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Algorithms and Data Structures

Lecture 4 Asymptotic notations

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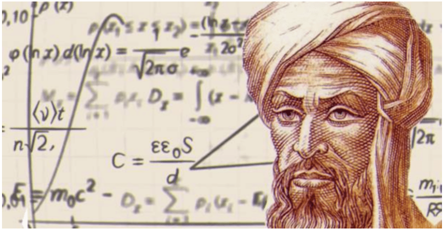


Comparing Algorithms by Running Time

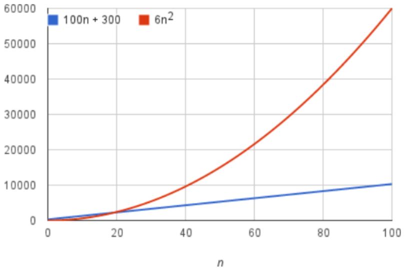
What we have learned from the previous lectures:

- The running time of the algorithm over an input is the number of elementary operations execute by the algorithm.
- The growth rate of the running time refers to how fast the running time grows with the input size.
- We only care about the situation when the input is very big.
- We can “speed-up” an algorithm by running it on a faster computer by a constant factor.

Question. Say we have two algorithms A, B that solve the same problem:

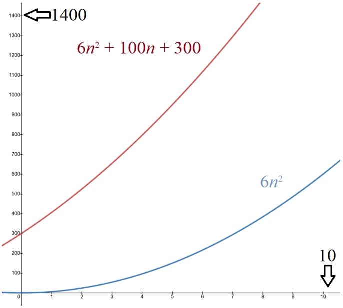
- A has running time $6n^2 + 100n + 300$
- B has running time $100n + 300$

What can we say about their running time?



How do we compare real-valued functions? We need a framework that "unifies" them.

Step 1. Compare the curves: $f_1(n) = 6n^2 + 100n + 300$ and $f_2(n) = 6n^2$:



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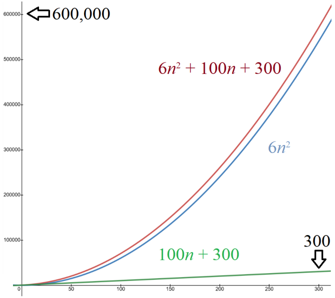

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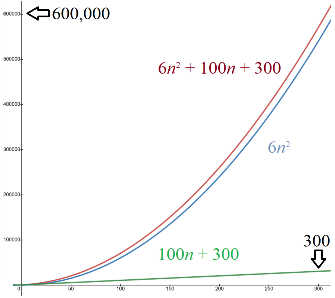
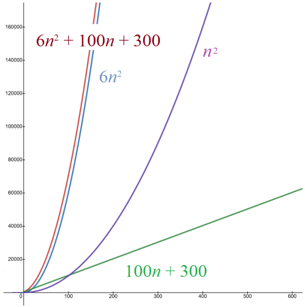


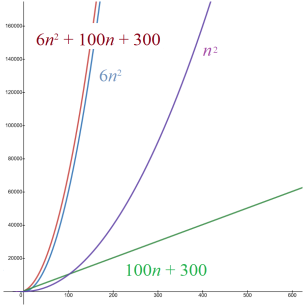
- The $6n^2$ term becomes larger than the remaining terms $100n + 300$ when n is sufficiently large.
- As n grows, $100n + 300$ becomes insignificant compared to $6n^2$, and $6n^2$ increasingly resembles $6n^2 + 100n + 300$.
- We can thus say that $6n^2$ dominates $100n + 300$.

Rule 1. When expressing the growth of running time of an algorithm, keep only the dominant term.

Step 2. Compare the curves: $f_2(n) = 6n^2$, $f_3(n) = 100n + 300$, and $f_4(n) = n^2$



Step 2. Compare the curves: $f_2(n) = 6n^2$, $f_3(n) = 100n + 300$, and $f_4(n) = n^2$



- The dominance of $6n^2$ over $100n + 300$ is not caused by the leading coefficient 6.
- The function n^2 also dominates $100n + 300$.

Rule 2. When expressing the growth of running time of an algorithm, drop the constant coefficient.

Dominance Relation

To express the growth rate of running time of algorithm, we essentially "categorise" them into "classes".

Example.

- $\log n$, $3\log_5 n$, $\log_3 n^2$, $10\lg n + 100$ can all be seen as "$\log n$".
- n, $10n$,$n+3$, $100n + 20000$ can all be seen as "n".
- n^2, $0.5n^2$, $n^2 + 100n + 3$ can all be seen as "n^2".
- n^3, $0.1n^3 + 5n^2 + 100n + 20000$ can be seen as "n^3".

The dominance relation on these classes form a chain:

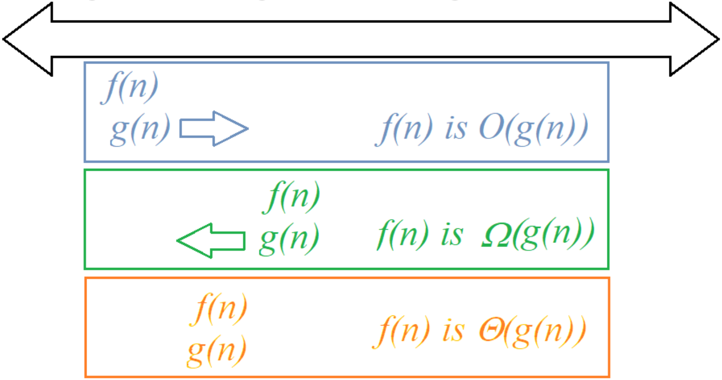
$$1 \prec \log n \prec (\log n)^2 \prec n \prec n\log n \prec n^2 \prec n^2\log n \prec n^3 \prec \cdots \prec 2^n \prec n!$$

where $f \prec g$ denotes that f is dominated by g.

Asymptotic Notation

Asymptotic notations such as big-O, big-Omega, and big-Theta are used to compare the limiting behaviours of functions:

$$1 < \log n < n < n \log n < n^2 < n^2 \log n < n^3 < \cdots < 2^n < n!$$



Suppose f and g are functions from $\mathbb{N}$ to $\mathbb{R}$, which take on non-negative values.

Definition [O]

- We say that $f(n)$ is $O(g(n))$ if there is some $c > 0$ and some $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have $f(n) \leq cg(n)$.
- In this case, we also say $g(n)$ is an asymptotic upper bound for $f(n)$. Loosely, $f(n)$ is $O(g(n))$ iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} < \infty$$

Examples.

- $5n - 2$ is $O(n)$.

  Why? $f(n) = 5n - 2$, and $g(n) = n$.

  Take $c = 5$ and $n_0 = 1$, then for any $n > n_0$, $5n - 2 \leq cn$.
- $5n - 2$ is also $O(n^2)$.
- $4n^2$ is NOT $O(n)$.

Definition [Ω]

- We say that $f(n)$ is $\Omega(g(n))$ if there are $c > 0$ and $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have $f(n) \geq cg(n)$.
- In this case, we also say that $g(n)$ is an asymptotic lower bound for $f(n)$. Loosely, $f(n)$ is $\Omega(g(n))$ iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$$

Note. f is $\Omega(g)$ if and only if g is $O(f)$.

Examples.

- $2n + 2$ is $\Omega(10n + 20)$.

  Why? $f(n) = 2n + 2$, and $g(n) = 10n + 20$.

  Take $c = 6$ and $n_0 = 4$. Then for any $n > n_0$,
  $6f(n) = 12n + 12 \geq 10n + 20 = g(n)$.
- $3n^2$ is $\Omega(n)$.

Definition [Θ]

- We say that $f(n)$ is $\Theta(g(n))$ if $f(n)$ is $O(g(n))$ and $g(n)$ is $O(f(n))$.
- In this case, we also say that $f(n)$ grows at the same rate as $g(n)$. Loosely, $f(n)$ is $\Omega(g(n))$ iff

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} \text{ is a positive real number}$$

Examples.

- $5n - 2$ is $\Theta(16n + 33)$.
- $\log_{10} n$ is $\Theta(\log n)$.
- $2n^2 + 3n$ is $\Theta(n^2)$.

Note.

- Θ-notation captures our categorisation of growth functions above. If f is $\Theta(g)$ then they belong to the same class.
- We sometimes write "$f \in O(g)$", "$f \in \Omega(g)$", "$f \in \Theta(g)$" with the same meaning as "f is $O(g)$", "f is $\Omega(g)$", "f is $\Theta(g)$", respectively.

Example. Every linear function $f(n) = an + b$, $a > 0$, is $O(n)$.

Proof. Take a linear function $f(n) = an + b$ where $a > 0$.
For any $n \geq 1$, we have

$$an + b \leq an + |b| \leq (a + |b|)n$$

So if we set $c = a + |b|$ and $n_0 = 1$, the condition in the definition holds.
Thus $an + b$ is $O(n)$. $\square$

Alternative proof.

$$\lim_{n \to \infty} \frac{an + b}{n} = a$$

Thus by definition $f(n)$ is $O(n)$. $\square$

Example. If $f(n) = n$, $g(n) = n^2/2$, then f is $O(g)$ and g is not $O(f)$.

Proof.

- Note that $f(n) \leq 2g(n)$ for any $n \geq 0$. Thus $f(n)$ is $O(g(n))$.
- Conversely, we prove that $g(n)$ is not $O(f(n))$ by contradiction.

  Suppose, for a contradiction, that eventually $n^2 \leq Cn$ for some constant C.

  Then $n \leq C$ for all sufficiently large n.

  This is a contradiction. Thus $g(n)$ is not $O(f(n))$.

$\square$

Alternative proof.

$$\lim_{n\to\infty} \frac{n}{n^2/2} = 0$$

So n is $O(n^2/2)$ but is not $\Omega(n^2/2)$.

Example. Show that $n \lg n$ is $O(2^{-10}n^2)$.

Proof. Set $c = 2^{-10}$ and $n_0 = 2$.
Then for any $n > n_0$, we have

$$2^{-10} n \lg n \leq 2^{-10} n^2.$$

Thus $n \lg n$ is $O(2^{-10}n^2)$. □

Alternative proof.

$$\lim_{n\to\infty} \frac{n \lg n}{2^{-10}n^2} = \lim_{n\to\infty} \frac{\lg n}{2^{-10}n} = 0 \qquad \text{(apply L'Hopital Rule)}$$

Thus by definition $n \lg n$ is $O(2^{-10}n^2)$. □

- **Reflexivity**:
  $f(n)$ is $O(f(n))$, $\Omega(f(n))$ and $\Theta(f(n))$
- **Symmetry**:
  $f(n)$ is $\Theta(g(n)) \Leftrightarrow g(n)$ is $\Theta(f(n))$
- **Transitivity**:
  $f(n)$ is $O(g(n))$ and $g(n)$ is $O(h(n)) \Rightarrow f(n)$ is $O(h(n))$.
  $f(n)$ is $\Omega(g(n))$ and $g(n)$ is $\Omega(h(n)) \Rightarrow f(n)$ is $\Omega(h(n))$.
  $f(n)$ is $\Theta(g(n))$ and $g(n)$ is $\Theta(h(n)) \Rightarrow f(n)$ is $\Theta(h(n))$.
- **Sum rule**: If f_1 is $O(g_1)$ and f_2 is $O(g_2)$, then $f_1 + f_2$ is $O(\max\{g_1, g_2\})$.
- **Product rule**: If f_1 is $O(g_1)$ and f_2 is $O(g_2)$ then $f_1 f_2$ is $O(g_1 g_2)$.

More Examples

- $\log_a(n)$ is $\Theta(\lg n)$ for each $a > 1$.
  Why? $\log_a(n) = \lg n / \lg a$.

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- $\log_a(n)$ is $\Theta(\lg n)$ for each $a > 1$.
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- $n \lg n$ is $O(n^2)$ and $n \lg n$ is not $\Omega(n^2)$.
  Why?

$$\lim_{n\to\infty} \frac{n \lg n}{n^2} = 0$$

More Examples

- $\log_a(n)$ is $\Theta(\lg n)$ for each $a > 1$.
  Why? $\log_a(n) = \lg n / \lg a$.
- $n \lg n$ is $O(n^2)$ and $n \lg n$ is not $\Omega(n^2)$.
  Why?
  $$\lim_{n\to\infty} \frac{n \lg n}{n^2} = 0$$
- 2^n is $\Omega(n^{100})$.
  Why?
  $$\lim_{n\to\infty} \frac{2^n}{n^{100}} = \infty$$

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- $\log_a(n)$ is $\Theta(\lg n)$ for each $a > 1$.
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- 2^n is $\Omega(n^{100})$.
  Why?

$$\lim_{n\to\infty} \frac{2^n}{n^{100}} = \infty$$

- $10^{-100}n^2 + 10^{100}n$ is $\Omega(n)$ but NOT $O(n)$.
  Why?

$$\lim_{n\to\infty} \frac{10^{-100}n^2 + 10^{100}n}{n} = \infty$$

More Examples

- $\log_a(n)$ is $\Theta(\lg n)$ for each $a > 1$.
  Why? $\log_a(n) = \lg n / \lg a$.
- $n \lg n$ is $O(n^2)$ and $n \lg n$ is not $\Omega(n^2)$.
  Why?
  $$\lim_{n\to\infty} \frac{n \lg n}{n^2} = 0$$
- 2^n is $\Omega(n^{100})$.
  Why?
  $$\lim_{n\to\infty} \frac{2^n}{n^{100}} = \infty$$
- $10^{-100}n^2 + 10^{100}n$ is $\Omega(n)$ but NOT $O(n)$.
  Why?
  $$\lim_{n\to\infty} \frac{10^{-100}n^2 + 10^{100}n}{n} = \infty$$
- $1 + (-1)^n$ is $O(1)$ but not $\Omega(1)$.
  Why?
  For any $n \geq 0$, $1 + (-1)^n \leq 1$. So $1 + (-1)^n$ is $O(1)$.
  Conversely, $1 + (-1)^n = 0$ for any odd $n \in \mathbb{N}$. Thus no matter what c is $c(1 + (-1)^n) = 0 < 1$ for infinitely many n. Thus $1 + (-1)^n$ is not $\Omega(1)$.

Summary

Here is a list of the main points covered in this lecture

- When comparing the running time of algorithms, the focus is to compare the growth rate of the function of running time with respect to input size.
- When expressing the growth rate of running time, we apply two rules:
  - **Rule 1.** Keep only the dominant term
  - **Rule 2.** Drop the constant coefficient
- One can categorise the growth rate functions into a number of classes, which form a chain under the dominance relation:

$$1 \prec \log n \prec (\log n)^2 \prec n \prec n \log n \prec n^2 \prec n^2 \log n \prec n^3 \prec \cdots \prec 2^n \prec n!$$

- Asymptotic notations are used to compare the limiting behaviours of functions.
  - f is $O(g)$: g is an asymptotic upper bound for f
  - f is $\Omega(g)$: g is an asymptotic lower bound for f
  - f is $\Theta(g)$: g and f have the same growth rate.
- Properties of asymptotic notations


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Math Time

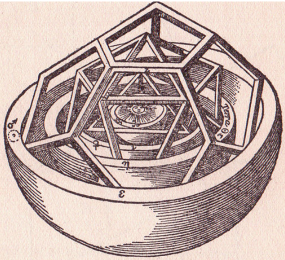

Fig. 38 —Inner Portion of Kepler's Cosmographicum. (See Fig. 37.)

Evaluating a Sum

Suppose $\varphi\colon \mathbb{N} \to \mathbb{R}$ is a function defined on the natural numbers,

$$\varphi(0), \varphi(1), \varphi(2), \varphi(3), \varphi(4), \ldots$$

where each $\varphi(i) \geq 0$.

How do we analyse the limiting behaviours of the sum

$$\sum_{i=0}^{n} \varphi(i) = \varphi(0) + \varphi(1) + \varphi(2) + \cdots + \varphi(n)?$$

Example. Say $\varphi(n) = \sqrt{n} + 1$, and we want to evaluate the limiting behaviour of

$$\sum_{i=0}^{n} \varphi(i) = \sqrt{0} + 1 + \sqrt{1} + 1 + \sqrt{2} + 1 + \cdots \sqrt{n} + 1$$

Integral Approximation

Idea: Approximate the upper bound and lower bound of $\sum_{i=0}^{n} \varphi(i)$ using integration.

Example. Say $\varphi(n) = \sqrt{n} + 1$.

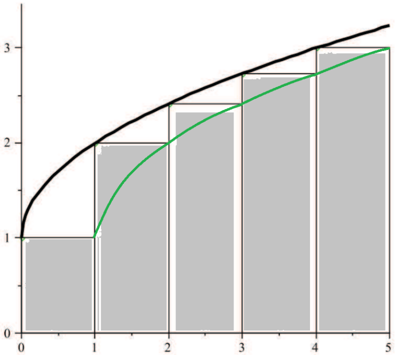

Figure: $\sum_{i=0}^{4}(\sqrt{i} + 1)$

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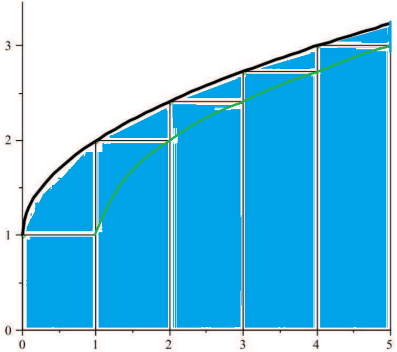

Figure: (Upper bound) $\sum_{i=0}^{4}(\sqrt{i} + 1) \leq \int_0^5 \sqrt{x} + 1 dx$

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Idea: Approximate the upper bound and lower bound of $\sum_{i=0}^{n} \varphi(i)$ using integration.

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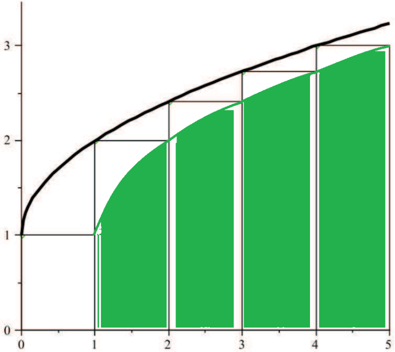

Figure: (Lower bound) $\sum_{i=0}^{4}(\sqrt{i}+1) \geq \int_0^4 \sqrt{x}+1dx$

Example. Say $\varphi(n) = \sqrt{n} + 1$.

$$\int_0^n \sqrt{x} + 1dx \leq \sum_{i=0}^{n}(\sqrt{i} + 1) \leq \int_0^{n+1} \sqrt{x} + 1dx$$

We know that

$$\int \sqrt{x} + 1dx = \frac{2}{3}x^{3/2} + x + c$$

and thus

$$\int_0^n \sqrt{x} + 1dx = \frac{2}{3}n^{3/2} + n \leq \sum_{i=0}^{n}(\sqrt{i} + 1) \leq \frac{2}{3}(n+1)^{3/2} + n$$

So $\sum_{i=0}^{n}(\sqrt{i} + 1)$ is $\Theta(n^{3/2})$.

Example. Analyse the limiting behaviour of the sum

$$\sum_{i=1}^{n} 1/i$$

Using integral approximation, we have

$$\int_{1}^{n} 1/x dx \leq \sum_{i=1}^{n} 1/i \leq \int_{0}^{n+1} 1/x dx$$

Since $\int 1/x dx = \ln x + c$, we have

$$\ln n \leq \sum_{i=1}^{n} 1/i \leq \ln(n+1) dx$$

Thus $\sum_{i=1}^{n} 1/i$ is $\Theta(\log n)$.

Note. The sum $1/1 + 1/2 + 1/3 + \cdots + 1/n$ is called the nth harmonic number and is denoted by H_n. Our analysis shows H_n is $\Theta(\log n)$.

Example. Analyse the limiting behaviour of the function $\ln n!$.

- $\ln n! = \ln(1 \times 2 \times \cdots \times n) = \ln 1 + \ln 2 + \cdots + \ln n$.
- It is easy to see that $\ln n! \leq n \ln n$, so $\ln n!$ is $O(n \ln n)$.
- Apply integral approximation, we get

$$\int_1^n \ln x dx \leq \ln n!$$

- We note $\int \ln x dx = x \ln x - x + c$. Thus

$$\int_1^n \ln x dx = n \ln n - n + 1$$

- This means that

$$\ln n! \geq n \ln n - n + 1$$

We now need to show that $n \ln n - n + 1$ is $\Omega(n \ln n)$.

Continue from the example above.

- Now let $h(x) = x \ln x/(x \ln x - x)$. We know that after $x = 3$ the function h is strictly decreasing.
- Let $c = h(3) = 3 \ln 3/(3 \ln 3 - 3)$ and $n_0 = 3$.
- For any $n > n_0$, $c \geq h(n) = n \ln n/(n \ln n - n)$.
- Then $c(n \ln n - n) \geq n \ln n$.
- Thus $c(n \ln n - n + 1) \geq n \ln n$.

This means that $\ln n!$ is $\Omega(n \ln n)$. In conclusion we have

$$\ln n! \text{ is } \Theta(n \ln n).$$


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