : X \to Y$ be any continuous map. Define the homotopy $F : X \times [0,1] \to Y$ by

$$F(x,t) := G(f(x),t).$$

Then:

$$F(x,0) = G(f(x),0) = f(x), \quad F(x,1) = G(f(x),1) = y_0 \quad \text{for all } x \in X.$$

Hence, $f \simeq c$, where $c(x) := y_0$ is the constant map. Thus, f is homotopic to a constant map.

□