Problem 1 (a)

(i)

$P(\underline{x}|s_i) = N(\underline{x}, m_i, \underline{\Sigma}_i)$, $i=1,2,3$

so, $g_i(x) = -\frac{1}{2}\ln|\underline{\Sigma}_i| - \frac{1}{2} d_m^2(\underline{x}, \underline{m_i}) + \ln \lambda_i \qquad i=1,2,3$

(ii)

$g_i(\underline{x}) = -\frac{1}{2}\ln|\underline{\Sigma}_i| - \frac{1}{2}(\underline{x}-\underline{m_i})^T \underline{\Sigma}_i^{-1}(\underline{x}-\underline{m_i}) + \ln P(s_i)$

$g_i(\underline{x}) = -\frac{1}{2}\ln|\underline{\Sigma}_i| - \frac{1}{2}\underline{x}^T \underline{\Sigma}_i^{-1}\underline{x} + \underline{m_i}^T \underline{\Sigma}_i^{-1}\underline{x} - \frac{1}{2}\underline{m_i}^T\underline{\Sigma}_i^{-1}\underline{m_i} + \ln P(s_i)$

so, $\left\{\begin{array}{ll} g_i(\underline{x}) = \underline{x}^T \underline{W}_i \underline{x} + \underline{w}_i^T \underline{x} + w_0^{(i)} & \\ \underline{w}_i = (m_i^T \underline{\Sigma}_i^{-1})^T & i=1.2.3 \\ w_0 = -\frac{1}{2}\underline{m_i}^T \underline{\Sigma}_i^{-1}\underline{m_i} + \ln\lambda_i - \frac{1}{2}\ln|\underline{\Sigma}_i| & i=1.2.3 \\ \underline{W}_i = -\frac{1}{2}\underline{\Sigma}_i^{-1} & i=1.2.3 \end{array}\right.$

(b) $g_k(\underline{x}) > g_j(\underline{x}) \quad \forall j \neq k \Rightarrow x \in S_k$

it is quadratic, because it is quadratic function of $\underline{x}$

(c) (i) because $\underline{\Sigma}_1 = \underline{\Sigma}_2 = \underline{\Sigma}_3 = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

so, $g_i(\underline{x}) = \underline{m_i}^T \underline{\Sigma}^{-1}\underline{x} - \frac{1}{2}\underline{m_i}^T \underline{\Sigma}^{-1}\underline{m_i} + \ln P(s_i)$

$\underline{\Sigma}^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

so, $g_1(\underline{x}) = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\underline{x} - \frac{1}{2}\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \ln\lambda_1$

$$g_2(\underline{x}) = \begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\underline{x} - \frac{1}{2}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1\\ -1\end{bmatrix} + \ln \pi_2$$

$$g_3(\underline{x}) = \begin{bmatrix}-2 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\underline{x} - \frac{1}{2}\begin{bmatrix}-2 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}-2\\ 2\end{bmatrix} + \ln \pi_3$$

so, $\begin{cases} g_1(\underline{x}) = 4x_1 + 3x_2 - 5 + \ln \pi_1 \\ g_2(\underline{x}) = x_1 - \frac{1}{2} + \ln \pi_2 \\ g_3(\underline{x}) = -2x_1 - 2 + \ln \pi_3 \end{cases}$

classifiers are linear

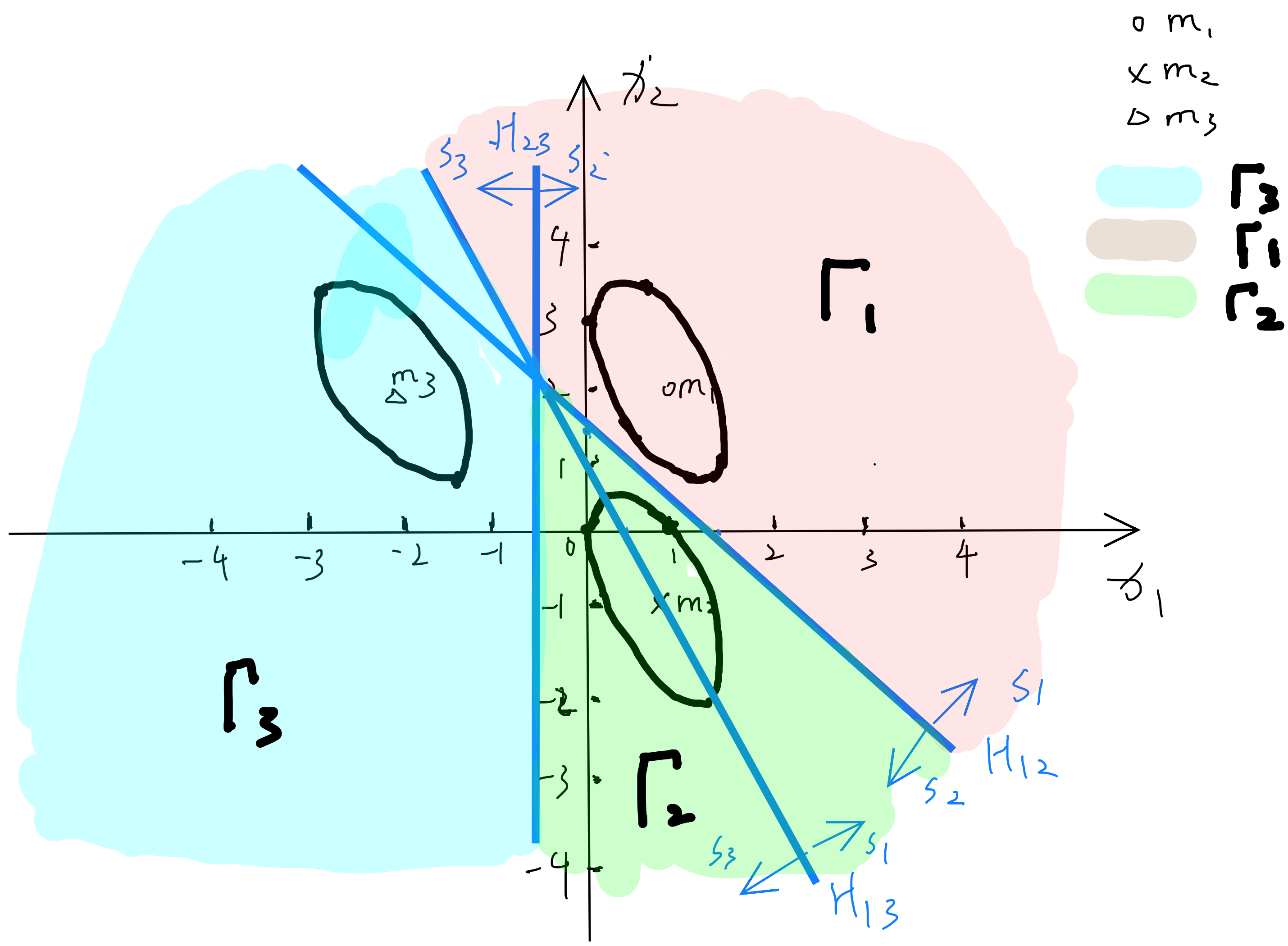
(ii) $g_i(\underline{x}) = -(\underline{x}-m_i)^T \underline{\underline{\Sigma}}_i^{-1}(\underline{x}-m_i) = -d_m^2(\underline{x}, \underline{m_i})$

$\begin{cases} g_1(\underline{x}) = 4x_1 + 3x_2 - 5 \\ g_2(\underline{x}) = x_1 - \frac{1}{2} \\ g_3(\underline{x}) = -2x_1 - 2 \end{cases}$

so $\begin{cases} H_{12}: & x_1 + x_2 - \frac{3}{2} = 0 \\ H_{23}: & x_1 + \frac{1}{2} = 0 \\ H_{13}: & 2x_1 + x_2 - 1 = 0 \end{cases}$

(iii)

$\begin{cases} d_m^2(\underline{x}, m_1) = 1 \Rightarrow & 2x_1^2 + x_2^2 + 2x_1x_2 - 8x_1 - 6x_2 + 9 = 0 \\ d_m^2(\underline{x}, m_2) = 1 \Rightarrow & 2x_1^2 + 2x_1x_2 + x_2^2 - 2x_1 = 0 \\ d_m^2(\underline{x}, m_3) = 1 \Rightarrow & 2x_1^2 + 2x_1x_2 + x_2^2 + 4x_1 + 3 = 0 \end{cases}$


o m1
x m2
▷ m3
Γ3
Γ1
Γ2
y2
y1
H23
S3
S2
H12
S1
S2
H13
S1
S3
Γ1
Γ2
Γ3
om1
xm2
m3

Problem 2

(a) $g_i(x) = -\frac{1}{2}\ln|\Sigma_i| - \frac{1}{2}(x-m_i)^T\Sigma_i^{-1}(x-m_i) + \ln P(s_i)$

$$g_i(x) = -\frac{1}{2}\ln|\Sigma_i| - \frac{1}{2}x^T\Sigma_i^{-1}x + m_i^T\Sigma_i^{-1}x - \frac{1}{2}m_i^T\Sigma_i^{-1}m_i + \ln\alpha_i$$

$$g_i(x) = -\frac{1}{2}\ln|\Sigma_i| - \frac{1}{2}[x_1\ x_2]\begin{bmatrix}\left(\frac{1}{\sigma_1(i)^2}\right) & 0\\ 0 & \left(\frac{1}{\sigma_2(i)^2}\right)\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} +$$

$$[m_1^{(i)}\ m_2^{(i)}]\begin{bmatrix}\left(\frac{1}{\sigma_1(i)^2}\right) & 0\\ 0 & \left(\frac{1}{\sigma_2(i)^2}\right)\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} - \frac{1}{2}[m_1^{(i)}\ m_2^{(i)}]\begin{bmatrix}\left(\frac{1}{\sigma_1(i)^2}\right) & 0\\ 0 & \left(\frac{1}{\sigma_2(i)^2}\right)\end{bmatrix}\begin{bmatrix}m_1^{(i)}\\m_2^{(i)}\end{bmatrix}$$

$$+\ln P(s_i)$$

$$g_i(x) = -\frac{1}{2}\ln(\sigma_1(i))^2(\sigma_2(i))^2 - \frac{1}{2}x_1^2\left(\frac{1}{\sigma_1(i)^2}\right) - \frac{1}{2}x_2^2\left(\frac{1}{(\sigma_2(i))^2}\right) + x_1 m_1(i)\left(\frac{1}{\sigma_1(i)^2}\right) + x_2 m_2(i)\left(\frac{1}{\sigma_2(i)^2}\right)$$

$$-\frac{1}{2}(m_1(i))^2\left(\frac{1}{(\sigma_1(i))^2}\right) - \frac{1}{2}(m_2(i))^2\left(\frac{1}{(\sigma_2(i))^2}\right) + \ln\alpha_i$$

$w_{11}(i) = -\frac{1}{2}\left(\frac{1}{\sigma_1(i)^2}\right)$

$w_{12}(i) = 0$

$w_{22}(i) = -\frac{1}{2}\left(\frac{1}{\sigma_2(i)^2}\right)$

$w_1(i) = m_1(i)\left(\frac{1}{\sigma_1(i)^2}\right)$

$w_2(i) = m_2(i)\left(\frac{1}{\sigma_2(i)^2}\right)$

$w_0(i) = -\frac{1}{2}(m_1(i))^2\frac{1}{\sigma_1(i)^2} - \frac{1}{2}(m_2(i))^2\frac{1}{\sigma_2(i)^2} - \frac{1}{2}\ln(\sigma_1(i))^2(\sigma_2(i))^2 + \ln\alpha_i$

(b) it is quadratic

(c) (i) $\underline{\underline{\Sigma}}_1 = \underline{\underline{\Sigma}}_2 = \underline{\underline{\Sigma}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

so, $g_i(\underline{x}) = \underline{m_i}^T \underline{\underline{\Sigma}}^{-1} \underline{x} - \frac{1}{2} \underline{m_i}^T \underline{\underline{\Sigma}}^{-1} \underline{m_i} + \ln P(s_i)$

$\Sigma^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

so, $g_1(\underline{x}) = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \underline{x} - \frac{1}{2} \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \ln \pi_1$

$g_2(\underline{x}) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \underline{x} - \frac{1}{2} \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \ln \pi_2$

$g_3(\underline{x}) = \begin{bmatrix} -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \underline{x} - \frac{1}{2} \begin{bmatrix} -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} + \ln \pi_3$

so $\begin{cases} g_1(\underline{x}) = x_1 + x_2 - \frac{3}{2} + \ln \pi_1 \\ g_2(\underline{x}) = x_1 - \frac{1}{2} x_2 - \frac{3}{4} + \ln \pi_2 \\ g_3(\underline{x}) = -2x_1 + x_2 - 3 + \ln \pi_3 \end{cases}$

it is linear

(ii) $g_i(\underline{x}) = -(\underline{x} - m_i)^T \underline{\underline{\Sigma}}_i^{-1} (\underline{x} - m_i) = -d_m^2(\underline{x}, \underline{m_i})$

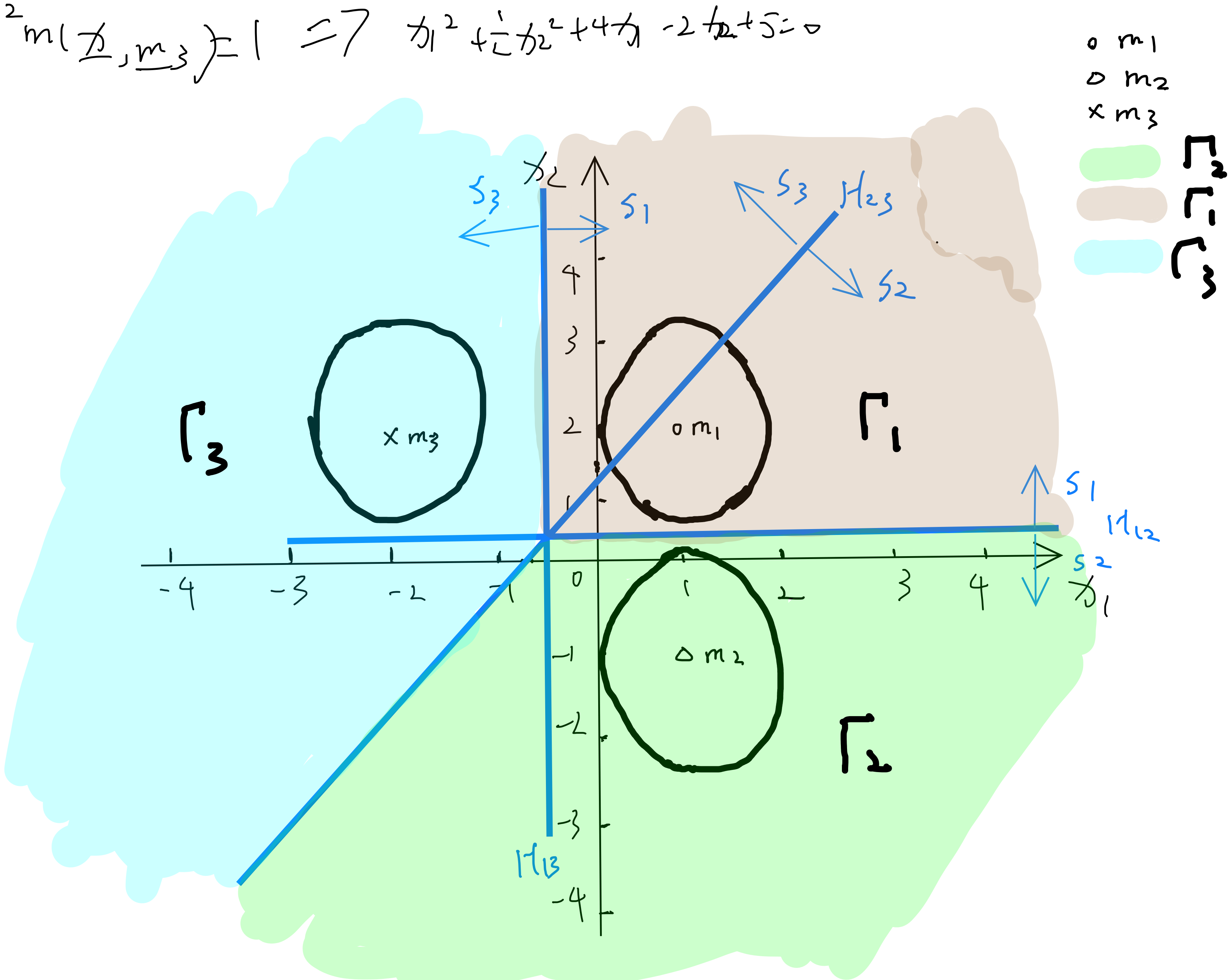
$\begin{cases} g_1(\underline{x}) = x_1 + x_2 - \frac{3}{2} \\ g_2(\underline{x}) = x_1 - \frac{1}{2} x_2 - \frac{3}{4} \\ g_3(\underline{x}) = -2x_1 + x_2 - 3 \end{cases}$

so, $\begin{cases} H_{12}: \; x_2 - \frac{1}{2} = 0 \\ H_{23}: \; x_1 - \frac{1}{2} x_2 + \frac{3}{4} = 0 \\ H_{13}: \; x_1 + \frac{1}{2} = 0 \end{cases}$

(iii) $d_m^2(\underline{x}, \underline{m}_1) = 1 \Rightarrow x_1^2 + \frac{1}{2}x_2^2 - 2x_1 - 2x_2 + 2 = 0$

$d_m^2(\underline{x}, \underline{m}_2) = 1 \Rightarrow x_1^2 + \frac{1}{2}x_2^2 - 2x_1 + x_2 + \frac{1}{2} = 0$

$d_m^2(\underline{x}, \underline{m}_3) = 1 \Rightarrow x_1^2 + \frac{1}{2}x_2^2 + 4x_1 - 2x_2 + 5 = 0$



(d) compare the plot 2(c)(iii) with plot of 1(c)(iii)

it make a substantial difference in the decision boundaries and regions. because we use Σ_i and it will change according to the data

(3) (a)

$$\left.\begin{aligned} R(\alpha_1|\underline{x}) &= \lambda_{11}P(s_1|\underline{x}) + \lambda_{12}P(s_2|\underline{x}) \\ R(\alpha_2|\underline{x}) &= \lambda_{21}P(s_1|\underline{x}) + \lambda_{22}P(s_2|\underline{x}) \end{aligned}\right\} \Rightarrow R(\alpha_1|\underline{x}) \underset{s_1}{\overset{s_2}{\gtrless}} R(\alpha_2|\underline{x})$$

So, we have $\lambda_{11}P(s_1|\underline{x}) + \lambda_{12}P(s_2|\underline{x}) \underset{s_1}{\overset{s_2}{\gtrless}} \lambda_{21}P(s_1|\underline{x}) + \lambda_{22}P(s_2|\underline{x})$

$$\Rightarrow \frac{P(\underline{x}|s_1)}{P(\underline{x}|s_2)} \underset{s_2}{\overset{s_1}{\gtrless}} \frac{\lambda_{12}-\lambda_{22}}{\lambda_{21}-\lambda_{11}} \frac{P(s_2)}{P(s_1)}$$

because $P(\underline{x}|s_i) = N(\underline{x}, \underline{m}_i, \underline{\Sigma}_i)$

so, we have

$$\frac{|\underline{\Sigma}_1|^{\frac{1}{2}}}{|\underline{\Sigma}_2|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}\left[(\underline{x}-m_1)^T\underline{\Sigma}_1^{-1}(\underline{x}-m_1) - (\underline{x}-m_2)^T\underline{\Sigma}_2(\underline{x}-m_2)\right]\right\} \underset{s_2}{\overset{s_1}{\gtrless}} \frac{\lambda_{12}-\lambda_{22}}{\lambda_{21}-\lambda_{11}} \frac{P(s_2)}{P(s_1)}$$

because $d_m^2(\underline{x}, \underline{m}) = (\underline{x}-\underline{m})^T\underline{\Sigma}^{-1}(\underline{x}-\underline{m})$

so, finally,

$$\frac{|\underline{\Sigma}_1|^{\frac{1}{2}}}{|\underline{\Sigma}_2|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}\left(dm^2(\underline{x}, \underline{m}_1) - dm^2(\underline{x}, m_2)\right)\right\} \underset{s_2}{\overset{s_1}{\gtrless}} \frac{\lambda_{12}-\lambda_{22}}{\lambda_{21}-\lambda_{11}} \frac{P(s_2)}{P(s_1)}$$

(b) $P(s_1) = 0.8$
$P(s_2) = 0.2$

(c) $\Sigma^{-1} = \begin{pmatrix} \frac{8}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} \end{pmatrix}$

$dm^2(\underline{x}, \underline{m}_1) = (\underline{x}-\underline{m}_1)^T \Sigma^{-1}(\underline{x}-\underline{m}_1)$

$\Rightarrow dm^2(\underline{x}, m_1) = \frac{8}{3}x_1^2 + \frac{2}{3}x_2^2 + \frac{4}{3}x_1x_2 - \frac{32}{3}x_1 - \frac{20}{3}x_2 + \frac{56}{3}$

$$\Rightarrow d_m^2(x, m_2) = \frac{8}{3}x_1^2 + \frac{2}{3}x_2^2 + \frac{4}{3}x_1 x_2 - 24x_1 - 8x_2 + 56$$

so, we can write

$$\exp\left\{-\frac{1}{2}\left(d_m^2(x,m_1) - d_m^2(x,m_2)\right)\right\} \underset{s_2}{\overset{s_1}{\gtrless}} \frac{\lambda_{12}-\lambda_{22}}{\lambda_{21}-\lambda_{11}} \frac{P(s_2)}{P(s_1)}$$

$$\exp\left\{-\frac{1}{2}\left(\frac{40}{3}x_1 + \frac{4}{3}x_2 - \frac{112}{3}\right)\right\} \underset{s_2}{\overset{s_1}{\gtrless}} \frac{10}{1}\cdot\frac{1}{4}$$

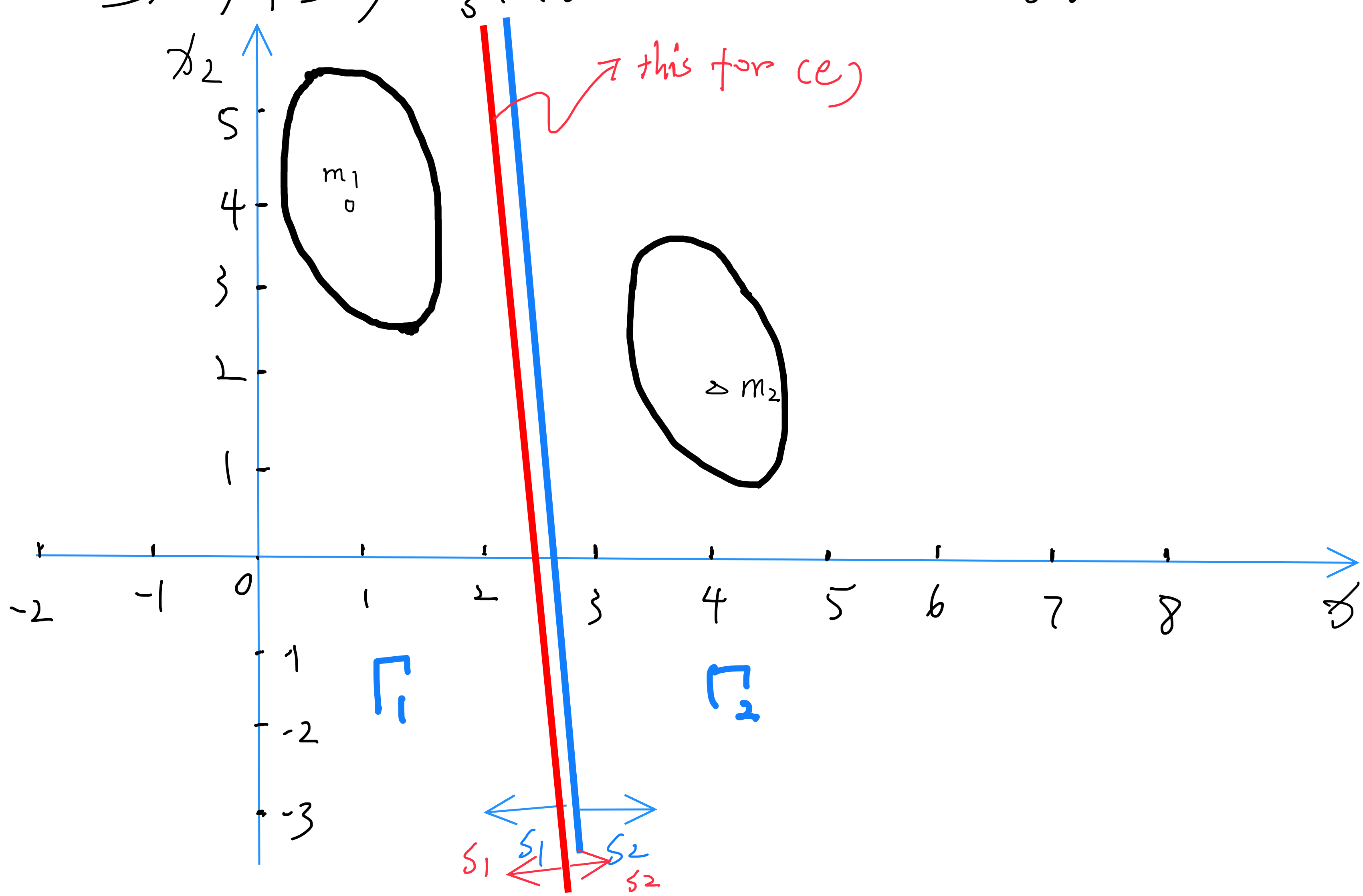
$$-\frac{20}{3}x_1 - \frac{2}{3}x_2 + \frac{56}{3} \underset{s_2}{\overset{s_1}{\gtrless}} \ln\frac{5}{2}$$

so, finally $-20x_1 - 2x_2 + 56 - 3\ln\frac{5}{2} \underset{s_2}{\overset{s_1}{\gtrless}} 0$

(d)

$$d_m^2(x,m_1)=1 \Rightarrow \frac{8}{3}x_1^2 + \frac{2}{3}x_2^2 + \frac{4}{3}x_1 x_2 - \frac{32}{3}x_1 - \frac{20}{3}x_2 + \frac{53}{3} = 0$$

$$d_m^2(x,m_2)=1 \Rightarrow \frac{8}{3}x_1^2 + \frac{2}{3}x_2^2 + \frac{4}{3}x_1 x_2 - 24x_1 - 8x_2 + 55 = 0$$



(e)

$$-\frac{20}{3}x_1 - \frac{2}{3}x_2 + \frac{56}{3} \underset{S_2}{\overset{S_1}{\gtrless}} \ln 10$$

the boundary like this: $-20x_1 - 2x_2 + 56 - 3\ln 10 \underset{S_2}{\overset{S_1}{\gtrless}} 0$

as plot in red in (d), the boundary will move a little to the left