Homework 2

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Problem 1. **Weierstrass' calssical counterexample from 1870.** Consider the minimum problem

$$F(u) := \int_{-1}^{1} (xu'(x))^2 \mathrm{d}x = \min! \quad u \in C^1[-1,1] \quad u(-1) = 0 \quad u(1) = 1$$

Use the sequence

$$u_n(x) := \frac{1}{2} + \frac{1}{2}\frac{\arctan nx}{\arctan n} \quad n = 1, 2, \cdots$$

in order to show that this variational problem has **no solution**. Recall that $C^1[-1,1]$ denotes the space of continuously differentiable functions $u : [-1,1] \to \mathbb{R}$.

Solution. Set $M \equiv \{u \in C^1[-1,1] : u(-1) = 0 \text{ and } u(1) = 1\}$. Then, the problem is equivalent to

$$F(u) = \min! \quad u \in M$$

Since $u_n(-1) = 0$ and $u_n(1) = 1$, we get $u_n \in M$ for all n. We calculate

$$\begin{aligned} F(u_n) = \int_{-1}^{1} (xu_n'(x))^2 \mathrm{d}x \leq \int_{-1}^{1} (x^2 + \frac{1}{n^2})(u_n'(x))^2 \mathrm{d}x \\ = \frac{1}{4(\arctan n)^2} \int_{-1}^{1} \frac{1+n^2x^2}{n^2} \cdot \frac{n^2}{(1+n^2x^2)^2} \mathrm{d}x \\ = \frac{1}{4(\arctan n)^2} \int_{-n}^{n} \frac{\mathrm{d}y}{1+y^2} \\ = \frac{1}{2n \cdot \arctan n} \end{aligned}$$

Hence, $F(u_n) \to 0$ as $n \to \infty$. Since $F(u) \geq 0$ for all $u \in M$, this implies

$$\mathop{inf}_{u \in M} F(u) = 0$$

Suppose now that u is a solution of the minimum problem. Then

$$F(u) = 0 \quad u \in M$$

and hence

$$xu'(x) = 0 \quad \text{for all } x \in [-1, 1]$$

This implies $u'(x) = 0$ on $[-1, 1]$, i.e., $u(x) = \text{const}$. But this contradicts with the side condition $u(-1) = 0$ and $u(1) = 1$. □

NOTE OF PROBLEM 1. This example was given by Weierstrass to show that a minimum problem in the calculus of variations need not always have a solution, namely.

The infimum of the functional F on the set M is not attained at some point u of M.

PROBLEM 2. **The classical Hilbert space $l_2^{\mathbb{K}}$.** By definition, the space $l_2^{\mathbb{K}}$ consists of all the seqences $(u_n)_{n\geq 1}$ with $u_n \in \mathbb{K}$ for all $n \in \mathbb{N}$

and
$$\sum_{n=1}^{\infty} |u_n|^2 < \infty$$
Show that $l_2^{\mathbb{K}}$ is an infinite-dimensional Hilbert space over $\mathbb{K}$ equipped with the inner product
$$\langle u, v\rangle := \sum_{n=1}^{\infty} \bar{u}_n v_n$$
where $u := (u_n)$ and $v := (v_n)$.

Solution. We have shown $l_2^{\mathbb{K}}$ is a infinite-dimensional linear space over $\mathbb{K}$.

Step 1: Show that $\langle\ ,\ \rangle$ is an inner product.

(1) For any $u \in l_2^{\mathbb{K}}$, we have
$$\langle u, u\rangle = \sum_{n=1}^{\infty} \bar{u}_n u_n = \sum_{n=1}^{\infty} |u_n|^2 \geq 0$$
and $\langle u, u\rangle = 0$ iff $u_n = 0$ for each $n = 1, 2, \cdots$, that is, $u = 0$.

(2) For any $u, v, w \in l_2^{\mathbb{K}}$, and $\alpha, \beta \in \mathbb{K}$, we have
$$\begin{aligned}\langle u, \alpha v + \beta w\rangle &= \sum_{n=1}^{\infty} \bar{u}_n(\alpha v_{n+\beta w_n}) \\ &= \alpha \sum_{n=1}^{\infty} \bar{u}_n v_n + \beta \sum_{n=1}^{\infty} \bar{u}_n w_n \\ &= \alpha\langle u, v\rangle + \beta\langle u, w\rangle\end{aligned}$$

(3) For any $u, v \in l_2^{\mathbb{K}}$
$$\overline{\langle u, v\rangle} = \overline{\sum_{n=1}^{\infty} \bar{u}_n v_n} = \sum_{n=1}^{\infty} \bar{v}_n u_n = \langle v, u\rangle$$

Choose a Cauchy sequence $(u_n^{(k)})$ in $l_2^{\mathbb{K}}$, which means for $\forall \varepsilon > 0$, there exists $N > 0$, such that $\forall k_1, k_2 \geq N$, we have

$$\begin{aligned} \|(u_n^{(k_1)} - u_n^{(k_2)})\| &= \langle u^{k_1} - u^{k_2}, u^{k_1} - u^{k_2} \rangle^{\frac{1}{2}} \\ &= \Big(\sum_{n=1}^{\infty} |u_n^{(k_1)} - u_n^{(k_2)}|^2 \Big)^{\frac{1}{2}} < \varepsilon \end{aligned}$$

Since

$$|u_n^{(k_1)} - u_n^{(k_2)}| < \|(u_n^{(k_1)} - u_n^{(k_2)})\| < \varepsilon \quad \text{for every } n$$

By applying the classical Cauchy convergence theorem, $u_n^{(k)}$ converges to u_n^* and $u^* = (u_n^*)$. It suffices to show that $u^* \in l_2^{\mathbb{K}}$ and $\|u^{(k)} - u^*\| \to 0$. Restricting the summation to $n \leq N$ and letting $k_2 \to \infty$, we obtain

$$\Big(\sum_{n=1}^{N} |u_n^{(k_1)} - u_n^*|^2 \Big)^{\frac{1}{2}} < \varepsilon$$

Letting $N \to \infty$, we get

$$\|u^{(k_1)} - u^*\| < \varepsilon$$

That is, $\|u^{(k)} - u^*\| \to 0$. and

$$\|u^*\| \leq \|u^* - u^{(k)}\| + \|u^{(k)}\| \in l_2^{\mathbb{K}}$$

$\square$

Problem 3. **The Banach space** $C[a, b]$ Let $-\infty < a < b < \infty$. Show that the Banach space $C[a, b]$ equipped with the usual maximum norm

$$\|u\| = \max_{a \leq x \leq b} |u(x)|$$

is *not* a Hilbert space.

SOLUTION. Suppose it is a Hilbert space, then the **parallelogram identity** holds:

$$2\|u\|^2 + 2\|v\|^2 = \|u+v\|^2 + \|u-v\|^2 \quad \forall u, v \in C[a,b]$$

However

$$\begin{aligned}\|u+v\|^2 + \|u-v\|^2 &= \max_{a\le x\le b} |u(x)+v(x)| + \max_{a\le x\le b} |u(x)-v(x)| \\ &\le 2\max_{a\le x\le b} |u(x)| + 2\max_{a\le x\le b} |v(x)| = 2\|u\|^2 + 2\|v\|^2\end{aligned}$$

Thus the parallelogram identity is violated. □

PROBLEM 4. **The Ritz method.** By Section 2.7.1, the variational problem

$$\int_0^\pi \left(2^{-1}u'^2 - u\cos x\right) dx = \min!, \quad u \in C^2[0,\pi], \quad u(0) = u(\pi) = 0 \quad \text{(V)}$$

is equivalent to the boundary-value problem

$$u''(x) + \cos x = 0 \quad \text{on } [0,\pi], \quad u(0) = u(\pi) = 0, \qquad \text{(B)}$$

which has a unique solution u. Explicitly,

$$u(x) = \cos x + 2\pi^{-1}x - 1.$$

Use the Ritz method in order to compute an approximate solution u_{2n} of (V), by making the ansatz

$$u_{2n}(x) = \sum_{k=1}^{2n} c_k \sin kx.$$

Determine the coefficients $c_1, \ldots, c_{2n}$. Show that (u_{2n}) converges uniformly on $[0,\pi]$ to the solution u of (V).

Solution. The Ritz method yields an approximate solution

$$u_{2n}(x) = \sum_{k=1}^{2n} c_k \sin kx$$

where the unknown coefficients c_k are determined by the minimum problem

$$F(c) := \int_0^\pi (2^{-1}u_{2n}'^2 - u_{2n}\cos x)\mathrm{d}x = \min!$$

We compute

$$u_{2n}' = \sum_{k=1}^{2n} kc_k \cos kx$$

and

$$F(c) = \int_0^\pi \big(\frac{1}{2}(\sum_{k=1}^{2n} kc_k \cos kx)^2 - \sum_{k=1}^{2n} c_k \sin kx \cos x\big)\mathrm{d}x$$

To solve the minimum problem, we set derivative with respect to each c_k

$$\frac{\partial F}{\partial c_k} = 0$$

So

$$\begin{aligned}
\frac{\partial F}{\partial c_k} &= \int_0^\pi \big((\sum_{j=1}^{2n} jc_j \cos jx) \cdot k\cos kx - \sin kx \cos x\big)\mathrm{d}x \\
&= \begin{cases} \int_0^\pi k^2 c_k \cos^2 kx\mathrm{d}x - \dfrac{2k}{k^2-1} & k \text{ even} \\ \int_0^\pi k^2 c_k \cos^2 kx\mathrm{d}x - 0 & k \text{ odd} \end{cases} \\
&= \begin{cases} \dfrac{\pi k^2}{2}c_k - \dfrac{2k}{k^2-1} & k \text{ even} \\ \dfrac{\pi k^2}{2}c_k - 0 & k \text{ odd} \end{cases}
\end{aligned}$$

Thus we determine the coefficients c_k

$$c_k = \begin{cases} \dfrac{2}{\pi} \cdot \dfrac{1}{r(4r^2-1)} & k = 2r \\ 0 & k = 2r-1 \end{cases}$$

That is ,

$$u_{2n} = \frac{2}{\pi}\sum_{r=1}^{n}\frac{\sin 2rx}{r(4r^2-1)}$$

As $n \to \infty$, this series converges uniformly on $[0, \pi]$ to the exact solution $u(x) = \cos x - 2\pi^{-1}x - 1$ by the convergence of Ritz method. $\square$

Before moving on to the next problem, we introduce an important *smoothing technique* first.

Smoothing of functions by using mean values (Friedrichs' mollification). The point of departure is the integral

$$u_\varepsilon(x) := \int_{\mathbb{R}^N} \phi_\varepsilon(x-y)u(y)\,dy,$$

where $\phi_\varepsilon(x) := \varepsilon^{-N}\phi(\varepsilon^{-1}x)$ along with

$$\phi(x) := \begin{cases} ce^{-(1-|x|^2)^{-1}} & \text{if } x \in \mathbb{R}^N \text{ and } |x| < 1, \\ 0 & \text{if } x \in \mathbb{R}^N \text{ and } |x| \geq 1. \end{cases}$$

Then

(i) $\phi \in C_0^\infty(\mathbb{R}^N)$.

(ii) $\phi \geq 0$ on $\mathbb{R}^N$.

(iii) $\int_{\mathbb{R}^N} \phi(x)\,dx = 1$ for a suitable choice of the constant $c > 0$.

Hence:

(i*) $\phi_\varepsilon \in C_0^\infty(\mathbb{R}^N)$ and $\phi_\varepsilon(x) = 0$ if $|x| \geq \varepsilon$ for all $\varepsilon > 0$.

(ii*) $\phi_\varepsilon \geq 0$ on $\mathbb{R}^N$ for all $\varepsilon > 0$.

(iii*) $\int_{\mathbb{R}^N} \phi_\varepsilon(x)\,dx = 1$ (see Figure 2.17 for $N = 1$).

Let $u \in L_2(G)$, where G is a nonempty open set in $\mathbb{R}^N$, $N \geq 1$. We set $u(x) = 0$ outside G. Then

(α) $u_\varepsilon \in C^\infty(\mathbb{R}^N)$ for all $\varepsilon > 0$.

(β) $u_\varepsilon \in L_2(G)$ for all $\varepsilon > 0$.

(γ) $u_\varepsilon \to u$ in $L_2(G)$ as $\varepsilon \to +0$.

Now we will use the technique to show the following problems.

PROBLEM 5. **Density** (Proof of Proposition 7 in Section 2.2). Let G be a nonempty open set in $\mathbb{R}^N$, $N \geq 1$.

(a) Show that the set $C^\infty(G)$ is *dense* in $L_2(G)$.

(b) Show that $C_0^\infty(G)$ is *dense* in $L_2(G)$.

(c) Show that $C(\overline{G})$ is *dense* in $L_2(G)$.

SOLUTION.

(a) SOLUTION: **Step 1**: First we show $u_\varepsilon \in C^\infty(G)$. Consider the ball

$$B := \{x \in G \subseteq \mathbb{R}^N : |x - x_0| < 1\}$$

around the given point x_0, and consider the set

$$B_\varepsilon := \{y \in G : \mathrm{dist}(B, y) \leq \varepsilon\}.$$

Since $\phi_\varepsilon(x - y) = 0$ for all points $x, y \in \mathbb{R}^N$ with $|x - y| \geq \varepsilon$

$$u_\varepsilon(x) = \int_{B_\varepsilon} \phi_\varepsilon(x - y) u(y)\, dy \quad \text{for all } x \in B.$$

By the *Schwarz inequality*, we obtain

$$\int_{B_\varepsilon} |u(y)|\, dy = \int_{B_\varepsilon} 1 \cdot |u(y)|\, dy \leq \left(\int_{B_\varepsilon} dy\right)^{\frac{1}{2}} \left(\int_{B_\varepsilon} |u(y)|^2\, dy\right)^{\frac{1}{2}} < \infty,$$

since $\int_{B_\varepsilon} dy = |B_\varepsilon| < \infty$ and $u \in L_2(G)$ implies $u \in L_2(B_\varepsilon)$. Thus, $u \in L(B_\varepsilon)$.

First let $N = 1$. For all $x \in B$, $y \in B_\varepsilon$, and $k = 0, 1, 2, \ldots$, $\varepsilon > 0$, we obtain

$$\left|\phi_\varepsilon^{(k)}(x-y)u(y)\right| \leq \text{const}(k, \varepsilon)|u(y)|, \tag{114}$$

where $\phi_\varepsilon^{(k)}$ denotes the k-th derivative. In this connection, note that the function $\phi_\varepsilon^{(k)}$ is continuous on $\mathbb{R}$, and hence it is bounded on compact sets by the Weierstrass theorem. In particular, $\phi_\varepsilon^{(k)}$ is bounded on each ball.

Applying standard theorems on parameter integrals, the continuous derivative $u_\varepsilon^{(k)}$ exists on B, where

$$u_\varepsilon^{(k)}(x) := \int_{B_\varepsilon} \phi_\varepsilon^{(k)}(x-y)u(y)\, dy \quad \text{for all } x \in B, \quad k = 0, 1, \ldots.$$

Since the center x_0 of the ball B is arbitrary, this implies $u_\varepsilon \in C^\infty(G)$.

Step 2: We show $u_\varepsilon \to u$ in $L_2(G)$ as $\varepsilon \to +0$.

Let $B := \{z \in \mathbb{R}^N : |z| < 1\}$. Recall that $\phi = 0$ outside and $\int_B \phi(z)\, dz = 1$. Set $z = \varepsilon^{-1}(x-y)$, we have

$$u_\varepsilon(x) = \int_B u(x - \varepsilon z)\phi(z)\, dz,$$

and hence

$$u_\varepsilon(x) - u(x) = \int_B (u(x-\varepsilon z) - u(x))\phi(z)\, dz.$$

The *Schwarz inequality* yields

$$\begin{aligned} |u_\varepsilon(x) - u(x)|^2 &= |\int_B (u(x-\varepsilon z) - u(x))\phi(z)\, dz|^2 \\ &\leq C \int_B |u(x-\varepsilon z) - u(x)|^2\, dz \end{aligned}$$

where C is a positive constant. By the *p-mean continuity* of the Lebesgue integral with $p = 2$, for each η, there is an $\varepsilon_0 > 0$ such that

$$\int_G |u(x - \varepsilon z) - u(x)|^2 \, dx < \eta$$

for all $z \in B$ and all $\varepsilon : 0 < \varepsilon \leq \varepsilon_0$. Thus, it follows from the *Fubini–Tonelli theorem* that

$$\begin{aligned}
\int_G |u_\varepsilon(x) - u(x)|^2 \, dx &\leq C \int_G \left(\int_B |u(x - \varepsilon z) - u(x)|^2 \, dz \right) dx \\
&= C \left(\int_B \int_G |u(x - \varepsilon z) - u(x)|^2 \, dx \right) dz \\
&\leq C \, |B| \cdot \eta
\end{aligned}$$

for all $\varepsilon : 0 < \varepsilon \leq \varepsilon_0$. Hence

$$\int_G |u_\varepsilon(x) - u(x)|^2 \, dx \to 0 \quad \text{as } \varepsilon \to +0.$$

This is $u_\varepsilon \to u$ in $L_2(G)$ as $\varepsilon \to +0$.

Therefore, $C^\infty(G)$ is dense in $L_2(G)$.

(b) **Case A**: The nonempty open set G is bounded. Let C be a compact set with $C \subset G$, and let $u \in L_2(G)$. We set

$$v(x) := \begin{cases} u(x) & \text{on } C, \\ 0 & \text{on } G - C. \end{cases}$$

Then

$$\int_G |u - v|^2 \, dx = \int_{G-C} |u|^2 \, dx.$$

By the *absolute continuity* of the integral, the right-hand integral is arbitrarily small provided the measure of the set $G - C$ is sufficiently small. Thus, for each given η, we can choose the set C in such a way that

$$\|u - v\| = \left(\int_G |u - v|^2 \, dx \right)^{\frac{1}{2}} < \eta.$$

By smoothing technique , there is a function $v_\varepsilon \in C^\infty(\mathbb{R}^N)$ such that

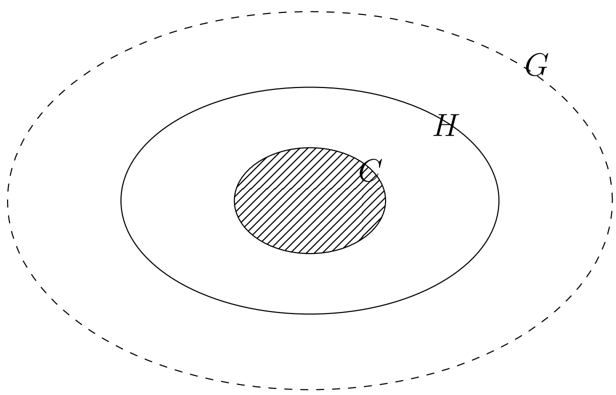
$$\|v - v_\varepsilon\| < \eta \quad \text{for all } \varepsilon : 0 < \varepsilon \le \varepsilon_0.$$

Next, let us show that $v_\varepsilon \in C_0^\infty(G)$ for sufficiently small ε. In fact, since $v = 0$ on $G - C$

$$v_\varepsilon(x) = \int_C \phi_\varepsilon(x - y) v(y)\, dy.$$

Hence $v_\varepsilon(x) = 0$ for all $x \in G$ with $\mathrm{dist}(x, C) > \varepsilon$ because $\phi_\varepsilon(x - y) = 0$ for $|x - y| \ge \varepsilon$. Since C is a compact subset of the open set G, there is an open set H such that

$$C \subset H \subset \overline{H} \subset G$$



Consequently, if we choose the number ε sufficiently small, then $\mathrm{dist}(x, C) > \varepsilon$ for all $x \in G - \overline{H}$, and hence

$$v_\varepsilon(x) = 0 \quad \text{for all } x \in G - \overline{H},$$

i.e., $v_\varepsilon \in C_0^\infty(G)$. Summarizing,

$$\|u - v_\varepsilon\| \le \|u - v\| + \|v - v_\varepsilon\| < 2\eta,$$

i.e., $C_0^\infty(G)$ is dense in $L_2(G)$.

Case B: The open set G is unbounded. Then, for each $\eta > 0$, there is an open ball B such that

$$\int_{G-H} |u|^2\, dx < \eta^2,$$

where $H := G \cap B$ and $H \neq \emptyset$.

Applying **Case A** to the nonempty *bounded* open set H, there is a function $v_\varepsilon \in C_0^\infty(H)$, and hence $v_\varepsilon \in C_0^\infty(G)$, such that

$$\int_H |u - v_\varepsilon|^2\, dx < \eta^2.$$

Since $v_\varepsilon = 0$ on $G - H$, we get

$$\|u - v_\varepsilon\|^2 = \int_{G-H} |u|^2\, dx + \int_H |u - v_\varepsilon|^2\, dx < \eta^2,$$

i.e., $C_0^\infty(G)$ is dense in $L_2(G)$.

(c) Since $C_0^\infty(G) \subseteq C(\overline{G})$, it follows directly from (b).

$\square$

PROBLEM 6. **Separability** (Proof of Corollary 8 in Section 2.2).

(a) Let $G = [a, b]$ be a bounded open interval in $\mathbb{R}$. Show that $L_2(G)$ is *separable*.

(b) Let G be an *unbounded* open interval in $\mathbb{R}$, e.g., $G = \mathbb{R}$. Show that $L_2(G)$ is *separable*.

SOLUTION.

(a) Let $u \in L_2(G)$ and $\varepsilon > 0$ be given. Since$C(\overline{G})$ is dense in $L_2(G)$ for any nonempty open set $G \in \mathbb{R}^{\mathbb{N}}$, the set $C[a,b]$ is dense in $L_2(G)$, i.e., there is a function $v \in C[a,b]$ such that

$$\|u - v\| = \left(\int_a^b |u - v|^2 \, dx \right)^{\frac{1}{2}} < \varepsilon.$$

By the *Weierstrass approximation theorem*, the set of polynomials with real coefficients is dense in the Banach space $C[a,b]$, i.e., there is a real polynomial p such that

$$\|v - p\|_* := \max_{a \le x \le b} |v(x) - p(x)| < \varepsilon.$$

Let us introduce

$$\mathcal{M} := \text{set of all polynomials with rational coefficients.}$$

For each real number a_j and each $\varepsilon > 0$, there is a rational number r_j such that

$$|a_j - r_j| < \varepsilon$$

Thus for each polynomial p, there is a polynomial $q \in \mathcal{M}$ such that

$$\|p - q\|_* < \sum_{j=0}^{n} |a_j - r_j| (\max_{a \le x \le b} |x|)^j \le \text{const} \cdot \varepsilon.$$

Hence $\|v - q\|_* \le \|v - p\|_* + \|p - q\|_* < 2\varepsilon$. This implies

$$\|v - q\| = \left(\int_a^b |v - q|^2 \, dx \right)^{\frac{1}{2}} \le (b-a)^{\frac{1}{2}} \|v - q\|_* < (b-a)^{\frac{1}{2}} 2\varepsilon.$$

Summarizing, for each $\varepsilon > 0$, there is a $q \in \mathcal{M}$ such that

$$\|u - q\| \le \|u - v\| + \|v - q\| < \varepsilon + (b-a)^{\frac{1}{2}} 2\varepsilon.$$

That is, the set $\mathcal{M}$ is dense in $L_2(G)$. Since the set $\mathcal{M}$ is countable, the space $L_2(G)$ is *separable.*

(b) There exists a sequence (G_n) of *bounded* open intervals in G such that $G_1 \subseteq G_2 \subseteq \cdots \subseteq G$ and

$$G = \bigcup_{n=1}^{\infty} G_n.$$

Define

$$\chi_n(x) := \begin{cases} 1 & \text{if } x \in G_n, \\ 0 & \text{if } x \in \mathbb{R} - G_n, \end{cases}$$

and

$$\mathcal{M}_\infty := \{\chi_n q : q \in \mathcal{M} \text{ and } n = 1, 2, \dots\}.$$

Where

$$\chi_n q := \chi_n(x) \cdot q(x) = \begin{cases} q(x) & \text{if } x \in G_n \\ 0 & \text{if } x \in \mathbb{R} - G_n \end{cases}$$

Since $\mathcal{M}$ is countable, it suffices to show $\mathcal{M}$ is dense in $L_2(G)$. Let $u \in L_2(G)$ and $\varepsilon > 0$ be given. There exists a *bounded* interval J with $J \subseteq G$ and

$$\int_{G-J} |u|^2 \, dx < \varepsilon^2,$$

by a well-known property of the Lebesgue integral. Choose some interval G_n such that $J \subseteq G_n \subseteq G$. Then

$$\int_{G-G_n} |u|^2 \, dx \leq \int_{G-J} |u|^2 \, dx < \varepsilon^2.$$

By (a), for any bounded set G_n, there is a polynomial $q \in \mathcal{M}$ such that

$$\int_{G_n} |u - q|^2 \, dx < \varepsilon^2.$$

Hence

$$\|u - \chi_n q\|^2 = \int_{G-G_n} |u|^2 \, dx + \int_{G_n} |u - q|^2 \, dx < 2\varepsilon^2.$$

Consequently, the *countable* set $\mathcal{M}_\infty$ is dense in $L_2(G)$, i.e., $L_2(G)$ is *separable.*

□