

Lecture 13: Z Scores

z scores

Z scores are great. Z scores allow us to avoid the hassle of memorizing algebra rules. At the end of lecture 12 “Normal Distribution,” we took 4 different scores and placed them on a single distribution so we could compare them. That was a lot of work. Thankfully we have a formula that allows us to quickly compare scores with different means and SD’s...

As long as we have a mean and standard deviation, z scores allow us to compare individual scores on a single distribution where the mean = 0 and each standard deviation is expressed as +1, -1, etc. A z score of +1 = 1 SD above the mean. A z score of -1 = 1 SD below the mean. A z score of 0 = the score is right on the mean.

The Standard Normal Distribution (SND) or “normal curve”

Recall from the readings, lecture 12 “normal distributions,” that the standard normal distribution is the same as the normal curve.

The standard normal distribution is special type of normal distribution for a continuous random variable Z with a common dimension. This is the distribution upon which our z-scores convert to, if you will.

In the standard normal distribution

- mean=median=mode
- sd=1 and mean =0

In the SND we also have a formula to convert any “raw score” to a z score. The most common z score formula is for population data below but we have more

$$z = \frac{x - \mu}{\sigma}$$

Z score formulas

Population data (most common)

$$z = \frac{x - \mu}{\sigma}$$

x = the raw score or the “test score”

μ = population mean

σ = population standard deviation

Test Scores from Lecture 12

Area	test score x	Mean μ	SD σ	
management	90	90	5	
service	75	80	10	
computers	60	50	5	
writing	81	76	10	

We did a whole lot of algebra at the end of lecture 12 (Normal Curve). And figured out that the person's score were best in computers, followed by writing, management, and service

1. computers
2. writing
3. management
4. customer service

But we can use the z formula to avoid all that algebra. Simply plug the information in to the z formula.

z-scores for all 4 tests

Below I plug the information for all 4 tests into the z formula:

$$z = \frac{x - \mu}{\sigma}$$

management

score (x)=	90	z=	$\frac{x - \text{mean}}{\text{sd}}$	$\frac{90 - 90}{5}$	=	$\frac{0}{5}$	=	0
mean=	90							
sd=	5							

service

score (x)=	75	z=	$\frac{x - \text{mean}}{\text{sd}}$	$\frac{75 - 80}{10}$	=	$\frac{-5}{10}$	=	-0.5
mean=	80							
sd=	10							

computers

score (x)=	60	z=	$\frac{x - \text{mean}}{\text{sd}}$	$\frac{60 - 50}{5}$	=	$\frac{10}{5}$	=	2
mean=	50							
sd=	5							

writing

score (x)=	81	z=	$\frac{x - \text{mean}}{\text{sd}}$	$\frac{81 - 76}{10}$	=	$\frac{5}{10}$	=	0.5
mean=	76							
sd=	10							

So to summarize those calculations:

$$z(\text{management}) = 0$$

$$z(\text{service}) = -0.5$$

$$z(\text{computers}) = 2$$

$$z(\text{writing}) = 0.5$$

What do these z-scores tell us?

These z –scores allow us to compare all 4 tests on “the z distribution” where the mean =0 and each unit of standard deviation =1. Converting those 4 tests to z scores allows us to skip all that math from the end of lecture 12! And we can do the same thing: compare all the test scores (when the mean and standard deviation is the same).

The sign (+ or -) is important! Here is what those z scores mean in plain English.

z (management) = 0 this z score says the test score was right on the mean (look at the first table in this lecture. Recall their score was 90 and the mean =90).

z(service) = -0.5 this z score says that the test score is one half of one standard deviation BELOW the mean. Recall this z score is NEGATIVE. That is why it is BELOW the mean.

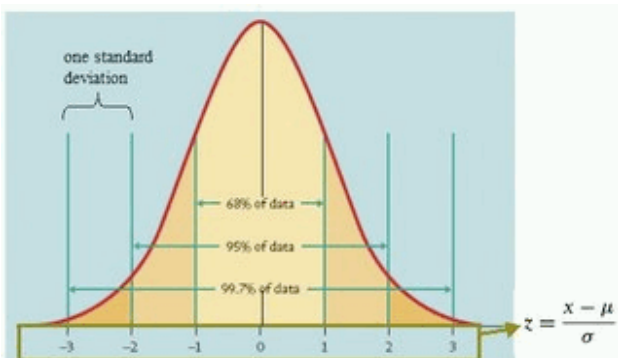
z(computers)= 2 this z score says that the test score is 2 standard deviation ABOVE the mean. Recall this z score is POSITIVE. That is why it is ABOVE the mean.

z(writing)= 0.5 this z score says that the test score is one half of one standard deviation ABOVE the mean. Recall this z score is POSITIVE. That is why it is ABOVE the mean.

Another explanation of z scores.

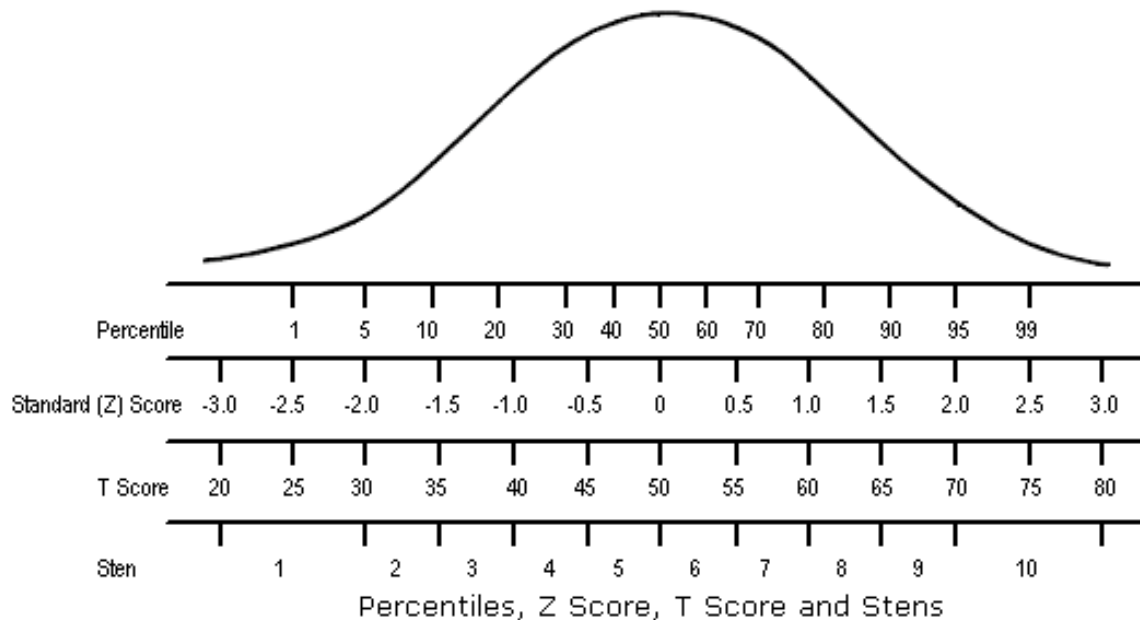
Here is another way to understand z scores. A z of 1 corresponds to one SD. A z of 0.5 corresponds to $\frac{1}{2}$ of a SD, a z of 0.1 corresponds to $\frac{1}{10}$ of a SD and so on and so on. Negative z scores indicate that the item is “below” the mean and positive z scores indicate it is above the mean. A z score of 0 indicates that the score is right on the mean.

So, a z score of +1 = 1 SD above the mean. A z score of -1 = 1 SD below the mean. A z score of 0 = the score is right on the mean. See the book for the best pictures, but one is included below.



source: <http://www.sci.sdsu.edu/class/psychology/psy271/Weeks/psy271week06.htm>

Look at the picture below. Notice how the z-score and “raw score” can be on the same dimension. In the picture below the z-score of 0 is the same as the mean for the t-score line of 50. Thus pretend the t-score line is the “x” or “raw score” dimension where the mean = 50 and a SD=10. On the z dimension the mean =0 and every standard deviation is expressed as a unit of one. That is why the z-score of +1 (1 SD above the mean) is equivalent to a raw score (t-score) of 60. In the raw score dimension the mean is 50 and every standard deviation is 10. So the mean +1 SD = 50+10=60.



source: www.psychometric-success.com/images/PT0502.gif

Practice computing a few z-scores

Here is some data to practice doing a few z scores.

$$X=50$$

$$\mu=50$$

$$\sigma=10$$

$$z = \frac{x - \mu}{\sigma} = \frac{50 - 50}{10} = \frac{0}{10} = 0 \text{ (see how z-score of 0 falls right on the mean?)}$$

$$X=60$$

$$\mu=50$$

$$\sigma= 10$$

$$z = \frac{x - \mu}{\sigma} = \frac{60 - 50}{10} = \frac{10}{10} = 1 \quad (\text{see how z-score of +1 is 1 SD above the mean?})$$

$$X=40$$

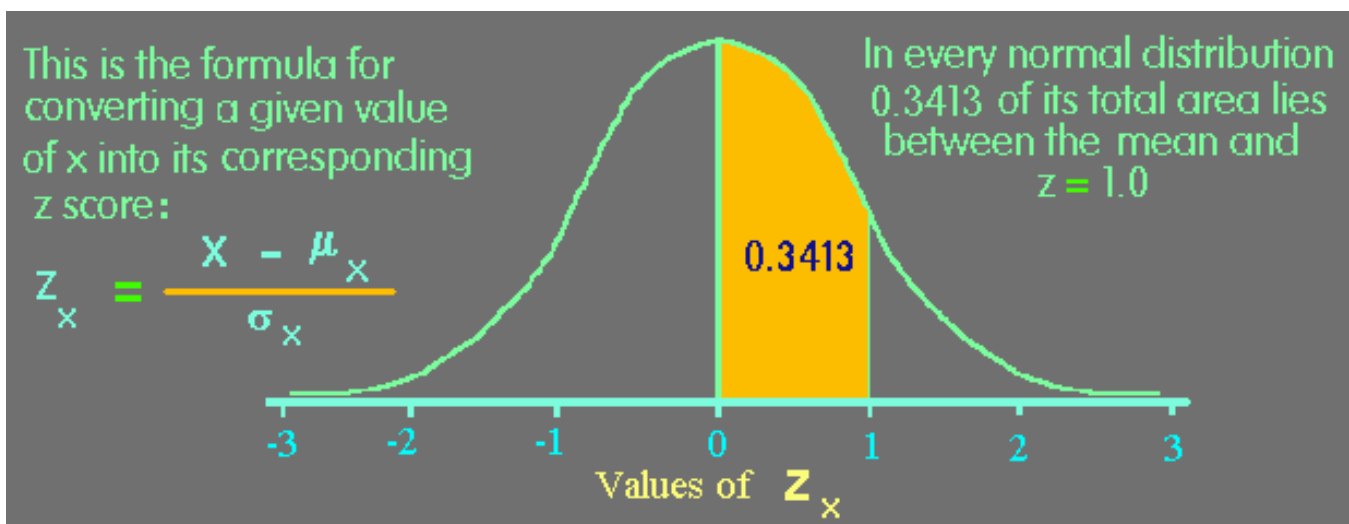
$$\mu=50$$

$$\sigma= 10$$

$$z = \frac{x - \mu}{\sigma} = \frac{40 - 50}{10} = \frac{-10}{10} = -1 \quad (\text{see how z-score of -1 is 1 SD below the mean?})$$

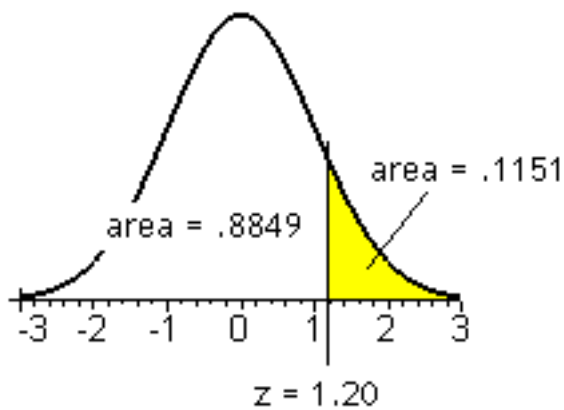
z scores and probability

Once you have a z score you can look it up in a table and find out the area between the z score and the mean under the standard normal distribution curve (normal curve). Look at our book. Some z score tables give you the area under the curve between the mean and the z-score. In the picture below .3413 or 34.13% of the whole curve lies between the mean and a z score of +1.



source: <http://genome.imim.es/~eeyras/courses/MasterUPF/SplicingRegulation/>

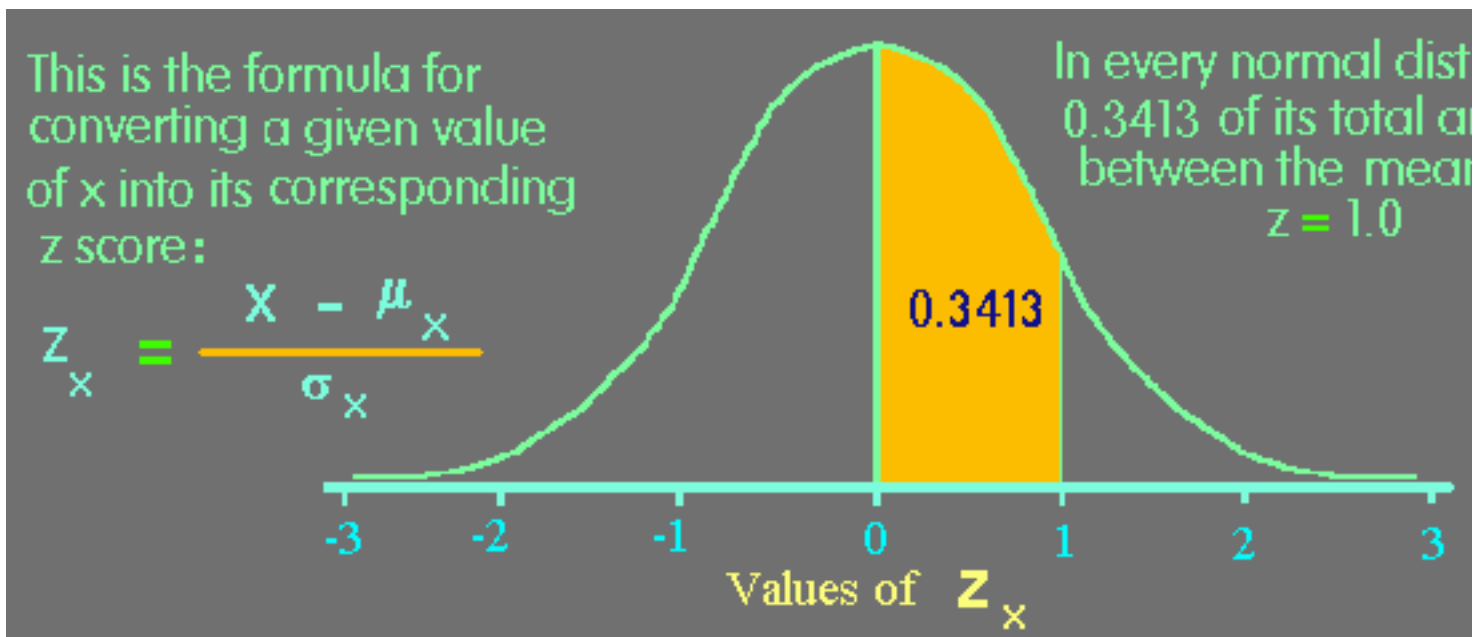
Some z tables give the area under the curve from the z-score out towards the tails of the curve. In the picture below, the area from a score of 1.2 out to the tails of the curve is 0.1151 or 11.51% of the whole area under the curve.



Area under the normal curve is also probability!

As you can see in the picture below, the area between $z=+1$ and the mean ($z=0$) = 0.3413 or 34.13%. That means that 34.13% of the curve lies in that area. We can use the area under the normal curve to make probability statements.

Pretend there is a test where the scores are “normally distributed.” For example, there is a 34.13% chance that a score picked at random will fall between the mean [$z=0$] and 1 sd above the mean [$z=+1$].



normal curve is symmetrical so negative z scores the same as positive z scores

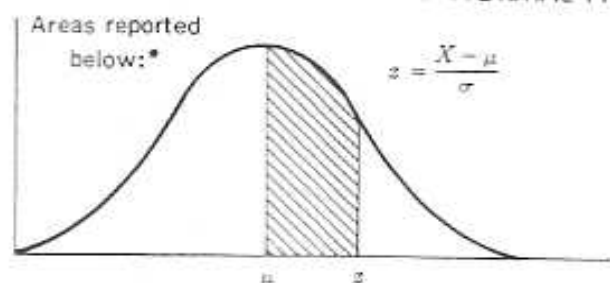
I do not have a picture of a negative z-score, but because the normal curve is symmetrical, the area between a the mean and $z=+1$ is the same area as a that between the mean and $z=-1$. The area between mean and $z= -1$ is also 34.13%.

So the opposite of the above test example is also true. Pretend there is a test where the scores are “normally distributed.” There is a 34.13% chance that a score picked at random will fall between the mean [$z=0$] and 1 sd BELOW the mean [$z=+1$].

FOR INPERSON CLASS (online can ignore but you will become familiar with these z-scores as you do practice problems in this class:
two very important z scores are 95 and 99%
95% $z=1.96 = .4750$ draw
99% $z=2.575 = .4950$ draw
(note problem of 3 digits)

Z table

AREAS UNDER THE STANDARD NORMAL PROBABILITY DISTRIBUTION



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549
0.7	.2580	.2612	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4014
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4983	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4989	.4989	.4989	.4989	.4990	.4990
3.5	.4997									
4.0	.4999683									

* Example: For $z = 1.96$, the shaded area is 0.4750 out of the total area of 1.0000.

Practice

Everything that appears in these lecture notes are fair game for the test. They are the best “study guide” I can provide. It is impossible to provide a “list” that is more comprehensive than the lecture notes above. There are practice problems here in this lecture and additional practice problems in the [lecture 13b: practice problems \(13b_practice.pdf\)](#) associated with this lecture.

z-table and z scores mini exercise

The purpose of this exercise is to

1. learn how to read the z-table. In *Statistics: A First Course* by Sanders and Smidt the z-table is inside the front cover (and in Appendix 2) and is called “Areas Under the Standard Normal Probability Distribution.” Here is a link to a [z table](#) and there is also a link in our course schedule. You could also google the term “z table” and come up with a ton of them.

2. and to do various exercises using that table.

When you are done with this exercise you should be able to:

- locate a z-score on the table and tell how much area (**or probability!**) is between that score and the mean.
- tell how much area is between two z-scores -- or variations on that theme.
- compute a z-score for a value given that value, its mean, and it’s standard deviation.

Reading the Table

This should be pretty straight forward if you use a little common sense. The z-scores are given by increments of 0.1 (or tenths) down the left hand vertical column and for increments of 0.01 (or hundredths) across the top horizontal row. Use the various combinations of tenths and hundreds to find the exact z-score you are looking for. For example, for a z-score of .2 the area between that score and the mean is .0793. For a z-score of .23 the area between that score and the curve is .0910. Here is a portion of the table:

z	.00	.01	.02	.03
0.0	.0000	.0040	.0080	.0120
0.1	.0398	.0438	.0478	.0517
0.2	.0793	.0832	.0871	.0910

Find the z score for .4500 or 45%

As you will see the area of .4500 or 45% will be used a lot in this course. But you can't find it in a z-table that only gives you two digits (100ths). The area for .4500 is somewhere between the z-scores of 1.64 and 1.65. See part of the table below:

z	.00	.01	.02	.03	.04	.05
1.6	.4452	.4463	.4474	.4484	.4495	.4505

What do you do? You take a mean of the two scores. I will show you the math below, but the easy way is to add a 0.005 to the end of the smaller z-score: $1.64 + 0.005 = 1.645$. So “add a five” to the end of the smallest z-score. Here is the math

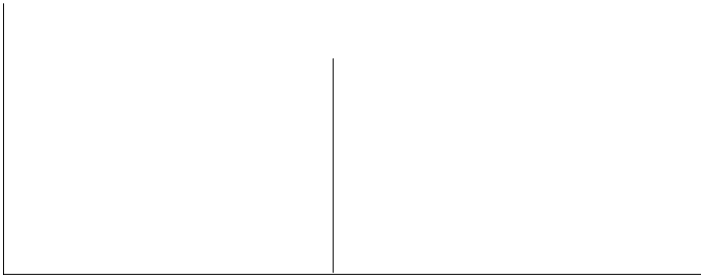
$$1.64 + 1.65 / 2 = 3.29 / 2 = 1.645.$$

Areas (or Probabilities) Under the Curve

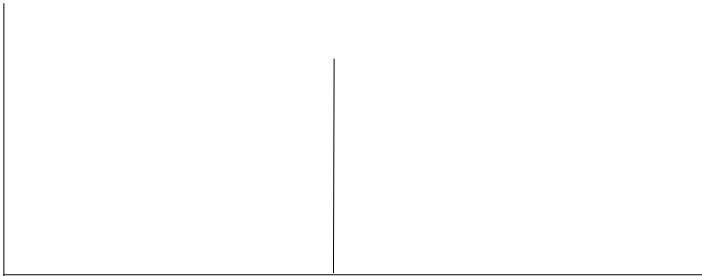
When given a z-score we can find out the area under the curve (between that score and the mean) . For example, for a z-score of 1.0 the area under the curve is .3413 or 34.13% of the whole. A picture of this concept is provided below:

Answers to the 4 questions that follow are below.

1. What is the area under the curve for a z-score of -1.0? (draw a picture)



2. What is the area under the curve between .5 and 1.2? (Hint: you have to use subtraction to solve this problem.)



3. Which z-score will have a greater area under the curve -- .2 or 1.5? (Hint: draw a picture)

4. Convert these “raw scores” to z-scores. For all problems, the mean = 5 and the standard deviation =10.

10, 20, 30, 40.

answers:

1. 34.13% 2. $.3849 - .1915 = .1934$ 3. 1.5 4. .5, 1.5, 2.5, 3.5

More practice z-score exercises (answers below):

Recall the additional practice problems [lecture 13b: practice problems \(13b_practice.pdf\)](#) linked in our course schedule.

- The scores on the SAT test are normally distributed and thus resemble the shape of the z table. What is probability that a SAT test score selected at random will fall within plus or minus 1 sd from the mean? How about for SAT test score of plus or minus two 2?
- We have a randomly selected raw score=50 mean=40 sd=10 what is the z score and what is area under the curve to right of this score?
- Random raw score 80 mean =50 sd=10 This test score is better than what percentage of all scores?
 - $x_1=80$ mean=60 sd=10 $x_2=40$ mean=45 sd=5 which score is better?
 - using the answers in 3a., what is the area under the curve between these two z scores?
- The area of .1700 lies under the curve of a NEGATIVE z score. What is that z score?
- The C&C of Honolulu regularly gives fitness tests to applicants to be lifeguards. They only want to take fittest 20% of all applicants. The scores are normally distributed with a mean = 120 and sd 10. What is the cutoff score for these applicants. $x = \mu + z \sigma$
- The scores on the SAT test are normally distributed with mean =500 and sd=100. Your professor in this class scored 500 in math and 450 in verbal. What are his z scores for each test?
- What percentage of people who took the SAT scored better than your professor on math? Conversely, what percentage of people did your professor score better than on math? What percentage of people who took the SAT scored better than your professor in verbal? Conversely, what percentage of people did your professor score better than on verbal?

Answers to practice z-score exercises

1. z of plus or minus 1 = 68% z of plus or minus 2 = 95.44%

2. ANSWER $z=1$ 15.87% area to the right ($100-84.13=15.87$) or another way to figure the answer out is $50-34.13=15.87$

3. test score the $z=3$ and it is better than 99.87% of all scores ($.4987+.5=99.87$)

3a. x_1 is better score because its $z=2$ and it's two SD above mean. x_2 has a z score of -1.

3b. $.4772+.3413=.8185$ or 81.85% [$Z_1=+2(.4772)$ $Z_2=-1(.3413)$.]

4. -.44

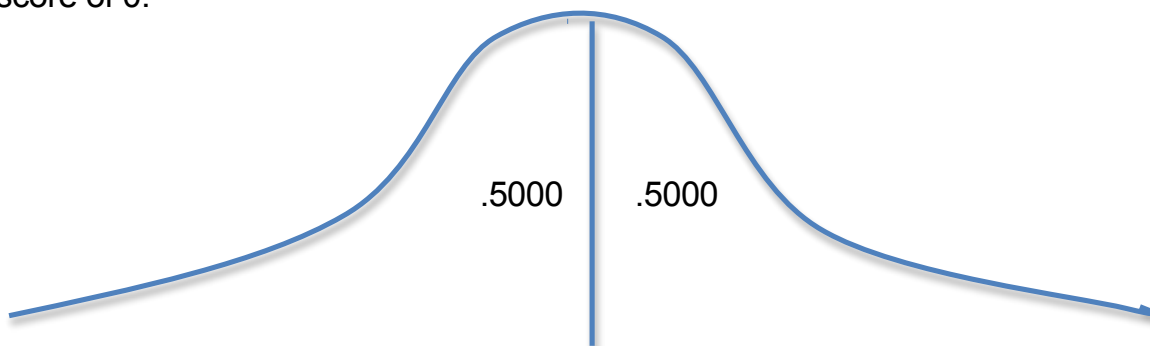
5. $x = \mu + z \sigma$

$x = 120 + .845 (10) = 128.45$

128.45 is the test score cut off

6. math: $\frac{500-500}{100} = \frac{0}{100} = 0$ verbal: $\frac{450-500}{100} = \frac{0-50}{100} = -.5$

7. math: professor scored worse than 50% and better than 50% of people who SAT math because his math z score = 0 and is right on the mean or the middle of bell curve. Look at z score of 0.



verbal: z-score of -0.5. Area from table = .1915 or 19.15%. So professor scored better than people to the left of that. $50\% - .1915 = 30.85\%$ He scored worse than 19.15% + the 50% on the right side of curve = $50 + 19.15 = 69.15\%$

