Quadratic modeling for "Directed Feedback Node Set" problem

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For any integer t less than given k, we are able to eliminate t numbers of nodes in our directed graph to derive another directed graph without cycle. So we need to declare bunch of variables. Here is an array of acceptable ts, we name it k:

$$k = [1, 2, 3, ..., k]$$

And x_i represents our nodes. if $x_i = 0$ that node has been selected to be deleted. and otherwise $x_i = 1$ is a servivor node. There is a matrice E for primary graph representing edges. for every $e_{i,j}$ in matrice E if $e_{i,j} = 1$ it means node i is directed to node j.

The function below makes charges for eliminated node number beeing more than k:

$$k - \sum_{i=1}^{n} 1 - x_i \tag{1}$$

And for every cycle of length L we have the cost functions represented as shown:

$$L = 3 \tag{2}$$

$$\sum_{l=1}^{n}\sum_{m=1}^{n}\sum_{p=1}^{n} e_{l,m}e_{m,p}e_{l,p}(x_l * x_m + x_m * x_p + x_p * x_l) \tag{3}$$

$$L = 4 \tag{4}$$

$$\sum_{l=1}^{n}\sum_{m=1}^{n}\sum_{p=1}^{n}\sum_{q=1}^{n} e_{l,m}e_{m,p}e_{p,q}e_{q,l}(x_l * x_m + x_m * x_p + x_p * x_q + x_q * x_l) \tag{5}$$

$$\vdots \tag{6}$$

$$L = n - k \tag{7}$$

$$\sum_{l=1}^{n}\sum_{m=1}^{n}\cdots\sum_{p=1}^{n} e_{l,m}\cdots e_{p,l}(x_l * x_m + \cdots + x_p * x_l) \tag{8}$$

Applying coefficients the total cost function will be like:

$$A(k-\sum_{i=1}^{n}(1-x_i)) \tag{9}$$

$$+B(\sum_{l=1}^{n}\sum_{m=1}^{n}\sum_{p=1}^{n}e_{l,m}e_{m,p}e_{l,p}(x_l*x_m+x_m*x_p+x_p*x_l)) \tag{10}$$

$$\vdots \tag{11}$$

$$+F(\sum_{l=1}^{n}\sum_{m=1}^{n}\cdots\sum_{p=1}^{n}e_{l,m}\cdots e_{p,l}(x_l*x_m+\cdots+x_p*x_l)) \tag{12}$$

$$\tag{13}$$

Which the condition $A > B > \cdots > F$ must be considered. because it might charge the cost for cycles existing in primary graph but there are not in the new one (but it is less than the cost of same length that are actually present but it might be more than the cycles of smaller length), so because it is less possible of greater number of nodes to make an actuall cycle (because it is more likely to be a deleted node among them) we can evaluate the latter coefficients less than former coefficients. This way the cost of abscent cycles of greater length won't be greater than cost of present cycles of smaller length.